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A SOLUTION TO A PROBLEM OF GRÜNBAUM AND MOTZKIN AND OF ERDŐS AND PURDY ABOUT BICHROMATIC CONFIGURATIONS OF POINTS IN THE PLANE

BY

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ABSTRACT

Let P be a set of n blue points in the plane, not all on a line. Let R be a set of m red points such that $P \cap R = \emptyset$ and every line determined by P contains a point from R. We provide an answer to an old problem by Grünbaum and Motzkin [9] and independently by Erdős and Purdy [6] who asked how large must m be in terms of n in such a case? More specifically, both [9] and [6] were looking for the best absolute constant c such that $m \geq cn$. We provide an answer to this problem and show that $m \geq (n-1)/3$.

1. Introduction

A beautiful result of Motzkin [14], Rabin, and Chakerian [3] states that any set of non-collinear red and blue points in the plane determines a monochromatic line. Grünbaum and Motzkin [9] initiated the study of biased coloring, that is, coloring of the points such that no purely blue line is determined. The intuition behind this study is that if the number of blue points is much larger than the number of red points, then unless the set of blue points is collinear the set of blue and red points should determine a monochromatic blue line.

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The same problem was considered independently by Erdős and Purdy [6] who stated it in a slightly different way.

Problem A: Given a non-collinear set P of n points in the plane we wish to stab all the lines determined by this set by another set R of m red points such that $P \cap R = \emptyset$. Give a lower bound for m in terms of n.

It will be more convenient for us to consider the equivalent dual problem (by point-line duality in the plane) for lines in the plane:

Problem B: Given a set $\mathcal{L}$ of n non-concurrent blue lines in the real projective plane we wish to find a set $\mathcal{R}$ of m red lines different from the blue ones such that every intersection point of blue lines is incident to a red line. Give a lower bound for m in terms of n.

An $\Omega(n)$ bound for the cardinality of R in Problem B follows from the so-called "weak Dirac's conjecture". In 1951 Dirac [5] conjectured that in any set of n non-concurrent lines there exists a line incident to at least $\frac{n}{2} - O(1)$ intersection points with other lines in the set. Szemerédi and Trotter [20] and Beck [1] proved a weaker result which is that under the above conditions there exists a line incident to $\Omega(n)$ intersection points. This result is known as the "weak Dirac's conjecture". The proofs in [1, 20] used the upper bound of Szemerédi and Trotter [20] for the number of incidences between points and lines in the plane.

Notice that any lower bound for the problem of Dirac implies immediately the same lower bound for Problem B. Indeed, if there exists a blue line ℓ incident to m intersection points with other blue lines, then clearly m distinct red lines are required just to ensure that every intersection point on ℓ is incident to a red line.

In the original papers [20] and [1] lower bounds of $10^{-186}n$ and $2^{-1000}n$, respectively, are shown for the "weak Dirac's conjecture". Today much better bounds in terms of the multiplicative constant are known (see [17] for the best bound) for the number of incidences between points and lines in the plane. Consequently also the constant in the "weak Dirac's conjecture" is improved. Very recently, Payne and Wood [18] carried out this calculation of the best constant in the "weak Dirac's conjecture". They combined the above-mentioned progress on the number of point-line incidences in the plane and with some more ideas showed that the lower bound in the "weak Dirac's conjecture" can be improved to $(n-3)/76$.

On the other hand and also very recently, Lund, Purdy, and Smith [13] showed that Dirac's conjecture is false if we replace lines by pseudolines. They constructed examples of n pseudolines where each one is incident to at most $\frac{4}{9}n$ intersection points. This means in particular that one cannot hope for a better lower bound than $\frac{4}{9}n$ for (the pseudo-line version of) Problem B through finding a blue line with many intersection points on it.

