

PUISEUX SERIES DYNAMICS OF QUADRATIC RATIONAL MAPS

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ABSTRACT

We give a complete description for the dynamics of quadratic rational maps with coefficients in the completion of the field of formal Puiseux series.

Introduction

The field of formal Laurent series $\mathbb{C}((\tau))$ with complex coefficients is naturally endowed with a non-Archimedean valuation induced by the order of vanishing at $\tau = 0$ (see Section 1.1). In this paper we study iterations of quadratic rational maps with coefficients on a smallest algebraically closed and complete extension of $\mathbb{C}((\tau))$, which we denote by \mathbb{L} .

The results of this paper are in the context of the recent developments in the study of rational maps acting on the projective line over a non-Archimedean field (e.g., see [BR10] and references therein). The main motivation arises from the interplay established in [Kiw06] between dynamics over \mathbb{L} and the study of rational maps with complex coefficients acting on the Riemann sphere. Our motivation is reinforced with the recent results by De Marco and Faber [DeMFab] where they establish measure theoretic manifestations of this interplay.

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The main purpose of this paper is to give a full description for the dynamics of quadratic rational maps with coefficients in \mathbb{L} . The interplay with dynamics over \mathbb{C} is also manifested here. More precisely, our description of the geometry and topological dynamics of a large class of maps is based on the construction of a “model” for the dynamics which finds its origin in the study of the complex quadratic polynomial family $z \mapsto z^2 + c$.

The natural space to study the dynamics of a rational map $\varphi \in \mathbb{L}(z)$ is the Berkovich projective line $\mathbb{P}_{\mathbb{L}}^{1,an}$ which contains the projective line $\mathbb{P}_{\mathbb{L}}^1$ as a dense subset (see Section 1). In contrast with $\mathbb{P}_{\mathbb{L}}^1$, the Berkovich line $\mathbb{P}_{\mathbb{L}}^{1,an}$ is a compact and arcwise connected topological space. Moreover, $\mathbb{P}_{\mathbb{L}}^{1,an}$ is a non-metric tree. As in complex dynamics, the Berkovich line is divided into two complementary sets: the Fatou and Julia sets (see Section 2). The Julia set $J(\varphi)$ is either a singleton or uncountable. If $J(\varphi)$ is a singleton, then we say that φ is simple.

Our description of the dynamical properties of quadratic rational maps $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$ is organized according to the character and number of periodic orbits in $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$. We refer the reader to Section 2.2 for the basics about periodic points. A periodic point x in $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$ must be a topological branched point of $\mathbb{P}_{\mathbb{L}}^{1,an}$. In a certain sense, the action of φ^p on the branches growing from a period p point x is encoded by a complex rational map $T_x\varphi^p : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$.

THEOREM 1: *Let $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$ be a quadratic rational map which is not simple. Then one and exactly one of the following holds:*

- (1) $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1 = \emptyset$, that is, $J(\varphi) \subset \mathbb{P}_{\mathbb{L}}^1$.
- (2) $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$ is the grand orbit of an indifferent periodic orbit.
- (3) $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$ is the grand orbit of one repelling periodic orbit \mathcal{O} of period $q \geq 2$. For all $x \in \mathcal{O}$, the map $T_x\varphi^q : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ is a quadratic rational map with a multiple fixed point (i.e., a multiple root of $T_x\varphi^q(z) - z$).
- (4) $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$ is the grand orbit of two distinct periodic orbits $\mathcal{O}, \mathcal{O}'$ of periods $q, q' \geq 2$, respectively, where $q' > q$. The map $T_x\varphi^q : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ is a quadratic rational map with a multiple fixed point, for all $x \in \mathcal{O}$. The map $T_x\varphi^{q'} : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ is (modulo choice of coordinates) a quadratic polynomial, for all $x \in \mathcal{O}'$.

In [Kiw12] we use this result to establish a complete description of the so-called “rescaling limits” of complex quadratic rational maps. In fact, the repelling periodic orbit \mathcal{O} , in (3) and (4) above, is related to a limiting behavior

of complex quadratic rational maps, close to infinity in moduli space, studied by Stimson [Sti93], Epstein [Eps00] and De Marco [DeM07].

Next we describe the dynamics and geometry of Fatou set components. Given a periodic orbit \mathcal{O} , the set of points $x \in \mathbb{P}_{\mathbb{L}}^{1,an}$ such that $\varphi^n(x)$ converges to \mathcal{O} is called the basin of \mathcal{O} . According to Rivera-Letelier’s classification [RL03a], every fixed Fatou component U is either a component of the basin of an attracting fixed point $z \in \mathbb{P}_{\mathbb{L}}^1$, or U is the intersection of finitely many open balls where φ acts as a bijection. The above classification extends naturally to periodic Fatou components. In the latter case, we say that U is a “Rivera domain”. For a Rivera domain U , fixed under φ , which is not a ball, the convex hull \mathcal{A}_U of ∂U in $\mathbb{P}_{\mathbb{L}}^{1,an}$ may be regarded as an invariant finite simplicial tree. If \mathcal{A}_U contains at most one branched point and exactly one fixed point, then we say that U is a **starlike Rivera domain**. (See Section 2.3.)

It should be mentioned that, for dynamics over \mathbb{L} , the basin of a periodic orbit $\mathcal{O} \subset J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^{1,an}$ is always non-empty and contains uncountably many Fatou components, even if \mathcal{O} is a “repelling” periodic orbit.

THEOREM 2: *Let $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$ be a quadratic rational map which is not simple. If U is a Fatou component, then U is eventually periodic or U is a ball or an annulus contained in the basin of a periodic orbit $\mathcal{O} \subset J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$. Moreover, one and exactly one of the following holds:*

- (1) $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1 = \emptyset$ and the Fatou set consists of one totally invariant component which is the basin of an attracting fixed point $z \in \mathbb{P}_{\mathbb{L}}^1$.
- (2) $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$ is the grand orbit of an indifferent fixed point. Every periodic Fatou component is a fixed Rivera domain consisting of an open ball.
- (3) $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$ is the grand orbit of an indifferent periodic orbit of period at least 2. There exists a unique periodic Fatou component. This Fatou component is a starlike Rivera domain.
- (4) $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$ is the grand orbit of one or two repelling periodic orbits. There is exactly one fixed Fatou component. This fixed Fatou component is a starlike Rivera domain whose boundary is the repelling orbit of least period in $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$. Moreover, periodic Fatou components of higher periods are open balls.

Let us agree that a Fatou component U is called **wandering** if $\varphi^n(U) \neq \varphi^m(U)$ for all non-negative integers $n \neq m$ and U is not contained in the basin of a periodic orbit. According to Sullivan [Sul85], every Fatou component of a

complex rational map is eventually periodic. However, Benedetto [Ben02] has shown that wandering Fatou components do arise in polynomial dynamics over the p -adic field \mathbb{C}_p . Our previous theorem rules out the existence of wandering Fatou components for quadratic rational maps over \mathbb{L} . Hence, Theorem 2 can be regarded as an analogue, just for quadratic rational maps over \mathbb{L} , of Sullivan's result described above. For polynomials with coefficients in fields such as \mathbb{L} , Trucco [Tru09] has already ruled out the existence of wandering Fatou components. Thus, there is some supporting evidence for the following.

CONJECTURE: *Let L be a field endowed with a non-Archimedean absolute value which is complete with respect to the induced metric. Assume that the residue field of L has characteristic zero. Suppose that L contains a field F with discrete value group such that the elements of L which are algebraic over F are dense in L . If $\varphi \in L(\zeta)$ is a rational map of degree at least 2, then every Fatou component of φ is eventually periodic or is contained in the basin of a periodic orbit.*

Finally, we describe the dynamics over the Julia set. For quadratic rational maps with a repelling periodic orbit in $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$ we will construct a model for the dynamics over the convex hull $\widehat{J(\varphi)}$ of $J(\varphi)$. These models arise from "abstract α -laminations" as discussed in Section 6. An abstract α -lamination is an equivalence relation in \mathbb{R}/\mathbb{Z} closely related to the landing pattern of dynamical external rays of complex quadratic polynomials, and they are a slight generalization of the α -laminations defined by McMullen in [McM94]. From an abstract α -lamination λ we build an inverse system of finite simplicial trees whose limit $\mathcal{T}^\infty(\lambda)$ is naturally endowed with a dynamics $m_2 : \mathcal{T}^\infty(\lambda) \rightarrow \mathcal{T}^\infty(\lambda)$ inherited from multiplication by 2 acting on \mathbb{R}/\mathbb{Z} .

THEOREM 3: *Let $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$ be a quadratic rational map which is not simple. Then exactly one of the following holds:*

- (1) $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1 = \emptyset$ and $\varphi : J(\varphi) \rightarrow J(\varphi)$ is topologically conjugate to the full shift on two symbols.
- (2) $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$ contains an indifferent periodic orbit and $\varphi : J(\varphi) \rightarrow J(\varphi)$ is topologically conjugate to a subshift of finite type with topological entropy $\log 2$.
- (3) $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$ contains a repelling periodic orbit and the convex hull $\widehat{J(\varphi)}$ of $J(\varphi)$ is totally invariant under φ . Let U_0 be the unique fixed Rivera domain.

- (a) *If no critical point eventually maps into U_0 , then $\varphi : \widehat{J(\varphi)} \rightarrow \widehat{J(\varphi)}$ is topologically conjugate to the action of m_2 on the tree of an abstract α -lamination.*
- (b) *If a critical point eventually maps into U_0 , then $\varphi : \widehat{J(\varphi)} \rightarrow \widehat{J(\varphi)}$ is topologically semiconjugate to the action of m_2 on the tree of an α -lamination. The topological semiconjugacy restricts to a topological conjugacy on $J(\varphi)$.*

A more detailed description of the topological conjugacies mentioned in the above theorem is contained in Section 4 for cases (1) and (2) as well as in Section 7 for case (3).

In the complex setting, a complete description for the topological dynamics of quadratic rational maps seems difficult, with the available techniques, even if one restricts attention only to quadratic polynomials. After Yoccoz, satisfactory descriptions exist for complex quadratic polynomials which are either hyperbolic or non-renormalizable (e.g., see [Hub93]). Theorem 3 above (and its proof) says that, from the viewpoint of the Julia set over \mathbb{L} , the description of the analogue of complex quadratic renormalizable polynomials is much simpler and thus feasible with the available techniques. One should also regard Theorem 3 as a description of possible dynamical behavior of complex quadratic rational maps close to infinity in moduli space. In this sense, our results in part (3) are reminiscent of the so-called “Levy cycles” studied by Tan in [Tan92] and by Rees in [R03].

The fact that $\mathbb{C}((\tau)) \subset \mathbb{L}$ has discrete valuation group plays no significant role in the proof of our results. However, we rely heavily on the “tameness” of rational maps over \mathbb{L} . A rational map over a complete non-Archimedean field L is called **tame** if it is locally one-to-one in the complement of a topologically finite subtree of $\mathbb{P}_L^{1,an}$ (i.e., a topological tree with finitely many vertices and edges). For a tame quadratic rational map φ over a field with positive characteristic residue field, say a p -adic field \mathbb{C}_p , it would be interesting to understand if a complete description of its dynamics is within reach. It is worth pointing out that part of our analysis is based on an intermediate result which is not valid over \mathbb{C}_p , namely, that when a quadratic rational map over \mathbb{L} has a non-rigid repelling periodic orbit then a rigid critical point is attracted to a periodic orbit of this type (see (4) in Lemma 5.1). Here is where the fact that \mathbb{L} has a characteristic zero residue field is essential.

For higher degree rational maps over \mathbb{L} we expect that the class of bicritical rational maps (maps with two rigid critical points) can be studied with similar techniques but leading to slightly different descriptions. However, for maps with more than two critical points our techniques break down.

We briefly outline the organization of the paper.

In Section 1 we summarize some basic facts about the field \mathbb{L} and the Berkovich projective line $\mathbb{P}_{\mathbb{L}}^{1,an}$. Section 2 is devoted to the basics of iterations of rational maps on $\mathbb{P}_{\mathbb{L}}^{1,an}$.

In Section 3 we establish two preliminary results about quadratic rational maps: a first rough classification according to their Julia periodic orbits in $\mathbb{P}_{\mathbb{L}}^{1,an} \setminus \mathbb{P}_{\mathbb{L}}^1$ (Proposition 3.1), and the number and geometry of fixed Rivera domains is described in Proposition 3.5.

Section 4 contains a detailed description of the Julia and Fatou dynamics for maps φ without a repelling periodic orbit in $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$, namely, the case in which $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1 = \emptyset$ is covered by Proposition 4.1 and the case in which $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$ is the grand orbit of an indifferent periodic orbit is covered by Proposition 4.2.

Section 5 is devoted to the more difficult case of maps possessing a repelling periodic orbit in $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$. Here we study the geometry of these maps by introducing the “filled Julia set” and “dynamical pieces”. We establish Proposition 5.3 which is the last ingredient needed for the proofs of Theorems 1 and 2. These proofs are the content of Section 5.7.

In Section 6 we introduce and discuss the basic properties of abstract α -laminations. Then in Section 7 we discuss how these laminations are employed to describe the dynamics of quadratic rational maps possessing a repelling periodic orbit in $J(\varphi) \setminus \mathbb{P}_{\mathbb{L}}^1$. That is, we prove Propositions 7.1 and 7.3 which are the last ingredient needed for the proof of Theorem 3.

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1. Berkovich projective line over \mathbb{L}

In this section, after introducing the field \mathbb{L} , we summarize notations, definitions and facts related to the Berkovich projective line over \mathbb{L} . We must warn the reader that this is not a self-contained exposition. We refer to [BR10, Ber90, Duc07a, Duc07b, RL06a] and the references therein for a more detailed account on Berkovich spaces.

1.1. THE FIELD OF FORMAL PUISEUX SERIES AND EXTENSIONS. The field of **formal Laurent series** $\mathbb{C}((\tau))$ with coefficients in \mathbb{C} is naturally endowed with the valuation:

$$\left| \sum_{n \geq n_0} c_n \tau^n \right|_o = \exp(-\min\{n \in \mathbb{Z} \mid c_n \neq 0\}).$$

This valuation is non-Archimedean and restricts to the trivial valuation on \mathbb{C} . An algebraic closure of $\mathbb{C}((\tau))$ is the **field of formal Puiseux series** $\mathbb{C}\langle\langle\tau\rangle\rangle$ (e.g., see [CA00]). The field $\mathbb{C}\langle\langle\tau\rangle\rangle$ is the injective limit of the fields $\mathbb{C}((\tau^{1/m}))$, where $m \in \mathbb{N}$ and the injective limit is taken with respect to the obvious inclusions. The valuation $|\cdot|_o$ in $\mathbb{C}((\tau))$ extends uniquely to $\mathbb{C}\langle\langle\tau\rangle\rangle$ (e.g., see [Cas86]). However, $\mathbb{C}\langle\langle\tau\rangle\rangle$ endowed with this valuation is not complete. We let \mathbb{L} be the field obtained as the completion of $\mathbb{C}\langle\langle\tau\rangle\rangle$. It follows that \mathbb{L} is algebraically closed (e.g., see [Cas86]). Moreover, each element ζ of \mathbb{L} can be represented by a series of the form

$$\zeta = \sum_{n \geq 0} c_n \tau^{\lambda_n},$$

where $\lambda_n \in \mathbb{Q}$ and $\lambda_n \rightarrow +\infty$ as $n \rightarrow +\infty$. The value group $|\mathbb{L}^\times|_o$ of \mathbb{L} is $\exp(\mathbb{Q})$.

1.2. BALLS AND AFFINOIDS IN \mathbb{L} AND IN $\mathbb{P}_{\mathbb{L}}^1$. Basic geometric objects of \mathbb{L} and/or $\mathbb{P}_{\mathbb{L}}^1$, such as balls, annuli and affinoids, will be relevant to our discussion.

Given $r > 0$ and $\zeta \in \mathbb{L}$ we let

$$B_r^-(\zeta) = \{\xi \in \mathbb{L} \mid |\xi - \zeta|_o < r\}$$

and

$$B_r(\zeta) = \{\xi \in \mathbb{L} \mid |\xi - \zeta|_o \leq r\}.$$

If $r \in |\mathbb{L}^\times|_o$, we say that $B_r^-(\zeta)$ is an **open ball** in \mathbb{L} and $B_r(\zeta)$ is a **closed ball** in \mathbb{L} . If $r \notin |\mathbb{L}^\times|_o$, then $B_r^-(\zeta) = B_r(\zeta)$ is an **irrational ball** in \mathbb{L} . However,

every ball is both open and closed in the topology of \mathbb{L} . The closed unit ball $B_1(0)$ is a local ring $\mathfrak{O}_{\mathbb{L}}$, called the **ring of integers of \mathbb{L}** , whose maximal ideal is $\mathfrak{M}_{\mathbb{L}} = B_1^-(0)$. The **residue field $\mathfrak{O}_{\mathbb{L}}/\mathfrak{M}_{\mathbb{L}}$ of \mathbb{L}** , usually denoted $\tilde{\mathbb{L}}$, is canonically identified with \mathbb{C} .

Given two nested balls $B \subset B'$ of radii $0 < r < r'$, respectively, we say that $A = B' \setminus B$ is an **annulus** in \mathbb{L} with **modulus**

$$\text{mod } A = \log r' - \log r.$$

The modulus of an annulus is preserved under affine transformations.

As usual, we regard \mathbb{L} as a subset of $\mathbb{P}_{\mathbb{L}}^1$ via the identification of $\zeta \in \mathbb{L}$ with $[\zeta : 1] \in \mathbb{P}_{\mathbb{L}}^1$. Moreover, we let $\infty = [1 : 0]$. Thus, we may identify $\mathbb{P}_{\mathbb{L}}^1$ with $\mathbb{L} \cup \{\infty\}$.

By definition, a subset of $\mathbb{P}_{\mathbb{L}}^1$ is called an **open** (resp. **closed, irrational**) **ball** of $\mathbb{P}_{\mathbb{L}}^1$ if it is an open (resp. closed, irrational) ball contained in \mathbb{L} or its complement is a closed (resp. open, irrational) ball contained in \mathbb{L} .

A non-empty intersection A of closed balls of $\mathbb{P}_{\mathbb{L}}^1$ is called an **affinoid of $\mathbb{P}_{\mathbb{L}}^1$** . Projective transformations of $\mathbb{P}_{\mathbb{L}}^1$ map an open (resp. closed, irrational) ball onto a ball of the same type. Hence, the same holds for affinoids. A subset A' of $\mathbb{P}_{\mathbb{L}}^1$ which is the image under a projective transformation of an annulus A of \mathbb{L} is called an **annulus of $\mathbb{P}_{\mathbb{L}}^1$** and we let $\text{mod } A' = \text{mod } A$.

1.3. BERKOVICH PROJECTIVE LINE. The aim of this section is to agree on some terminology and notation regarding the Berkovich projective line $\mathbb{P}_{\mathbb{L}}^{1,an}$ over \mathbb{L} . The Berkovich projective line is a non-metric real tree (e.g., see Favre–Jonsson [FJ04]), and may be endowed with the strong or the weak topology. With respect to the weak topology, our default topology for $\mathbb{P}_{\mathbb{L}}^{1,an}$, the Berkovich projective line is compact, sequentially compact and Hausdorff but it is not metrizable (e.g., see [BR10, RL06a]).

1.3.1. *Points, balls, annuli, affinoids and basic open sets of $\mathbb{P}_{\mathbb{L}}^{1,an}$.* The classical projective line $\mathbb{P}_{\mathbb{L}}^1$ is a dense subset of $\mathbb{P}_{\mathbb{L}}^{1,an}$. Points of $\mathbb{P}_{\mathbb{L}}^1$ in the Berkovich projective line $\mathbb{P}_{\mathbb{L}}^{1,an}$ are called **type I**, or **rigid points**, or **classical points**. Rigid points are endpoints in the tree structure of $\mathbb{P}_{\mathbb{L}}^{1,an}$.

A set $B \subset \mathbb{P}_{\mathbb{L}}^{1,an}$ which is the closure in $\mathbb{P}_{\mathbb{L}}^{1,an}$ of a closed (resp. irrational) ball of $\mathbb{P}_{\mathbb{L}}^1$ is called a **closed** (resp. **closed irrational**) **ball of $\mathbb{P}_{\mathbb{L}}^{1,an}$** . Given a ball $B \subset \mathbb{P}_{\mathbb{L}}^{1,an}$ the topological boundary of B in $\mathbb{P}_{\mathbb{L}}^{1,an}$, denoted by ∂B , is a singleton $\{\xi\}$. We say that ξ is the **point associated to B** . Points associated

to closed balls are called **type II** or **rational points** and are ramification points in the tree structure. Several closed balls may have the same associated point. However, given a type II point ξ there exists a unique closed ball B such that ξ is the point associated to B and $B \cap \mathbb{P}_{\mathbb{L}}^1 \subset \mathbb{L}$. The point associated to $\mathfrak{D}_{\mathbb{L}}$ is called the **Gauss point**. Points associated to irrational balls are called **type III** or **irrational points** and are regular points in the tree structure. Every point in $\mathbb{P}_{\mathbb{L}}^{1,an}$ which is not of type I, II or III is an endpoint in the tree structure and it is called a **type IV** or **singular point**.

Other important geometric objects in $\mathbb{P}_{\mathbb{L}}^{1,an}$ are described as follows. An **open ball** of $\mathbb{P}_{\mathbb{L}}^{1,an}$ is the complement of a closed ball. A connected component of the complement of finitely many points is called a **basic open sets** of $\mathbb{P}_{\mathbb{L}}^{1,an}$. An **affinoid** of $\mathbb{P}_{\mathbb{L}}^{1,an}$ is a non-empty intersection of finitely many Berkovich closed balls.

Berkovich balls as well as affinoids and basic open sets are convex and connected. The collection of basic open sets forms a basis for the (weak) topology of $\mathbb{P}_{\mathbb{L}}^{1,an}$. The intersections of balls and affinoids of $\mathbb{P}_{\mathbb{L}}^{1,an}$ with the classical projective line $\mathbb{P}_{\mathbb{L}}^1$ are balls (of the same type) and affinoids of $\mathbb{P}_{\mathbb{L}}^1$, respectively. Given an affinoid or a basic open set U the convex hull of ∂U is a (finite) simplicial tree embedded in $\mathbb{P}_{\mathbb{L}}^{1,an}$ called the **skeleton** of U and denoted by \mathcal{A}_U . There is a canonical retraction $\pi_U : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathcal{A}_U$ that maps a point $x \notin \mathcal{A}_U$ to the boundary point of the connected component of $\mathbb{P}_{\mathbb{L}}^{1,an} \setminus \mathcal{A}_U$ which contains x .

An **annulus** A of $\mathbb{P}_{\mathbb{L}}^{1,an}$ is the intersection of two balls B, B' with distinct boundary points such that $B \cup B' = \mathbb{P}_{\mathbb{L}}^{1,an}$. For any annulus $A \subset \mathbb{P}_{\mathbb{L}}^{1,an}$ we have that $A \cap \mathbb{P}_{\mathbb{L}}^1$ is an annulus of $\mathbb{P}_{\mathbb{L}}^1$. The **modulus** of A is, by definition, $\text{mod}(A \cap \mathbb{P}_{\mathbb{L}}^1)$.

1.3.2. *Set of directions at points.* Following Favre and Jonsson [FJ04], given $\zeta \in \mathbb{P}_{\mathbb{L}}^{1,an}$ we say that the set $T_{\zeta} \mathbb{P}_{\mathbb{L}}^{1,an}$ formed by the connected components of $\mathbb{P}_{\mathbb{L}}^{1,an} \setminus \{\zeta\}$ is the **set of directions** at ζ . This definition coincides with the Favre–Jonsson tree theoretical definition and with the intuition that each “branch coming out” of ζ corresponds to a direction at ζ (cf. [BR10, Appendix B.6]).

At any given type I or IV point ζ there is only one direction, namely $\mathbb{P}_{\mathbb{L}}^{1,an} \setminus \{\zeta\}$. The directions at any given type II or III point ζ are in one-to-one correspondence with the open balls with boundary point ζ (equivalently, with associated point ζ). At a type III point ζ , the set of directions $T_{\zeta} \mathbb{P}_{\mathbb{L}}^{1,an}$ consists of two

directions which correspond to the two open irrational balls whose boundary is $\{\zeta\}$. At a type II point ζ , the set of directions is isomorphic to $\mathbb{P}_{\mathbb{C}}^1$. That is, there exists a natural bijection from $T_{\zeta}\mathbb{P}_{\mathbb{L}}^{1,an}$ onto $\mathbb{P}_{\mathbb{C}}^1$ which is unique up to postcomposition by a (complex) Möbius transformation.

For any $\zeta \in \mathbb{P}_{\mathbb{L}}^{1,an}$, we denote by $D_{\zeta} : \mathbb{P}_{\mathbb{L}}^{1,an} \setminus \{\zeta\} \rightarrow T_{\zeta}\mathbb{P}_{\mathbb{L}}^{1,an}$ the map which assigns to a point ξ its direction. When ζ is the Gauss point, the restriction of D_{ζ} to $\mathbb{P}_{\mathbb{L}}^1$ is the usual reduction. Thus, the direction at ζ containing a point $\xi \in \mathbb{P}_{\mathbb{L}}^{1,an} \setminus \{\zeta\}$ will be denoted by $D_{\zeta}(\xi)$. We will systematically abuse notation by simultaneously regarding a direction D at ζ as an element of $T_{\zeta}\mathbb{P}_{\mathbb{L}}^{1,an}$ and as a subset of $\mathbb{P}_{\mathbb{L}}^{1,an}$.

1.3.3. *Hyperbolic space.* We say that $\mathbb{H}_{\mathbb{L}} := \mathbb{P}_{\mathbb{L}}^{1,an} \setminus \mathbb{P}_{\mathbb{L}}^1$ is the **hyperbolic space over \mathbb{L}** . That is, the points in $\mathbb{H}_{\mathbb{L}}$ are of types II, III or IV. Hyperbolic space is endowed with a metric $\text{dist}_{\mathbb{H}_{\mathbb{L}}}$ which is defined as follows. Given two distinct type II or III points $\zeta_0, \zeta_1 \in \mathbb{H}_{\mathbb{L}}$,

$$A(\zeta_0, \zeta_1) = D_{\zeta_0}(\zeta_1) \cap D_{\zeta_1}(\zeta_0)$$

is an annulus. The **hyperbolic distance between ζ_0 and ζ_1** is defined by

$$\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\zeta_0, \zeta_1) = \text{mod } A(\zeta_0, \zeta_1).$$

The unique continuous extension of $\text{dist}_{\mathbb{H}_{\mathbb{L}}}$ to $\mathbb{H}_{\mathbb{L}} \times \mathbb{H}_{\mathbb{L}}$ endows hyperbolic space with a metric which we also denote by $\text{dist}_{\mathbb{H}_{\mathbb{L}}}$. With this metric $\mathbb{H}_{\mathbb{L}}$ is a \mathbb{R} -tree. The classical projective line $\mathbb{P}_{\mathbb{L}}^1$ is the boundary at infinity of $\mathbb{H}_{\mathbb{L}}$. For future reference we state and prove a simple fact.

