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# KRONECKER MULTIPLICITIES IN THE $(k, \ell)$ HOOK ARE POLYNOMIALLY BOUNDED

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ABSTRACT

The problem of decomposing the Kronecker product of $S_n$ characters is one of the last major open problems in the ordinary representation theory of the symmetric group $S_n$. In this note $\lambda$ and $\mu$ are partitions of $n$, $n$ goes to infinity, and we prove upper and lower polynomial bounds for the multiplicities of the Kronecker product $\chi^\lambda \otimes \chi^\mu$, where for some fixed $k$ and $\ell$ both partitions $\lambda$ and $\mu$ are in the $(k, \ell)$ hook.

## 1. Introduction

We assume that the characteristic of the base field is zero: $char(F) = 0$. As usual, $S_n$ is the $n$-th symmetric group. To the partition $\lambda \vdash n$ corresponds the irreducible $S_n$ character $\chi^\lambda$ of degree $\deg(\chi^\lambda) = f^\lambda$ [5, 7, 9]. Let $\varphi, \psi$ be two $S_n$ characters (same $n$). Their Kronecker (or inner) product $\varphi \otimes \psi$ is defined via $(\varphi \otimes \psi)(\sigma) = \varphi(\sigma)\psi(\sigma)$ where $\sigma \in S_n$. Then $\varphi \otimes \psi$ is an $S_n$ character. Since $char(F) = 0$, $\varphi \otimes \psi$ is a non-negative integer combination of the irreducibles $\chi^\lambda$, $\lambda \vdash n$.

*Definition 1.1:* Let $\lambda, \mu \vdash n$; then denote

$$\chi^\lambda \otimes \chi^\mu = \sum_{\rho \vdash n} \kappa(\lambda, \mu, \rho) \cdot \chi^\rho. \tag{1}$$

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The problem of calculating the Kroecker multiplicities $\kappa(\lambda, \mu, \rho)$ is one of the last major open problems in the ordinary representation theory of the symmetric group $S_n$. Algorithms for such calculations are given, for example, in [3, 4], but these algorithms become very involved when applied to general partitions.

Let $H(k, \ell; n)$ denote the following subset of partitions:

$$H(k, \ell; n) = \{\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n \mid \lambda_{k+1} \leq \ell\} \quad \text{and} \quad H(k, \ell) = \bigcup_n H(k, \ell; n).$$

We remark that $H(k, \ell)$ parametrizes an important subset of the irreducible representations of the General Linear Lie Superalgebra $pl(k, \ell)$ [1]. Our main result is the following theorem.

THEOREM 1.2: *Given $0 \leq k, \ell \in \mathbb{Z}$, there exist $a = a(k, \ell)$, $b = b(k, \ell)$, satisfying the following condition: For any $n$, any $\lambda, \mu \in H(k, \ell; n)$ and any $\rho \vdash n$, $\kappa(\lambda, \mu, \rho) \leq a \cdot n^b$. Namely, these Kronecker multiplicities are polynomially bounded above.*

We also show that some of these multiplicities are bounded below by a polynomial growth.

One of the main tools for proving Theorem 1.2 is a recursive formula for calculating $\kappa(\lambda, \mu, \rho)$, a formula due to Dvir [3, Theorem 2.3 ]; see Theorem 4.1 below.

Let $\lambda \vdash m$, $\mu \vdash n$ and let $\chi^\lambda \hat{\otimes} \chi^\mu$ denote the outer tensor product:

$$\chi^\lambda \hat{\otimes} \chi^\mu = (\chi^\lambda \times \chi^\mu) \uparrow_{S_m \times S_n}^{S_{m+n}} . \tag{2}$$

Then $\chi^\lambda \hat{\otimes} \chi^\mu$ is an $S_{m+n}$ character, hence

$$\chi^\lambda \hat{\otimes} \chi^\mu = \sum_{\nu \vdash m+n} r(\lambda, \mu, \nu) \cdot \chi^\nu. \tag{3}$$

Note that by (2),

$$\deg(\chi^\lambda \hat{\otimes} \chi^\mu) = \binom{m+n}{n} \deg(\chi^\lambda) \cdot \deg(\chi^\mu) = \binom{m+n}{n} f^\lambda f^\mu. \tag{4}$$

The multiplicities $r(\lambda, \mu, \nu)$ are given by the Littlewood–Richardson rule and we call them the L-R multiplicities. An important tool in proving Theorem 1.2 is the fact, proved below, that in the $(k, \ell)$-hook, these L-R multiplicities are polynomially bounded .