An important special case of Problem A was solved in [21] and extended in [15]: In 1970 Scott [19] conjectured that any set of n non-collinear points in the plane determines at least $2\lfloor\frac{n}{2}\rfloor$ lines with distinct directions. In the same paper [19], Scott also includes an analogous conjecture in three dimensions. Scott's conjecture in the plane was proved by Ungar [21]. Notice that this is equivalent to saying that given a set of n blue points in the plane and a set of m points on the line at infinity (therefore, in fact, on any given line) such that there is no monochromatic blue line, then $m \geq 2\lfloor\frac{n}{2}\rfloor$. This bound is best possible.

In [15] this result is extended as follows: Suppose that P is a set of n non-collinear blue points in the plane and R is a set of m red points such that $P \cap R = \emptyset$ and every line determined by P contains a red point that is extreme on that line (with respect to its incident blue points); then $m \geq 2\lfloor\frac{n}{2}\rfloor$. (This result is later used in [15] and [16] to solve Scott's conjecture in three dimensions.)

It is evident, however, that the answer to Problems A and B is different. Constructions found by Grünbaum show that m can be as small as $n-4$ in Problems A and B, and there are sporadic constructions (that is, for small values of n) in which $|R|$ is equal to $|P|-6$ (see [10]).

In this paper we provide the following partial answer to Problems A and B which improves significantly on the bound of $(n-3)/76$ that can be deduced by using the bound on the "weak Dirac's conjecture" in [18]. Our proof uses a purely combinatorial argument that does not rely on the asymptotic bounds on the number of point-line incidences in the plane:

Theorem 1: *Let $\mathcal{L}$ be a set of n non-concurrent blue lines and let $\mathcal{R}$ be a set of m red lines in the real projective plane. If $\mathcal{L} \cap \mathcal{R} = \emptyset$ and there is a line from $\mathcal{R}$ through every intersection point of lines in $\mathcal{L}$, then $m \geq (n-1)/3$.*

2. Proof of Theorem 1

The idea of the proof is to estimate in two different ways the cardinality of the following set T of special triples (r, e, c) such that:

- e is an edge in the arrangement $\mathcal{A}(\mathcal{L})$ delimited by two vertices, say W and Z,
- c is a line in $\mathcal{L}$ passing through the vertex W, and
- r is a line in $\mathcal{R}$ passing through the vertex Z.

See Figure 1 for an example of a triple in T. We note that throughout all of our drawings below, lines in $\mathcal{L}$ are drawn solid while lines in $\mathcal{R}$ are drawn dashed. A simple lower bound for $|T|$ is argued as follows. Consider any two lines b and c in $\mathcal{L}$. Let W be the intersection point of b and c. Then there are precisely two edges e of $\mathcal{A}(\mathcal{L})$ on the line b that are incident to W. (Here we use the fact that not all the lines in $\mathcal{L}$ pass through the same point. Notice also that the two edges incident to W may have the same other vertex Z, in the case where all the lines in $\mathcal{L}$, but one, are concurrent.) For each of these two edges there is at least one red line r in $\mathcal{R}$ passing through the vertex of the edge distinct from W. Therefore, for every (ordered) pair of blue lines b and c we obtain two distinct triples in T, so that no triple arises more than once in this manner. This implies that $|T| \geq 2n(n-1)$ (see Figure 1).

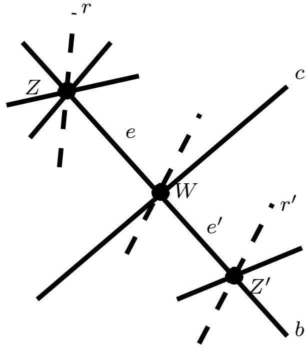

Figure 1. Every $b, c \in \mathcal{L}$ give rise to two triples in T.

To obtain a good lower bound for m in terms of n it will therefore be helpful to bound from above the number of triples in T to which a given red line r

belongs. We denote by $T(r)$ the set of triples in T in which the red line is r. The following lemma provides an upper bound for the cardinality of $T(r)$. As the reader may notice this is closely related to the so called Zone Theorem (see [2, 4]). Our proof is indeed inspired by the proof in [4]. As we shall comment later, it is very well possible that one may be able to use a more elaborated argument, as used in [2] for the zone theorem, to provide an improved bound for the lemma:

LEMMA 1: *For every line $r \in \mathcal{R}$ we have $|T(r)| \leq 6n$.*

Proof. Let r be the given red line and assume without loss of generality that r is horizontal. That is, we consider an affine picture of the projective plane in which r is horizontal. We can also assume, by applying a suitable projective transformation, that no two lines in $\mathcal{L} \cup \mathcal{R}$ are parallel in this affine picture.