LEMMA 1.1: *Let $\{X_{\ell}\}_{\ell \in \mathbb{N}}$ be a decreasing sequence of closed subsets of $\mathbb{P}_{\mathbb{L}}^{1,an}$ such that ∂X_{ℓ} is a singleton contained in $\mathbb{H}_{\mathbb{L}}$ for all $\ell \in \mathbb{N}$. If*

$$\sum_{\ell \in \mathbb{N}} \text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial X_{\ell}, \partial X_{\ell+1}) = \infty,$$

then

$$X = \bigcap_{\ell \in \mathbb{N}} X_{\ell}$$

is a singleton contained in $\mathbb{P}_{\mathbb{L}}^1$.

Proof. The compactness of $\mathbb{P}_{\mathbb{L}}^{1,an}$ implies that X is not empty. Without loss of generality assume that $\partial X_{\ell} \neq \partial X_{\ell+1}$ for all $\ell \in \mathbb{N}$. Given $\ell \in \mathbb{N}$, let $B_{\ell+1}$ be the ball which is the complement of the direction at $\partial X_{\ell+1}$ containing ∂X_{ℓ} . It follows that $\partial B_{\ell+1} = \partial X_{\ell+1}$ and $B_{\ell+1} \supset X_{\ell+1}$. Moreover, $Y = \bigcap_{\ell} B_{\ell+1}$

is contained in $\mathbb{P}_{\mathbb{L}}^1$. Otherwise, for any $\zeta \in Y \cap \mathbb{H}_{\mathbb{L}}$ we would have that $\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\zeta, \partial B_1) > \text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial X_{\ell+1}, \partial X_1) \rightarrow \infty$ as $\ell \rightarrow \infty$. Therefore, $Y \supset X$ is a singleton, since Y is connected and $\mathbb{P}_{\mathbb{L}}^1$ is totally disconnected. ■

1.4. ACTION OF RATIONAL MAPS ON $\mathbb{P}_{\mathbb{L}}^{1,an}$. The action of a rational map $\varphi \in \mathbb{L}(\zeta)$ on $\mathbb{P}_{\mathbb{L}}^1$ extends continuously to an action on $\mathbb{P}_{\mathbb{L}}^{1,an}$. The extended action, which we also denote by $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$, preserves the Berkovich type of the points. Moreover, it is an open map. Furthermore, at any given $\xi \in \mathbb{P}_{\mathbb{L}}^{1,an}$, the map $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$ has a well defined (local) degree $\text{deg}_{\xi} \varphi$. The number of preimages of any point $\zeta \in \mathbb{P}_{\mathbb{L}}^{1,an}$, under φ (counting multiplicities), coincides with the degree of $\varphi : \mathbb{P}_{\mathbb{L}}^1 \rightarrow \mathbb{P}_{\mathbb{L}}^1$. (See [BR10, sections 2.3, 2.4, 9.1].)

Also, for any rational map $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$, the preimage of an affinoid (resp. basic open set) A is the finite union of affinoids (resp. basic open sets) A_1, \dots, A_k and $\varphi : A_j \rightarrow A$ has a well defined degree d_j for all j .

1.4.1. *Möbius transformations.* From the above, it follows that the group of linear fractional transformations $\text{PSL}(2, \mathbb{L})$ acts on $\mathbb{P}_{\mathbb{L}}^{1,an}$. The stabilizer of the Gauss point is $\text{PSL}(2, \mathfrak{O}_{\mathbb{L}})$. The action of $\text{PSL}(2, \mathbb{L})$ is transitive on type II points (see [BR10, Section 2.3]).

1.4.2. *The action on the space of directions.* A rational map $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$ also acts on the set of directions. More precisely, according to Rivera [RL03a] (compare with [BR10, Section 9.3]), given a direction D at a type II or III point ζ the following hold:

- (1) There exists a closed ball $B \subset D$ such that $\varphi(D \setminus B_0)$ is an annulus for all closed balls B_0 for which $D \supset B_0 \supset B$.
- (2) There exists $k \in \mathbb{N}$ such that $\varphi : D \setminus B \rightarrow \varphi(D \setminus B)$ is k -to-one.
- (3) $\varphi(D \setminus B)$ is contained in one direction D' at $\varphi(\zeta)$. Moreover, $D' \setminus \varphi(D \setminus B)$ is a closed ball.
- (4) $\varphi(D) = D'$ or $\varphi(D) = \mathbb{P}_{\mathbb{L}}^{1,an}$. In the former case we say that D is a **good direction**, in the latter we say that D is a **bad direction**.
- (5) For all $\xi \in D \setminus B_0$,

$$\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\varphi(\zeta), \varphi(\xi)) = k \cdot \text{dist}_{\mathbb{H}_{\mathbb{L}}}(\zeta, \xi).$$

It follows that $T_{\zeta} \varphi : D \mapsto D'$ defines the **tangent map**

$$T_{\zeta} \varphi : T_{\zeta} \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow T_{\varphi(\zeta)} \mathbb{P}_{\mathbb{L}}^{1,an}.$$

We say that k is **the degree of $T_\zeta\varphi$ at D** , and denote this number by $\deg_D T_\zeta\varphi$.

- (6) There exists a number $n \geq 0$ such that any point in $\mathbb{P}_\mathbb{L}^{1,an} \setminus T_\zeta\varphi(D)$ has n preimages (counting multiplicities) in D and any point in $T_\zeta\varphi(D)$ has $n + \deg_D T_\zeta\varphi$ preimages (counting multiplicities) in D . Following Faber [Fab11], n is called the **surplus multiplicity**.
- (7) At a type III point ζ , the tangent map $T_\zeta\varphi$ is a bijection. The degree of $T_\zeta\varphi$ is the same in both directions and coincides with the degree of φ at ζ .
- (8) At a type II point ζ , the tangent map $T_\zeta\varphi : T_\zeta\mathbb{P}_\mathbb{L}^{1,an} \rightarrow T_{\varphi(\zeta)}\mathbb{P}_\mathbb{L}^{1,an}$ is a rational map (in the corresponding $\mathbb{P}_\mathbb{C}^1$ -structures) of degree at least 1. The degree of $T_\zeta\varphi$ at D as a rational map over \mathbb{C} and the degree of $T_\zeta\varphi$ in the direction D (as defined above) coincide. Moreover, the local degree of φ at ζ coincides with the degree of $T_\zeta\varphi$ as a rational map over \mathbb{C} .

1.4.3. *Counting critical points.* Since the residue field of \mathbb{L} has characteristic zero, the degree of a map on a ball is closely related to the number of critical points in it. (Compare with [BR10, Appendix A.10] and [Fab11], [Fab12])

LEMMA 1.2: *Let $B \subset \mathbb{P}_\mathbb{L}^{1,an}$ be a closed ball and φ a rational map in $\mathbb{L}(\zeta)$. If $\varphi(B)$ is a ball, then the number of (rigid) critical points of φ in $B \cap \mathbb{P}_\mathbb{L}^1$ is (counting multiplicities) $\deg_{\partial B} \varphi - 1$.*

Proof. Note that by the above discussion $\deg_{\partial B}$ coincides with the degree of $\varphi : B \rightarrow \varphi(B)$. We may restrict to the case in which $B \cap \mathbb{P}_\mathbb{L}^1$ and $\varphi(B) \cap \mathbb{P}_\mathbb{L}^1$ are balls which contain the origin in \mathbb{L} . It follows that the Newton polygon (e.g., see [Cas86]) of $d\varphi/d\zeta$ is obtained from that of φ by translation towards the left. Thus, subtracting one from the number of zeros of φ in $B \cap \mathbb{P}_\mathbb{L}^1$ we obtain the number of zeros of $d\varphi/d\zeta$ in φ in $B \cap \mathbb{P}_\mathbb{L}^1$. ■

2. Rational dynamics over the Berkovich projective line

2.1. THE JULIA AND FATOU SETS. Let $\varphi : \mathbb{P}_\mathbb{L}^{1,an} \rightarrow \mathbb{P}_\mathbb{L}^{1,an}$ be a rational map of degree $\deg \varphi \geq 2$. A point $\zeta \in \mathbb{P}_\mathbb{L}^{1,an}$ belongs to the **Julia set** $J(\varphi)$ if for all neighborhoods U of ζ we have that $\cup \varphi^n(U)$ omits at most two points. The complement of the Julia set is the **Fatou set** $F(\varphi)$ (see [BR10, Section 10.5]).

These definitions agree with the original ones by Hsia. More precisely, $\zeta \in \mathbb{P}_{\mathbb{L}}^1$ lies in the Julia set defined by Hsia [Hsi00] if and only if $\zeta \in J(\varphi) \cap \mathbb{P}_{\mathbb{L}}^1$.

2.2. PERIODIC POINTS. Given a rational map $\varphi \in \mathbb{L}(\zeta)$ we say that $\zeta_0 \in \mathbb{P}_{\mathbb{L}}^{1,an}$ is **periodic** if $\varphi^n(\zeta_0) = \zeta_0$ for some $n \geq 1$. The minimal n such that $\varphi^n(\zeta_0) = \zeta_0$ is called the **period** of ζ_0 .

When $\zeta_0 \in \mathbb{P}_{\mathbb{L}}^1$ is a period p rigid point, the **multiplier** of ζ_0 is, by definition,

$$\lambda = \frac{d\varphi^p}{d\zeta}(\zeta_0) \in \mathbb{L}.$$

As usual, the periodic point ζ_0 is called **attracting** (resp. **neutral** or **indifferent**, **repelling**) according to whether the multiplier λ belongs to $\mathfrak{M}_{\mathbb{L}}$ (resp. $\mathfrak{D}_{\mathbb{L}} \setminus \mathfrak{M}_{\mathbb{L}}$, $\mathbb{L} \setminus \mathfrak{D}_{\mathbb{L}}$). Type I repelling periodic points belong to the Julia set and type I non-repelling periodic points belong to the Fatou set. According to Benedetto [Ben98], every rational map $\varphi \in \mathbb{L}(\zeta)$ has at least one non-repelling fixed point in $\mathbb{P}_{\mathbb{L}}^1$.

Following Rivera-Letelier, a non-rigid periodic point $\zeta \in \mathbb{P}_{\mathbb{L}}^{1,an}$ of period p is called **repelling** if the local degree of φ^p at ζ is at least 2. Otherwise, it is called **neutral** or **indifferent**. Non-rigid repelling periodic points must be type II points. Also, non-rigid periodic points which belong to the Julia set must be type II points (see [RL03b]). The following result is the content of a personal communication with Juan Rivera-Letelier [RL06b].

THEOREM 2.1 (Rivera-Letelier): *Let ζ be a non-rigid period p periodic point of a rational map $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$ of degree at least 2. The point ζ belongs to $J(\varphi)$ if and only if one of the following hold:*

- (1) $\deg_{\zeta} \varphi^p \geq 2$.
- (2) $\deg_{\zeta} \varphi^p = 1$ and there exists a bad direction at ζ with infinite forward orbit under $T_{\zeta} \varphi^p$.

Proof. Since every non-rigid repelling periodic point belongs to $J(\varphi)$ (see [BR10, Section 10.7]), we may assume that $p = 1$ and $\deg_{\zeta} \varphi = 1$. Note that $T_{\zeta} \varphi$ is a bijection. Thus a bad direction D has infinite forward orbit under $T_{\zeta} \varphi$ if and only if D has infinite backward orbit ($\{D' \in \mathbb{P}_{\mathbb{C}}^1 \mid T_{\zeta} \varphi^n(D') = D \text{ for some } n \geq 0\}$).

In the case that there exists a bad direction D at ζ which has infinite backward orbit, every neighborhood U of ζ contains a direction D' in the backward orbit of a bad direction. Hence, there exists $n \geq 0$ such that $\mathbb{P}_{\mathbb{L}}^{1,an} = \varphi^n(D') \subset \varphi^n(U)$. Therefore $\zeta \in J(\varphi)$.

Now we consider the case in which every bad direction has finite backward orbit. That is, every bad direction is periodic. Thus, after removing appropriate closed balls in the orbit of the bad directions, we obtain a neighborhood U of ζ which is invariant under φ (i.e., $\varphi(U) = U$). Hence, $\zeta \in F(\varphi)$. ■

It follows that non-rigid indifferent periodic points may belong to the Julia set or the Fatou set. However, indifferent periodic points in the Julia set must be of type II.

The **basin of a periodic orbit** \mathcal{O} is the interior of the set of points ξ such that the omega limit of ξ is \mathcal{O} . Non-rigid periodic orbits contained in the Julia set always have non-empty basin. In fact, given a point ζ of type II there are uncountably many directions D at ζ such that its image under $T_\zeta\varphi^n$ is a good direction, for all $n \geq 0$, since there are only finitely many bad directions. If ζ belongs to a periodic orbit \mathcal{O} , say of period p , which lies in $J(\varphi)$, then for uncountably many directions D we have that $\{\varphi^{np}(D)\}$ are pairwise distinct directions at ζ . Thus, each of these directions is a component of the Fatou set contained in the basin of \mathcal{O} .

2.3. PERIODIC FATOU COMPONENTS. A connected component U of the Fatou set $F(\varphi) \subset \mathbb{P}_{\mathbb{L}}^{1,an}$ is called a **Fatou component**. Since φ is an open map, the image of a Fatou component is again a Fatou component.

If a Fatou component U contains an attracting periodic point ξ , then U is a component of the basin of attraction of the orbit of ξ . In this case, U is called the **immediate basin of ξ** .

We say that a period p Fatou component U is a **Rivera domain** if $\varphi^p : U \rightarrow U$ is a bijection.

THEOREM 2.2 (Rivera [RL03a]): *Let $\varphi \in \mathbb{L}(\zeta)$ be a rational map of degree at least 2. Then a periodic Fatou component is either the immediate basin of an attracting periodic point or a Rivera domain.*

THEOREM 2.3 (Rivera [RL03a]): *Let $\varphi \in \mathbb{L}(\zeta)$ be a rational map of degree at least 2. If U is a Rivera domain, then U is a basic open set and every point in ∂U is a periodic point in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$.*

If U is a component of an immediate basin of an attracting periodic orbit, then U is an open ball or ∂U is a Cantor set.

It follows that, if U is a Rivera domain, then the retraction $\pi_U : U \rightarrow \mathcal{A}_U$ commutes with φ (see Section 1.3.1 for the definition of π_U and \mathcal{A}_U). In particular, $\varphi : \mathcal{A}_U \rightarrow \mathcal{A}_U$ has finite order.

Let $\varphi \in \mathbb{L}(\zeta)$ be a degree $d \geq 2$ rational map. Assume that U is a fixed Rivera domain for φ . Let $\xi \in \partial U$ and denote by D the direction at ξ that contains U . We let $\eta(\xi) \geq 0$ be the multiplicity of D as a fixed point of $T_\xi \varphi$.

THEOREM 2.4 (Rivera [RL03a]): *Let $\varphi \in \mathbb{L}(\zeta)$ be a degree $d \geq 2$ rational map. Assume that U is a fixed Rivera domain for φ . Let $N \geq 0$ be the number of fixed points in $U \cap \mathbb{P}_{\mathbb{L}}^1$. Then*

$$N = 2 + \sum_{\xi \in \partial U, \varphi(\xi) = \xi} (\eta(\xi) - 2).$$

Although the above results were originally proved in the context of p -adic dynamics, the techniques apply, with some modifications, to the context of dynamics over \mathbb{L} .

3. Julia periodic orbits in $\mathbb{H}_{\mathbb{L}}$ and fixed Rivera domains

The aim of this section is to give a rough classification of quadratic rational maps in $\mathbb{L}(\zeta)$ according to their Julia non-rigid periodic orbits and to describe the geometry of fixed Rivera domains.

3.1. JULIA PERIODIC ORBITS IN $\mathbb{H}_{\mathbb{L}}$. As a first step towards proving Theorem 1 we study periodic orbits in $\mathbb{H}_{\mathbb{L}}$.

PROPOSITION 3.1: *Let $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$ be a quadratic rational map which is not simple. Then one and only one of the following holds:*

- (1) *There are no periodic points in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$.*
- (2) *There exists exactly one indifferent periodic orbit \mathcal{O} in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$ and \mathcal{O} is the unique periodic orbit in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$.*
- (3) *There is at least one repelling periodic orbit in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$.*

We state and establish two necessary lemmas before proving this proposition.

Recall that a **critical point** of a rational map $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$ is a rigid point $\zeta \in \mathbb{P}_{\mathbb{L}}^1$ where the derivative of $\varphi : \mathbb{P}_{\mathbb{L}}^1 \rightarrow \mathbb{P}_{\mathbb{L}}^1$ vanishes. A **critical value** is the image of a critical point under φ .

LEMMA 3.2: *Let $\varphi \in \mathbb{L}(\zeta)$ be a quadratic rational map. Suppose that $\xi_0 \in \mathbb{H}_{\mathbb{L}}$ and $\varphi(\xi_0) = \xi_1$. Then the following are equivalent:*

- (1) $\deg T_{\xi_0} \varphi = 1$.
- (2) *The critical points of φ belong to the same direction D_0 in $T_{\xi_0} \mathbb{P}_{\mathbb{L}}^{1,an}$.*
- (3) *The critical values of φ belong to the same direction D_1 in $T_{\xi_1} \mathbb{P}_{\mathbb{L}}^{1,an}$.*

If the above holds, then $T_{\xi_0} \varphi(D_0) = D_1$ and D_0 is a bad direction.

Proof. The lemma holds trivially when ξ_0 is a type IV point. Thus, suppose that ξ_0 is a type II or III point.

(1) \implies (2). Assume that $\deg T_{\xi_0} \varphi = 1$. Counting preimages of a typical point with the aid of Section 1.4.2 it follows that there exists one and only one bad direction at ξ_0 , say D_0 . Since good directions are mapped injectively onto their image, the critical points of φ must belong to D_0 .

(2) \implies (3). Now assume that the two critical points belong to the direction D_0 at ξ_0 . In view of Section 1.4.2, if $\zeta \notin D_1 = T_{\xi_0} \varphi(D_0)$, then ζ has at most one preimage (counting multiplicities) in D_0 . In particular, ζ is not a critical value.

(3) \implies (1). Assume that both critical values lie in the same direction D_1 at ξ_1 and proceed by contradiction. If $\deg T_{\xi_0} \varphi = 2$, then every direction at ξ_0 is a good direction. Moreover, there are exactly two directions D, D' of degree two under $T_{\xi_0} \varphi$. By Lemma 1.2, each must contain a critical point. Then both D and D' are mapped under φ onto D_1 with degree 2. This is clearly impossible since points in D_1 do not have 4 preimages. ■

Remark 3.3: Suppose that there exists an open ball D contained in the Fatou set that contains both critical values. Then $\deg T_{\xi} \varphi = 1$, for all $\xi \in J(\varphi)$.

LEMMA 3.4: *Let $\varphi \in \mathbb{L}(\zeta)$ be a quadratic rational map. If φ has an indifferent periodic orbit $\mathcal{O} \subset J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$, then both critical values of φ belong to the same Fatou component D which is an open ball contained in the basin of \mathcal{O} . Moreover, $\partial D \subset \mathcal{O}$.*

Proof. Let ξ_0 be a point of the periodic orbit \mathcal{O} . Say that the period of \mathcal{O} is p . Consider a bad direction D_0 at ξ_0 for φ^p with infinite forward orbit under $T_{\xi_0} \varphi^p$ (see Theorem 2.1). By Lemma 3.2, there is exactly one bad direction at $\varphi^n(\xi_0)$, for all $n \geq 0$. This guarantees the existence of a smallest integer $k \geq 1$ such that $T_{\xi_0} \varphi^{n+k}(D_0)$ is a good direction for all $n \geq 0$. Again by Lemma 3.2,

$D = T_{\xi_0} \varphi^k(D_0)$ contains both critical values. Therefore, $\{\varphi^{np}(D)\}_{n \geq 0}$ is a collection of pairwise distinct directions at $\varphi^k(\xi_0)$. Hence, for all $\zeta \in D$, the forward orbit of ζ under φ^p converges to $\varphi^k(\xi_0)$. It follows that D is a Fatou component contained in the basin of \mathcal{O} . Moreover, $\partial D \subset \mathcal{O}$. ■

Proof of Proposition 3.1. Suppose that (1) does not hold. The presence of an indifferent periodic orbit in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$ implies that all the periodic orbits in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$ are indifferent, by Remark 3.3 and Lemma 3.4. That is, (3) does not hold. Moreover, also by Lemma 3.4, there is at most one indifferent periodic orbit in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$. ■

3.2. STARLIKE RIVERA DOMAIN. Here we establish that a fixed Rivera domain U of a quadratic rational map φ is an open ball or a starlike domain. Recall that, for us, a starlike Rivera domain U is the intersection of finitely many open balls such that its skeleton \mathcal{A}_U contains at most one topological branched point and a unique fixed point. In this case, the action of $\varphi : \mathcal{A}_U \rightarrow \mathcal{A}_U$ is a “rotation” around this fixed point. Note that an annulus U such that φ exchanges the endpoints of \mathcal{A}_U is a starlike domain.

PROPOSITION 3.5: *Let $\varphi \in \mathbb{L}(\zeta)$ be a quadratic rational map which is not simple. Then φ has at most two fixed Rivera domains.*

If φ has exactly one fixed Rivera domain U , then U is an open ball or a starlike domain. Moreover, the points in ∂U form one periodic orbit.

If φ has two fixed Rivera domains, then both are open balls which have the same boundary point.

Proof. Assume that a fixed Rivera domain U is an open ball. Then $\{\xi\} = \partial U$ is a fixed point. The degree of φ at ξ must be 1, for otherwise φ would be simple. Since a Möbius transformation $T_{\xi} \varphi : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ has at most two fixed points, ξ is in the boundary of at most two fixed Rivera domains. In view of Proposition 3.1 the boundary of any fixed Rivera domain must be $\{\xi\}$, since $\{\xi\}$ is the unique periodic orbit of the Julia set in $\mathbb{H}_{\mathbb{L}}$. Thus, if there is another fixed Rivera domain, then it is also an open ball.

If a fixed Rivera domain U is not an open ball, then ∂U consists of at least two points. By Proposition 3.1, if there is more than one orbit in ∂U , then all such orbits must be repelling. Note that there is exactly one component B of $\mathbb{P}_{\mathbb{L}}^{1,an} \setminus U$ that contains both critical points. If a periodic orbit in ∂U is repelling,

then this orbit must contain a point of B . But $\partial U \cap B$ is a singleton. Hence, there is only one such orbit.

We now show that a fixed Rivera domain U which is not an open ball has a starlike structure. That is, we let \mathcal{A}_U be the convex hull of ∂U and proceed to show that it contains at most one branched point and at most one fixed point ϑ . Observe that $\varphi : \mathcal{A}_U \rightarrow \mathcal{A}_U$ is a tree isomorphism and an isometry in the hyperbolic metric of $\mathbb{H}_{\mathbb{L}}$.

Let $q \geq 2$ be the period of the periodic orbit $\mathcal{O} = \partial U$. The domain U contains no rigid periodic point of period k such that $1 < k < q$. In fact, applying Theorem 2.4 for $1 \leq k < q - 1$, the number of fixed points of φ^k in U is $N = 2 + 0$. Hence, there are 2 fixed points of φ in U and every fixed point of φ^k in U is a fixed point of φ , for $2, \dots, q - 1$.

For us the vertices of \mathcal{A}_U consist of the topological branched points together with the endpoints of \mathcal{A}_U . We claim that every vertex of \mathcal{A}_U has period 1 or q . In fact, we proceed by contradiction and let v be a vertex of period k with $1 < k < q$. Endow the vertices of \mathcal{A}_U with the partial order inherited from the tree structure where v is a minimal element (i.e., the root). Let w be a maximal element among the vertices of \mathcal{A}_U with period k . It follows that φ^k fixes at most one branch of \mathcal{A}_U at w . Since w is not an endpoint of \mathcal{A}_U , we have that $T_w \varphi^k$ must have finite order strictly greater than 1 in order to permute the directions of the at least two branches at w which are not fixed by φ^k . That is, $T_w \varphi^k$ is conjugate to $z \mapsto \eta z$ where $\eta \neq 1$ is a root of unity. Hence, there exists a direction D at w , which is disjoint from \mathcal{A}_U , that is fixed under φ^k . Without loss of generality, assume that $D \cap \mathbb{P}_{\mathbb{L}}^1$ is the open unit ball. In $D \cap \mathbb{P}_{\mathbb{L}}^1$, we have that $\varphi^k(\zeta) = a_0 + a_1 \zeta + h.o.t$ where $|a_1 - \eta|_o < 1$ and $|a_0|_o < 1$. Therefore, $\varphi^k(\zeta) - \zeta$ has exactly one solution in $D \cap \mathbb{P}_{\mathbb{L}}^1$ which is a contradiction with the last sentence of the previous paragraph.

Finally, consider a point $\xi \in \partial U$ and let v be a fixed point of φ in \mathcal{A}_U such that $(v, \xi]$ contains no other fixed point. Let $k \geq 1$ be the smallest integer such that $\varphi^k((v, \xi]) \cap (v, \xi] \neq \emptyset$. Then $k \geq 2$ since $(v, \xi]$ is fixed point free. In fact $k = q$, for otherwise, there is a vertex in \mathcal{A}_U of period $1 < k < q$. It follows that \mathcal{A}_U is either an interval (and ξ has period 2) or a tree with v as the unique vertex which is not an endpoint. ■

4. Maps without a repelling periodic orbit in $\mathbb{H}_{\mathbb{L}}$

The description of the dynamics in the first two cases of Proposition 3.1 is rather simple. In Section 4.1 we deal with maps not having Julia periodic orbits in $\mathbb{H}_{\mathbb{L}}$ and, in Section 4.2, we consider maps with a Julia indifferent periodic orbit.

4.1. NOT SIMPLE WITH ATTRACTING FIXED POINT. Let us first state what occurs in the absence of Julia periodic orbits in $\mathbb{H}_{\mathbb{L}}$.

PROPOSITION 4.1: *Let φ be a quadratic rational map which is not simple. Then the following are equivalent:*

- (1) $J(\varphi) \subset \mathbb{P}_{\mathbb{L}}^1$.
- (2) φ has an attracting fixed point $\zeta \in \mathbb{P}_{\mathbb{L}}^1$.
- (3) There are no periodic orbits of φ in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$.

Moreover, if (1)–(3) hold, then the dynamics over $J(\varphi)$ is topologically conjugate to the one-sided shift on two symbols and $F(\varphi)$ is the immediate basin of attraction of ζ .

Proof. First we show, by contradiction, that (3) implies (2). If there is no attracting fixed point in $\mathbb{P}_{\mathbb{L}}^1$, then there exists an indifferent fixed point ζ' in $\mathbb{P}_{\mathbb{L}}^1$ (see [Ben98]). Therefore, ζ' belongs to a Rivera domain $U_{\zeta'}$ whose boundary, according to Theorem 2.3, contains a periodic orbit in $\mathbb{H}_{\mathbb{L}} \cap J(\varphi)$.

Now assume that (2) holds: φ has an attracting fixed point $\zeta \in \mathbb{P}_{\mathbb{L}}^1$. We simultaneously show that (1) holds ($J(\varphi) \subset \mathbb{P}_{\mathbb{L}}^1$) and that the dynamics is topologically equivalent to the one-sided shift on two symbols.

Let B be the union of all open balls B' containing ζ such that $\varphi(B') \subset B'$. It follows that B is a ball such that $\varphi(B) \subset B$. Now let ξ be the point associated to B . Since $D_{\xi}(\zeta)$ is contained in the basin of ζ , we have that $\varphi(D_{\xi}(\zeta)) = D_{\varphi(\xi)}(\zeta)$. By Schwartz Lemma, ξ is not a fixed point, for otherwise, $\varphi : D_{\xi}(\zeta) \rightarrow D_{\xi}(\zeta)$ would have degree 2 and we would also have that $\deg T_{\xi}\varphi = 2$, but φ is not simple. Hence ξ is contained in a direction D at $\varphi(\xi)$.