The L-R multiplicities $r(\lambda,\mu,\nu)$ also yield the decomposition of the skew character $\chi^{\lambda/\alpha}$; see [7, I.5]:

$$\chi^{\alpha}\hat{\otimes}\chi^{\nu}=\sum_{\lambda} r(\lambda,\alpha,\nu)\cdot\chi^{\lambda} \quad \text{if and only if} \quad \chi^{\lambda/\alpha}=\sum_{\nu} r(\lambda,\alpha,\nu)\cdot\chi^{\nu}. \tag{5}$$

In Section 5 we show that outside the hook the above theorems fail: we give an example where $\kappa(\lambda,\mu,\nu)$ grow as fast as $\sqrt{n!}$. We also give an example where the growth of the L-R multiplicities $r(\lambda,\mu,\nu)$ is $\geq$ exponential.

## 2. Preliminaries

Let $\varphi=\sum_{\lambda\vdash n} a_\lambda\chi^\lambda$ and $\psi=\sum_{\lambda\vdash n} b_\lambda\chi^\lambda$, two $S_n$ characters. We write $\varphi\leq\psi$ if all $a_\lambda\leq b_\lambda$. We write $\chi^\lambda\in\varphi$ if $\chi^\lambda\leq\varphi$.

**Young's rule:** Let $\lambda=(\lambda_1,\lambda_2,\ldots)\vdash n$ be a partition of $n$ and let $m\geq 0$. Let $Par(\lambda,m)=\{\mu=(\mu_1,\mu_2,\ldots)\vdash n+m \mid \mu_1\geq\lambda_1\geq\mu_2\geq\lambda_2\geq\cdots\}$. Then (the horizontal) Young rule says that

$$\chi^{\lambda}\hat{\otimes}\chi^{(m)}=\sum_{\mu\in Par(\lambda,m)}\chi^{\mu}. \tag{6}$$

Lemma 2.1: *Let $\varphi$ be an $S_m$ character supported on $H(k-1,\ell;m)$,*

$$\varphi=\sum_{\mu\in H(k-1,\ell;m)} c_\mu\cdot\chi^\mu,$$

*and assume all $c_\mu\leq M$. Let $0<u$ and write*

$$\varphi\hat{\otimes}\chi^{(u)}=\sum_{\nu\in H(k,\ell;m+u)} d_\nu\cdot\chi^\nu.$$

*Then all $d_\nu\leq 2^\ell\cdot M\cdot(u+1)^k$.*

*Proof.* We have

$$\varphi\hat{\otimes}\chi^{(u)}=\sum_{\mu\in H(k-1,\ell;m)} c_\mu\cdot\chi^\mu\hat{\otimes}\chi^{(u)}=\sum_{\nu\in H(k,\ell;m+u)} d_\nu\cdot\chi^\nu.$$

Let $\nu\in H(k,\ell;m+u)$ and let $L$ denote the number of partitions $\mu\in H(k-1,\ell;m)$ such that $\chi^\nu\in\chi^\mu\hat{\otimes}\chi^{(u)}$. Then the coefficient $d_\nu$ is the sum of $L$ coefficients $c_\mu$. We estimate $L$: By Young's rule the first $k$ rows contribute the factor (upper bound)

$$(\nu_1-\nu_2+1)(\nu_2-\nu_3+1)\cdots(\nu_k-\nu_{k+1}+1) \quad \text{(and} \quad \sum_i(\nu_i-\nu_{i+1})\leq u),$$

while the first $\ell$ columns contribute the factor (upper bound) $2^\ell$ (since $\chi^\nu \in \chi^\mu \hat{\otimes} \chi^{(u)}$, by Young's rule for each $1 \le j \le \ell$, either $\mu'_j = \nu'_j$ or $\mu'_j = \nu'_j - 1$). Here $\mu'$ is the conjugate partition of $\mu$. Now each $\nu_i - \nu_{i+1} \le u$ hence $L \le 2^\ell \cdot (u+1)^k$, and the proof follows. ∎

We shall need the following properties.