We denote by $T_1(r)$ the set of triples $(r, e, c) \in T(r)$ such that e lies above r; $T_2(r)$ will denote the complementary set of triples in $T(r)$, namely those triples $(r, e, c) \in T(r)$ such that e lies below r.

We show that $|T_1(r)| \leq 3n$; a symmetric bound holds for $|T_2(r)|$. We may assume without loss of generality that for every triple (r, e, c) in $T_1(r)$ the edge e is bounded. This is because we can apply a projective transformation that takes to the line at infinity a line r' parallel to r and located slightly below it.

Let $e_1, \ldots, e_s$ denote all the edges e_i such that (r, e_i, c) is in $T_1(r)$ for some $c \in \mathcal{L}$ (notice that also unbounded edges have two endpoints as they "wrap around infinity" in the projective plane). For every i let $b_i \in \mathcal{L}$ denote the line containing e_i and let Z_i be the vertex of e_i that is the intersection point of b_i and r. We assume that the indexing of the edges $e_1, \ldots, e_s$ is according to the location of $Z_1, \ldots, Z_s$ on r from left to right. Note that every vertex Z_i arises by at least two distinct blue lines (see, for example, the intersection point of a unique blue line with r in Figure 2, which is not a vertex in $\mathcal{A}(\mathcal{L})$). If two lines b_i and b_j meet r at the same point $Z_i = Z_j$, then we assume that if $i < j$, then above r, b_i is to the left of b_j. For every $1 \leq i \leq s$ we denote by W_i the vertex of e_i different from Z_i (see Figure 2).

Fix an index $1 \leq i \leq s$. A line $c \in \mathcal{L}$ through W_i, different from b_i, will be called a **left** line with respect to e_i if its intersection point with r lies to the left of Z_i. In a similar way we define a **right** line with respect to e_i. Also, c will be called an **extreme** line with respect to e_i, if its intersection point with r is extreme (leftmost or rightmost) on r among all the intersection points of r with

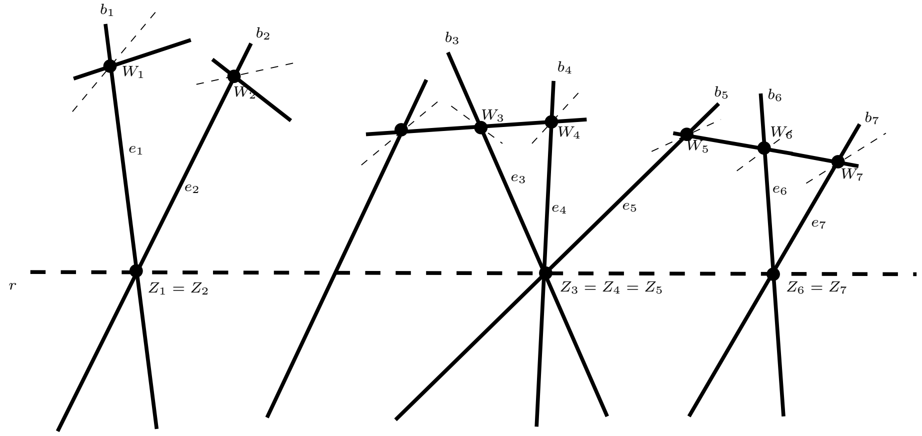

Figure 2. Notation used in the proof.

lines in $\mathcal{L}$ passing through W_i. If c is not extreme with respect to e_i it will be called **tame**. Observe that for every $1 \leq i \leq s$ there are at most two extreme lines with respect to e_i (see Figure 3).