We claim that $\varphi^{-1}(D)$ is the disjoint union of two open balls D_0, D_1 , each one compactly contained in D , each of which is mapped onto D in a one-to-one fashion. In fact, given a direction D' at ξ such that $T_{\xi}\varphi(D') = D$ we have that $\deg_{D'} T_{\xi}\varphi = 1$ and $\varphi(D') = D$; for otherwise, $\varphi(\mathbb{P}_{\mathbb{L}}^{1,an} \setminus D') \subset \mathbb{P}_{\mathbb{L}}^{1,an} \setminus D'$ which, by Section 1.4.2, contradicts the definition of B . Therefore $\varphi^{-1}(D)$ is

the disjoint union of two open balls D_0, D_1 , compactly contained in D , and each of them is mapped onto D in a one-to-one fashion.

Now we establish the conjugacy between $J(\varphi)$ and the full shift on two symbols. Observe that $J(\varphi) \subset D_0 \cup D_1$. Given $\varsigma \in J(\varphi)$, let $i_n(\varsigma) = 0$ or 1 if $\varphi^n(\varsigma) \in D_0$ or D_1 , respectively. It follows that the itinerary map

$$J(\varphi) \ni \varsigma \mapsto (i_n(\varsigma))_{n \geq 0} \in \{0, 1\}^{\mathbb{N} \cup \{0\}}$$

is continuous. To check that the itinerary map is one-to-one and onto, given an infinite symbol sequence $\mathbf{i} = (i_0, i_1, \dots)$, for each $n \geq 0$, let

$$D_n(\mathbf{i}) = \{\varsigma \mid \varphi^k(\varsigma) \text{ for all } 0 \leq k \leq n\}.$$

Now φ^{n+1} maps the annulus $D_n(\mathbf{i}) \setminus D_{n+1}(\mathbf{i})$ isomorphically onto $D \setminus D_{i_{n+1}}$, therefore

$$\text{dist}_{\mathbb{H}}(\partial D_n(\mathbf{i}), \partial D_{n+1}(\mathbf{i})) = \text{dist}(\partial D, \partial D_{i_{n+1}}).$$

From Lemma 1.1 we conclude that the intersection of the nested sequence $D_0(\mathbf{i}) \supset D_1(\mathbf{i}) \supset \dots$ is a singleton contained in $\mathbb{P}_{\mathbb{L}}^1$. It follows that the itinerary map is a conjugacy and that $J(\varphi) \subset \mathbb{P}_{\mathbb{L}}^1$. Moreover, $F(\varphi) = \mathbb{P}_{\mathbb{L}}^{1,an} \setminus J(\varphi)$ is connected and invariant. Therefore, $F(\varphi)$ is the immediate basin of ζ .

Since (1) implies (3) is obvious, we have proven the proposition. ■

4.2. INDIFFERENT PERIODIC ORBIT IN $\mathbb{H}_{\mathbb{L}}$. Now we describe the situation in the presence of an indifferent orbit in the Julia set.

Before stating our result let us denote the **grand orbit** of a periodic orbit \mathcal{O} by $GO(\mathcal{O})$. Also recall that the **renewal shift** is the (non-compact) subshift of the full shift on two symbols obtained after removing the grand orbit of a fixed point from the full shift.

PROPOSITION 4.2: *Suppose that φ has an indifferent periodic orbit \mathcal{O} of period $q \geq 1$ in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$. Then the following statements hold:*

- (1) $J(\varphi) \cap \mathbb{P}_{\mathbb{L}}^1$ is topologically conjugate to the renewal shift.
- (2) $J(\varphi) \cap \mathbb{H}_{\mathbb{L}} = GO(\mathcal{O})$.
- (3) The dynamics over $J(\varphi)$ is topologically conjugate to the subshift Σ_q of finite type in $2q$ symbols $\{X_0, \dots, X_{q-1}, Y_0, \dots, Y_{q-1}\}$ (subscripts mod q)

with the following allowed transitions:

$$\begin{aligned} X_j &\rightarrow X_{j+1}, && \text{for } j = 0, \dots, q-1. \\ X_{q-1} &\rightarrow Y_j, && \text{for } j = 0, \dots, q-1. \\ Y_j &\rightarrow X_j, && \text{for } j = 0, \dots, q-1. \\ Y_0 &\rightarrow Y_j, && \text{for } j = 0, \dots, q-1. \end{aligned}$$

Under the topological conjugacy, \mathcal{O} corresponds to the orbit of $(X_0X_1\dots X_{q-1})^\infty$.

- (4) There exists at least one fixed Rivera domain. The boundary of every Rivera domain is \mathcal{O} . If U is a periodic Fatou component, then U is a fixed Rivera domain.
- (5) If U is a Fatou component which is not eventually periodic, then U is an annulus or an open ball. Moreover, there exists $n_0 \geq 0$ such that $\varphi^n(U)$ is an open ball with $\partial\varphi^n(U) \subset \mathcal{O}$ for all $n \geq n_0$. Furthermore, U is contained in the basin of \mathcal{O} .

Remark 4.3: The above subshift factors onto the full shift on the symbols $\{X, Y\}$, where the factor map h is the one obtained by replacing the symbols X_j with X and Y_j with Y , for all j . The factor map h is one-to-one except over the backward orbit of X^∞ where it is q -to-one. In fact, note that the forward (periodic) orbit of $(X_0X_1\dots X_{q-1})^\infty$ is $h^{-1}(X^\infty)$.

Proof. Let V be the Fatou component that contains the critical values (see Lemma 3.4). Enumerate \mathcal{O} by $\xi_0, \xi_1, \dots, \xi_{q-1}$ respecting dynamics and so that $\partial V = \xi_1$. Subindices will be modulo q throughout this proof. Note that V is contained in the basin of \mathcal{O} .

Let U be the union of all fixed Rivera domains. Recall that U is an open ball, or a union of two open balls with the same boundary point, or a starlike domain (Proposition 3.5). In the first two cases, $q = 1$ and, in the last, $q > 1$. In particular, $L = \mathbb{P}_{\mathbb{L}}^{1,an} \setminus U$ is a “sphere” B_0 or the disjoint union of q closed balls B_0, \dots, B_{q-1} labeled such that $\{\xi_j\} = \partial B_j$. (For us a “sphere” is a subset of $\mathbb{P}_{\mathbb{L}}^{1,an}$ which, modulo a Möbius transformation, is the closure of the rigid unit “sphere” $\{|z|_o = 1\}$.) Now let W be the direction at ξ_0 containing the critical points and observe that $\varphi(W) = \mathbb{P}_{\mathbb{L}}^{1,an}$. In particular, $\varphi(B_0) = \mathbb{P}_{\mathbb{L}}^{1,an}$ and $\varphi : B_0 \setminus W \rightarrow B_1 \setminus V$ is a bijection. Moreover, if $q > 1$, then $\varphi : B_j \rightarrow B_{j+1}$ is a bijection for all $j \neq 0$.

It follows that ξ_1 has two preimages: $\xi_0 \in \mathcal{O}$ and $\xi'_0 \in W$. The degree of φ at each one of these preimages is 1. The bad direction at ξ_0 is W , and denote the bad direction at ξ'_0 by W' . Both directions contain the critical points, and their image direction is V . Thus, $A = W \cap W'$ is an open annulus which maps onto V . Therefore, A is a Fatou component contained in the basin of \mathcal{O} . It follows that each one of the closed balls $B = \mathbb{P}_{\mathbb{L}}^{1,an} \setminus W$ and $B' = \mathbb{P}_{\mathbb{L}}^{1,an} \setminus W'$ maps isomorphically onto $\mathbb{P}_{\mathbb{L}}^{1,an} \setminus V$. Denote by ξ'_j the preimage of ξ_{j+1} contained in B' .

Now consider the following level zero set

$$L_0 = L \setminus \bigcup_{n \geq 0} \varphi^n(V).$$

The set L_0 also has exactly q connected components Z_0, X_1, \dots, X_{q-1} where $Z_0 \subset B_0, X_1 \subset B_1, \dots, X_{q-1} \subset B_{q-1}$. Moreover, φ maps X_j isomorphically onto its image X_{j+1} for all $1 \leq j \leq q-2$. Also, X_{q-1} maps isomorphically onto Z_0 . However, Z_0 is the disjoint union of $X_0 = Z_0 \setminus W, A$ and B' . The set X_0 maps isomorphically onto X_1, A maps in a two-to-one fashion onto V and B' maps isomorphically onto $\mathbb{P}_{\mathbb{L}}^{1,an} \setminus V$.

Let $L_1 = \varphi^{-1}(L_0) \subset L_0$. The set L_1 has $2q$ connected components. The components X_0, \dots, X_{q-1} described above and the components Y_0, Y_1, \dots, Y_{q-1} are contained in $B' \subset Z_0$ such that $\varphi(Y_0) = Z_0$ and $\varphi(Y_j) = X_j$ for $j = 1, \dots, q-1$. Note that $\partial Y_j = \{\xi'_{j-1}\}$.

A key remark is that

$$\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial X_{q-1}, \partial(X_{q-1} \cap \varphi^{-1}(Y_j))) \geq \text{mod } A$$

for all $0 \leq j \leq q-1$, and

$$\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial Y_0, \partial(Y_0 \cap \varphi^{-1}(Y_0))) = \text{mod } A.$$

Since the complement of L_0 is contained in the Fatou set, we have that $J(\varphi) \subset L_1$, and we may introduce the itinerary function

$$J(\varphi) \ni \zeta \mapsto (i_k(\zeta))_{k \geq 0} \in \Sigma_q$$

if $\varphi^k(\zeta) \in i_k(\zeta)$. It follows that the itinerary map is well defined and continuous.

To check that the itinerary function is a bijection, consider a sequence $\mathbf{i} = (i_0, i_1, \dots) \in \Sigma_q$. For all $\ell \geq 0$, let

$$C_{\ell}(\mathbf{i}) = \{\zeta \in \mathbb{P}_{\mathbb{L}}^{1,an} \mid \varphi^k(\zeta) \in i_k \text{ for all } 0 \leq k \leq \ell\}.$$

Note that $C_\ell(\mathbf{i})$ is a non-empty and closed set. Moreover, for all ℓ , $C_\ell(\mathbf{i}) \supset C_{\ell+1}(\mathbf{i})$ and $C_\ell(\mathbf{i})$ has a unique boundary point which we denote by $\vartheta_\ell(\mathbf{i}) \in GO(\mathcal{O})$.

The arc $[\vartheta_\ell(\mathbf{i}), \vartheta_{\ell+1}(\mathbf{i})]$ is mapped isometrically, in the hyperbolic distance, onto $[\partial i_\ell, \partial(i_\ell \cap \varphi^{-1}(i_{\ell+1}))]$ by φ^ℓ . Therefore, if \mathbf{i} contains infinitely many symbols in $\{Y_0, \dots, Y_{q-1}\}$, then there are infinitely many $\ell \geq 0$ such that $i_\ell = X_{q-1}$ and $i_{\ell+1} = Y_j$ for some j or $i_\ell = i_{\ell+1} = Y_0$. From Lemma 1.1, $\bigcap C_\ell(\mathbf{i})$ is a singleton $\{\xi\}$ contained in $\mathbb{P}_\mathbb{L}^1$ and ξ is the unique point with itinerary \mathbf{i} . Moreover, $\xi \in J(\varphi)$ since it is the limit of $\vartheta_\ell(\mathbf{i}) \in GO(\mathcal{O}) \subset J(\varphi)$. If \mathbf{i} contains only finitely many symbols in $\{Y_0, \dots, Y_{q-1}\}$, then there exists k such that

$$(i_\ell)_{\ell \geq k} = (X_0 X_1 \cdots X_{q-1})^\infty$$

(the period q symbol sequence). Hence φ^k maps $\bigcap C_\ell(\mathbf{i})$ homeomorphically onto X , where X consists of B_0 with all the directions in the positive and negative orbit of W removed. Therefore, the unique point of X in the Julia set is ξ_0 and every direction at ξ_0 contained in X is a Fatou component contained in the basin of \mathcal{O} . It follows that $\bigcap C_\ell(\mathbf{i})$ contains exactly one point of the Julia set. Moreover, every Fatou component contained in $\bigcap C_\ell(\mathbf{i})$ is a ball contained in the basin of \mathcal{O} . Any Fatou component is either contained in some $\bigcap C_\ell(\mathbf{i})$, or eventually maps isomorphically onto a fixed Rivera domain, or eventually maps isomorphically onto A and statements (4) and (5) of the proposition follow. ■

5. Maps with a repelling periodic orbit in $\mathbb{H}_\mathbb{L}$

The aim of this section is to describe the dynamics of quadratic rational maps φ having a non-classical repelling periodic orbit.

Section 5.1 contains Lemma 5.1 which establishes the basic facts about these maps. Throughout this section as well as in Section 7 we will freely employ the content and notation of this Lemma. In Section 5.2 we introduce the filled Julia set $K(\varphi)$. We state Proposition 5.3 which describes its connected components and whose proof is completed, after some work, in Section 5.7. In Section 5.3 we introduce the basic structure to study $K(\varphi)$, namely “dynamical pieces” of integer levels. Section 5.4 deals with the easier case of maps for which only one critical point lies in $K(\varphi)$. In Section 5.6 we deal with the case in which both critical points lie in $K(\varphi)$, but before we need to introduce dynamical pieces

of “intermediate levels” in Section 5.5. Finally, we establish Proposition 5.3 in Section 5.7, which also contains the proof of Theorems 1 and 2.

5.1. THE BASICS. We start by stating the aforementioned lemma.

LEMMA 5.1: *Let $\varphi \in \mathbb{L}(\zeta)$ be a quadratic rational map which is not simple. Suppose that φ has a repelling periodic orbit in $\mathbb{H}_{\mathbb{L}}$. Then the following statements hold:*

(1) φ has exactly one fixed Fatou component U_0 which is a starlike Rivera domain. The boundary of U_0 is a repelling periodic orbit $\mathcal{O} = \{\xi_0, \xi_1 = \varphi(\xi_0), \dots, \xi_{q-1} = \varphi^{q-1}(\xi_0)\}$ of period $q > 1$. The skeleton \mathcal{A}_{U_0} contains a unique fixed point ϑ_0 which we will call the **center of U_0** . The Möbius transformation $T_{\vartheta_0} \mathbb{P}_{\mathbb{L}}^{1,an}$ is a rotation of order q .

(2) $L_0 = \mathbb{P}_{\mathbb{L}}^{1,an} \setminus U_0$ is the disjoint union of q closed balls B_0, \dots, B_{q-1} labeled such that $\partial B_j = \{\xi_j\}$. One of these balls, say B_0 , contains both critical points of φ . Moreover, the following holds:

(a)

$$\deg_{\xi_j} \varphi = \begin{cases} 1 & \text{if } j \neq 0, \\ 2 & \text{if } j = 0. \end{cases}$$

(b) For $j = 1, \dots, q - 1$, the bad direction at ξ_j is the direction that contains U_0 ; and

$$\varphi : B_j \rightarrow B_{j+1}$$

is a bijection (indices mod q).

(c) There exists a direction $D_0 \subset B_0$ at ξ_0 such that $\varphi(D_0)$ is the direction of U_0 at ξ_1 . Furthermore,

$$\varphi : B_0 \setminus D_0 \rightarrow B_1$$

is two-to-one (counting multiplicities) and

$$\varphi : D_0 \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an} \setminus B_1$$

is a bijection.

(3) For all $\xi \in \partial U_0$, the rational map

$$T_{\xi} \varphi^q : T_{\xi} \mathbb{P}_{\mathbb{L}}^{1,an} \cong \mathbb{P}_{\mathbb{C}}^1 \rightarrow T_{\xi} \mathbb{P}_{\mathbb{L}}^{1,an} \cong \mathbb{P}_{\mathbb{C}}^1$$

has a multiple fixed point in the direction of U_0 . All these maps are Möbius conjugate and will be denoted by $T_{\mathcal{O}} \varphi$.

- (4) *There exists a Fatou component containing a critical point which is an open ball D , contained in the basin of \mathcal{O} , with $\partial D \subset \mathcal{O}$.*
- (5) *Assume that φ has another non-rigid repelling periodic orbit \mathcal{O}' of period q' . Then $q' > q$ and, for all $\xi \in \mathcal{O}'$,*

$$T_\xi \varphi^{q'} : T_\xi \mathbb{P}_\mathbb{L}^{1,an} \cong \mathbb{P}_\mathbb{C}^1 \rightarrow T_\xi \mathbb{P}_\mathbb{L}^{1,an} \cong \mathbb{P}_\mathbb{C}^1$$

is (in the appropriate coordinate) a degree two polynomial. Moreover, there exists a critical point of φ that belongs to a Fatou component which is an open ball D' with $\partial D' \subset \mathcal{O}'$.

Proof. From Section 2.2 and Proposition 4.1 we conclude that φ has at least one classical indifferent periodic point. Hence, there exists a fixed Rivera domain U_0 . It follows from Proposition 3.1 that ∂U_0 is a repelling periodic orbit, say of period q . Since φ is not simple, $q > 1$. Now by Proposition 3.5, U_0 is the unique fixed Fatou component and it is a starlike domain. This proves part (1).

Observe that by Lemma 3.2 both critical points are contained in the same complementary ball of U_0 . Thus, in (2) we may assume that both critical points are in B_0 .

We now prove part (2a). Again, by Lemma 3.2, we have that $\deg_{\xi_j} \varphi = 1$ for all $j \neq 0$. But \mathcal{O} is repelling, thus $\deg_{\xi_0} \varphi = 2$ and (2a) follows.

In order to prove (2b), let $j \neq 0$. By Lemma 3.2, there is only one bad direction at ξ_j and this direction must be the one of U_0 , since it contains the critical points. Moreover, using the fact that $\deg_{\xi_j} \varphi = 1$, it follows that every direction contained in B_j is a good direction and cannot map to the direction $\mathbb{P}_\mathbb{L}^{1,an} \setminus B_{j+1}$. It follows that $\varphi : B_j \rightarrow B_{j+1}$ is a bijection.

For (2c), from Section 1.4.2 we conclude that the degree of $T_{\xi_0} \varphi$ in the direction $\mathbb{P}_\mathbb{L}^{1,an} \setminus B_0$ at ξ_0 is 1. Hence, there must exist another direction D_0 at ξ_0 which maps under $T_{\xi_0} \varphi$ onto $\mathbb{P}_\mathbb{L}^{1,an} \setminus B_1$. Since all the directions at ξ_0 are good directions, (2c) follows.

Observe that (3) follows from Theorem 2.4, which says that the direction of U_0 under $T_{\xi_0} \varphi^q$ is a multiple fixed point. That is,

$$T_{\xi_0} \varphi^q : T_{\xi_0} \mathbb{P}_\mathbb{L}^{1,an} \cong \mathbb{P}_\mathbb{C}^1 \rightarrow T_{\xi_0} \mathbb{P}_\mathbb{L}^{1,an} \cong \mathbb{P}_\mathbb{C}^1$$

has a multiple fixed point in the direction of U_0 .

For (4), we invoke a result from complex dynamics which guarantees that a map, such as $T_{\xi_0} \varphi^q$, with a multiple fixed point has a critical point, say D' , with infinite forward orbit (e.g., see [Mil99]). In particular, $T_{\xi_0} \varphi^n(D')$ is a good

direction at $\varphi^n(\xi)$ for all $n \geq 0$. Hence, $\varphi^n(D') = T_{\xi_0}\varphi^n(D')$ for all $n \geq 0$. We conclude that D' is a Fatou component, it contains a critical point of φ and the omega limit of every point in D' is ∂U_0 , since $\varphi^n(D') \neq \varphi^m(D')$ for all $n \neq m$.

Now for (5), note that another repelling periodic orbit

$$\mathcal{O}' = \{\xi'_0, \dots, \xi'_{q'-1}\}$$

(indices respecting dynamics and $\pmod{q'}$) is contained in $\mathbb{P}_{\mathbb{L}}^{1,an} \setminus \overline{U_0}$. For each $k = 0, \dots, q' - 1$, let B'_k be the closed ball with boundary $\{\xi'_k\}$ contained in $\mathbb{P}_{\mathbb{L}}^{1,an} \setminus \overline{U_0}$.

We claim that $\varphi(B'_k) = B'_{k+1}$. In fact, if $B'_k \subset B_j$ for some $j \neq 0$, then this follows trivially from (2b). If $B'_k \subset B_0 \setminus D_0$, then this also follows trivially but now from (2c). Finally, if $B'_k \subset D_0$, then $\varphi(B'_k)$ is disjoint from the direction at ξ'_{k+1} containing B_1 . Hence, $\varphi(B'_k) \subset B_j$ for some $j \neq 1$. Since $\varphi(B'_k)$ is a closed ball, it follows that $\varphi(B'_k) = B'_{k+1}$.

From the previous paragraph, every direction at ξ'_k contained in B'_k is a Fatou component. Therefore, the balls $B'_0, \dots, B'_{q'-1}$ are pairwise disjoint. In view of part (4), at most one of these balls, say B'_0 , contains a critical point. It follows that $\deg_{\xi'_0} \varphi^{q'} = 2$. Finally, $\varphi^{q'}(B'_0) = B'_0$ implies that

$$(T_{\xi'_0} \varphi^{q'})^{-1}(\mathbb{P}_{\mathbb{L}}^{1,an} \setminus B'_0) = \mathbb{P}_{\mathbb{L}}^{1,an} \setminus B'_0.$$

That is, $T_{\xi_0} \varphi^{q'}$ has a completely invariant direction; after putting this direction at ∞ , we have that $T_{\xi_0} \varphi^{q'}$ is a quadratic polynomial, which establishes part (5). ■

From the previous lemma it follows that for every non-classical repelling periodic orbit there exists a critical point contained in a Fatou component whose boundary is a point in the periodic orbit. Therefore:

COROLLARY 5.2: *If $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$ is a quadratic rational map, then φ has at most two non-classical repelling periodic orbits.*

5.2. THE FILLED JULIA SET. The standing assumption for the rest of this section is that the quadratic rational map φ has a fixed Rivera domain U_0 whose boundary is a period $q > 1$ repelling orbit \mathcal{O} .

We subdivide dynamical space according to whether a point eventually lands in the fixed Rivera domain or not. More precisely, we say that

$$K(\varphi) := \{\zeta \in \mathbb{P}_{\mathbb{L}}^{1,an} \mid \varphi^n(\zeta) \notin U_0 \text{ for all } n \geq 0\}$$

is the **filled Julia set** of φ .

PROPOSITION 5.3: *Let C be a connected component of $K(\varphi)$. Then one of the following holds:*

- (1) $C = \{\chi\}$ where χ is a rigid point.
- (2) C eventually maps onto a periodic closed ball B . Moreover, $\partial B \subset \mathcal{O}'$ where $\mathcal{O}' \neq \mathcal{O}$ is a repelling periodic orbit.
- (3) C consists of a closed ball with infinitely many directions removed. Moreover, C eventually maps onto a component C' whose unique boundary point lies in \mathcal{O} .

In all cases, ∂C is a singleton. Note that any non-trivial compact subsets X of $\mathbb{P}_{\mathbb{L}}^{1,an}$ having a single boundary point consists of the complement of a set U formed by the union of open balls which share their boundary point.

To prove this proposition we consider two different cases. The easier one is when there exists a critical point which is not in $K(\varphi)$. The proof in this easier case is contained in Section 5.4. The other case, when both critical points lie in $K(\varphi)$, requires more control on the geometry of $K(\varphi)$ and the corresponding proof is at the end of Section 5.6.

Before introducing the basic combinatorial structure which will allow us to study $K(\varphi)$, let us state and prove a basic fact.

LEMMA 5.4: $J(\varphi) = \partial K(\varphi)$.

Proof. The interior W of $K(\varphi)$ is contained in the Fatou set, since $\varphi^n(W) \cap U_0 = \emptyset$ for all $n \geq 0$. If $\zeta \in \partial K(\varphi)$ and U is a neighborhood of ζ , then $\varphi^n(U) \cap \mathcal{O} \neq \emptyset$ for some $n \geq 0$. Since $J(\varphi)$ is closed and totally invariant, we conclude that $J(\varphi) = \partial K(\varphi)$. ■

5.3. THE PUZZLE. By definition we say that the **dynamical level 0 set** L_0 is the complement of U_0 and each of its connected components is a **level 0 dynamical piece**.

Recursively, for all $\ell \geq 0$ we let

$$L_{\ell+1} = \varphi^{-1}(L_{\ell})$$

be the **dynamical level $\ell + 1$ set**. Note that $L_{\ell+1} \subset L_{\ell}$. Each connected component of L_{ℓ} is called a **level ℓ dynamical piece**.

Given a point $\zeta \in L_\ell$ and an integer $\ell \geq 0$, we denote by $P_\ell(\zeta)$ the level ℓ piece which contains ζ .

LEMMA 5.5: *The following statements hold:*

- (1) *For all $\ell \geq 1$, each level ℓ piece is contained in a unique level $\ell - 1$ piece.*
- (2) *For all $\ell \geq 1$, the image under φ of each level ℓ piece P is a level $\ell - 1$ piece P' . Moreover, $\varphi(\partial P) = \partial P'$.*
- (3) *Every piece of level ℓ is an affinoid, for all $\ell \geq 0$.*
- (4) *If $\ell \geq 0$ and P_ℓ is a level ℓ piece, then $\partial P_\ell \subset GO(\mathcal{O})$.*
- (5) $K(\varphi) = \bigcap_{\ell \geq 0} L_\ell$.

Proof. Note that (1) and (5) are an immediate consequence of the definitions. Moreover, (2) follows from the fact that φ is an open map. Statement (3) is a consequence of the fact that each connected component of the preimage of an affinoid is an affinoid. Finally, (4) follows from induction, since φ must map the boundary of a piece onto the boundary of a piece of lower level. ■

A decreasing sequence of pieces is called a **dynamical end**. More precisely, a dynamical end E is a collection

$$E = \{P_\ell\}_{\ell \geq 0}$$

such that, for all ℓ , we have that P_ℓ is a level ℓ piece and $P_{\ell+1} \subset P_\ell$. The **intersection E^\cap of the end E** is by definition

$$E^\cap = \bigcap P_\ell.$$

Note that every connected component of $K(\varphi)$ is the intersection of an end and vice versa.

Let us now discuss the geometry of the puzzle pieces. Recall from Lemma 5.1 that the level 0 set $L_0 = \mathbb{P}_{\mathbb{L}}^{1,an} \setminus U_0$ is the disjoint union of q closed balls labeled by B_0, \dots, B_{q-1} where:

- B_0 contains both critical points.
- For $j = 0, \dots, q - 1$, we have that $\varphi(\partial B_j) = \partial B_{j+1}$, subindices $\pmod q$.

Hence, there are exactly q level 0 pieces and each of them is a closed ball.

Preserving the notation of Lemma 5.1 and applying parts (2b) and (2c) we conclude that there are exactly $2q - 1$ pieces of level 1. In fact, let

$$B'_1, \dots, B'_{q-1}$$

be the preimage of B_2, \dots, B_q (respectively) under the bijection

$$\varphi : D_0 \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an} \setminus B_1.$$

Hence

$$B_0 \setminus D_0, B_1, \dots, B_{q-1}, B'_1, \dots, B'_{q-1}$$

is a complete list of the level 1 pieces. With the exception of the first one, all of them are closed balls. Nevertheless, observe that the boundary of each of them is a singleton.