*Remark 2.2:* Let $\lambda \in H(k, \ell; n)$; then the number of sub-partitions $\alpha \subseteq \lambda$ is $\le (n+1)^{k+\ell}$.

*Proof.* For each $1 \le i \le k$ there are $\le n+1$ possible values for $\alpha_i$ (namely the values $0, 1, 2, \ldots, n$). Similarly for the first $\ell$ columns. ∎

PROPOSITION 2.3 ([1]): *Let $\lambda \in H(k_1, \ell_1; n)$, $\mu \in H(k_2, \ell_2; n)$, and let $k = k_1 k_2 + \ell_1 \ell_2$ and $\ell = k_1 \ell_2 + k_2 \ell_1$. Then $\chi^\lambda \otimes \chi^\mu$ is supported on $H(k, \ell; n)$:*

$$\chi^\lambda \otimes \chi^\mu = \sum_{\nu \in H(k,\ell;n)} \kappa(\lambda, \mu, \nu) \cdot \chi^\nu.$$

## 3. Polynomial upper bound for the L-R coefficients in the hook

As a first step towards proving Theorem 1.2 we prove such a bound for the L-R multiplicities.

LEMMA 3.1: *Given $0 \le k_1, \ell_1, k_2, \ell_2$, there exist $a = a(k_1, \ell_1, k_2, \ell_2)$ and $b = b(k_1, \ell_1, k_2, \ell_2)$ satisfying the following condition:*

*Let $\lambda \in H(k_1, \ell_1)$ and $\mu \in H(k_2, \ell_2)$ and write*

$$\chi^\lambda \hat{\otimes} \chi^\mu = \sum_{\nu \vdash |\lambda| + |\mu|} r(\lambda, \mu, \nu) \cdot \chi^\nu.$$

*Then $r(\lambda, \mu, \nu) \le a(|\lambda| + |\mu|)^b$, namely, the $r(\lambda, \mu, \nu)$ are polynomially bounded.*

*Proof.* By induction on $k_1 + k_2 + \ell_1 + \ell_2 = q$. We do the induction step: w.l.o.g. we can assume that $k_1 > 0$ and that $\lambda \notin H(k_1 - 1, \ell_1)$ (otherwise the claim follows by induction). Thus, with $\lambda = (\lambda_1, \lambda_2, \ldots)$ we assume that $\lambda_{k_1} \ge \ell_1 + 1$. Construct

$$\bar{\lambda} = (\lambda_1, \ldots, \lambda_{k_1 - 1}, \ell_1, \lambda_{k_1+1}, \lambda_{k_1+2}, \ldots).$$

Note that $\bar{\lambda} \in H(k_1 - 1, \ell_1)$, hence we can use induction on the tuple $(k_1 - 1, k_2, \ell_1, \ell_2)$. By Young's rule, $\chi^\lambda \in \chi^{\bar{\lambda}} \hat{\otimes} \chi^{(u)}$ where $u = \lambda_{k_1} - \ell_1$ (check the diagrams). It follows that

$$\chi^{\lambda}\hat{\otimes}\chi^{\mu} \le (\chi^{\bar{\lambda}}\hat{\otimes}\chi^{\mu})\hat{\otimes}\chi^{(u)}.$$

By induction the coefficients in $\chi^{\bar{\lambda}}\hat{\otimes}\chi^{\mu}$ are polynomially bounded, hence by Lemma 2.1 the coefficients in $\chi^{\lambda}\hat{\otimes}\chi^{\mu}$ are polynomially bounded. ∎