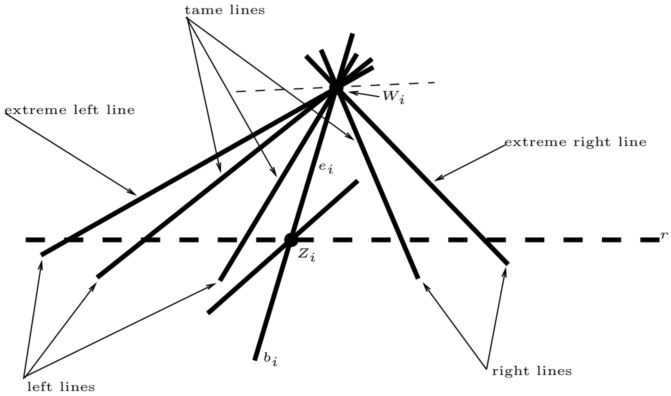

Figure 3. Right, left, and extreme lines with respect to e_i.

Claim 1: *If $W_i = W_j$ and $i < j$, then $j = i + 1$.*

Proof. Assume not; then one of the (at least two) lines in $\mathcal{L}$ passing through Z_{i+1} must intersect either the relative interior of e_i or the relative interior of e_j, contrary to the assumption that these are two edges in the arrangement $\mathcal{A}(\mathcal{L})$ (see Figure 4). ∎

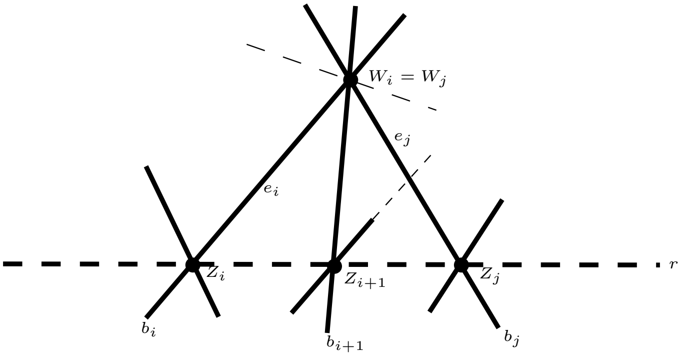

Figure 4. Illustrating the proof of Claim 1.

As a simple consequence of Claim 1, there are no three distinct indices i_1, i_2, i_3 such that $W_{i_1} = W_{i_2} = W_{i_3}$.

Claim 2: *If a line c in $\mathcal{L}$ is tame with respect to two edges e_i, e_j, then $W_i = W_j$. Consequently, because of Claim 1, a line in $\mathcal{L}$ can be tame with respect to at most two edges.*

Proof. Assume without loss of generality that $i < j$ and that $W_i \neq W_j$. If c is a right line with respect to e_i and a left line with respect to e_j, then either b_i crosses the relative interior of the edge e_j, or b_j crosses the relative interior of the edge e_i, a contradiction (see Figure 5 (a) and (b), respectively).

Note that because $i < j$, it is not possible that c is a left line with respect to e_i and a right line with respect to e_j.

Assume that c is a left line with respect to both e_i and e_j; then W_i is closer to r along c than W_j, and then the extreme left line with respect to e_i crosses the relative interior of the edge e_j, a contradiction (see Figure 5 (c)). A symmetric argument applies if c is a right line with respect to both e_i and e_j. ∎

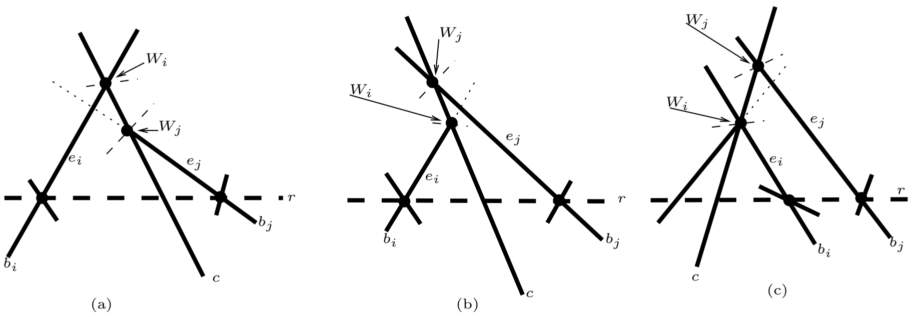

Figure 5. Illustrating the proof of Claim 2.