Now we proceed to study the geometry of pieces of arbitrary level.

LEMMA 5.6: *Let P be a level $\ell \geq 0$ piece. Then P is an affinoid and ∂P is a singleton contained in $GO(\mathcal{O})$.*

Proof. We proceed by induction on the level of the piece P . The claim is clearly true for level 0 pieces. Now suppose that the claim is true for level $\ell - 1$ pieces. Let P be a level ℓ piece which maps onto P' . Denote by χ' the unique boundary point of P' . By the induction hypothesis, every direction at χ' is either contained in or disjoint from P' . We may assume that χ' has two preimages, for otherwise the claim is trivial. Hence, there exists a direction D'_v at χ' containing both critical values. Observe that since the critical values of φ belong to distinct directions at $\xi_1 \in \mathcal{O}$, D'_v must contain U_0 and therefore is disjoint from P' . Consider $\chi \in \partial P$. If a direction at χ contains the critical points, then it is disjoint from P , by Section 1.4.2 combined with Lemma 3.2. If a direction D at χ is critical point free, then D is a good direction and either $\varphi(D) \subset P'$ or $\varphi(D) \cap P' = \emptyset$. Hence, directions at χ are either contained in P or disjoint from P . It follows that $\partial P = \{\chi\}$. ■

For any $\zeta \in L_\ell$ we will denote by $D_\ell(\zeta)$ the direction at $\partial P_\ell(\zeta)$ that contains ζ . From the previous lemma it follows that for all $\zeta \in L_\ell$, $D_\ell(\zeta) \subset P_\ell(\zeta)$. Moreover, $\varphi(D_\ell(\zeta)) = D_{\ell-1}(\varphi(\zeta))$ provided that $\ell \geq 1$.

LEMMA 5.7: *Consider an integer $\ell \geq 0$ and two nested pieces $P_{\ell+1} \subset P_\ell$ of levels $\ell + 1$ and ℓ . If $\partial P_{\ell+1} \neq \partial P_\ell$, then*

- (1) $P_{\ell+1}$ is a closed ball and
- (2) $\partial\varphi(P_{\ell+1}) \neq \partial\varphi(P_\ell)$.

Proof. Since every direction at ∂P_ℓ contained in P_ℓ is a good direction, (2) follows. For (1) we proceed by induction. In view of our discussion of the geometry of level 0 and 1 pieces, (1) holds when $\ell = 0$. By (2) we may assume that $\varphi(P_{\ell+1})$ is the ball whose complement is the direction at $\partial\varphi(P_{\ell+1})$ containing U_0 . Then the inductive step is a consequence of the observation that if P is a piece such that $\partial P \neq \partial B_0$, then the image of every direction at ∂P disjoint from U_0 is a direction disjoint from U_0 . ■

COROLLARY 5.8: *Let E be a dynamical end. Then E^\cap is either a nested intersection of closed balls or a closed ball with infinitely (countable) many directions removed. In the latter case, ∂E^\cap is a point in $GO(\mathcal{O})$.*

Proof. Let $E = \{P_\ell\}$. By the previous lemma, if E^\cap is not the nested intersection of closed balls, then there exists ℓ_0 such that $\partial P_\ell = \partial P_{\ell_0}$ for all $\ell \geq \ell_0$. Since ∂P_{ℓ_0} is a point in $GO(\mathcal{O})$, we have that E^\cap eventually maps onto the intersection of $E(\xi_0)$ (recall that $\xi_0 \in \mathcal{O}$). The grand orbit of the direction of U_0 under $T_{\xi_0}\varphi^q$ is infinite, since this direction is a multiple fixed point of $T_{\xi_0}\varphi^q$ (see Lemma 5.1). Therefore, $E^\cap(\xi_0)$ is a ball with a countably infinite set of directions removed and the same occurs for E^\cap . ■

5.4. WHEN $K(\varphi)$ CONTAINS ONLY ONE CRITICAL POINT. In this case, we show that the intersections of ends which have infinitely many closed balls are singletons in the classical line.

LEMMA 5.9: *Assume that there exists one critical point ω of φ such that $\omega \notin K(\varphi)$. Let $E = \{P_\ell\}$ be a dynamical end such that P_ℓ is a closed ball for infinitely many ℓ . Then $E^\cap = \{\zeta\} \subset \mathbb{P}_\mathbb{L}^1$.*

Proof. Let $\ell_0 \geq 1$ be such that ω is not a level ℓ_0 point. Let

$$a = \min\{\text{dist}_{\mathbb{H}_\mathbb{L}}(\partial P_{\ell_0}(\zeta), \partial P_{\ell_0+1}(\zeta)) \mid \partial P_{\ell_0}(\zeta) \neq \partial P_{\ell_0+1}(\zeta) \text{ and } \zeta \in L_{\ell_0+1}\}.$$

Consider $\ell \geq \ell_0 + 1$ and suppose that P'_ℓ and $P'_{\ell+1}$ are pieces of levels ℓ and $\ell + 1$ such that $\partial P'_\ell \neq \partial P'_{\ell+1}$. We claim that

$$\text{dist}_{\mathbb{H}_\mathbb{L}}(\partial P'_\ell, \partial P'_{\ell+1}) = \text{dist}_{\mathbb{H}_\mathbb{L}}(\partial\varphi(P'_\ell), \partial\varphi(P'_{\ell+1})).$$

In fact, the direction at $\partial P'_\ell$ that contains $P'_{\ell+1}$ maps isomorphically onto its image. Therefore, φ maps $]\partial P'_\ell, \partial P'_{\ell+1}[$ isometrically onto $]\partial\varphi(P'_\ell), \partial\varphi(P'_{\ell+1})[$. It follows that

$$\text{dist}_{\mathbb{H}_\mathbb{L}}(\partial P'_\ell, \partial P'_{\ell+1}) \geq a.$$

For the dynamical end $E = \{P_\ell\}$, there are infinitely many $\ell \geq 0$ such that $\partial P_\ell \neq \partial P_{\ell+1}$; we conclude that there are infinitely many levels such that $\text{dist}_{\mathbb{H}_\mathbb{L}}(\partial P_\ell, \partial P_{\ell+1}) \geq a$. From Lemma 1.1, we conclude that E^\cap is a singleton contained in $\mathbb{P}_\mathbb{L}^1$. ■

5.5. PIECES OF INTERMEDIATE LEVEL AND MARKED GRIDS. It will be convenient to simultaneously consider another collection of pieces.

Recall that ϑ_0 is the center of the starlike fixed Rivera domain U_0 , that is, the unique fixed point in the skeleton \mathcal{A}_{U_0} (see Lemma 5.1). Let $\vartheta_{1/2}$ be the unique preimage of ϑ_0 in D_0 . Consider the closed ball $B_{1/2} \subset D_0$ defined by $B_{1/2} = \mathbb{P}_\mathbb{L}^{1,an} \setminus D_{\vartheta_{1/2}}(\vartheta_0)$, that is, the ball obtained after removing the direction of U_0 at $\vartheta_{1/2}$ from the Berkovich line. We define the **level 1/2 set** as

$$L_{1/2} = B_{1/2} \cup B_1 \cup \dots \cup B_{q-1}.$$

Recursively, for $\mathbb{Z} \ni \ell \geq 0$ we let

$$L_{\ell+3/2} = \varphi^{-1}(L_{\ell+1/2}).$$

Each connected component of $L_{\ell+1/2}$ is called a **level $\ell+1/2$ dynamical piece**.

To avoid confusion let us be precise about the notation related to natural numbers:

$$\begin{aligned} \mathbb{N} &= \{n \in \mathbb{Z} \mid n \geq 1\}, \\ \mathbb{N}_0 &= \{n \in \mathbb{Z} \mid n \geq 0\}, \\ \mathbb{N}/2 &= \{n/2 \in \mathbb{Q} \mid n \in \mathbb{N}\}, \\ \mathbb{N}_0/2 &= \{n/2 \in \mathbb{Q} \mid n \in \mathbb{N}_0\}. \end{aligned}$$

Note that for all $\ell \in \mathbb{N}_0/2$,

$$L_{\ell+1} \subset L_{\ell+1/2} \subset L_\ell.$$

Hence every piece of level $\ell + 1/2$ is contained in a unique piece of level ℓ . Consider a piece $P_{\ell+1/2}$ of level $\ell+1/2$. It follows that $\partial P_{\ell+1/2} \subset GO(\mathcal{O} \cup \{\vartheta_0\})$. Moreover, if $\ell \geq 1$, then $\varphi(P_{\ell+1/2})$ is a level $\ell - 1/2$ piece.

LEMMA 5.10: *Let $\ell \geq 0$ be an integer. Consider an intermediate piece $P_{\ell+1/2}$ of level $\ell + 1/2$. Let P_ℓ be the level ℓ piece containing $P_{\ell+1/2}$.*

- (1) *If $\partial P_\ell = \partial P_{\ell+1/2}$, then $P_\ell = P_{\ell+1/2}$.*
- (2) *If $\partial P_\ell \neq \partial P_{\ell+1/2}$, then $P_{\ell+1/2}$ is a closed ball.*

Proof. By the definition of level $1/2$ pieces, both assertions are clearly true if $\ell = 0$. We proceed by induction.

For (1), given $\ell \geq 1$, if $\partial P_\ell = \partial P_{\ell+1/2}$, then $\partial\varphi(P_\ell) = \partial\varphi(P_{\ell+1/2})$. By the inductive hypothesis, $\varphi(P_\ell) = \varphi(P_{\ell+1/2})$. It follows that $P_\ell = P_{\ell+1/2}$.

For (2), consider $\ell \geq 1$, assume that $\partial P_\ell \neq \partial P_{\ell+1/2}$. Then $P_{\ell+1/2}$ is contained in a direction D at ∂P_ℓ which must be a good direction. Hence $\partial\varphi(P_\ell) \neq \partial\varphi(P_{\ell+1/2})$. By the inductive hypothesis, $\varphi(P_{\ell+1/2})$ is a closed ball. Given $\zeta \in \partial P_{\ell+1/2}$, every direction at ζ , contained in D , must map into $\varphi(D)$, and therefore into the ball $\varphi(P_{\ell+1/2})$. Hence $P_{\ell+1/2}$ contains all the directions at ζ , with the exception of the one that is not contained in D . ■

LEMMA 5.11: *Assume that both critical points are level $\ell \in \mathbb{N}/2$ points. Let $P_{\ell-1/2}$ be a level $\ell - 1/2$ piece and D be a direction at $\partial P_{\ell-1/2}$ contained in $P_{\ell-1/2}$ but not contained in L_ℓ . Then the following holds:*

- (1) D contains a unique level ℓ piece P_ℓ ,
- (2) $\varphi : D \setminus P_\ell \rightarrow \varphi(D \setminus P_\ell)$ is a degree one or two map between annuli. The degree is two if and only if P_ℓ contains a critical point. Equivalently, the degree of φ at the boundary point of P_ℓ is two.

Proof. We assume that both critical points are level $k \geq \ell$ points and proceed by induction on ℓ . For $\ell = 1/2$ and $\ell = 1$, the statement follows from the description of the pieces up to level 1. Assume that (1) and (2) are true for $\ell - 1$. Thus $\varphi(D)$ is a direction at $\partial P_{\ell-3/2}$ contained in $P_{\ell-3/2}$ but not contained in $L_{\ell-1}$. Let P be the unique piece of level $L_{\ell-1}$ contained in $\varphi(D)$. By the previous lemma P is a closed ball.

If $\varphi : D \rightarrow \varphi(D)$ has degree one, then the preimage of P is a unique closed ball P' . Moreover, D is critical point free and so is P' .

If $\varphi : D \rightarrow \varphi(D)$ has degree two, then ∂P has a unique preimage ζ in D , since $D \setminus P$ is critical value free and therefore P contains one critical value. It follows that the preimage of P is the unique closed ball $P' \subset D$ with boundary point ζ . Moreover, P' contains a critical point and $D \setminus P'$ maps under φ onto its image as a degree two map. ■

Recall from Lemma 5.1 (4) that there exists an open ball U containing a critical point which is a component of the basin of \mathcal{O} with boundary point in \mathcal{O} . Such boundary ∂U must be ∂B_0 in view of part (2) of the same lemma. Hence we may introduce a labelling of the critical points as follows.

Definition 5.12: Denote by ω' a critical point that belongs to a Fatou component $U(\omega')$ which is an open ball in the basin of \mathcal{O} such that $\partial U(\omega') = \partial B_0$. Denote by ω the other critical point. If there are two choices for ω' just make an arbitrary one. We say that ω is the **active critical point**.

Note that $\partial U(\omega') = \partial B_0$ and, therefore, $\partial P_\ell(\omega') = \partial B_0$ for all $\ell \geq 0$. We emphasize that there might be a choice involved.

Definition 5.13: Let $\ell \in \mathbb{N}/2$. Assume that the active critical point ω is a level ℓ point. Let ζ be a point of level ℓ . For $m \in \mathbb{N}_0/2, n \in \mathbb{N}_0$ such that $m + n \leq \ell$, we let

$$M_{n,m}(\zeta) = \begin{cases} 2 & \text{if } P_m(\varphi^n(\zeta)) = P_m(\omega), \\ 1 & \text{otherwise.} \end{cases}$$

The **marked grid of ζ of level ℓ** is the array of 1's and 2's given by

$$\mathbf{M}_\ell(\zeta) = (M_{n,m}(\zeta)).$$

If $M_{n,m}(\zeta) = 2$ we say that (n, m) is a **marked position**, otherwise we say that it is an **unmarked position**. We say that $\mathbf{M}_\ell(\omega)$ is the **critical marked grid**.

If $\omega, \zeta \in K(\varphi)$, then we may define $M_{n,m}(\zeta)$ as above for all $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0/2$. We say that the infinite array $\mathbf{M}(\zeta) = (M_{n,m}(\zeta))$ is the **marked grid of ζ** .

From the definition it follows that:

- (Ma) If $M_{n,\ell}$ is marked, then $M_{n,j}$ is marked for all $j \leq \ell$.
- (Mb) If $M_{n,\ell}$ is marked, then $M_{n+j,\ell-j} = M_{j,\ell-j}(\omega)$ for $0 \leq j \leq \ell$.

Marked grids are introduced to help us keep track of the hyperbolic distance between the boundary of the pieces.

LEMMA 5.14: *Assume that both critical points are level ℓ points. Let ζ be a level ℓ point and $\zeta_n = \varphi^n(\zeta)$. If $m + n \leq \ell$ and $m - 3/2 \geq 0$, then*

$$M_{n,m}(\zeta) \cdot \text{dist}_{\mathbb{H}_\mathbb{L}}(\partial P_{m-1/2}(\zeta_n), \partial P_m(\zeta_n)) = \text{dist}_{\mathbb{H}_\mathbb{L}}(\partial P_{m-3/2}(\zeta_{n+1}), \partial P_{m-1}(\zeta_{n+1})).$$

Proof. First observe that $\partial P_{m-1/2}(\zeta_n) = \partial P_m(\zeta_n)$ if and only if

$$\partial P_{m-3/2}(\zeta_{n+1}) = \partial P_{m-1}(\zeta_{n+1}).$$

Now if $\partial P_{m-1/2}(\zeta_n) \neq \partial P_m(\zeta_n)$, the required identity follows from (2) of the previous lemma. ■

We must warn the reader that the degree d of $\varphi : P_\ell(\zeta_n) \rightarrow P_{\ell-1}(\zeta_{n+1})$, in general, does not agree with $M_{n,\ell}(\zeta)$. However, always $M_{n,\ell}(\zeta) \leq d$ but strict inequality might hold. That is, it might occur that $\omega \notin P_\ell(\zeta_n)$ while $d = 2$. Nevertheless, for future reference we record, without proof, a situation in which the degree and the value of the marked grid coincide.

LEMMA 5.15: *Assume that both critical points are level ℓ' points. Let ζ be a level ℓ' point. If there exist $\ell \leq \ell'$ such that $P_\ell(\zeta)$ is a critical point free closed ball, then $M_{0,\ell'}(\zeta) = 1$ and the degree of $\varphi : P_{\ell'}(\zeta) \rightarrow P_{\ell'-1}(\varphi(\zeta))$ is also 1.*

5.6. BOTH CRITICAL POINTS IN $K(\varphi)$. Let us assume that both critical points belong to $K(\varphi)$. It will be convenient to say that $\ell_0 \in \mathbb{N}_0/2$ is the depth of the n_0 -th column of a marked grid $(M_{n,\ell})$ if M_{n_0,ℓ_0} is marked but $M_{n_0,\ell_0+1/2}$ is not marked.

We distinguish marked grids \mathbf{M} as follows:

- (1) If there exists p such that $M_{pn,\ell}$ is marked for all $\ell \geq 0$ and all $n \in \mathbb{N}_0$, then we say that \mathbf{M} is **periodic**. The smallest such $p \geq 1$ is called the period of \mathbf{M} .
- (2) Assume that \mathbf{M} is not periodic. If there exists $k \geq 1$ and $p \geq 1$ such that for all $n \geq 0$ we have that $M_{np+k,\ell}$ is marked for all $\ell \geq 0$, then we say that \mathbf{M} is **strictly preperiodic**.
- (3) If a marked grid is periodic or strictly preperiodic, then we say that it is **eventually periodic**.
- (4) If the critical marked grid $\mathbf{M}(\omega)$ has columns of arbitrarily large depth, then we say that $\mathbf{M}(\omega)$ is **critically recurrent**.

The marked grids involved in our study of quadratic rational maps over \mathbb{L} have levels in $\mathbb{N}_0/2$ in contrast with the marked grids that appear in the literature related to complex quadratic and cubic polynomial dynamics where levels are indexed by \mathbb{N}_0 (see [BH88, Mil00a]). The introduction of mid-level positions allow us to avoid the so called “semi-critical” positions which arise in complex quadratic polynomial dynamics. However, “degenerate annuli” do appear in our context, just as in the study of quadratic polynomials.

Given $\zeta \in K(\varphi)$ we are interested on deciding whether

$$S(\zeta) := \sum_{\ell \in \mathbb{N}/2} \text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P_{\ell-1/2}(\zeta), \partial P_\ell(\zeta))$$

converges or diverges.

If $\partial P_{\ell-1/2}(\zeta) = \partial P_\ell(\zeta)$ for all $\ell \geq \ell_0$, then the above sum clearly converges. Otherwise, $\partial P_{\ell-1/2}(\zeta) \neq \partial P_\ell(\zeta)$ for infinitely many values of $\ell \in \mathbb{N}/2$ and the following result establishes exactly when the corresponding series is convergent.

THEOREM 5.16: *Assume that both critical points belong to $K(\varphi)$. Consider $\zeta \in K(\varphi)$. If $\partial P_{\ell-1/2}(\zeta) \neq \partial P_\ell(\zeta)$ for infinitely many values of $\ell \in \mathbb{N}_0/2$, then $S(\zeta)$ converges if and only if $\mathbf{M}(\zeta)$ is eventually periodic.*

The proof of the theorem is outlined after the two lemmas below. We will adapt the original techniques of Branner–Hubbard and Yoccoz (e.g., see [BH88, Mil00a]). The only difference with the usual techniques is that, in our context, only a weaker version of the so-called “third tableau rule” holds:

LEMMA 5.17 (Weak third rule): *Let $\zeta \in K(\varphi)$. Consider $\ell \in \mathbb{N}/2$ and assume that $\partial P_{\ell-1/2}(\omega) \neq \partial P_\ell(\omega)$. For all $k \in \mathbb{N}$ and $\ell_0 \in \mathbb{N}/2$ such that $\ell_0 \geq \ell + k$, the following holds:*

If $M_{j,\ell_0-j}(\omega)$ is not marked for all $j = 1, \dots, k - 1$, $M_{k,\ell_0-k+1/2}(\omega)$ and $M_{n,\ell_0}(\zeta)$ are marked but $M_{n,\ell_0+1/2}(\zeta)$ is not marked, then $M_{n+k,\ell_0-k+1/2}(\zeta)$ is not marked.

Proof. We proceed by contradiction and assume that $M_{n+k,\ell_0-k+1/2}(\zeta)$ is marked. That is,

$$P_{\ell_0-k+1/2}(\zeta_{n+k}) = P_{\ell_0-k+1/2}(\omega).$$

By Lemma 5.15, $\varphi^k : P_{\ell_0}(\omega) \rightarrow P_{\ell_0-k}(\omega)$ has degree 2, since $P_{\ell_0-k-j}(\varphi^j(\omega))$ is a critical point free ball for $1 \leq j < k$. Now $P_{\ell_0+1/2}(\omega)$ maps with degree 2 in k iterates onto $P_{\ell_0-k+1/2}(\omega)$. Therefore, the unique level $\ell_0 + 1/2$ piece contained in $P_{\ell_0}(\omega) = P_{\ell_0}(\zeta_n)$ which maps in k iterates onto $P_{\ell_0-k+1/2}(\omega)$ is $P_{\ell_0+1/2}(\omega)$. We conclude that $P_{\ell_0+1/2}(\omega) = P_{\ell_0+1/2}(\zeta_n)$ which contradicts the fact that $M_{n,\ell_0+1/2}(\zeta)$ is unmarked. ■

The key to prove the theorem is the following result.

LEMMA 5.18: *If $\mathbf{M}(\omega)$ is recurrent but not periodic, then $S(\omega)$ diverges.*

Proof. Since $\mathbf{M}(\omega)$ is not periodic, there exists $\ell' \in \mathbb{N}/2$ such that $\partial P_{\ell'}(\omega) \neq \partial B_0$. Hence, there exists $\ell_0 \in \mathbb{N}/2$ such that $\partial P_{\ell_0-1/2}(\omega) \neq \partial P_{\ell_0}(\omega)$.

Given $d \in \mathbb{N}/2$ such that $\partial P_{d-1/2}(\omega) \neq \partial P_d(\omega)$, we claim that there exist integers $m > k > 0$ such that $P_{d+k}(\omega)$ maps under φ^k onto $P_d(\omega)$ with degree 2 and $P_{d+m}(\omega)$ maps under φ^m onto $P_d(\omega)$ with degree 2. Thus, we will obtain

that

$$\begin{aligned} \text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P_{d-1/2}(\omega), \partial P_d(\omega)) &= \text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P_{d+k-1/2}(\omega), \partial P_{d+k}(\omega)) \\ &\quad + \text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P_{d+m-1/2}(\omega), \partial P_{d+m}(\omega)). \end{aligned}$$

Let $k \geq 1$ be such that the k -th column of $\mathbf{M}(\omega)$ is the first one with depth at least d . For $0 \leq j < k$, we have that $\partial P_{d-1/2+k-j}(\omega_j) \neq \partial P_{d+k-j}(\omega_j)$ since, under φ^{k-j} , these singletons map onto the singletons $\partial P_{d-1/2}(\omega) \neq \partial P_d(\omega)$. By Lemma 5.15, it follows that k has the desired properties.

To obtain m we follow the original proof, as presented in [Mil00a, Lemma 1.3 (b)], and we outline how the “weak third rule” maybe applied instead of the “third rule”. Denote the depth of the k -th column by $d' - 1/2$. Consider the largest integer n such that $d' - kn > d$. Taking into account that $\partial P_d(\omega) \neq \partial P_{d-1/2}(\omega)$, we may apply $(n - 1)$ -times Lemma 5.17, just as in [Mil00a], to prove that $M_{d'-j, k+j}(\omega)$ is unmarked for all j such that $1 < j \leq kn$. Now let m be the smallest integer greater than $(n + 1)k$ such that the m -th column has depth at least d . It follows that m has the desired properties.

The level $d + k$ and $d + m$ critical pieces are usually called children of $P_d(\omega)$. Repeating the argument one obtains 2^t descendants of the t -th generation such that the total contribution of each generation to the sum is

$$\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P_{d-1/2}(\omega), \partial P_d(\omega)).$$

Thus, the sum $S(\omega)$ diverges. ■

Proof of Theorem 5.16. Let $\zeta \in K(\varphi)$ be as in the statement of the theorem.

First we consider the case in which there exists $\ell_0 \in \mathbb{N}/2$ such that $\partial P_{\ell_0}(\omega) = \partial P_{\ell}(\omega)$ for all $\ell \geq \ell_0$. If $\partial P_{d-1/2}(\zeta) \neq \partial P_d(\zeta)$, then $\partial P_{d-1/2-j}(\zeta_j) \neq \partial P_{d-j}(\zeta_j)$ for all $j \leq d - 1/2$. Thus, taking $k \in \mathbb{N}$ such that $d - k = \ell_0 + 1/2$ or $d - k = \ell_0 + 1$, we obtain that $P_{d-j}(\zeta_j) \neq P_{d-j}(\omega)$ for all j such that $j \leq k$. Therefore, $\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P_{d-1/2}(\zeta), \partial P_d(\zeta))$ takes one of the finitely many positive values $\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P, \partial P')$ where P (resp. P') is a level $d - k - 1/2$ (resp. $d - k$) piece such that $P \supset P'$ and $\partial P \neq \partial P'$. Hence the sum $S(\zeta)$ diverges.

Now we consider the case in which there exists ℓ_0 such that all the positions of depth $\ell \geq \ell_0$ in $\mathbf{M}(\zeta)$ are unmarked. For all $d > \ell_0$, taking $k \in \mathbb{N}$ such that $d - k = \ell_0 + 1/2$ or $d - k = \ell_0 + 1$, we obtain that $P_{d-j}(\zeta_j) \neq P_{d-j}(\omega)$ for all j such that $j \leq k$. Again, if $\partial P_{d-1/2}(\zeta) \neq \partial P_d(\zeta)$, then $\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P_{d-1/2}(\zeta), \partial P_d(\zeta))$ takes finitely many positive values. Thus, the sum $S(\zeta)$ diverges.

Next we consider the case in which $S(\omega)$ diverges. By the above we may assume that $\mathbf{M}(\zeta)$ has columns of arbitrarily large depth. For each ℓ , let $n(\ell)$ be the first column in $\mathbf{M}(\zeta)$ with depth at least ℓ . Since all the positions, $M_{\ell+n(\ell)-j,j}(\zeta)$ are unmarked, for $j = 0, \dots, n(\ell)$, we obtain that

$$\text{dist}_{\mathbb{H}_{\mathbb{L}}}(P_{\ell-1/2}(\omega), P_{\ell}(\omega)) = \text{dist}_{\mathbb{H}_{\mathbb{L}}}(P_{\ell+n(\ell)-1/2}(\zeta), P_{\ell+n(\ell)}(\zeta)).$$

It follows that the sum diverges.