Corollary 3.2: *Given $H(k,\ell)$, there exist $a = a(k,\ell)$ and $b = b(k,\ell)$ satisfying the following condition: Let $\lambda \in H(k,\ell)$ and $\alpha \subseteq \lambda$, and write $\chi^{\lambda/\alpha} = \sum_{\rho} r(\alpha,\rho,\lambda)\cdot\chi^{\rho}$; then all $r(\alpha,\rho,\lambda) \le a|\lambda|^{b}$.*

*Proof.* Note that by (5), these coefficients also satisfy the relation $\chi^{\alpha}\hat{\otimes}\chi^{\rho} = \sum_{\lambda} r(\alpha,\rho,\lambda)\cdot\chi^{\lambda}$. Here $\alpha,\rho \subseteq H(k,\ell)$. Thus, in the above lemma, choose $a = a(k,k,\ell,\ell)$ and $b = b(k,k,\ell,\ell)$ and get $r(\alpha,\rho,\lambda) \le a\cdot(|\alpha|+|\rho|)^{b} = a\cdot|\lambda|^{b}$. ∎

## 4. Kronecker multiplicities, polynomial bounds

4.1. Upper bound. Recall [7, page 114] the form $\langle\lambda,\mu\rangle = \langle\chi^{\lambda},\chi^{\mu}\rangle$ equals 1 if $\lambda = \mu$ and equals zero otherwise. Ignoring the negative terms in [3, Theorem 2.3], deduce the following inequality.

Theorem 4.1 ([3]): *Let $\lambda,\mu,\rho \vdash n$ and let*

$$\chi^{\lambda}\otimes\chi^{\mu} = \sum_{\rho}\kappa(\lambda,\mu,\rho)\cdot\chi^{\rho}. \tag{7}$$

*Then:*

(1)

$$\kappa(\lambda,\mu,\rho) \le \sum_{\alpha\vdash\rho_1,\ \alpha\subseteq\lambda\cap\mu}\langle\chi^{\lambda/\alpha}\otimes\chi^{\mu/\alpha},\chi^{(\rho_2,\rho_3,\ldots)}\rangle. \tag{8}$$

(2) *Tensoring (7) by $\chi^{(1^n)}$ implies that $\chi^{\lambda}\otimes\chi^{\mu'} = \sum_{\rho}\kappa(\lambda,\mu,\rho)\cdot\chi^{\rho'}$; then by part 1,*

$$\kappa(\lambda,\mu,\rho) \le \sum_{\alpha\vdash\rho'_1,\ \alpha\subseteq\lambda\cap\mu'}\langle\chi^{\lambda/\alpha}\otimes\chi^{\mu'/\alpha},\chi^{(\rho'_2,\rho'_3,\ldots)}\rangle. \tag{9}$$

*Note that $(\rho'_2,\rho'_3,\ldots) = (\rho_1-1,\rho_2-1,\ldots)'$.*

*Proof.* For each $1 \le i \le k$ there are $\le n+1$ possible values for $\alpha_i$, namely the values $0,1,\ldots,n$. Similarly for the possible values of $\alpha'_j$, $1 \le j \le \ell$. ∎

Theorem 4.2: *Given the $(k,\ell)$ hook, there exist $a = a(k,\ell)$ and $b = b(k,\ell)$ satisfying the following condition: Given any partitions $\lambda, \mu \in H(k,\ell;n)$ and $\rho \vdash n$, the multiplicities $\kappa(\lambda,\mu,\rho)$ satisfy $\kappa(\lambda,\mu,\rho) \le a \cdot n^b$. Namely, in the $(k,\ell)$ hook, these Kronecker multiplicities are polynomially bounded.*