CLAIM 3: *Suppose an edge e_i has two extreme lines with respect to it (a left line and a right line). Then b_i may be tame with respect to at most one edge e_j. In the latter case b_i and b_j meet at W_i.*

Proof. Suppose that b_i is tame with respect to two edges e_j, e_k. By Claim 2, $W_j = W_k$ and therefore, by Claim 1, we may assume $k = j + 1$.

It cannot be that $W_i = W_j = W_k$, as a consequence of Claim 1. Therefore, $W_i \neq W_j = W_k$, and then W_i lies in the relative interior of the segment between W_j and Z_i on b_i. If $i < j$, then the left extreme line with respect to e_i must cross the edge e_j (see Figure 6), and if $j < i$, then the right extreme line with respect to e_i must cross the edge e_j. These contradictions complete the proof. ∎

Given any triple $(r, e, c) \in T_1(r)$, either c is an extreme line with respect to e or it is tame with respect to e. For every $i = 1, \ldots, s$ we denote by x_i the number of extreme lines with respect to e_i. We denote by t_i the number of times b_i is tame with respect to another edge e_j. The number of triples in $T_1(r)$ is therefore $\sum_{i=1}^{s}(x_i + t_i)$.

Clearly $x_i \leq 2$ for every i. By Claim 3, if $x_i = 2$, then $t_i = 1$. Recall that, by Claim 2, $t_i \leq 2$ for every i. Therefore, we have $x_i + t_i \leq 3$ for every $1 \leq i \leq s$. This proves the upper bound of $3n$ for the cardinality of $T_1(r)$. Symmetric arguments show that $|T_2(r)| \leq 3n$, thus proving Lemma 1. ∎

Lemma 1 implies an upper bound of $6mn$ for the cardinality of T. Together with the lower bound of $2n(n-1)$ we get a lower bound for m, namely $m \geq (n-1)/3$, thus proving Theorem 1. ∎

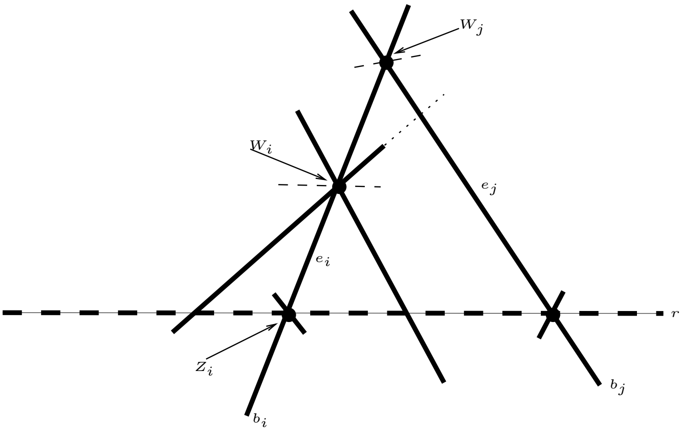

Figure 6. Illustrating the proof of Claim 3.

The best construction we are aware of where $|T_r|$ is large, is such that $|T_r|$ equals roughly $5n$. It is highly possible that this is the best upper bound one can take in Lemma 1 and consequently improve the lower bound in Theorem 1 to $m \geq \frac{2}{5}n$. So far we have indications that the bound in Lemma 1 is not best possible. However, our arguments to show this are a lot more technically involved than those presented here and would damage the presentation quite a bit (compare for this matter the argument in [4] with the more involved one in [2] for an upper bound of the complexity of a zone in the zone theorem). We therefore choose to leave this question open at the moment.

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