Note that if $S(\omega)$ converges, then either $\mathbf{M}(\omega)$ is periodic or there exists $\ell_0 \in \mathbb{N}/2$ such that $\partial P_{\ell_0}(\omega) = \partial P_{\ell}(\omega)$ for all $\ell \geq \ell_0$. Since we have already taken care of the latter, to finish the proof we have to consider the case in which $\mathbf{M}(\omega)$ is periodic and $\partial P_{\ell-1/2}(\omega) \neq \partial P_{\ell}(\omega)$ for infinitely many ℓ . It is not difficult to prove that if $\mathbf{M}(\omega)$ is periodic, then $S(\omega)$ converges. Thus, if $\mathbf{M}(\zeta)$ is eventually periodic, then $S(\zeta)$ converges. It only remains to show that if $\mathbf{M}(\zeta)$ is not eventually periodic, then $S(\zeta)$ diverges. This is the other instance in which the “(weak) third rule” is applied. We follow the exposition of Milnor ([Mil00a, Theorem 2.4, Case 2]). Consider N such that all columns of $\mathbf{M}(\omega)$ with depth at least N have infinite depth. That is, these columns are multiples of p , where p is the period of $\mathbf{M}(\omega)$. Without loss of generality we may assume that $\partial P_{N-1/2}(\omega) \neq \partial P_N(\omega)$. There are infinitely many pairs (m, d) with $d \in \mathbb{N}/2$, $d \geq N$, $m \in \mathbb{N}$ such that the m -th column of $\mathbf{M}(\zeta)$ has depth exactly $d - 1/2$ and the m -th column is the first to have depth at least $d - 1/2$. Taking into account that $\partial P_{N-1/2}(\omega) \neq \partial P_N(\omega)$ we may apply the “weak third rule” to conclude that the columns $m + jp$ have depth $d - 1/2 - jp$ as long as $d - 1/2 - jp \geq N$. Let k be the largest integer such that $d - 1/2 - kp \geq N$. Now consider the smallest integer $m' > m + kp$ such that $M_{m',N}(\zeta)$ is marked. It follows that all the positions $M_{m'+N-j,j}(\zeta)$ are unmarked for all $j = 0, \dots, m' - 1$. Thus

$$\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P_{m'+N-1/2}(\zeta), \partial P_{m'+N}(\zeta)) = \text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P_{N-1/2}(\omega), \partial P_N(\omega)).$$

Note that $m' + N > m + d - 1/2 - p$. So we may recursively choose pairs (m, d) to obtain infinitely many values of $\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P_{\ell-1/2}(\zeta), \partial P_{\ell}(\zeta))$ which agree with $\text{dist}_{\mathbb{H}_{\mathbb{L}}}(\partial P_{N-1/2}(\omega), \partial P_N(\omega))$. Thus $S(\zeta)$ diverges. ■

COROLLARY 5.19: *Assume that both critical points are in $K(\varphi)$. Let $\zeta \in K(\varphi)$ and suppose that $\partial P_{\ell}(\zeta) \neq \partial P_{\ell+1/2}(\zeta)$ for infinitely many values of ℓ . Then one of the following holds:*

- (1) $\bigcap P_{\ell}(\zeta) = \{\zeta\}$ and ζ is a rigid point.

- (2) $\mathbf{M}(\zeta)$ is eventually periodic and $\bigcap P_\ell(\zeta)$ is an eventually periodic closed ball B . Moreover, $\mathbf{M}(\omega)$ is periodic and $\bigcap P_\ell(\omega)$ is a periodic closed ball B' . The point associated to B' lies in a non-rigid repelling periodic orbit \mathcal{O}' and B eventually maps onto B' .

Proof. If $S(\zeta)$ diverges, then $\bigcap P_\ell(\zeta) = \{\zeta\}$ and ζ is a rigid point, by Lemma 1.1. That is, (1) holds.

Assume that $S(\zeta)$ converges. Then $\mathbf{M}(\zeta)$ is eventually periodic, by the previous theorem. From Corollary 5.8, we have that $\bigcap P_\ell(\zeta)$ is a closed ball B . Denote the eventual period of $\mathbf{M}(\zeta)$ by p . The definition of an eventually periodic marked grid implies the existence of a smallest $k \geq 0$ such that the active critical point ω lies in $P_\ell(\varphi^{pn+k}(\zeta))$ for all $n \geq 0$ and all $\ell \geq 0$. Therefore, $B' = \varphi^k(B) = \bigcap P_\ell(\omega)$ is a period p closed ball. Moreover, $\varphi^p : B' \rightarrow B'$ has degree 2 and therefore the point associated to B' is a type II repelling period p point. ■

5.7. PROOFS OF PROPOSITION 5.3, THEOREMS 1 AND 2.

Proof of Proposition 5.3. From Corollary 5.8 and Lemma 5.9 the proposition follows when there exists a critical point $\omega \notin K(\varphi)$. Now, Corollaries 5.8 and 5.19 establish the proposition when both critical points are contained in $K(\varphi)$. ■

Proof of Theorems 1 and 2. According to Proposition 3.1 we have three possibilities:

- (a) $J(\varphi) \cap \mathbb{H}_\mathbb{L}$ is periodic point free.
- (b) There exists an indifferent periodic orbit \mathcal{O} in $J(\varphi) \cap \mathbb{H}_\mathbb{L}$.
- (c) There exists a repelling periodic orbit \mathcal{O} in $J(\varphi) \cap \mathbb{H}_\mathbb{L}$.

From Proposition 4.1, (a) is equivalent to $J(\varphi) \subset \mathbb{P}_\mathbb{L}^1$. In this case, $F(\varphi) = \mathbb{P}_\mathbb{L}^{1,an} \setminus J(\varphi)$ is connected and coincides with the basin of an attracting fixed point. Thus, (1) of Theorems 1 and 2 hold.

Assume that (b) holds. From Proposition 4.2 (2), we have that $J(\varphi) \cap \mathbb{H}_\mathbb{L} = GO(\mathcal{O})$. That is, Theorem 1 (2) holds. Moreover, Proposition 4.2 (4) implies that every periodic Fatou component is a fixed Rivera domain with boundary \mathcal{O} . From Proposition 3.5, Rivera domains are either balls or starlike. In the former, the boundary must be a fixed point and Theorem 2 (2) holds. In the latter, this starlike domain is unique and Theorem 2 (3) holds. Also observe

that Proposition 4.2 (5) implies that Fatou components are eventually periodic or balls/annuli contained in the basin of \mathcal{O} .

Now assume that (c) holds. From Proposition 5.3 and Lemma 5.4 we conclude that every point in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$ is contained in the grand orbit of a non-rigid repelling orbit. According to Corollary 5.2 there are at most two such orbits which, by Lemma 5.1 (3) and (5), are such that Theorem 1 (3) or (4) holds. Hence, we have established Theorem 1. If U is a Fatou component, then either U eventually maps onto the fixed Rivera domain or U is contained in a connected component C of $K(\varphi)$. According to Proposition 5.3, U must be a direction at the unique Julia set point ζ in C . Moreover, ζ belongs to the grand orbit of a non-rigid repelling periodic orbit. Thus, U is either an eventually periodic open ball (of period at least 2) or it is contained in the basin of a periodic orbit. That is, Theorem 2 (4) holds.

From the above, we conclude that Fatou components are eventually periodic or balls/annuli contained in the basin of a periodic orbit. Thus, we have also established the statement of Theorem 2. ■

6. Quadratic laminations and trees

Invariant laminations were introduced in complex polynomial dynamics by Thurston [Thu09] and have been widely used to describe the dynamical space as well as the parameter space of complex polynomials. In [McM94, Section 8.5], McMullen defines the α -lamination of a quadratic polynomial. Here we introduce a variation of these α -laminations which will allow us to describe quadratic rational maps over \mathbb{L} . More precisely, we will introduce abstract α -laminations and construct a tree associated to each such abstract α -lamination. In a certain sense, this tree is related both to the Yoccoz puzzle used to study the dynamics of complex quadratic polynomials (e.g., see [Hub93, Mil00a]), as well as to the obstructions which arise in the “mating” of two complex quadratic polynomials [Tan92].

In Section 6.1 we introduce abstract α -laminations as well as illustrate the definition with those that arise from complex quadratic polynomial dynamics. Their geometric representation as “laminations” in $\mathbb{P}_{\mathbb{C}}^1$ is discussed in Section 6.3. The trees of integer levels associated to an abstract α -lamination and their basic properties are discussed in Section 6.4. Then, in Section 6.5, we

introduce the associated infinite tree with its corresponding dynamics. In Section 6.6 we construct all abstract α -laminations via symbolic dynamics. Finally, in Section 6.7, we define degree 2 branched maps of trees and prove a lifting lemma.

Throughout, we denote the multiplication by 2 map acting on \mathbb{R}/\mathbb{Z} by m_2 .

6.1. ABSTRACT α -LAMINATIONS. Let $\{t_0, \dots, t_{q-1}\} \subset \mathbb{R}/\mathbb{Z}$ be a period $q \geq 2$ periodic orbit under m_2 labeled $\text{mod } q$ and respecting cyclic order in \mathbb{R}/\mathbb{Z} . Such a periodic orbit has **rotation number** $p/q \in \mathbb{R}/\mathbb{Z}$ if $m_2(t_j) = t_{j+p}$ for all j . For example, $\{1/3, 2/3\}$ and $\{1/7, 2/7, 4/7\}$ have rotation numbers $1/2$ and $1/3$, respectively. According to Bullett and Sentenac [BS94], for each pair of co-prime positive integers p and q , there is exactly one periodic orbit of m_2 with rotation number p/q .

Definition 6.1: An equivalence relation λ in \mathbb{R}/\mathbb{Z} is called an **abstract α -lamination** if the following hold:

- (1) **Invariant.** If A is a class, then $m_2(A)$ is a class.
- (2) **Finite.** Every class contains finitely many elements.
- (3) **Unlinked.** If A and B are two distinct classes, then A is contained in a connected component of $\mathbb{R}/\mathbb{Z} \setminus B$.
- (4) **Consecutive preserving.** If A is a class and $]t, s[$ is a connected component of $\mathbb{R}/\mathbb{Z} \setminus A$, then $]m_2(t), m_2(s)[$ is a connected component of $\mathbb{R}/\mathbb{Z} \setminus m_2(A)$.
- (5) **α -supported.** There exists a class A_0 with at least two elements such that $m_2(A_0) = A_0$ and, for all classes A with at least two elements, there exists $\ell \geq 0$ such that $A_0 = m_2^\ell(A)$.

We say that A_0 is the **fixed class of λ** . It follows that A_0 as above has a well defined rotation number, say p/q , and we say that λ is an **abstract α -lamination in the p/q -limb**.

Throughout, we let $1 \leq p < q$ be relatively prime integers.

Definition 6.2: Let $A_0 \subset \mathbb{R}/\mathbb{Z}$ be the unique set of q arguments such that $m_2(A_0) = A_0$ and A_0 has rotation number p/q . According to Milnor [Mil00b], among the connected components of $\mathbb{R}/\mathbb{Z} \setminus A_0$ there is one with smallest length, say $I_{p/q} =]\theta_0, \theta_1[$. The interval $I_{p/q}$ is called the **p/q -characteristic interval**.

6.2. α -LAMINATIONS THAT ARISE FROM COMPLEX DYNAMICS. Examples of abstract α -laminations arise in the context of iteration of quadratic complex polynomials. We refer the reader to [Mil99] for a detailed exposition about the basics of iteration of complex rational and polynomial maps.

Consider the quadratic family

$$Q_c(z) = z^2 + c$$

where $c \in \mathbb{C}$. The **Mandelbrot set** \mathcal{M} consists of all parameters c for which the corresponding Julia set J_c is connected. Let $A_0 \subset \mathbb{R}/\mathbb{Z}$ be the periodic orbit under m_2 with rotation number p/q . The p/q -**limb** of \mathcal{M} , denoted $L_{p/q}$, consists of all parameters $c \in \mathcal{M}$ such that the external rays with arguments in A_0 land at a fixed point of Q_c . This fixed point is called the α -**fixed point** of Q_c .

Recall that the Douady–Hubbard map $\Phi : \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus \mathcal{M}$ is the unique conformal isomorphism tangent to the identity at infinity. The image of $]1, +\infty[\exp(2\pi it)$ under Φ is called the **parameter ray** $R_{\mathcal{M}}^t$ at argument t . Parameter rays at arguments $t \in \mathbb{Q}/\mathbb{Z}$ have a well defined limit (i.e., land) as they approach the Mandelbrot set. If t is periodic under m_2 , then the landing point c is such that Q_c has a multiple periodic point. If t is strictly preperiodic under m_2 , then the landing point c is such that the critical point is preperiodic under Q_c (e.g., see [Mil00b]). Moreover, denote by $I_{p/q} =]\theta_0, \theta_1[$ the p/q -characteristic interval. Then, the parameter rays $R_{\mathcal{M}}(\theta_0)$ and $R_{\mathcal{M}}(\theta_1)$ land at the same point c_0 , which is the unique parameter in the boundary of the main cardioid for which Q_{c_0} has a fixed point with multiplier $\exp(2\pi ip/q)$ [DH85]. The limb $L_{p/q}$ can be described as follows. These parameter rays together with c_0 cut the complex plane into two sectors (connected components). Denote by S the sector not containing the main cardioid; see Figure 1. It follows that $L_{p/q} = \overline{S} \cap \mathcal{M} = (S \cap \mathcal{M}) \cup \{c_0\}$ (see [Ate92, Mil00b]). There is a unique parameter $c_{p/q}$ in $L_{p/q}$ with a period q critical orbit [DH85]. This parameter $c_{p/q}$ is called the **center** of $L_{p/q}$.

A large class of examples of abstract α -laminations in the p/q -limb arise as the equivalence relations that encode the landing pattern of external rays at the grand orbit of the α -fixed point of Q_c with $c \in L_{p/q}$.

Definition 6.3: For $c \in L_{p/q}$ we let the α -**lamination** $\lambda_\alpha(Q_c)$ of Q_c be the equivalence relation in \mathbb{R}/\mathbb{Z} that identifies two distinct arguments t, s if and

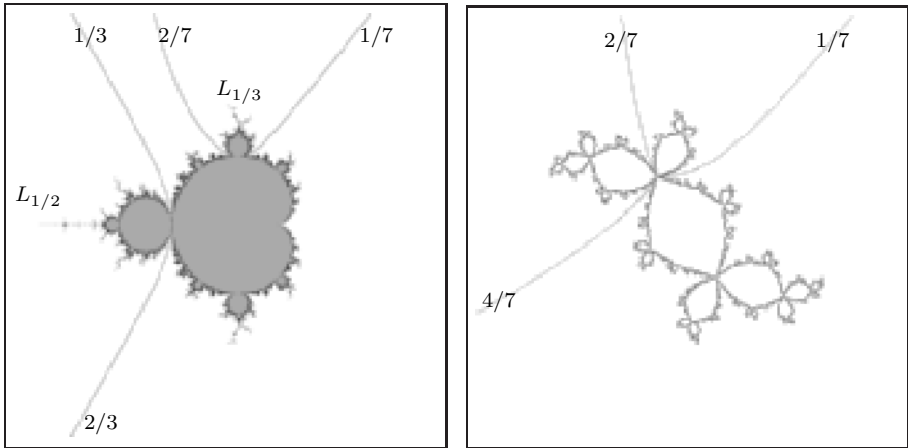


Figure 1. Left: the parameter rays with arguments in $\partial I_{1/2}$ and $\partial I_{1/3}$, and the limbs $L_{1/2}$ and $L_{1/3}$ of the Mandelbrot set. Right: the Julia set of the center $c_{1/3}$ of $L_{1/3}$ together with the rays landing at the α -fixed point.

only if the external rays of Q_c with arguments t, s land at a common point in the grand orbit of the α -fixed point of Q_c .

From [McM94], it follows that $\lambda_\alpha(Q_c)$ is an abstract α -lamination in the p/q -limb. We will show that not all abstract α -laminations arise in quadratic polynomial dynamics (see Remark 6.13). If $\lambda = \lambda_\alpha(Q_c)$ for some c , we simply say that λ is an **α -lamination**.

The α -lamination of a quadratic polynomial Q_c such that the critical point $z = 0$ eventually maps onto its α -fixed point is called a **critically prefixed α -lamination**. For reasons that will be apparent after Proposition 6.14 and Remark 6.13, an abstract α -lamination which is not the α -lamination of a quadratic polynomial is called an **almost critically prefixed lamination**.

For our purpose, it will be also convenient to construct α -laminations via symbolic dynamics (see Section 6.6 below).

6.3. GEOMETRIC REPRESENTATION OF AN α -LAMINATION. Following Thurston, it is useful to have a geometric representation of laminations. We identify \mathbb{R}/\mathbb{Z} with the boundary of the unit disk $\mathbb{D} \subset \mathbb{C}$ via $t \mapsto \exp(2\pi it)$. Given a finite set $A \subset \mathbb{R}/\mathbb{Z}$, denote by $\text{Convex}(A) \subset \overline{\mathbb{D}}$ the convex hull of $A \subset \partial\mathbb{D}$ with respect to

the hyperbolic metric. From the unlinked property of α -laminations, it follows that convex hulls of distinct classes of an abstract α -lamination are disjoint.

Throughout, we identify $\mathbb{P}_{\mathbb{C}}^1$ with $\mathbb{C} \cup \{\infty\}$ via $[z : 1] \mapsto z$ and $[0 : 1] \mapsto \infty$.

Definition 6.4: Consider an equivalence relation λ in \mathbb{R}/\mathbb{Z} such that equivalence classes are finite and pairwise unlinked (see Definition 6.1). We say that the **geometric lamination** $\mathcal{L}_0(\lambda) \subset \mathbb{P}_{\mathbb{C}}^1$ of λ **centered at the origin** is the set formed by the union of all $\text{Convex}(A)$ such that A is a non-trivial λ -class.

Consider the reflection $M(z) = 1/z$ around the unit circle. We say that $\mathcal{L}_{\infty}(\lambda) = M(\mathcal{L}_0(\lambda)) \subset \mathbb{P}_{\mathbb{C}}^1$ is the **geometric lamination of λ centered at infinity**.

For our construction of a tree associated to an abstract α -lamination λ we will “saturate” the “support” of λ by finite sets A_{ℓ} . More precisely, let A_0 be the fixed class of λ and, for all $\ell \geq 1$, let

$$A_{\ell} = m_2^{-\ell}(A_0).$$

For all $\ell \geq 1$, note that $A_{\ell-1} \subset A_{\ell}$ and that each λ -class is either contained in or disjoint from A_{ℓ} . We say that the restriction $\lambda^{(\ell)}$ of λ to A_{ℓ} is the **level ℓ restriction of λ** and that A_{ℓ} is the **level ℓ support of λ** .

If A_{ℓ} is the level ℓ support of a lamination in the p/q -limb, then $-A_{\ell} = M(A_{\ell})$ is the level ℓ support of a lamination in the $-p/q$ -limb. We will work under the agreement that the $-p/q$ -limb denotes the $(q - p)/q$ -limb since $-p/q = (q - p)/q \in \mathbb{R}/\mathbb{Z}$.

The following definition is closely related to the “mating construction” [Dou83], more precisely, to the “Levy cycle” of “obstructed matings” [Tan92].

Definition 6.5: Let λ be an abstract α -lamination in the p/q -limb and λ_* be the α -lamination of the center of the $-p/q$ -limb. Denote by $\mathcal{L}_0(\lambda^{(\ell)})$ the level ℓ geometric lamination of λ centered at the origin and $\mathcal{L}_{\infty}(\lambda_*^{(\ell)})$ the level ℓ geometric lamination of λ_* centered at infinity. We say that the **level ℓ geometric lamination of λ** is

$$\mathcal{L}(\lambda^{(\ell)}) = \mathcal{L}_0(\lambda^{(\ell)}) \cup \mathcal{L}_{\infty}(\lambda_*^{(\ell)}).$$

See the lower left of Figure 2 for the illustration of a geometric lamination.

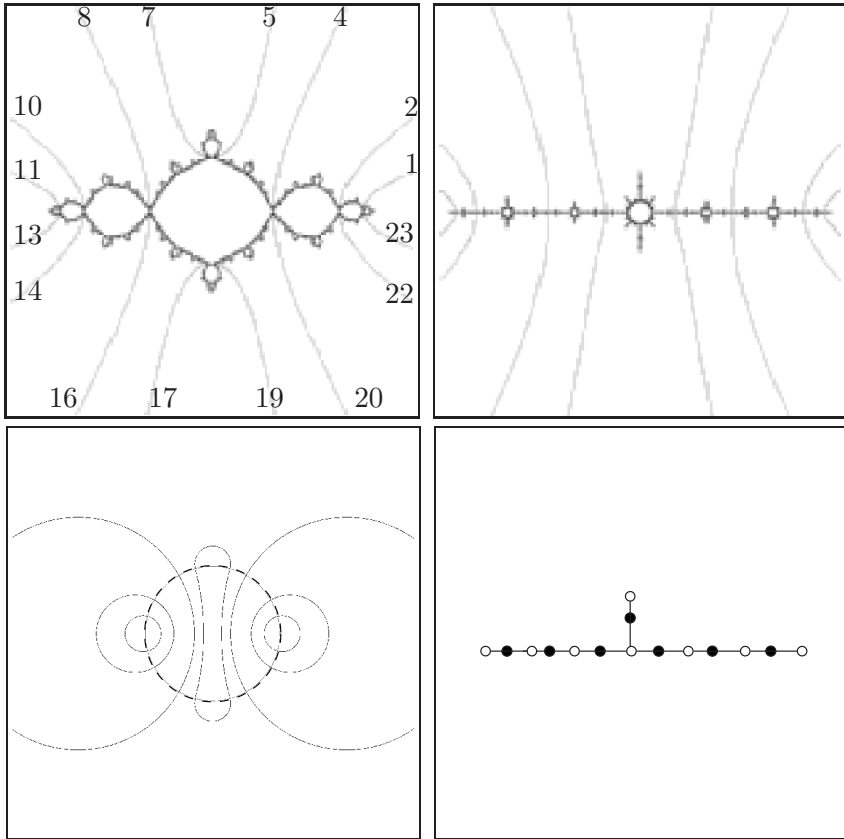


Figure 2. Top left: the Julia sets of the center of the $-1/2 = 1/2$ limb (the basilica). Top right: the Julia set of Q_c for $c = -1.754877666\dots$ (the airplane: $z = 0$ has period 3). In these two figures, we illustrate external rays that land at points which map onto the α -fixed point in at most 3 iterates with angles labelled so that one unit corresponds to $1/24$ of a turn around \mathbb{R}/\mathbb{Z} . Bottom left: the geometric lamination of level $\ell = 3$ of $\lambda_\alpha(Q_c)$ (as a reference the unit circle is illustrated by a broken line). Bottom right: the associated tree of level 3. Dots corresponding to Γ -vertices are filled while Y -vertices are illustrated with unfilled dots.

6.4. THE TREE OF LEVEL ℓ OF AN ABSTRACT α -LAMINATION. The dual tree to a level ℓ geometric lamination is the tree of level ℓ of the α -lamination. (Compare with the discussion and definitions of [McM94, Section 8.5].)

Definition 6.6: Let λ be an abstract α -lamination. Each connected component C of $\mathcal{L}(\lambda^{(\ell)})$ is called a Γ -**vertex of level ℓ** . The collection of all Γ -vertices of level ℓ is denoted by $\Gamma(\lambda^{(\ell)})$. Each connected component U of $\mathbb{P}_\mathbb{C}^1 \setminus \mathcal{L}(\lambda^{(\ell)})$ is called a Y -**vertex of level ℓ** . The collection of all Y -vertices of level ℓ is denoted by $Y(\lambda^{(\ell)})$. The **level ℓ tree $\mathcal{T}(\lambda^{(\ell)})$** is the finite simplicial graph with vertices $\Gamma(\lambda^{(\ell)}) \cup Y(\lambda^{(\ell)})$ such that an edge of the graph joins the vertices $U \in Y(\lambda^{(\ell)})$ and $C \in \Gamma(\lambda^{(\ell)})$ if and only if

$$C \cap \partial U \neq \emptyset.$$

No edges join vertices of the same type.

It will be convenient to agree on a concrete realization of the abstract finite simplicial complex $\mathcal{T}(\lambda^{(\ell)})$. More precisely, let $V = \Gamma(\lambda^{(\ell)}) \cup Y(\lambda^{(\ell)})$. Then we regard V as a subset of Euclidean space $\mathbb{R}^V = \{\tau : V \rightarrow \mathbb{R}\}$ by identifying $v \in V$ with the characteristic function $\delta_v : V \rightarrow \mathbb{R}$, where $\delta_v(w) = 0$ if $w \neq v$, and $\delta_v(v) = 1$. An edge joining v and w is identified with the 1-simplex given by a line segment \mathbb{R}^V joining δ_v and δ_w . We regard this 1-simplex as parametrized by barycentric coordinates: $s \mapsto s\delta_v + (1 - s)\delta_w$, where $s \in [0, 1]$. In order to ease notation we simply write $\mathcal{T}(\lambda^{(\ell)})$ for the abstract simplicial complex or for the concrete simplicial complex, as well as sometimes it will only denote the underlying topological space.

Note that, by the Jordan curve theorem, $\mathcal{T}(\lambda^{(\ell)})$ is a finite simplicial tree. That is, the underlying topological space is contractible.

It follows that $\mathcal{T}(\lambda^{(0)})$ has exactly one Γ -vertex, namely

$$\text{Convex}(A_0) \cup M(\text{Convex}(-A_0)),$$

q Y -vertices, and q edges joining the Y -vertices to the Γ -vertices. That is, $\mathcal{T}(\lambda^{(0)})$ is a starlike tree.

For all $\ell \geq 0$, we will consider two simplicial maps $\pi_{\ell+1}$ and m_2 from $\mathcal{T}(\lambda^{(\ell+1)})$ onto $\mathcal{T}(\lambda^{(\ell)})$. One is of topological nature while the other is of dynamical origin. Recall that a simplicial map between trees is determined by its action on vertices. A map f between vertices of trees is simplicial if, given any edge in the domain, say joining v and w , then $f(v) = f(w)$, or $f(v)$ and $f(w)$ are the

endpoints of an edge in the range. Simplicial maps extend uniquely in a linear fashion to a (simplicial) map between the corresponding concrete simplicial complexes and their underlying spaces.

For short we say that a vertex of $\mathcal{T}(\lambda^{(\ell)})$ is a **vertex of level ℓ** . To simplify notation, vertices of level ℓ will be regarded as elements of a tree and as subsets of $\mathbb{P}_{\mathbb{C}}^1$, according to convenience. Also, when λ is clear from context, we let

$$\mathcal{L}^{(\ell)} = \mathcal{L}(\lambda^{(\ell)}), \mathcal{T}^{(\ell)} = \mathcal{T}(\lambda^{(\ell)}), \dots$$

LEMMA 6.7: *Let λ be an abstract α -lamination. Consider $\ell \geq 0$ and let v be a vertex of level $\ell + 1$. Then the following hold:*

- (1) *There exists a unique vertex $\pi_{\ell+1}(v)$ of level ℓ such that $v \subset \pi_{\ell+1}(v)$.*
- (2) *There exists a unique vertex u of level ℓ such that $m_2(v \cap \mathbb{R}/\mathbb{Z}) = u \cap \mathbb{R}/\mathbb{Z}$. We say that $u = m_2(v)$.*

Moreover, with the above notation:

- (3) *If v and w are endpoints of an edge of level $\ell + 1$, then $\pi_{\ell+1}(v) = \pi_{\ell+1}(w)$ or $\pi_{\ell+1}(v)$ and $\pi_{\ell+1}(w)$ are endpoints of an edge of level ℓ .*
- (4) *If v and w are endpoints of an edge of level $\ell + 1$, then $m_2(v)$ and $m_2(w)$ are endpoints of an edge of level ℓ .*

Thus, $\pi_{\ell+1}$ and m_2 extend uniquely to simplicial maps from $\mathcal{T}(\lambda^{(\ell+1)})$ onto $\mathcal{T}(\lambda^{(\ell)})$. Furthermore, $\pi_{\ell+1}$ is monotone and

$$m_2 \circ \pi_{\ell+1} = \pi_{\ell} \circ m_2.$$

Proof. (1) Since vertices of level ℓ are pairwise disjoint, uniqueness of $\pi_{\ell+1}(v)$ follows. For the existence, note that Γ -vertices of level ℓ are also Γ -vertices of level $\ell + 1$. Hence, a Γ -vertex v of level $\ell + 1$ is already a vertex of level ℓ or is disjoint from the Γ -vertices of level ℓ . Thus, $\pi_{\ell+1}(v) = v$ in the former case, and v is contained in a connected component $v' = \pi_{\ell+1}(v)$ of $\mathbb{P}_{\mathbb{C}}^1 \setminus \mathcal{L}^{(\ell)}$ in the latter. Similarly, every Y -vertex of level $\ell + 1$ is contained in a Y -vertex of level ℓ .