*Proof.* Since $\lambda \in H(k,\ell,;n)$, the number of sub-partitions $\alpha$, $\alpha \subseteq \lambda \cap \mu \subseteq \lambda$ in (8) is bounded by $(n+1)^{k+\ell}$, and similarly for $\alpha$, $\alpha \subseteq \lambda \cap \mu' \subseteq \lambda$ in (9). Hence it suffices to show that each summand $\langle \chi^{\lambda/\alpha} \otimes \chi^{\mu/\alpha}, \chi^{(\rho_2,\rho_3,\ldots)}\rangle$ in (8), and each summand $\langle \chi^{\lambda/\alpha} \otimes \chi^{\mu'/\alpha}, \chi^{(\rho'_2,\rho'_3,\ldots)}\rangle$ in (9), are polynomially bounded. By Corollary 3.2, $\chi^{\lambda/\alpha} = \sum_\pi r(\alpha,\pi,\lambda) \cdot \chi^\pi$ with $r(\alpha,\pi,\lambda)$ polynomially bounded. Similarly for $\chi^{\mu/\alpha} = \sum_\theta r(\alpha,\theta,\mu) \cdot \chi^\theta$, with $r(\alpha,\theta,\mu)$ polynomially bounded.

Thus the summand $\langle \chi^{\lambda/\alpha} \otimes \chi^{\mu/\alpha}, \chi^{(\rho_2,\rho_3,\ldots)}\rangle$ is replaced by polynomially many summands $\langle \chi^{\pi} \otimes \chi^{\theta}, \chi^{(\rho_2,\rho_3,\ldots)}\rangle$ (we call this "a D step"), while the summand $\langle \chi^{\lambda/\alpha} \otimes \chi^{\mu'/\alpha}, \chi^{(\rho'_2,\rho'_3,\ldots)}\rangle$ is replaced by polynomially many summands $\langle \chi^{\pi} \otimes \chi^{\theta}, \chi^{(\rho'_2,\rho'_3,\ldots)}\rangle$ (we call this "a D$'$ step").

Since $\lambda, \mu \in H(k,\ell;n)$, the character $\chi^\lambda \otimes \chi^\mu$ is supported on $H(k^2+\ell^2, 2k\ell)$, so we calculate the coefficient $\kappa(\lambda,\mu,\rho)$ where $\rho \in H(k^2+\ell^2, 2k\ell)$. Note that a D step decreases the arm by 1, while a D$'$ step decreases the leg by 1. After at most $k^2+\ell^2$ many D steps and $2k\ell$ many D$'$ steps we reach $\kappa(\emptyset,\emptyset,\emptyset) = 1$. Thus, after at most $k^2+\ell^2+2k\ell = (k+\ell)^2$ steps we arrive at at-most polynomially many summands, each equal to $\kappa(\emptyset,\emptyset,\emptyset) = 1$. This completes the proof. ∎

4.2. Lower bound. For sequences $a_n, b_n$ of non-negative numbers we denote $a_n \cong b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$.

Lemma 4.3: *Let $n = kw$, $\lambda = (w,w,\ldots,w) = (w^k) \in H(k,0;n)$. Thus $\chi^\lambda \otimes \chi^\lambda$ is supported on $H(k^2,0)$. Fix $k$ and let $w$ go to infinity (hence also $n$ goes to infinity). Let $\epsilon > 0$. Then there exist partitions $\nu \vdash n$ such that*

$$\kappa(\lambda,\lambda,\nu) \ge n^{(k^2-4)(k^2-1)/4-\epsilon},$$

*which is a polynomial lower bound.*

*Proof.* Since $w$ goes to infinity, by Stirling's formula, for some constant $A$

$$f^\lambda \cong A \cdot \left(\frac{1}{n}\right)^{(k^2-1)/2} \cdot k^n, \quad \text{hence} \quad (f^\lambda)^2 \cong A^2 \cdot \left(\frac{1}{n}\right)^{k^2-1} \cdot k^{2n}. \tag{10}$$

Replacing $A$ by a slightly smaller constant $B$ we have

$$(f^\lambda)^2 > B^2 \cdot \left(\frac{1}{n}\right)^{k^2-1} \cdot k^{2n} \tag{11}$$

if $n$ is large enough.

