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YET ANOTHER SHORT PROOF OF BOURGAIN'S DISTORTION ESTIMATE FOR EMBEDDING OF TREES INTO UNIFORMLY CONVEX BANACH SPACES

BY

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ABSTRACT

We use a self-improvement argument to give a very short and elementary proof of the result of Bourgain saying that infinite regular trees do not admit bi-Lipschitz embeddings into uniformly convex Banach spaces.

Let T_n be the binary rooted tree of depth n and let $c_B(A)$ denote the distortion of the metric space A in B, that is to say the infimum of all numbers D such that there is a number $s > 0$ and a map $\varphi : A \to B$ such that

$$sd(x, y) \leqslant d(\varphi(x), \varphi(y)) \leqslant sDd(x, y)$$

for all $x, y \in A$.

The modulus of (uniform) convexity $\delta_X(\varepsilon)$ of a Banach space X with norm $|\cdot|$ is defined as

$$\inf \left\{ 1 - \frac{|x+y|}{2} \;\middle|\; |x| = |y| = 1 \text{ and } |x-y| \geqslant \varepsilon \right\}$$

for $\varepsilon \in (0, 2]$. The space X is said to be p-uniformly convex if $\delta_X(\varepsilon) \geqslant c\varepsilon^p$ for some $c > 0$. Note that in particular, for $q \in (1, \infty)$ the L_q spaces are p-uniformly convex with $p = \max(2, q)$. See, for example, [1] for more details on uniform convexity.

Our goal is to provide a simple and elementary proof of the following result.

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Theorem 1 (Bourgain): *If X is p-uniformly convex then*

$$c_X(T_n) \gtrsim (\ln n)^{\frac{1}{p}}.$$

Several proofs of this result have been given over the years, either in the general case or for Hilbert space; see notably [2, 6, 4, 7, 5]. As we discovered after writing a first draft of this paper, the method we use is very close to Johnson and Schechtman's proof of the distortion estimate for diamond graphs [3] (see also [8]). However, it seems not to have been noticed before that this method gives such an effective proof of Bourgain's estimate.

Proof. The first step is similar to previous proofs, notably the one by Matoušek [6]. Let Y be the four-vertices tree with one root a_0 which has one child a_1 and two grand-children a_2, a_2'.

Lemma 2: *There is a constant $K = K(X)$ such that if $\varphi : Y \to X$ is D-Lipschitz and distance non-decreasing, then either*

$$|\varphi(a_0) - \varphi(a_2)| \leqslant 2\Big(D - \frac{K}{D^{p-1}}\Big)$$

or

$$|\varphi(a_0) - \varphi(a_2')| \leqslant 2\Big(D - \frac{K}{D^{p-1}}\Big).$$

We provide the proof below for the sake of completeness.

Let now $\varphi : T_n \to X$ be a D-Lipschitz, distance non-decreasing map; we shall construct a less distorded embedding of a smaller tree.

Given any vertex of T_n that does not belong to the last generation, let us name arbitrarily one of its two children its daughter and the other its son. We select two grandchildren of the root in the following way: we pick the grandchild mapped closest to the root by φ among its daughter's children and the grandchild mapped closest to the root by φ among its son's children (ties are resolved arbitrarily). Then we select inductively, in the same way, two grandchildren for all previously selected vertices up to generation $n-2$.

The set of selected vertices, endowed with half the distance induced by the tree metric, is then isometric to $T_{\lfloor \frac{n}{2} \rfloor}$, and the lemma ensures that the restriction of φ to this set has distortion at most

$$f(D) = D - \frac{K}{D^{p-1}}.$$

We can iterate these restrictions $\lfloor \log_2(n) \rfloor$ times to get an embedding of T_1 whose distortion is at most

$$D - \lfloor \log_2(n) \rfloor \frac{K}{D^{p-1}}$$

since each iteration improves the distortion by at least K/D^{p-1}. Since the distortion of any embedding is at least 1, we get

$$\log_2(n) \lesssim D^p$$

which is Theorem 1. ∎

Remark 3: Working out the constants gives the more precise result that

(1) $$c_X(T_n) \geqslant \left(\frac{pc}{2}\right)^{\frac{1}{p}} (\log_2 n)^{\frac{1}{p}} + \text{l.o.t.}$$

where c can be replaced by $\liminf \delta_X(\varepsilon)\varepsilon^{-p}$. In particular, we get

$$c_{\ell^2}(T_n) \geqslant \frac{1}{2\sqrt{2}} (\log_2 n)^{\frac{1}{2}} + \text{l.o.t.}$$

but one can get the better $\frac{1}{2}(\log_2 n)^{\frac{1}{2}}$ from the Linial–Saks proof [5].

Proof of Lemma 2. Assume $\varphi(a_0) = 0$ and let $x_1 = \varphi(a_1)$, $x_2 = \varphi(a_2)$ and $x_2' = \varphi(a_2')$.

Suppose that $|x_2| \geqslant 2(D - \eta)$ for some η to be chosen afterward; then by the triangle inequality, $|x_1|$ and $|x_2 - x_1|$ are at least $D - 2\eta$.

Define $v = \frac{|x_1|}{|x_2 - x_1|}(x_2 - x_1)$; then

$$|x_1 + v - x_2| = ||x_1| - |x_2 - x_1|| \leqslant 2\eta$$

and

$$|x_1 + v| \geqslant |x_2| - |x_1 + v - x_2| \geqslant 2D - 4\eta.$$

The vectors $x_1/|x_1|$ and $v/|x_1|$ have unit norm and their average has norm at least $1 - 2\eta/D$; letting $\varepsilon = (2\eta/cD)^{\frac{1}{p}}$ the convexity assumption therefore yields $|x_1 - v| \leqslant \varepsilon D$. It follows that

$$|2x_1 - x_2| \leqslant |x_1 + v - x_2| + |x_1 - v| \leqslant 2\eta + \varepsilon D.$$

Suppose that also $|x_2'| \geqslant 2(D - \eta)$; then the same reasoning yields the same estimate on $|x_2' - 2x_1|$ so that

$$|x_2 - x_2'| \leqslant 4\eta + 2D\left(\frac{2\eta}{cD}\right)^{\frac{1}{p}}.$$

Now we can choose $\eta = K/D^{p-1}$ with K small enough to ensure that the above inequality implies $|x_2 - x'_2| < 2$. This contradicts the hypothesis that φ is distance non-decreasing, therefore as desired $|x_2|$ or $|x'_2|$ must be smaller than $2(D - \eta)$. ∎

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