Let us prove statement (2). From the invariance property of α -laminations, if $v \in \Gamma^{(\ell+1)}$, then $m_2(v \cap \mathbb{R}/\mathbb{Z})$ is the intersection of a vertex in $\Gamma^{(\ell)}$ with the unit circle. Now, if $U \in Y^{(\ell+1)}$, then the connected components in $U \cap \mathbb{R}/\mathbb{Z}$ can be joined by a sequence of hyperbolic geodesics contained in U (either through \mathbb{D} or through $\mathbb{P}_{\mathbb{C}}^1 \setminus \overline{\mathbb{D}}$) arbitrarily close to the boundary of U . Assume that two such components of $U \cap \mathbb{R}/\mathbb{Z}$, say $]a_0, b_0[$ and $]a_1, b_1[$, can be joined by a hyperbolic

geodesic inside \mathbb{D} or outside $\overline{\mathbb{D}}$, without crossing $\mathcal{L}^{(\ell+1)}$, and with endpoints arbitrarily close to b_0 and a_1 . By the consecutive preserving property of λ and λ_* , the intervals $]2a_0, 2b_0[$ and $]2a_1, 2b_1[$ may be joined by a hyperbolic geodesic inside \mathbb{D} or outside $\overline{\mathbb{D}}$ without crossing $\mathcal{L}^{(\ell)}$. It follows that $m_2(U \cap \mathbb{R}/\mathbb{Z})$ is contained in a Y -vertex of level ℓ , say V . Moreover, the boundary of V is formed by geodesics having as endpoints the image of endpoints of geodesics in the boundary of U . Hence, $m_2(U \cap \mathbb{R}/\mathbb{Z}) = V \cap \mathbb{R}/\mathbb{Z}$.

(3) and (4), as well as the functional relation between m_2 and π_ℓ , are a straightforward consequence of (1) and (2). ■

To show that there is a natural inclusion of trees of lower levels into the trees of higher levels, we need to establish the following property of the α -lamination of the center of a limb.

LEMMA 6.8: *Let λ_* be the α -lamination of the center of a limb. For all $\ell \geq 0$, if A is a level $\ell + 1$ class which is not a level ℓ class, then the connected component of $\mathbb{R}/\mathbb{Z} \setminus A$ of length greater than $1/2$ contains all level $\ell + 1$ non-trivial classes B such that $B \neq A$.*

Remark 6.9: From Lemma A.6 in [Kiw01], given a non-trivial class A of an abstract α -lamination there exists a connected component of $\mathbb{R}/\mathbb{Z} \setminus A$ of length greater than $1/2$ or $m_2 : A \rightarrow m_2(A)$ is two-to-one. Thus the existence of a connected component as in the lemma is automatically guaranteed.

Proof. Let Q_c be a quadratic polynomial which is the center of a limb.

Note that the Fatou component V of Q_c containing the critical point $z = 0$ contains the α fixed point in its boundary. Given a non-trivial class A of $\lambda_* = \lambda_\alpha(Q_c)$, denote by I_A the connected component of $\mathbb{R}/\mathbb{Z} \setminus A$ of length greater than $1/2$. We claim that the fixed class A_0 is contained in I_A , provided that $A \neq A_0$. In fact, the extreme points of I_A correspond to the arguments of external rays which, together with their common landing point z_A , cut the complex plane into two regions. The one containing all the rays with arguments in I_A also contains the critical point (e.g., see [Mil00b]), therefore it contains the Fatou component V as well as the α fixed point and the rays landing at it. Thus, $A_0 \subset I_A$. Similarly, if $A \neq A_0 + 1/2$, then $A_0 + 1/2 \subset I_A$.

To prove the lemma we proceed by induction. Since the non-trivial level 1 classes are the fixed class A_0 and the prefixed class $A_0 + 1/2$, the statement clearly holds for $\ell = 0$. We assume that the statement is true for $\ell \geq 0$ and, by

contradiction, we prove that the statement is true for $\ell + 1$. That is, let A be a non-trivial class of level $\ell + 2$ which is not of level $\ell + 1$. Assume that there exists a level $\ell + 2$ class $B \neq A$ such that $B \subset \mathbb{R}/\mathbb{Z} \setminus I_A$. It follows that m_2 is injective on $\mathbb{R}/\mathbb{Z} \setminus I_A$, hence $m_2(B)$ is a class of level $\ell + 1$ contained in $m_2(\mathbb{R}/\mathbb{Z} \setminus I_A)$. But since A_0 and $A_0 + 1/2$ are contained in I_A , we must have that A_0 is contained in the interval $\mathbb{R}/\mathbb{Z} \setminus m_2(\mathbb{R}/\mathbb{Z} \setminus I_A)$. Thus $I_{m_2(A)} = m_2(\mathbb{R}/\mathbb{Z} \setminus I_A)$, which contradicts the inductive hypothesis since $m_2(B)$ is not contained in $I_{m_2(A)}$. ■

Now we show that the map $\pi_{\ell+1}$ is, in fact, a deformation retract. We will also prove that edges are not “subdivided” as we increase the level.

LEMMA 6.10: *Let λ be an abstract α -lamination. Let $\ell \geq 0$ and v be a Y -vertex of level ℓ . Then there exists a unique Y -vertex $\iota_\ell(v)$ of level $\ell + 1$ such that:*

$$\begin{aligned} \pi_{\ell+1}(\iota_\ell(v)) &= v, \\ \partial v &\subset \partial \iota_\ell(v). \end{aligned}$$

Given a Γ -vertex v of level ℓ define $\iota_\ell(v) = v \in \Gamma^{(\ell+1)}$. Then ι_ℓ determines an injective simplicial map $\iota_\ell : \mathcal{T}(\lambda^{(\ell)}) \rightarrow \mathcal{T}(\lambda^{(\ell+1)})$. Moreover, $\pi_{\ell+1} \circ \iota_\ell = \text{id}$ and $\iota_\ell \circ \pi_{\ell+1}$ is homotopic to the identity $\text{rel } \mathcal{T}(\lambda^{(\ell)})$.

Proof. Consider a Y -vertex v of level ℓ . From the previous lemma, it follows that two connected components of ∂v cannot be separated by a connected component of $\mathcal{L}^{(\ell+1)}$. Hence, there exists a Y -vertex $\iota_\ell(v)$ of level $\ell + 1$ which is obtained from v after removing finitely many topological disks completely contained in v . This vertex $\iota_\ell(v)$ has the desired properties.

Observe that given any two endpoints $v \in Y^{(\ell)}$ and $w \in \Gamma^{(\ell)}$ of an edge in $\mathcal{T}^{(\ell)}$, by definition we have that $\partial v \cap w \neq \emptyset$. From the previous paragraph, $\partial \iota_\ell(v) \cap \iota_\ell(w) = \partial \iota_\ell(v) \cap w \neq \emptyset$. Therefore, $\iota_\ell(v)$ and $\iota_\ell(w)$ are endpoints of an edge in $\mathcal{T}^{(\ell+1)}$. That is, ι_ℓ determines a simplicial map between the corresponding trees. It is not difficult to see that ι_ℓ has the desired properties. ■

Note that ι_ℓ is an inclusion of simplicial trees. Hence, it uniquely defines an inclusion of the underlying spaces. So one may regard the trees $\mathcal{T}^{(\ell)}$ as an increasing sequence of simplicial trees and/or topological spaces. It is important to stress that edges of level ℓ are not subdivided when included in the tree of level $\ell + 1$ via the map ι_ℓ .

6.5. THE FULL TREE OF AN α -LAMINATION. From Lemma 6.7, we conclude that

$$(\mathcal{T}(\lambda^{(\ell+1)}), \pi_{\ell+1})$$

is an inverse system of (contractible) topological spaces.

Definition 6.11: Let λ be an abstract α -lamination. The **full tree of λ** is the topological space

$$\mathcal{T}^\infty(\lambda) = \varprojlim (\mathcal{T}^{(\ell+1)}(\lambda), \pi_{\ell+1}).$$

From Lemma 6.7, the action of m_2 extends to a continuous map

$$m_2 : \mathcal{T}^\infty(\lambda) \rightarrow \mathcal{T}^\infty(\lambda).$$

We will show that in a great variety of cases the action of a quadratic rational map $\varphi : \mathbb{P}_{\mathbb{L}}^{1,an} \rightarrow \mathbb{P}_{\mathbb{L}}^{1,an}$ on the (Berkovich) convex hull of its Julia set is topologically conjugate to $m_2 : \mathcal{T}^\infty(\lambda) \rightarrow \mathcal{T}^\infty(\lambda)$ for an appropriate λ . The Julia set of φ will correspond to the inverse limit of Y -vertices.

6.6. CONSTRUCTION OF α -LAMINATIONS VIA SYMBOLIC DYNAMICS. Our aim here is to show how via symbolic dynamics of $m_2 : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ we may construct abstract α -laminations in the p/q -limb. Recall that $I_{p/q}$ denotes the characteristic interval of the p/q -limb (see Definition 6.2). For each argument $\theta \in \overline{I_{p/q}}$ we will produce at least one and at most three abstract α -laminations.

For any $\theta \in \overline{I_{p/q}}$ we consider two partitions $\{I_0^-(\theta), I_1^-(\theta)\}$ and $\{I_0^+(\theta), I_1^+(\theta)\}$ of \mathbb{R}/\mathbb{Z} into semicircles where

$$I_0^+(\theta) = [\theta/2 + 1/2, \theta/2[, \quad I_0^-(\theta) =]\theta/2 + 1/2, \theta/2],$$

and

$$I_1^+(\theta) = [\theta/2, \theta/2 + 1/2[, \quad I_1^-(\theta) =]\theta/2, \theta/2 + 1/2].$$

It follows that A_0 is contained in exactly one element of each partition, for all $\theta \in \overline{I_{p/q}}$.

We will construct abstract α -laminations using the itinerary of m_2 -orbits according to each one of the two partitions of \mathbb{R}/\mathbb{Z} . Let $\Sigma = \{0, 1\}^{\mathbb{N} \cup \{0\}}$ and, for $\epsilon = +$ or $-$, define

$$it_\theta^\epsilon(t) = (i_0, i_1, \dots) \in \Sigma$$

if $m_2^k(t) \in I_{i_k}^\epsilon(\theta)$. The equivalence relation $\lambda^\epsilon(\theta)$ is the relation that identifies two distinct arguments s and t if and only if s, t belong to the m_2 -grand orbit of A_0 and $it_\theta^\epsilon(t) = it_\theta^\epsilon(s)$.

One may produce a new equivalence relation from $\lambda^\pm(\theta)$. Namely, we let $\lambda(\theta)$ be the smallest equivalence relation that contains both $\lambda^+(\theta)$ and $\lambda^-(\theta)$.

LEMMA 6.12: *If $\theta \in I_{p/q} =]\theta_0, \theta_1[$, then $\lambda(\theta)$, $\lambda^+(\theta)$ and $\lambda^-(\theta)$ are abstract α -laminations in the p/q -limb. Moreover, denote by λ_* the α -lamination of the center of the p/q -limb. Then $\lambda_* = \lambda^+(\theta_1) = \lambda^-(\theta_0)$.*

Proof. We start proving that $\lambda^\pm(\theta)$ are abstract α -laminations, for all $\theta \in I_{p/q}$. For $j = 0, 1$, the map $m_2 : I_j^\pm(\theta) \rightarrow \mathbb{R}/\mathbb{Z}$ preserves cyclic order and is onto. Hence, $\lambda^\pm(\theta)$ satisfies the invariant and consecutive preserving property. Thus, if $m_2^\ell(t)$ is periodic, for some $\ell \geq 0$, then the same holds for all arguments s which are $\lambda^\pm(\theta)$ -equivalent to t . It follows that $\lambda^\pm(\theta)$ -classes are finite. A pair of distinct classes B_1 and B_2 are either contained in the same half circle or unlinked. In the former case, B_1 and B_2 are unlinked if and only if $m_2(B_1)$ and $m_2(B_2)$ are unlinked. The unlinked property for $\lambda^\pm(\theta)$ follows.

Now we show that $\lambda(\theta)$ is an abstract α -lamination, for all $\theta \in I_{p/q}$. The only relevant case is when $\theta \notin A_0$ and $m_2^{\ell_0}(\theta) \in A_0$ for some $\ell_0 \geq 1$. The $\lambda(\theta)$ -class of $\theta/2$ is the union of its $\lambda^+(\theta)$ -class with the $\lambda^+(\theta)$ -class of $\theta/2 + 1/2$. It follows that a $\lambda(\theta)$ -class consists of points with the same itinerary or it eventually maps onto the class of $\theta/2$ through a cyclic order preserving map. Therefore, we may apply a similar reasoning to conclude that $\lambda(\theta)$ is an abstract α -lamination. (According to [BFH92, DH85] the lamination $\lambda(\theta)$ is the α -lamination of a quadratic polynomial where the critical point eventually maps to the α fixed point, that is, a critically prefixed lamination.)

Finally, the α -lamination λ_* of the quadratic polynomial in the p/q -limb with a period q critical orbit is $\lambda_* = \lambda^+(\theta_1) = \lambda^-(\theta_0)$, according to Poirier’s description [Poi09]. ■

If $m_2^n(\theta) \notin A_0$ for all $n \geq 0$, then it is fairly easy to check that $\lambda(\theta) = \lambda^+(\theta) = \lambda^-(\theta)$.

If $m_2^n(\theta) \in A_0$ for some smallest $n \geq 1$, then $\lambda^+(\theta) \neq \lambda^-(\theta)$. In fact, the $\lambda^+(\theta)$ -class of 2θ and the $\lambda^-(\theta)$ -class of 2θ agree. Denote this common class by B . It follows that the $\lambda^+(\theta)$ -class of θ is $m_2^{-1}(B) \cap [\theta, \theta + 1/2[$ while the $\lambda^-(\theta)$ -class of θ is $m_2^{-1}(B) \cap]\theta/2 + 1/2, \theta]$.

Remark 6.13: According [DH85], given $\theta \in I_{p/q}$, if $c \in \mathcal{M}$ is the accumulation point of the parameter ray $R_{\mathcal{M}}^\theta$, then $\lambda_\alpha(Q_c) = \lambda(\theta)$. Moreover, every

α -lamination is the α -lamination of a quadratic polynomial which is the accumulation point of some parameter ray. It follows that an abstract α -lamination λ is not an α -lamination if $\lambda = \lambda^+(\theta)$ or $\lambda = \lambda^-(\theta)$ for some $\theta \in I_{p/q}$ which eventually maps into the fixed class.

With little more work, given an abstract α -lamination λ , it is possible to find θ such that one of the laminations $\lambda(\theta), \lambda^\pm(\theta)$ coincides with λ :

PROPOSITION 6.14: *Let λ be an abstract α -lamination in the p/q -limb. Denote by $I_{p/q}$ the corresponding characteristic interval. Then there exists $\theta \in \overline{I_{p/q}}$ such that $\lambda = \lambda(\theta)$, or $\lambda = \lambda^+(\theta)$, or $\lambda = \lambda^-(\theta)$.*

First we show that given an abstract α -lamination λ and a level ℓ , we may find $\theta \in \mathbb{R}/\mathbb{Z}$ such that the level ℓ restriction of $\lambda(\theta)$ and λ coincide:

LEMMA 6.15: *Let λ be an abstract α -lamination. Denote by $\lambda^{(\ell)}$ its level ℓ restriction. Consider*

$$I^{(\ell)} = \{\theta \in \mathbb{R}/\mathbb{Z} \mid \text{Convex}(\{\theta/2, \theta/2 + 1/2\}) \cap \mathcal{L}^{(\ell)} = \emptyset\}.$$

Then exactly one of the following holds:

- (1) $I^{(\ell)} \neq \emptyset$. In this case,

$$\lambda^+(\theta)^{(\ell)} = \lambda^-(\theta)^{(\ell)} = \lambda^{(\ell)}$$

for all $\theta \in I^{(\ell)}$.

- (2) $I^{(\ell)} = \emptyset$ and there exists a $\lambda^{(\ell)}$ -class B such that $m_2 : B \rightarrow m_2(B)$ is two-to-one. In this case, for all $\theta \in m_2(B)$,

$$\lambda(\theta)^{(\ell)} = \lambda^{(\ell)}.$$

Proof. From Lemma A.6 in [Kiw01], either $I^{(\ell)} \neq \emptyset$ or there exists a $\lambda^{(\ell)}$ -class B such that $m_2 : B \rightarrow m_2(B)$ is two-to-one.

In case (1), since $\text{Convex}(\{\theta/2, \theta/2 + 1/2\})$ is disjoint from $\mathcal{L}^{(\ell)}$, we have that each $\lambda^{(\ell)}$ -class is contained in $] \theta/2, \theta/2 + 1/2[$ or in $] \theta/2 + 1/2, \theta/2[$. Therefore, $\lambda^{(\ell)} \subset \lambda^\pm(\theta)^{(\ell)}$. Taking into account that every class of $\lambda^{(\ell)}$ as well as every class of $\lambda^\pm(\theta)^{(\ell)}$ has exactly q elements, we conclude that $\lambda^{(\ell)} = \lambda^\pm(\theta)^{(\ell)}$.

In case (2), since $\{\theta/2, \theta/2 + 1/2\} \subset B$, each $\lambda^{(\ell)}$ -class different from B is contained in $] \theta/2, \theta/2 + 1/2[$ or in $] \theta/2 + 1/2, \theta/2[$. Hence, as in the previous case, every $\lambda^{(\ell)}$ -class that is not eventually mapped onto B is a $\lambda(\theta)^{(\ell)}$ -class.

Note that $it_\theta^+(\theta/2) = it_\theta^-(\theta/2 + 1/2)$. Since $it_\theta^+(\theta/2)$ coincides with $it_\theta^+(t)$ for all $t \in B \cap I_1^+(\theta)$ and $it_\theta^+(\theta/2 + 1/2)$ coincides with $it_\theta^+(t)$ for all $t \in B \cap I_0^+(\theta)$, it follows that B is contained in a $\lambda(\theta)^{(\ell)}$ -class. Taking into account that $\lambda(\theta)^{(\ell)}$ -classes have at most $2q$ elements, it follows that B is a $\lambda(\theta)^{(\ell)}$ -class.

Now we assume that $A \neq B$ is a $\lambda^{(\ell)}$ -class such that $m_2(A)$ is a $\lambda(\theta)^{(\ell)}$ -class. We claim that A is a $\lambda(\theta)^{(\ell)}$ -class. By the unlinked property of λ , the class A is contained in the interior of $I_0^+(\theta)$ or of $I_1^+(\theta)$. Again by the unlinked property, if $A \ni t$ and $t \in I_0^+(\theta)$ (resp. $t \in I_1^+(\theta)$) then the $\lambda(\theta)^{(\ell)}$ -class A' of t is contained in $I_0^+(\theta)$ (resp. $I_1^+(\theta)$). By the invariance property of both laminations, $m_2(A') = m_2(A)$. Hence, $A' = A$.

Since every $\lambda^{(\ell)}$ -class eventually maps to B or it is a $\lambda(\theta)^{(\ell)}$ -class, it follows that $\lambda^{(\ell)} = \lambda(\theta)^{(\ell)}$. ■

Proof of Proposition 6.14. Given an abstract α -lamination λ in p/q -limb, for all $\ell \geq 0$, consider the decreasing collection $\{I^{(\ell)}\}$ of subsets of \mathbb{R}/\mathbb{Z} as in the previous lemma. If $I^{(\ell)} \neq \emptyset$ for all ℓ , choose $\theta_\ell \in I^{(\ell)}$ so that the sequence $\{\theta_\ell\}$ is monotone. Without loss of generality we assume that this sequence converges to θ and, to fix ideas, suppose that it is increasing (with respect to the order in the interval $]\theta - 1, \theta[$). It follows that, for all $t \in \mathbb{R}/\mathbb{Z}$,

$$\lim_{\ell \rightarrow \infty} it_{\theta_\ell}^+(t) = it_\theta^+(t).$$

In particular, $\lambda^+(\theta)^{(\ell)} = \lambda^{(\ell)}$ for all $\ell \geq 0$. Therefore, $\lambda^+(\theta) = \lambda$.

Now if $I^{(\ell_0)} = \emptyset$ for some ℓ_0 , then B as in part (2) of the previous lemma coincides for all $\ell \geq \ell_0$. Choosing $\theta \in B$ it follows that $\lambda(\theta) = \lambda$. ■

We will also need to establish the following properties of the α -lamination of the center of the p/q -limb.

LEMMA 6.16: *Let λ_* be the lamination of the center of the p/q -limb. For $\ell \geq 0$, denote by $\mathcal{T}_*^{(\ell)}$ its tree of level ℓ and by $v_*^{(\ell)}(x)$ the vertex containing $x \in \mathbb{C} \cup \{\infty\}$. Then, for all $\ell \geq 0$,*

$$v_*^{(\ell+1)}(0) = v_*^{(\ell+1)}(\infty).$$

Moreover, for all $\theta \in m_2(v_*^{(\ell+1)}(0))$, we have that

$$\mathcal{T}_*^{(\ell+1)} = \mathcal{T}^{(\ell+1)}(\theta).$$

Proof. First we observe that A is a class of the α -lamination of the center of the $-p/q$ if and only if $-A$ is a class of λ_* . It follows that $\mathcal{L}(\lambda_*^{(\ell+1)})$ is invariant under $1/\bar{z}$.

Consider $\theta \in m_2(v_*^{(\ell+1)}(0))$. Since $\theta/2$ and $\theta/2 + 1/2$ lie in $v_*^{(\ell+1)}(0)$, there is a curve $\gamma \subset v_*^{(\ell+1)}(0)$ connecting $\theta/2$ and $\theta/2 + 1/2$ formed by concatenating geodesics in \mathbb{D} or outside $\bar{\mathbb{D}}$. By the invariance of $\mathcal{L}(\lambda_*^{(\ell+1)})$ under $1/\bar{z}$, we may replace the geodesics outside $\bar{\mathbb{D}}$ in γ by geodesics contained in \mathbb{D} . Thus, we can connect $\theta/2$ and $\theta/2 + 1/2$ through a sequence of geodesic paths contained in $v_*^{(\ell+1)}(0) \cap \mathbb{D}$. By convexity, it follows that the diameter joining $\theta/2$ and $\theta/2 + 1/2$ is disjoint from $\mathcal{L}(\lambda_*^{(\ell+1)})$. By $1/\bar{z}$ -invariance, $v_*^{(\ell+1)}(0) = v_*^{(\ell+1)}(\infty)$. From Lemma 6.15 we obtain that $\lambda_*^{(\ell+1)} = \lambda^{(\ell+1)}(\theta)$. ■

6.7. BRANCHED MAPS OF TREES. Multiplication by 2 on \mathbb{R}/\mathbb{Z} acts on level ℓ trees as a two-to-one map ramified over an interval.

Definition 6.17: Consider finite simplicial trees $\mathcal{T}, \mathcal{T}'$ and let $g : \mathcal{T} \rightarrow \mathcal{T}'$ be a simplicial map. Let $I \subset \mathcal{T}'$ be a simplicial subcomplex homeomorphic to a closed interval or a singleton. We say that g is a **degree two map branched over I** if the following holds: there exists a tree automorphism $\gamma : \mathcal{T} \rightarrow \mathcal{T}$ of order 2 (i.e., a simplicial involution) such that $I = g(\text{Fix}(\gamma))$, and $g(x) = g(y)$ if and only if $x = y$ or $y = \gamma(x)$. We say that $J = g^{-1}(I)$ is the **critical interval of g** and I is the **critical value interval of g** .

As mentioned above our definition is tailored to include the action of multiplication by 2 on trees associated to α -laminations.

LEMMA 6.18: *Let λ be an abstract α -lamination and, for $\ell \geq 0$, let $\mathcal{T}^{(\ell)}$ be the associated level ℓ tree. Denote by $v_\ell(0)$ and $v_\ell(\infty)$ the vertices of $\mathcal{T}^{(\ell)}$ containing 0 and ∞ , respectively. Then $m_2 : \mathcal{T}^{(\ell+1)} \rightarrow \mathcal{T}^{(\ell)}$ is a degree 2 branched map over the interval $I = [m_2(v_{\ell+1}(0)), m_2(v_{\ell+1}(\infty))]$.*

Proof. Recall that the vertices of $\mathcal{T}^{(\ell)}$ are subsets of $\mathbb{P}_{\mathbb{C}}^1 = \mathbb{C} \cup \{\infty\}$. For all $\ell \geq 0$, we have that v is a vertex of $\mathcal{T}^{(\ell+1)}$ if and only if $-v$ is a vertex of $\mathcal{T}^{(\ell+1)}$. Moreover, $m_2(v) = m_2(-v)$.

Now let $\gamma : \mathcal{T}^{(\ell+1)} \rightarrow \mathcal{T}^{(\ell+1)}$ be the involution induced by $v \mapsto -v$. We claim that given a level $\ell + 1$ vertex v , we have that $v = -v$ if and only if one of the following occurs:

- (1) ∂v separates 0 and ∞ .

- (2) $0 \in v$.
- (3) $\infty \in v$.

The claim is rather immediate when $\{0, \infty\} \cap v \neq \emptyset$ so we assume that $\{0, \infty\} \cap v = \emptyset$. We let W denote the connected component of $\mathbb{P}_{\mathbb{C}}^1 \setminus v$ that contains ∞ . Now $v = -v$ if and only if $\partial W = \partial(-W)$, which implies that 0 and ∞ are separated by ∂W . Conversely, if ∂W separates 0 and ∞ , then $\partial W \cap \partial(-W) \neq \emptyset$. Since ∂W and $\partial(-W)$ are equal or disjoint the claim follows.

The vertices that separate 0 from ∞ together with the ones that contain 0 or ∞ form a (possibly degenerate) interval, and the lemma follows. ■

We will need the following “lifting” property in the process of establishing a topological conjugacy between maps acting on trees associated to abstract α -laminations and the dynamics of some degree two rational maps acting on the convex hull of their Julia sets in $\mathbb{P}_{\mathbb{L}}^{1,an}$.

LEMMA 6.19: Consider simplicial trees \mathcal{A}' and \mathcal{T}' with subtrees $\mathcal{A} \subset \mathcal{A}'$ and $\mathcal{T} \subset \mathcal{T}'$. Suppose that $\varphi : \mathcal{A}' \rightarrow \mathcal{A}$ and $m : \mathcal{T}' \rightarrow \mathcal{T}$ are degree two branched maps over $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{I}_{\mathcal{T}}$, respectively, such that $\varphi(\mathcal{A}) \subset \mathcal{A}$ and $m(\mathcal{T}) \subset \mathcal{T}$. Also, suppose that $\varphi : \mathcal{A} \rightarrow \varphi(\mathcal{A})$ and $m : \mathcal{T} \rightarrow m(\mathcal{T})$ are degree two branched maps.