By Proposition 2.3, $\chi^\lambda \otimes \chi^\lambda$ is supported on the $k^2$-strip $H(k^2, 0)$. Taking degrees we get

$$(f^\lambda)^2 = \sum_{\nu \in H(k^2,0;n)} \kappa(\lambda, \lambda, \nu) \cdot f^\nu. \tag{12}$$

Denote $g = (k^2-4)(k^2-1)/4 - \epsilon$, and assume all these $\kappa(\lambda, \lambda, \nu) \leq n^g$; then

$$(f^\lambda)^2 \leq n^g \cdot \sum_{\nu \in H(k^2,0;n)} f^\nu.$$

By [8, 4.5.1],

$$\sum_{\nu \in H(k^2,0;n)} f^\nu \cong C \cdot \left(\frac{1}{n}\right)^{k^2(k^2-1)/4} \cdot k^{2n}$$

for some constant $C$. Let $C < C'$; then for $n$ large enough

$$(f^\lambda)^2 \leq n^g C' \left(\frac{1}{n}\right)^{k^2(k^2-1)/4} \cdot k^{2n}. \tag{13}$$

Combining (11) and (13) we deduce that

$$B^2 \cdot \left(\frac{1}{n}\right)^{k^2-1} \cdot k^{2n} < n^g C' \left(\frac{1}{n}\right)^{k^2(k^2-1)/4} \cdot k^{2n}.$$

Forming $l.h.s./r.h.s.$ deduce that for the constant $c = B^2/C'$

$$c \cdot n^{(k^2-4)(k^2-1)/4-g} = cn^\epsilon < 1$$

for all large $n$, which is a contradiction. ∎

## 5. Outside the hook

We now give examples outside the hook, where the Littlewood–Richardson and Kronecker multiplicities are not polynomially bounded.

5.1. A $\sqrt{n!}$ LOWER BOUND FOR SOME KRONECKER MULTIPLICITIES.

*Example 5.1:* Here we show that outside the hook, some Kronecker multiplicities grow at least as fast as $(n/e)^{n/2}$, so roughly, as $\sqrt{n!}$.

Let $\varepsilon > 0$, assume that as $n$ goes to infinity all Kronecker multiplicities $\kappa(\lambda,\mu,\nu)$ satisfy

$$\kappa(\lambda,\mu,\nu) < \left(\frac{n}{e}\right)^{\frac{n}{2}(1-\varepsilon)}, \tag{14}$$

and we derive a contradiction. Let $\lambda = \mu \vdash n$ be the Vershik–Kerov Logan–Shepp partition which maximizes $f^\lambda$ [10, Theorem 1]; see also [6]. Then there exist constants $c_0, c_1 > 0$ such that for $n$ large enough

$$e^{-c_1\sqrt{n}}\sqrt{n!} \le f^\lambda \le e^{-c_0\sqrt{n}}\sqrt{n!}. \tag{15}$$

Hence, by a slight abuse of notations, for a maximizing $\lambda$ we write

$$f^\lambda \cong e^{-c\sqrt{n}}\sqrt{n!}. \tag{16}$$

By Stirling's we similarly have

$$\deg(\chi^\lambda \otimes \chi^\lambda) = (f^\lambda)^2 \cong e^{-C\sqrt{n}} n!, \tag{17}$$

where $C = 2c > 0$. On the other hand, the assumption (14) implies that

$$\deg(\chi^\lambda \otimes \chi^\lambda) < \left(\frac{n}{e}\right)^{\frac{n}{2}(1-\varepsilon)} \sum_{\nu\vdash n} f^\nu. \tag{18}$$

It follows from the RSK correspondence [9] that the sum $\sum_{\nu\vdash n} f^\nu$ equals $T_n$, the number of involutions in $S_n$. It was proved in [2] (see also in the Math. Review) that

$$T_n \cong \frac{e^{\sqrt{n}}\cdot\sqrt{n!}}{(\pi n)^{1/4}\cdot q},$$

where $q = \sqrt{2}\cdot e^{1/4}$, so for large $n$,

$$T_n < e^{\sqrt{n}}\cdot\sqrt{n!}. \tag{19}$$

Then (17), (18) and (19) imply that

$$e^{-C\sqrt{n}}\cdot n! < \left(\frac{n}{e}\right)^{\frac{n}{2}(1-\varepsilon)} e^{\sqrt{n}}\cdot\sqrt{n!}, \quad \text{so} \quad \sqrt{n!} < \left(\frac{n}{e}\right)^{\frac{n}{2}(1-\varepsilon)} e^{(C+1)\sqrt{n}}. \tag{20}$$