Assume that $h : \mathcal{A} \rightarrow \mathcal{T}$ is a tree isomorphism such that the following hold:

- (1) The following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\varphi} & \mathcal{A} \\
 h \downarrow & & h \downarrow \\
 \mathcal{T} & \xrightarrow{m} & \mathcal{T}.
 \end{array}$$

- (2) $h(\mathcal{I}_{\mathcal{A}}) = \mathcal{I}_{\mathcal{T}}$.

Then there exists a tree isomorphism $h' : \mathcal{A}' \rightarrow \mathcal{T}'$ such that the following hold:

- (1) The following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{A}' & \xrightarrow{\varphi} & \mathcal{A} \\
 h' \downarrow & & h \downarrow \\
 \mathcal{T}' & \xrightarrow{m} & \mathcal{T}.
 \end{array}$$

- (2) $h'|_{\mathcal{A}} = h$.

Proof. Without loss of generality we may assume that \mathcal{A} and \mathcal{T} contain $\varphi^{-1}(\mathcal{I}_{\mathcal{A}})$ and $m^{-1}(\mathcal{I}_{\mathcal{T}})$, since both φ and m are bijections over these sets. Now consider two connected components $\mathcal{A}_0, \mathcal{A}_1$ of $\mathcal{A}' \setminus \mathcal{A}$ such that $\varphi(\mathcal{A}_0) = \varphi(\mathcal{A}_1)$. Let $\{x_j\} = \overline{\mathcal{A}_j} \cap \mathcal{A}$ for $j = 0, 1$. Note that m has two inverse branches m_0 and m_1 defined on $h(\varphi(\overline{\mathcal{A}_0}))$ and $h(\varphi(\overline{\mathcal{A}_1}))$ which are continuous bijections into \mathcal{T}' such that $h(x_j) = m_j(h(\varphi(x_j)))$. Define h' on \mathcal{A}_j as $h' = m_j \circ h \circ \varphi$. It is not difficult to check that h' has the desired properties. ■

7. Topological model

For any subset $X \subset \mathbb{P}_{\mathbb{L}}^{1,an}$ we denote by \widehat{X} its convex hull in $\mathbb{P}_{\mathbb{L}}^{1,an}$. Namely \widehat{X} is the union of all arcs $[x_0, x_1]$ where $x_0, x_1 \in X$.

Here we give a complete description of $\widehat{J(\varphi)}$ for quadratic rational maps possessing a non-rigid repelling periodic orbit. We will show that $\widehat{J(\varphi)}$ is completely invariant under φ (see Lemma 7.5 below). Moreover, we will describe the structure and dynamics over $\widehat{J(\varphi)}$ with the aid of the tree associated to an appropriate abstract α -lamination (see Section 6).

The (simultaneous) proof of the following two propositions is given in Section 7.4.

PROPOSITION 7.1: *Let φ be a quadratic rational map over \mathbb{L} which is not simple. Assume that φ has a non-rigid repelling periodic orbit and that both critical points belong to the filled Julia set. Then there exists an abstract α -lamination λ which is not critically prefixed such that $\varphi : \widehat{J(\varphi)} \rightarrow \widehat{J(\varphi)}$ is topologically conjugate to $m_2 : \mathcal{T}^{(\infty)}(\lambda) \rightarrow \mathcal{T}^{(\infty)}(\lambda)$.*

Definition 7.2: Let φ be a quadratic rational map over \mathbb{L} which is not simple. Assume that φ has a non-rigid repelling periodic orbit and that one critical point ω is not in $K(\varphi)$. Denote by $\pi_{U_0} : U_0 \rightarrow \mathcal{A}_0$ the projection of the fixed Rivera domain U_0 into its skeleton (see Section 1.3.1). Then

$$\mathcal{O}_0 = \pi_{U_0}(\mathcal{O}(\omega) \cap U_0)$$

is a periodic orbit (see Section 2.3). We define \sim to be the equivalence relation in $\widehat{J(\varphi)}$ that identifies two distinct elements ζ_1, ζ_2 if and only if ζ_1, ζ_2 lie in the same connected component of $\varphi^{-n}(\widehat{\mathcal{O}_0})$ for some $n \geq 0$.

PROPOSITION 7.3: *Let φ be a quadratic rational map over \mathbb{L} which is not simple. Assume that φ has a non-rigid repelling periodic orbit and that one critical point ω is not in $K(\varphi)$. Then the following map is well defined:*

$$\psi : \widehat{J(\varphi)} / \sim \rightarrow \widehat{J(\varphi)} / \sim$$

$$[\zeta] \mapsto [\varphi(\zeta)]$$

where $[\zeta]$ denotes the \sim -class of ζ .

Moreover, there exists a critically prefixed α -lamination λ and a homeomorphism $h : \widehat{J(\varphi)} / \sim \rightarrow \mathcal{T}^{(\infty)}(\lambda)$ which conjugates the action of ψ with that of m_2 (i.e., $m_2 \circ h = h \circ \psi$).

Remark 7.4: Note that \sim classes are either trivial or contained in a Fatou component which eventually maps onto U_0 . Thus h is a conjugacy over the Julia set. Sometimes $\widehat{\mathcal{O}}_0$ will be simply the singleton $\{\vartheta_0\}$ (see Lemma 5.1). In that case, \sim is the trivial relation and the above proposition gives a topological model for the dynamics over $\widehat{J(\varphi)}$.

In Section 7.1 we show that $\widehat{J(\varphi)}$ is completely invariant and simultaneously introduce an increasing sequence of trees $\mathcal{A}^{(\ell)} \subset \widehat{J(\varphi)}$ whose union is dense and such that $\varphi : \mathcal{A}^{(\ell+1)} \rightarrow \mathcal{A}^{(\ell)}$ is a degree two branched map. The construction of the (semi)conjugacies will rely on choosing a decreasing sequence of subsets $v_{\mathcal{M}}^{(\ell)}$ of \mathbb{R}/\mathbb{Z} so that for all $\theta \in v_{\mathcal{M}}^{(\ell)}$ we are able to construct (semi)conjugacies with domain $\mathcal{A}^{(\ell)}$ and range $\mathcal{T}(\lambda(\theta))^{(\ell)}$. In Section 7.2 we prepare to treat at once the cases in which $\text{Crit}(\varphi) \subset K(\varphi)$ and $\text{Crit}(\varphi) \not\subset K(\varphi)$. To spread conjugacies defined on $\mathcal{A}^{(\ell)}$ to $\mathcal{A}^{(\ell+1)}$ we employ Lemma 6.19, but before we need some control over the branched intervals. This control is achieved in Section 7.3 by introducing appropriate subsets of the trees involved which contain the “post-branched” points. Finally, Section 7.4 contains the proof of Propositions 7.1 and 7.3 as well as the proof of Theorem 3.

7.1. TREES OF LEVEL ℓ AND $\widehat{J(\varphi)}$. The proofs of Propositions 7.1 and 7.3 rely on the following basic facts about $\widehat{J(\varphi)}$.

We will continue to work under the standing assumption that the quadratic rational map φ has a fixed Rivera domain U_0 whose boundary is a period $q > 1$ repelling orbit \mathcal{O} . We will use the notation of Lemma 5.1. Also, we let ω and ω' be the critical points of φ where ω is the active critical point (see Definition 5.12). Furthermore, let $\text{Crit}(\varphi) = \{\omega, \omega'\}$.

LEMMA 7.5: Let $\mathcal{A}^{(0)}$ be the skeleton of U_0 . For $\ell \geq 0$ recursively define

$$\mathcal{A}^{(\ell+1)} = \varphi^{-1}(\mathcal{A}^{(\ell)}).$$

The following statements hold for all $\ell \geq 0$:

- (1) $\mathcal{A}^{(\ell)} \subset \mathcal{A}^{(\ell+1)}$.
- (2) $\mathcal{A}^{(\ell)}$ is the convex hull of $\varphi^{-\ell}(\mathcal{O})$, in particular, $\mathcal{A}^{(\ell)}$ is connected.
- (3) If $x \in \varphi^{-(\ell+1)}(\mathcal{O}) \setminus \varphi^{-\ell}(\mathcal{O})$, then x is an endpoint of $\mathcal{A}^{(\ell+1)}$.
- (4) For all $\zeta \in \mathcal{A}^{(\ell)} \setminus GO(\mathcal{O})$, if D is a direction at ζ disjoint from $\mathcal{A}^{(\ell)}$, then $\varphi^\ell(D) \subset U_0$.
- (5)

$$\widehat{J(\varphi)} = \bigcup_{\ell \geq 0} \overline{\mathcal{A}^{(\ell)}} \supset \overline{GO(\mathcal{O})} = J(\varphi).$$

- (6) $\widehat{J(\varphi)}$ is completely invariant.

Proof. Statement (1) is a consequence of the forward invariance of the initial skeleton (i.e., $\varphi(\mathcal{A}^{(0)}) = \mathcal{A}^{(0)}$).

For (2) we claim that $\mathcal{A}^{(\ell)}$ is connected for all ℓ and prove this claim by induction. Since φ is an open map, every connected component of $\mathcal{A}^{(\ell+1)} = \varphi^{-1}(\mathcal{A}^{(\ell)})$ maps onto $\mathcal{A}^{(\ell)}$, but the preimage of the singleton $\partial B_1 \subset \mathcal{A}^{(0)}$ is the singleton ∂B_0 . Hence, $\mathcal{A}^{(\ell+1)}$ is connected and the claim follows. Now the endpoints of $\mathcal{A}^{(0)}$ are contained in \mathcal{O} , hence the endpoints of $\mathcal{A}^{(\ell)}$ are contained in $\varphi^{-\ell}(\mathcal{O})$. Since every closed and connected subset of $\mathbb{P}_{\mathbb{L}}^{1,an}$ is convex, we have established (2).

For (3) we proceed by contradiction. Suppose that $x \in \varphi^{-(\ell+1)}(\mathcal{O}) \setminus \varphi^{-\ell}(\mathcal{O})$ is not an endpoint of $\mathcal{A}^{(\ell+1)}$. Let C be a connected component of $\mathcal{A}^{(\ell+1)} \setminus \{x\}$ not containing $\mathcal{A}^{(\ell)}$. Then \overline{C} contains a component \mathcal{A}_U of $\varphi^{-(\ell+1)}(\mathcal{A}^{(0)})$ such that $x \in \mathcal{A}_U$. All the endpoints of \mathcal{A}_U lie in $\mathcal{A}^{(\ell+1)} \setminus \mathcal{A}^{(\ell)}$. Hence $\varphi^\ell(\mathcal{A}_U) \subset \mathcal{A}^{(1)} \setminus \mathcal{A}^{(0)}$, but then $\mathcal{A}^{(0)} = \varphi^{\ell+1}(\mathcal{A}_U)$ would be a subset of $\varphi(\mathcal{A}^{(1)} \setminus \mathcal{A}^{(0)}) = \mathcal{A}^{(0)} \setminus \{\xi_1\}$, which is a contradiction.

We prove (4) by induction. In fact, for $\ell = 0$ the statement holds trivially. For $\ell \geq 1$, given a direction D , as in (4), we have that D does not contain ξ_0 and therefore is a good direction at ζ . It follows that $\varphi(D)$ is a direction disjoint from $\mathcal{A}^{(\ell-1)}$ at $\varphi(\zeta) \in \mathcal{A}^{(\ell-1)} \setminus GO(\mathcal{O})$. By the inductive hypothesis, $\varphi^{\ell-1}(\varphi(D))$ is contained in U_0 .

For (5), note that by (2) we have

$$\widehat{J(\varphi)} \supset \overline{\bigcup_{\ell \geq 0} \mathcal{A}^{(\ell)}} \supset \overline{GO(\mathcal{O})} = J(\varphi),$$

since the closure of a convex set is convex. Taking the convex hulls of these three sets, the first \supset above may be replaced by equality.

In view of (1), $\overline{\bigcup \mathcal{A}^{(\ell)}}$ is completely invariant. Hence part (6) follows from (5). ■

Definition 7.6: For all $\ell \geq 0$, let $w^{(\ell)} \in \mathcal{A}^{(\ell)}$ be such that

$$[\xi_0, w^{(\ell)}] = [\omega', \omega] \cap \mathcal{A}^{(\ell)}.$$

By Lemma 3.2, given $x \in \mathcal{A}^{(\ell)}$ we have that $\deg_x \varphi = 2$ if and only if $x \in [\xi_0, w^{(\ell)}]$. Throughout, we let $v^{(\ell)} = \varphi(w^{(\ell+1)})$.

Below we prescribe a set of vertices $\mathcal{V}^{(\ell)}$ coherent with the topological structure of the topological tree $\mathcal{A}^{(\ell)}$. This set of vertices $\mathcal{V}^{(\ell)}$, as a subset of $\mathbb{P}_{\mathbb{L}}^{1,an}$, gives rise to an abstract simplicial complex in which two vertices are endpoints of an edge if and only if they can be joined in $\mathbb{P}_{\mathbb{L}}^{1,an}$ without crossing other vertices. Of course we regard $\mathcal{A}^{(\ell)}$ as the realization of this abstract simplicial complex where the edges are parametrized by the hyperbolic metric in $\mathbb{H}_{\mathbb{L}}$. That is, we will regard $\mathcal{A}^{(\ell)}$ as a simplicial complex with vertices $\mathcal{V}^{(\ell)}$ and edges parametrized by the distance in $\mathbb{H}_{\mathbb{L}}$.

LEMMA 7.7: If φ has a critical point $\omega \notin K(\varphi)$, let $\mathcal{O}_0 = \pi_{U_0}(\mathcal{O}(\omega) \cap U_0)$, otherwise let \mathcal{O}_0 be the empty set.

- (1) $\mathcal{A}^{(\ell)}$ is a topological tree such that its topological ramification points as well as endpoints are contained in

$$\mathcal{V}^{(\ell)} = (GO(\mathcal{O}) \cup GO(\vartheta_0) \cup GO(\mathcal{O}_0)) \cap \mathcal{A}^{(\ell)}.$$

- (2) For all $\ell \geq 0$, regarding $\mathcal{A}^{(\ell)}$ as simplicial trees with vertices $\mathcal{V}^{(\ell)}$, the map

$$\varphi : \mathcal{A}^{(\ell+1)} \rightarrow \mathcal{A}^{(\ell)}$$

is a degree two branched map over

$$\mathcal{I}_{\mathcal{A}}^{(\ell)} = [\xi_1, v^{(\ell)}] = [\varphi(\omega'), \varphi(\omega)] \cap \mathcal{A}^{(\ell)},$$

where $v^{(\ell)} = \varphi(w^{(\ell+1)}) \in \mathcal{V}^{(\ell)}$.

(3) $\mathcal{I}_A^{(\ell)} \subset \mathcal{I}_A^{(\ell+1)}$. Moreover, if $\mathcal{I}_A^{(\ell)} \subsetneq \mathcal{I}_A^{(\ell+1)}$, then

$$v^{(\ell+1)} \in GO(\mathcal{O}_0) \cup GO(\vartheta_0) \quad \text{or} \quad v^{(\ell+1)},$$

is an endpoint of $\mathcal{A}^{(\ell+1)}$.

(4) If $\text{Crit}(\varphi) \subset K(\varphi)$, then $v^{(\ell)} \in GO(\mathcal{O})$ for all ℓ . In this case, there exists a minimal $\ell_0 \geq 0$ such that $v^{(\ell)} = v^{(\ell_0)}$ for all $\ell \geq \ell_0$ if and only if the active critical value belongs to a Fatou component U such that $\partial U = \{v^{(\ell_0)}\}$.

(5) If $\text{Crit}(\varphi) \not\subset K(\varphi)$, then there exists ℓ such that $v^{(\ell)} \in GO(\mathcal{O}_0)$. In this case, if ℓ_0 is the minimal ℓ such that $v^{(\ell)} \in GO(\mathcal{O}_0)$, then $v^{(\ell)} = v^{(\ell_0)}$ for all $\ell \geq \ell_0$.

Proof. Observe that ω lies in a direction D at $w^{(\ell)}$ which is disjoint from $\mathcal{A}^{(\ell)}$. First we prove that $w^{(\ell)} \in \mathcal{V}^{(\ell)}$. Indeed, if $w^{(\ell)} \notin GO(\mathcal{O})$, then $\varphi^\ell(D) \subset U_0$, by Lemma 7.5 (4). Therefore, $\varphi^\ell(w^{(\ell)}) \in \mathcal{O}_0$ and $w^{(\ell)} \in \mathcal{V}^{(\ell)}$.

Assertion (1) is true for $\ell = 0$ and we proceed by induction to establish the assertion for arbitrary ℓ . Suppose that x is a topological ramification point of $\mathcal{A}^{(\ell+1)}$ and $\varphi(x)$ is not a topological ramification point of $\mathcal{A}^{(\ell)}$. Then the degree of φ at x must be two. Thus, the critical points belong to different directions at x . At least one of these directions is disjoint from $\mathcal{A}^{(\ell+1)}$, otherwise there would be at least three directions at $\varphi(x)$ containing points of $\mathcal{A}^{(\ell)}$. It follows that x is an endpoint of $[\omega, \omega'] \cap \mathcal{A}^{(\ell+1)}$. Therefore, $x = \xi_0$ or $x = w^{(\ell+1)}$ which belong to $\mathcal{V}^{(\ell+1)}$.

For (2), let $\gamma : \mathcal{A}^{(\ell+1)} \rightarrow \mathcal{A}^{(\ell+1)}$ be the involution defined by $\gamma(x) = x'$ if $\{x, x'\} = \varphi^{-1}(\varphi(x))$. It is not difficult to check that γ respects the simplicial structure of $\mathcal{A}^{(\ell+1)}$. Moreover, $\gamma(x) = x$ if and only if the degree of φ at x is 2, which is equivalent to $x \in [\omega', \omega] \cap \mathcal{A}^{(\ell+1)} = [\xi_0, w^{(\ell+1)}]$. Since φ restricted to $[\omega', \omega]$ is a bijection and $w^{(\ell)} \in \mathcal{V}^{(\ell)}$, assertion (2) follows.

From (2) we have that $\mathcal{I}_A^{(\ell)} \subset \mathcal{I}_A^{(\ell+1)}$. To prove (3), assume that $v^{(\ell+1)} \neq v^{(\ell)}$ and that $v^{(\ell+1)} \in GO(\mathcal{O})$. Hence $v^{(\ell+1)} \notin \mathcal{A}^{(\ell)}$. Thus

$$v^{(\ell+1)} \in \varphi^{-(\ell+1)}(\mathcal{O}) \setminus \varphi^{-\ell}(\mathcal{O}).$$

Therefore, $v^{(\ell+1)}$ is an endpoint of $\mathcal{A}^{(\ell+1)}$, by Lemma 7.5 (3).

For (4), if $\text{Crit}(\varphi) \subset K(\varphi)$, then $w^{(\ell+1)} \notin GO(\mathcal{O}_0) \cup GO(\vartheta_0)$, by Lemma 7.5(4). Thus, $w^{(\ell+1)} \in GO(\mathcal{O})$ and $v^{(\ell)} = \varphi(w^{(\ell+1)}) \in GO(\mathcal{O})$. In this case, assume that $v^{(\ell)} = v^{(\ell_0)}$ for all $\ell \geq \ell_0$. Let D be the direction at $v^{(\ell_0)}$ containing the active critical value. It follows that there are no points of $GO(\mathcal{O})$ in D , for

otherwise, $v^{(\ell)} \neq v^{(\ell_0)}$ for some ℓ . Since the closure of $GO(\mathcal{O})$ is the Julia set, it follows that D is a Fatou component. The converse is also straightforward.

For (5), assume that there exists a critical point $\omega \notin K(\varphi)$. Let ℓ_0 be the minimal ℓ such that $\varphi^{\ell+1}(\omega) \in U_0$. Denote by D the direction at $v^{(\ell_0)}$ containing $v = \varphi(\omega)$. Since $\pi_{\mathcal{A}^{(\ell)}} \circ \varphi = \varphi \circ \pi_{\mathcal{A}^{(\ell+1)}}$, for all ℓ , we have that the direction D is contained in a Fatou component which eventually maps onto U_0 . Therefore D is free of $GO(\mathcal{O})$ elements and $v^{(\ell)} = v^{(\ell_0)}$ for all $\ell \geq \ell_0$. ■

7.2. THE COLLAPSED TREES. To deal with the case $\text{Crit}(\varphi) \subset K(\varphi)$ and $\text{Crit}(\varphi) \not\subset K(\varphi)$ simultaneously it is convenient to make the following agreement.

Definition 7.8: If $\text{Crit}(\varphi) \not\subset K(\varphi)$, then let \sim be the relation in $\widehat{J(\varphi)}$ given by Proposition 7.3. Otherwise, let \sim be the trivial equivalence relation (no distinct points are identified). We denote the \sim -class of x by $[x]$.

LEMMA 7.9: *For all $\ell \geq 0$, if $x \in \mathcal{A}^{(\ell)}$, then $[x] = \{x\}$ or $[x]$ is a subtree of $\mathcal{A}^{(\ell)}$. Moreover, $\varphi([x]) = [\varphi(x)]$.*

Proof. Since $\varphi : \mathcal{A}^{(0)} \rightarrow \mathcal{A}^{(0)}$ is a bijection leaving $\widehat{\mathcal{O}}_0$ invariant and φ is an open map, it follows that $\varphi([x]) = [\varphi(x)]$ for all $x \in \widehat{J(\varphi)}$.

Now let $]v, v'[$ be an edge of $\mathcal{A}^{(\ell)}$. Since $\varphi^\ell : \mathcal{A}^{(\ell)} \rightarrow \mathcal{A}^{(0)}$ is a simplicial map and $\varphi : \mathcal{A}^{(0)} \rightarrow \mathcal{A}^{(0)}$ is a bijection leaving $\widehat{\mathcal{O}}_0$ invariant, we have that $\varphi^\ell(]v, v'[) \subset \widehat{\mathcal{O}}_0$ or $\varphi^n(]v, v'[)$ is disjoint from $\widehat{\mathcal{O}}_0$ for all $n \geq 0$. By Lemma 7.5, the endpoints of $\mathcal{A}^{(\ell)}$ lie in $GO(\mathcal{O})$. Thus, endpoints have trivial \sim -classes. Moreover, by definition, classes are connected and, using the fact that φ is an open map, classes are also closed. Therefore, for all $x \in \mathcal{A}^{(\ell)}$, the class $[x]$ is a closed, connected and simplicial subset of $\mathcal{A}^{(\ell)}$. That is, $[x]$ is a subtree of $\mathcal{A}^{(\ell)}$. ■

It follows that

$$\mathcal{B}^{(\ell)} = \mathcal{A}^{(\ell)} / \sim$$

is naturally endowed with a simplicial tree structure with vertices $\mathcal{W}^{(\ell)} = \mathcal{V}^{(\ell)} / \sim$. Since edges of $\mathcal{A}^{(\ell)}$ are either collapsed onto a point or mapped bijectively under the quotient map $\mathcal{A}^{(\ell)} \rightarrow \mathcal{B}^{(\ell)}$, edges of $\mathcal{B}^{(\ell)}$ inherit a parametrization from $\mathcal{A}^{(\ell)}$. Moreover, the induced map

$$\begin{aligned} \psi : \mathcal{B}^{(\ell+1)} &\rightarrow \mathcal{B}^{(\ell)} \\ [\zeta] &\mapsto [\psi(\zeta)] \end{aligned}$$

is a well defined degree two branched map of simplicial trees ramified over the interval $\mathcal{I}_{\mathcal{B}}^{(\ell)} = \mathcal{I}_{\mathcal{A}}^{(\ell)} / \sim$.

7.3. THE “POSTCRITICAL” TREES. In what follows we abuse notation and drop the brackets to write the \sim -equivalence classes. Thus, we simply write ϑ_0 for $[\vartheta_0]$, ξ_j for $[\xi_j]$, $w^{(\ell)}$ for $[w^{(\ell)}]$, etc. With this notation,

$$\mathcal{W}^{(\ell)} = GO(\mathcal{O}) \cup GO(\vartheta_0).$$

LEMMA 7.10: For $\ell \geq 0$, let

$$\mathcal{B}_c^{(\ell)} = \{x \in \mathcal{B}^{(\ell)} \mid w^{(\ell)} \notin [\vartheta_0, \psi^k(x)] \text{ for all } k \geq 0\}.$$

Then, for all $\ell \geq 0$, the following statements hold:

- (1) For all $x \in \mathcal{B}^{(\ell)}$, we have that $\psi([\vartheta_0, x]) \subset [\vartheta_0, \xi_1] \cup [\vartheta_0, \psi(x)]$.
- (2) $w^{(\ell)} \in \mathcal{B}_c^{(\ell)} \subset \mathcal{B}_c^{(\ell+1)}$.
- (3) $\psi(\mathcal{B}_c^{(\ell)}) \subset \mathcal{B}_c^{(\ell)}$.
- (4) $\mathcal{B}_c^{(\ell)}$ is a (connected) subtree of $\mathcal{B}^{(\ell)}$.
- (5) If $w^{(\ell)} \neq w^{(0)}$, then $\mathcal{I}_{\mathcal{B}}^{(\ell)} \subset \mathcal{B}_c^{(\ell)}$. Moreover, if $v^{(\ell)} \in GO(\mathcal{O})$ and v is a vertex of $\mathcal{B}^{(\ell)}$ such that $[v^{(\ell)}, v]$ is an edge, then $v \in \mathcal{B}_c^{(\ell)}$.
- (6) If $w^{(\ell+1)} = w^{(\ell)}$, then $\psi(\mathcal{B}_c^{(\ell+1)}) \subset \mathcal{B}_c^{(\ell)}$.
- (7) If $w^{(\ell+1)} \neq w^{(\ell)}$ and $w^{(\ell+1)} \in GO(\mathcal{O})$, then $\mathcal{B}_c^{(\ell+1)} = \mathcal{B}^{(\ell+1)}$.

Proof. Statement (1) is trivial for $\ell = 0$ so consider $\ell \geq 1$ and let ϑ'_0 be the unique point, different from ϑ_0 , such that $\psi(\vartheta'_0) = \vartheta_0$. Note that there are two intervals in $\mathcal{B}^{(\ell)}$ mapping onto $[\vartheta_0, \psi(x)]$, namely, $[\vartheta_0, z]$ and $[\vartheta'_0, z']$ where $\psi(z) = \psi(z') = \psi(x)$. Thus, $x = z$ or $x = z'$. In the former case,

$$\psi([\vartheta_0, z = x]) = [\vartheta_0, \psi(x)].$$

In the latter,

$$\psi([\vartheta_0, z = x']) \subset \psi([\vartheta_0, \vartheta'_0]) \cup \psi([\vartheta_0, z]) = [\vartheta_0, \xi_1] \cup [\vartheta_0, \psi(x)].$$

To establish (2), recall that ψ is a simplicial map. Thus, for all $k \geq 0$, the number of edges contained in $[\vartheta_0, \psi^k(w^{(\ell)})]$ is bounded above by the number of edges in $[\vartheta_0, w^{(\ell)}]$. It follows that $w^{(\ell)} \notin [\vartheta_0, \psi^k(w^{(\ell)})]$. Therefore, $w^{(\ell)} \in \mathcal{B}_c^{(\ell)}$. The rest of statement (2) as well as statement (3) follows directly from the definition.