By Stirling's formula $(n/e)^{n/2} < \sqrt{n!}$, therefore (20) implies that

$$\left(\frac{n}{e}\right)^{n/2} < \left(\frac{n}{e}\right)^{\frac{n}{2}(1-\varepsilon)}\cdot e^{(C+1)\sqrt{n}},$$

hence

$$\left(\frac{n}{e}\right)^{\frac{n}{2}\varepsilon} < e^{(C+1)\sqrt{n}}.$$

This is a contradiction, since the right-hand side is sub-exponential while the left hand side is essentially $(\sqrt{n!})^{\varepsilon}$, which grows to infinity faster than any exponential.

It should be interesting to find out if for any partitions $\lambda, \mu, \nu \vdash n$, all the Kronecker multiplicities $\kappa(\lambda, \mu, \nu)$ are bounded above by $\sqrt{n!}$.

### 5.2. Exponential lower bound for some L-R multiplicities.

*Example 5.2:* There exist partitions $\lambda \vdash n$, $n \to \infty$, such that some multiplicities in $\chi^\lambda \hat{\otimes} \chi^\lambda$ are not bounded above by $a^n$ for any $a < 2$.

*Proof.* As in Section 5.1, let $\lambda \vdash n$, $n = 1, 2, \ldots$ be a sequence of partitions maximizing $f^\lambda$ [6], [10]. Let $n$ be large and let $0 < a < 2$. We show that if one assumes that all $r(\lambda, \lambda, \nu) < a^n$, then one arrives at a contradiction. Hence some $r(\lambda, \lambda, \nu)$ grow at least exponentially.

That assumption $r(\lambda, \lambda, \nu) < a^n$ implies that for large $n$,

$$\deg(\chi^\lambda \hat{\otimes} \chi^\lambda) < a^n \cdot \sum_{\nu \vdash 2n} \deg(\chi^\nu).$$

Together with (19) it implies that

$$\deg(\chi^\lambda \hat{\otimes} \chi^\lambda) < a^n \cdot e^{\sqrt{2n}} \cdot \sqrt{(2n)!}. \tag{21}$$

By (4)

$$\deg(\chi^\lambda \hat{\otimes} \chi^\lambda) = (\deg(\chi^\lambda))^2 \cdot \binom{2n}{n}.$$

Combined with (16) we get

$$\deg(\chi^\lambda \hat{\otimes} \chi^\lambda) \cong e^{-2c\sqrt{n}} \cdot n! \cdot \binom{2n}{n} = e^{-2c\sqrt{n}} \cdot \frac{(2n)!}{n!}.$$

Together with (21) it implies

$$e^{-2c\sqrt{n}} \cdot \frac{(2n)!}{n!} < a^n \cdot e^{\sqrt{2n}} \cdot \sqrt{(2n)!},$$

so

$$\frac{(2n)!}{n! \cdot \sqrt{(2n)!}} < a^n \cdot e^{(2c+2)\sqrt{n}}.$$

Squaring both sides we get

$$\binom{2n}{n} < (a^2)^n \cdot e^{4(c+1)\sqrt{n}}. \tag{22}$$

By Stirling's formula $\binom{2n}{n} \cong \frac{1}{\sqrt{\pi n}} \cdot 4^n$, hence (22) implies that

$$\left(\frac{4}{a^2}\right)^n < \sqrt{\frac{\pi n}{2}} \cdot e^{4(c+1)\sqrt{n}}.$$

Since $a^2 < 4$, the left hand side grows exponentially with $n$ while the right hand side grows sub-exponentially, hence a contradiction. This completes the example. ∎

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