For (4), observe that $\mathcal{B}^{(0)} \subset \mathcal{B}_c^{(\ell)}$ for all ℓ . Hence, if $x \in \mathcal{B}_c^{(\ell)}$ and $y \in [\vartheta_0, x]$, then

$$[\vartheta_0, \psi^k(y)] \subset \psi^k([\vartheta_0, x]) \subset \mathcal{B}^{(0)} \cup [\vartheta_0, \psi^k(x)]$$

for all $k \geq 0$, by statement (1). Therefore, $w^{(\ell)} \notin [\vartheta_0, \psi^k(y)[$, for otherwise $x \notin \mathcal{B}_c^{(\ell)}$ or $w^{(\ell)} \in [\vartheta_0, \psi^k(y)[\subset \mathcal{B}^{(0)} \setminus \mathcal{O}$, which is impossible. It follows that $\mathcal{B}_c^{(\ell)}$ is a connected subtree.

For (5), assume that $w^{(\ell)} \neq w^{(0)}$. It is sufficient to show that $v^{(\ell)} \in \mathcal{B}_c^{(\ell)}$. Thus, we may assume that $w^{(\ell+1)} \neq w^{(\ell)}$. We proceed by contradiction. If $w^{(\ell)} \in [\vartheta_0, \varphi^k(w^{(\ell+1)})[$ for some $k \geq 1$, then $w^{(\ell)} \in [\varphi^k(w^{(\ell)}), \varphi^k(w^{(\ell+1)})[$, since the number of edges in $[\vartheta_0, w^{(\ell)}]$ is an upper bound for the number of edges in $[\vartheta_0, \varphi^k(w^{(\ell)})] \subset \varphi^k([\vartheta_0, w^{(\ell)}])$. However, $]w^{(\ell)}, w^{(\ell+1)}[$ contains only vertices in $GO(\vartheta_0)$. By Lemma 7.7 (5), we have that $w^{(\ell)} \notin GO(\vartheta_0)$, thus the same holds for its orbit, hence $w^{(\ell)} = \varphi^k(w^{(\ell)})$ is periodic. The periodic points of $\mathcal{B}^{(\ell)}$ belong to $\mathcal{B}^{(0)}$ and $w^{(\ell)} \notin U_0$. Therefore $w^{(\ell)} = \xi_0 = w^{(0)}$, which contradicts our assumption that $w^{(\ell)} \neq w^{(0)}$ so $v^{(\ell)} \in \mathcal{B}_c^{(\ell)}$.

Now assume that $v^{(\ell)} \in GO(\mathcal{O})$ and v is a vertex of $\mathcal{B}^{(\ell)}$ such that $[v^{(\ell)}, v]$ is an edge. If $v^{(\ell)} \neq v^{(\ell-1)}$, then there is a unique such vertex v by Lemma 7.7 (3) and v must lie in $[\vartheta_0, v^{(\ell)}] \subset \mathcal{B}_c^{(\ell)}$. If $v^{(\ell)} = v^{(\ell-1)}$ we proceed by contradiction. Suppose that $w^{(\ell)} \in [\vartheta_0, \varphi^{k-1}(v)[$ for some $k \geq 1$. As before, we conclude that $w^{(\ell)}$ lies in the interval $[\varphi^{k-1}(v^{(\ell-1)}), \varphi^{k-1}(v)[$ whose interior is an edge since $v^{(\ell)} = v^{(\ell-1)}$. Therefore, $w^{(\ell)} = \varphi^k(w^{(\ell)})$ is periodic and, as above, we obtain a contradiction.

Statement (6) follows from the definition and statement (7) is a consequence of Lemma 7.7 (3). ■

Now we consider $\theta \in I_{p/q}$. Given $x \in \mathbb{P}_{\mathbb{C}}^1 \cong \mathbb{C} \cup \{\infty\}$, denote by $v_{\theta}^{(\ell)}(x)$ the vertex of $\mathcal{T}^{(\ell)}(\theta) = \mathcal{T}(\lambda(\theta)^{(\ell)})$ which contains x . For $t \in \mathbb{R}/\mathbb{Z}$ we also denote by $v_{\theta}^{(\ell)}(t)$ the vertex containing $\exp(2\pi it)$ when no confusion arises. For $0 \leq m \leq \ell$, we denote by $\iota : \mathcal{T}^{(m)}(\theta) \rightarrow \mathcal{T}^{(\ell)}(\theta)$ the inclusion given by Lemma 6.10.

Recall from Lemma 6.18 that $m_2 : \mathcal{T}^{(\ell+1)}(\theta) \rightarrow \mathcal{T}^{(\ell)}(\theta)$ is a degree two branched map over the interval $\mathcal{I}_{\theta}^{(\ell)} = [m_2(v_{\theta}^{(\ell+1)}(\infty)), m_2(v_{\theta}^{(\ell+1)}(0))]$. Again to simplify notation, let

$$\begin{aligned} w_{\theta}^{(\ell+1)} &= v_{\theta}^{(\ell+1)}(0), \\ v_{\theta}^{(\ell)} &= m_2(w_{\theta}^{(\ell+1)}), \\ \beta_j &= \iota \circ m_2^j(v_{\theta}^{(\ell)}(\infty)) \in \mathcal{T}^{(\ell)}(\theta) \end{aligned}$$

for $j \geq 0$ (subscripts mod q). Finally, let $\alpha_0 \in \mathcal{T}^{(\ell)}(\theta)$ be the (inclusion of the) unique Γ -vertex of level 0.

LEMMA 7.11: Given $\theta \in I_{p/q}$ let

$$\mathcal{T}_c^{(\ell)}(\theta) = \{x \in \mathcal{T}^{(\ell)}(\theta) \mid w_\theta^{(\ell)} \notin [\alpha_0, \iota \circ m_2^k(x)] \text{ for all } k \geq 0\}.$$

For all $\ell \geq 0$, the following statements hold:

(1) For all $x \in \mathcal{T}^{(\ell)}(\theta)$,

$$\iota \circ m_2([\alpha_0, x]) = [\alpha_0, \beta_1] \cup [\alpha_0, \iota \circ m_2(x)].$$

(2) $w_\theta^{(\ell)} \in \mathcal{T}_c^{(\ell)}(\theta)$ and $\iota(\mathcal{T}_c^{(\ell)}(\theta)) \subset \mathcal{T}_c^{(\ell+1)}(\theta)$.

(3) $\iota \circ m_2(\mathcal{T}_c^{(\ell)}) \subset \mathcal{T}_c^{(\ell)}$.

(4) $\mathcal{T}_c^{(\ell)}(\theta)$ is a (connected) subtree of $\mathcal{T}^{(\ell)}(\theta)$.

We omit the proof of the previous lemma since it is, after changing notation, identical to that of Lemma 7.11 (1)–(4).

Denote by $\Gamma_c^{(\ell)}(\theta)$ the set of Γ -vertices of level ℓ which belong to $\mathcal{T}_c^{(\ell)}(\theta)$.

LEMMA 7.12: For all $\ell \geq 1$ and for all $\theta' \in v_\theta^{(\ell-1)}$,

$$\Gamma_c^{(\ell)}(\theta) = \Gamma_c^{(\ell)}(\theta').$$

Moreover, if v is a vertex of $\mathcal{T}_c^{(\ell)}(\theta)$ such that $v \subset v_\theta^{(\ell-1)}$, then ∂v is contained in the union of the vertices of $\Gamma_c^{(\ell)}(\theta)$. In particular, v is a vertex of $\mathcal{T}^{(\ell)}(\theta')$ for all $\theta' \in v_\theta^{(\ell-1)}$.

Proof. Let $\theta' \in v_\theta^{(\ell-1)}$. We first consider the case in which $v_\theta^{(\ell-1)}$ is a Γ -vertex and then deal with the case in which it is a Y -vertex.

Assume that $v_\theta^{(\ell-1)}$ is a Γ -vertex. Then $\lambda(\theta) = \lambda(\theta')$, by Lemma 6.15. In particular, $\Gamma_c^{(\ell)}(\theta) = \Gamma_c^{(\ell)}(\theta')$. Moreover, if $v \subset v_\theta^{(\ell-1)}$, then $v = v_\theta^{(\ell-1)}$ and the lemma follows in this case.

Assume that $v_\theta^{(\ell-1)}$ is a Y -vertex. Let V be the connected component of $\mathbb{P}_\mathbb{C}^1 \setminus w_\theta^{(\ell)}$ that contains α_0 . By Lemma 6.8, $W = \mathbb{P}_\mathbb{C}^1 \setminus V$ is such that $W \cap \mathbb{D}$ is a topological disk convex with respect to the hyperbolic metric. Hence, the diameter connecting $\theta'/2$ and $\theta'/2 + 1/2$ is completely contained in W . The vertices of $\mathcal{T}_c^{(\ell)}(\theta)$ are exactly those that belong to V and whose iterates also belong to V . Therefore, if A is a $\lambda(\theta)$ -class whose convex hull is contained in an element of $\Gamma_c(\theta)$, then A is unlinked with $\{\theta'/2, \theta'/2 + 1/2\}$. Since $m_2(\Gamma_c(\theta)) \subset \Gamma_c(\theta)$, the same holds for the $m_2^k(A)$. Hence A is a $\lambda(\theta')$ -class. It follows that $\Gamma_c(\theta) = \Gamma_c(\theta')$.

Now assume that v is a $Y^{(\ell)}$ -vertex which belongs to $\mathcal{T}_c^{(\ell)}(\theta)$ and that is contained in $v_\theta^{(\ell-1)}$. Let w be a $\Gamma^{(\ell)}(\theta)$ -vertex such that $\partial v \cap w \neq \emptyset$. Since

all the iterates $m_2^k(v)$ are contained in the closed set V , the same occurs with $m_2^k(w)$. Therefore, $w \in \mathcal{T}_c^{(\ell)}(\theta)$ and the lemma follows. ■

7.4. TOPOLOGICAL CONJUGACY. The simultaneous proof of Proposition 7.1 and Proposition 7.3 will rely on an inductive construction of conjugacies defined on the level ℓ trees $\mathcal{B}^{(\ell)}$.

The initial step consists of fixing a conjugacy $h_0 : \mathcal{B}^{(0)} \rightarrow \mathcal{T}^{(0)}(\theta)$ between the dynamics of $\psi : \mathcal{B}^{(0)} \rightarrow \mathcal{B}^{(0)}$ and that of $m_2 : \mathcal{T}^{(0)}(\theta) \rightarrow \mathcal{T}^{(0)}(\theta)$ for all θ in the characteristic interval $I_{p/q}$. In fact, the vertices of $\mathcal{B}^{(0)}$ are $\{\xi_0, \dots, \xi_{q-1}, \vartheta_0\}$ and $\mathcal{B}^{(0)}$ is the starlike tree obtained as the union of the intervals $[\vartheta_0, \xi_j]$. Note that $w^{(0)} = \xi_0$. As described before Lemma 7.11, for all $\theta \in I_{p/q}$ there is a unique Γ -vertex α_0 and the Y -vertices are $\beta_0, \dots, \beta_{q-1}$ where $\beta_0 = w_\theta^{(0)}$. It follows that the unique tree isomorphism $h_0 : \mathcal{B}^{(0)} \rightarrow \mathcal{T}^{(0)}(\theta)$ such that $h_0(\vartheta_0) = \alpha_0$ and $h_0(\xi_j) = \beta_j$ is a conjugacy.

Before proving the propositions we establish two necessary lemmas.

LEMMA 7.13: *Let λ_* be the α lamination of the center of the p/q -limb. Assume that $w^{(0)} = \dots = w^{(\ell-1)}$, for some $\ell \geq 1$. Then, for all $m = 1, \dots, \ell$, there exists a tree isomorphism $h_m : \mathcal{B}^{(m)} \rightarrow \mathcal{T}(\lambda_*^{(m)})$ such that*

$$h_m \circ \psi = \iota \circ m_2 \circ h_m$$

and the restriction of h_m to $\mathcal{B}^{(0)}$ is h_0 .

Proof. Assume that h_m has been already constructed and that $0 \leq m < \ell$. Since $\psi : \mathcal{B}^{(m+1)} \rightarrow \mathcal{B}^{(m+1)}$ (resp. $\iota \circ m_2 : \mathcal{T}(\lambda_*^{(m+1)}) \rightarrow \mathcal{T}(\lambda_*^{(m+1)})$) are degree two branched maps over the point ξ_1 (resp. β_1) and $h_m(\xi_1) = h_0(\xi_1) = \beta_1$, from Lemma 6.19, we may lift h_m to a conjugacy h_{m+1} with the desired properties. ■

LEMMA 7.14: *There exists a nested sequence*

$$v_{\mathcal{M}}^{(0)} \supset v_{\mathcal{M}}^{(1)} \supset \dots$$

of non-empty subsets of $I_{p/q}$ such that, for all $\ell \geq 1$, there exists a collection of simplicial tree isomorphisms

$$\{h_{\ell, \theta} : \mathcal{B}^{(\ell)} \rightarrow \mathcal{T}^{(\ell)}(\theta)\}_{\theta \in v_{\mathcal{M}}^{(\ell-1)}}$$

with the property that

$$v_{\mathcal{M}}^{(\ell)} = h_{\ell, \theta}(v_{\mathcal{M}}^{(\ell)})$$

for all $\theta \in v_{\mathcal{M}}^{(\ell-1)}$.

Moreover, the following statements hold:

- (1) The restriction of $h_{\ell,\theta}$ to $\mathcal{B}^{(0)}$ is h_0 , for all $\theta \in v_{\mathcal{M}}^{(\ell-1)}$.
- (2) For all $\theta \in v_{\mathcal{M}}^{(\ell-1)}$, the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{B}^{(\ell)} & \xrightarrow{\quad} & \mathcal{B}^{(\ell)} \\
 & \searrow \psi & \\
 \downarrow h_{\ell,\theta} & & \downarrow h_{\ell,\theta} \\
 \mathcal{T}^{(\ell)}(\theta) & \xrightarrow{\iota \circ m_2} & \mathcal{T}^{(\ell)}(\theta).
 \end{array}$$

- (3) For all $\theta, \theta' \in v_{\mathcal{M}}^{(\ell-1)}$,

$$h_{\ell,\theta}(v) = h_{\ell,\theta'}(v)$$

if $v \in \mathcal{B}_c^{(\ell)} \cap GO(\vartheta_0)$.

Proof. If $w^{(\ell)} = w^{(0)}$ for all ℓ , then the assertion follows from Lemmas 6.16 and 7.13 after declaring $v_{\mathcal{M}}^{(\ell-1)} = v_*^{(\ell-1)}$ (with the notation of Lemma 6.16) and $h_{\ell,\theta} = h_{\ell}$ (with the notation of Lemma 7.13) for all $\theta \in v_{\mathcal{M}}^{(\ell-1)}$.

If, for some $\ell_0 \geq 1$, we have that $w^{(\ell_0)} \neq w^{(0)} = w^{(1)} = \dots = w^{(\ell_0-1)}$, then for all $m = 1, \dots, \ell_0$, we let $v_{\mathcal{M}}^{(m-1)} = v_*^{(m-1)}$ (with the notation of Lemma 6.16) and $h_{m,\theta} = h_m$ as in Lemma 7.13. It is not difficult to check that $v_{\mathcal{M}}^{(\ell)} = h_{\ell,\theta}(v_{\theta}^{(\ell)})$ and that properties (1)–(3) hold for all $\ell < \ell_0$.

For $\ell \geq \ell_0$, we proceed by induction. That is, for all $m \leq \ell$, we suppose that $v_{\mathcal{M}}^{(m-1)}$ and $h_{m,\theta}$ have already been defined and that properties (1)–(3) are satisfied.

Note that by (1) and (2), the elements of $GO(\vartheta_0)$ map onto Γ -pieces and the elements of $GO(\mathcal{O})$ map onto Y -pieces under $h_{\ell,\theta}$. Also, the conjugacy implies that $h_{\ell,\theta}(w^{(\ell)}) = w_{\theta}^{(\ell)}$ for all $\theta \in v_{\mathcal{M}}^{(\ell-1)}$.

First we claim that $h_{\ell,\theta}(v^{(\ell)})$ is independent of $\theta \in v_{\mathcal{M}}^{(\ell-1)}$. In fact, under our assumption, $w^{(\ell)} \neq w^{(0)}$ for all $\ell \geq \ell_0$. By Lemma 7.10 (5), $\mathcal{I}_{\mathcal{B}}^{(\ell)} \subset \mathcal{B}_c^{(\ell)}$. In particular, $v^{(\ell)} \in \mathcal{B}_c^{(\ell)}$. If $v^{(\ell)} \in GO(\vartheta_0)$, then the claim follows from property (3). If $v^{(\ell)} \in GO(\mathcal{O})$, then for every $v \in \mathcal{B}_c^{(\ell)}$ such that $[v^{(\ell)}, v]$ is an edge, we have that $v \in \mathcal{B}_c^{(\ell)}$, by Lemma 7.10 (5). From property (3), every Γ -vertex which intersects non-trivially $\partial h_{\ell,\theta}(v^{(\ell)})$ is independent of $\theta \in v_{\mathcal{M}}^{(\ell-1)}$. Hence $h_{\ell,\theta}(v^{(\ell)})$ is independent of $\theta \in v_{\mathcal{M}}^{(\ell-1)}$.

Pick any $\theta \in v_{\mathcal{M}}^{(\ell-1)}$ and define $v_{\mathcal{M}}^{(\ell)} = h_{\ell,\theta}(v^{(\ell)})$. We must show that $v_{\mathcal{M}}^{(\ell)} \subset v_{\mathcal{M}}^{(\ell-1)}$. In fact, by the inductive hypothesis (2), branched values correspond under $h_{\ell,\theta}$

so we have that $v_\theta^{(\ell-1)} = \pi_\ell \circ h_{\ell,\theta}(v^{(\ell-1)})$, where $\pi_\ell : \mathcal{T}^{(\ell)}(\theta) \rightarrow \mathcal{T}^{(\ell-1)}(\theta)$ denotes the retraction of Lemma 6.7. Moreover, $\pi_\ell \circ h_{\ell,\theta}(v^{(\ell-1)}) = \pi_\ell \circ h_{\ell,\theta}(v^{(\ell)})$ since the interval $]v^{(\ell-1)}, v^{(\ell)}]$ is not contained in $\mathcal{B}^{(\ell-1)}$. From the definition of π_ℓ we conclude that $v_{\mathcal{M}}^{(\ell)} = h_{\ell,\theta}(v^{(\ell)})$ is contained in $v_\theta^{(\ell-1)} = h_{\ell-1,\theta}(v^{(\ell-1)}) = v_{\mathcal{M}}^{(\ell-1)}$.

Now we continue with the construction of the isomorphisms $h_{\ell+1,\theta}$ for all $\theta \in v_{\mathcal{M}}^{(\ell)}$.

To define $h_{\ell+1,\theta}$ for $\theta \in v_{\mathcal{M}}^{(\ell)}$ first we consider the case in which $v^{(\ell)} \in GO(\vartheta_0)$. It follows that $v_{\mathcal{M}}^{(\ell)} = h_{\ell,\theta}(v^{(\ell)})$ is a Γ -piece of level ℓ . By Lemma 6.15, for all $\theta, \theta' \in v_{\mathcal{M}}^{(\ell)}$, we have $\lambda(\theta) = \lambda(\theta')$. Thus, pick an element θ_ℓ of $v_{\mathcal{M}}^{(\ell)}$, and let $h_{\ell+1,\theta_\ell} : \mathcal{B}^{(\ell+1)} \rightarrow \mathcal{T}^{(\ell+1)}(\theta_\ell)$ be the lift obtained from h_{ℓ,θ_ℓ} after applying Lemma 6.19. Declare $h_{\ell+1,\theta} = h_{\ell+1,\theta_\ell}$ for all $\theta \in v_{\mathcal{M}}^{(\ell)}$. It is not difficult to check that properties (1)–(3) hold.

Now we assume that $w^{(\ell+1)} \in GO(\mathcal{O})$ and define $h_{\ell+1,\theta}$ for $\theta \in v_{\mathcal{M}}^{(\ell)}$ by studying two cases according to whether $w^{(\ell+1)} \neq w^{(\ell)}$ or $w^{(\ell+1)} = w^{(\ell)}$.

CASE 1. Suppose that $w^{(\ell+1)} \neq w^{(\ell)}$. By Lemma 7.10 (7) we have that $\mathcal{B}_c^{(\ell+1)} = \mathcal{B}^{(\ell+1)}$. Pick $\theta_\ell \in v_{\mathcal{M}}^{(\ell)}$ and apply Lemma 6.19 to $\psi : \mathcal{B}^{(\ell+1)} \rightarrow \mathcal{B}^{(\ell)}$ and $m_2 : \mathcal{T}^{(\ell+1)}(\theta_\ell) \rightarrow \mathcal{T}^{(\ell)}(\theta_\ell)$ to lift h_{ℓ,θ_ℓ} to a tree isomorphism

$$h_{\ell+1,\theta_\ell} : \mathcal{B}^{(\ell+1)} \rightarrow \mathcal{T}^{(\ell+1)}(\theta_\ell),$$

since by the choice of $v_{\mathcal{M}}^{(\ell)}$ we have that critical value intervals correspond under h_{ℓ,θ_ℓ} . It follows that $h_{\ell+1,\theta_\ell}(\mathcal{B}_c^{(\ell+1)}) = \mathcal{T}_c^{(\ell+1)}(\theta_\ell)$. Thus, we have that $\mathcal{T}^{(\ell+1)}(\theta_\ell) = \mathcal{T}_c^{(\ell+1)}(\theta_\ell)$. By Lemma 7.12, we conclude that $\mathcal{T}^{(\ell+1)}(\theta) = \mathcal{T}^{(\ell+1)}(\theta')$ for all $\theta, \theta' \in v_{\mathcal{M}}^{(\ell)}$. Thus, for all $\theta \in v_{\mathcal{M}}^{(\ell)}$, we may simply define $h_{\ell+1,\theta} = h_{\ell+1,\theta_\ell}$ and (1)–(3) clearly hold in this case.

CASE 2. Suppose that $w^{(\ell+1)} = w^{(\ell)}$. From Lemma 7.10 we have that

$$\psi(\mathcal{B}_c^{(\ell+1)}) \subset \mathcal{B}_c^{(\ell)}.$$

Let

$$\mathcal{B}_s^{(\ell+1)} = \psi^{-1}(\psi(\mathcal{B}_c^{(\ell+1)}))$$

and

$$\mathcal{T}_s^{(\ell+1)}(\theta) = m_2^{-1}(m_2(\mathcal{T}_c^{(\ell+1)}(\theta))).$$

Pick $\theta_\ell \in v_{\mathcal{M}}^{(\ell)}$. We may lift $h_{\ell,\theta_\ell} : \mathcal{B}_c^{(\ell)} \rightarrow \mathcal{T}_c^{(\ell)}(\theta_\ell)$ to a tree isomorphism $h_{\ell+1,\theta_\ell} : \mathcal{B}_s^{(\ell+1)} \rightarrow \mathcal{T}_s^{(\ell+1)}(\theta_\ell)$. Since the Γ -vertices of $\mathcal{T}_c^{(\ell+1)}(\theta)$ are independent of $\theta \in v_{\mathcal{M}}^{(\ell)}$ (Lemma 7.12), we have that they coincide with the Γ -vertices

$\mathcal{T}_c^{(\ell+1)}(\theta_\ell)$. We identify $\mathcal{T}_c^{(\ell+1)}(\theta)$ with $\mathcal{T}_c^{(\ell+1)}(\theta_\ell)$ via the unique tree isomorphism which preserves Γ -vertices. This tree isomorphism clearly extends to one from $\mathcal{T}_s^{(\ell+1)}(\theta)$ onto $\mathcal{T}_s^{(\ell+1)}(\theta_\ell)$. Thus, we may define $h_{\ell+1,\theta} : \mathcal{B}_s^{(\ell+1)} \rightarrow \mathcal{T}_s^{(\ell+1)}(\theta)$ as equal to $h_{\ell+1,\theta_\ell} : \mathcal{B}_s^{(\ell+1)} \rightarrow \mathcal{T}_s^{(\ell+1)}(\theta_\ell)$, for all $\theta \in v_{\mathcal{M}}^{(\ell)}$. Now we may apply Lemma 6.19 to extend $h_{\ell+1,\theta}$ to $\mathcal{B}^{(\ell+1)}$ by successive lifts to $(\psi|_{\mathcal{B}^{(\ell+1)}})^{-k}(\mathcal{B}_s^{(\ell+1)})$. Since every element of $\mathcal{B}^{(\ell+1)}$ eventually maps into $\mathcal{B}_s^{(\ell+1)}$, we obtain the desired isomorphism $h_{\ell+1,\theta} : \mathcal{B}^{(\ell+1)} \rightarrow \mathcal{T}^{(\ell+1)}(\theta)$. ■

Proof of Proposition 7.1 and Proposition 7.3. Let $v_{\mathcal{M}}^{(\ell)}$ and $h_{\ell,\theta}$ be as in the previous lemma. Take a monotone convergent sequence $\{\theta_n\} \subset \mathbb{R}/\mathbb{Z}$ such that $\theta_n \in v_{\mathcal{M}}^{(n)}$ for all n . Let θ_∞ denote the limit of θ_n . If the sequence is eventually constant, then let $\lambda = \lambda(\theta_\infty)$. Otherwise, let $\lambda = \lambda(\theta_\infty)^+$ when the sequence is increasing, and let $\lambda = \lambda(\theta_\infty)^-$ when the sequence is decreasing. It follows that for all $\ell \geq 0$, there exists $n(\ell)$ such that $\lambda^{(\ell)} = \lambda(\theta_n)^{(\ell)}$ for all $n \geq n(\ell)$. Since the number of isomorphisms between $\mathcal{B}^{(\ell)}$ and $\mathcal{T}(\lambda^{(\ell)})$ is finite, we may recursively extract subsequences $\{n_k(\ell)\}$ such that:

- (1) $n_k(0) = k$.
- (2) $\{n_k(\ell + 1)\}$ is a subsequence of $\{n_k(\ell)\}$.
- (3) $n_0(\ell) \geq n(\ell)$.
- (4) The restriction of $h_{n_k(\ell),\theta_{n_k(\ell)}}$ to $\mathcal{B}^{(\ell)}$ is constant, say equal to h_ℓ .

Passing to the inverse limit of $\{h_\ell\}$ we obtain a conjugacy

$$h : \widehat{J(\varphi)} / \sim \rightarrow \mathcal{T}^{(\infty)}(\lambda).$$

Since h maps $GO(\vartheta_0)$ onto the inverse limit of the $\Gamma^{(\ell)}(\lambda)$, it follows that $w^{(\ell_0)} \in GO(\vartheta_0)$ if and only if $h_{\ell_0}(v^{(\ell_0)})$ is a Γ -vertex containing θ_n for sufficiently large n . In this case, the sequence θ_∞ eventually maps onto the fixed class of λ . That is, λ is critically prefixed. ■

Proof of Theorem 3. According to Proposition 3.1 we have three possibilities:

- (a) $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$ is periodic point free.
- (b) There exists an indifferent periodic orbit \mathcal{O} in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$.
- (c) There exists a repelling periodic orbit \mathcal{O} in $J(\varphi) \cap \mathbb{H}_{\mathbb{L}}$.

From Proposition 4.1, we have that (a) implies that Theorem 3 (1) holds. From Proposition 4.2, we conclude that (b) implies Theorem 3 (2). From Proposition 7.1 and Proposition 7.3 we conclude that (c) implies Theorem 3 (3) (a) or (b). ■

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