ISRAEL JOURNAL OF MATHEMATICS **203** (2014), 1–22
DOI: 10.1007/s11856-012-0173-2

THE ISOTROPIC POSITION AND THE REVERSE SANTALÓ INEQUALITY

BY

APOSTOLOS GIANNOPOULOS

Department of Mathematics, National and Kapodistrian University of Athens
Panepistimioupolis 157-84, Athens, Greece
e-mail: apgiannop@math.uoa.gr

AND

GRIGORIS PAOURIS

Department of Mathematics, Texas A & M University
College Station, TX 77843, USA
e-mail: grigorios.paouris@gmail.com

AND

BEATRICE-HELEN VRITSIOU

Department of Mathematics, National and Kapodistrian University of Athens
Panepistimioupolis 157-84, Athens, Greece
e-mail: bevritsi@math.uoa.gr

Dedicated to the memory of Joram Lindenstrauss

ABSTRACT

We present proofs of the reverse Santaló inequality, the existence of M-ellipsoids and the reverse Brunn–Minkowski inequality, using purely convex geometric tools. Our approach is based on properties of the isotropic position.

Received December 16, 2011 and in revised form March 19, 2012

1. Introduction

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote the corresponding Euclidean norm by $\|\cdot\|_2$, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$.

A convex body K in $\mathbb{R}^n$ is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. We say that K is symmetric if $x\in K$ implies that $-x\in K$. We say that K is centered if its barycenter is at the origin, i.e. $\int_K\langle x,\theta\rangle\,dx=0$ for every $\theta\in S^{n-1}$. For every interior point x of K, we define the polar body $(K-x)^\circ$ of K with respect to x as follows:

$$(K-x)^\circ:=\{y\in\mathbb{R}^n:\langle z-x,y\rangle\leq 1\text{ for all }z\in K\}. \tag{1.1}$$

Note that $(K-x)^{\circ\circ}=K-x$.

The purpose of this article is to present an alternative route to some fundamental theorems of the asymptotic theory of convex bodies: the reverse Santaló inequality, the existence of M-ellipsoids and the reverse Brunn–Minkowski inequality. The starting point for our approach is the isotropic position of a convex body, which can be shown to simultaneously be an M-position for the body if its isotropic constant is bounded. The new ingredient in this paper is a way to also show, using only basic tools from the theory of convex bodies and log-concave measures, that every convex body with bounded isotropic constant satisfies the reverse Santaló inequality, and then that all bodies do.

We first recall the statements and the history of the results. The classical Blaschke–Santaló inequality states that for every symmetric convex body K in $\mathbb{R}^n$, the volume product $s(K):=|K||K^\circ|$ is less than or equal to the volume product $s(B_2^n)$, and equality holds if and only if K is an ellipsoid. More generally, for every convex body K, there exists a unique point z in the interior of K such that

$$|(K-z)^\circ|=\inf_{x\in\mathrm{int}(K)}|(K-x)^\circ|, \tag{1.2}$$

and for this point we have

$$|K||(K-z)^\circ|\leq s(B_2^n) \tag{1.3}$$

(with equality again if and only if K is an ellipsoid). This unique point is usually called the Santaló point of K and is characterized by the following property: the polar body $(K-z)^\circ$ of K with respect to the point z has its barycenter at the origin if and only if z is the Santaló point of K. Observe now that the body

$K - \mathrm{bar}(K)$ is centered and it is the polar body of $(K - \mathrm{bar}(K))^\circ$ with respect to the origin, hence 0 is the Santaló point of $(K - \mathrm{bar}(K))^\circ$. This means that for every centered convex body K,

$$s(K) = |K||K^\circ| = \inf_{x\in \mathrm{int}(K^\circ)} |K^\circ||(K^\circ - x)^\circ|, \tag{1.4}$$

and this allows us to restate the Blaschke–Santaló inequality in a more concise way: for every centered convex body K in $\mathbb{R}^n$, $s(K) \leq s(B_2^n)$, with equality if and only if K is an ellipsoid.

In the opposite direction, a well-known conjecture of Mahler states that $s(K) \geq 4^n/n!$ for every symmetric body K, and that $s(K) \geq (n+1)^{n+1}/(n!)^2$ in the not necessarily symmetric case. This has been verified for some classes of bodies, e.g., zonoids and 1-unconditional bodies (see [28], [18], [30] and [10]). The reverse Santaló inequality, or the Bourgain–Milman inequality, tells us that there exists an absolute constant $c > 0$ such that

$$\left(\frac{s(K)}{s(B_2^n)}\right)^{1/n} \geq c \tag{1.5}$$

for every convex body K in $\mathbb{R}^n$ which contains 0 in its interior. The inequality was first proved in [5] and answers the question of Mahler in the asymptotic sense: for every centered convex body K in $\mathbb{R}^n$, the affine invariant $s(K)^{1/n}$ is of the order of $1/n$. A few other proofs have appeared (see [20], [15], [25]), the most recent of which give the best lower bounds for the constant c and exploit tools from quite diverse areas: Kuperberg in [15] shows that in the symmetric case we have $c \geq 1/2$, and his proof uses tools from differential geometry, while Nazarov's proof [25] uses multivariable complex analysis and leads to the bound $c \geq \pi^2/32$. It should also be mentioned that Kuperberg had previously given an elementary proof [14] of the weaker lower bound $s(K)^{1/n} \geq c/(n \log n)$.

The original proof of the reverse Santaló inequality in [5] employed a dimension descending procedure which was based on Milman's quotient of subspace theorem. Thus, an essential tool was the MM^*-estimate which follows from Pisier's inequality for the norm of the Rademacher projection. In [20], Milman offered a second approach, which introduced an "isomorphic symmetrization" technique. This is a symmetrization scheme which is in many ways different from the classical symmetrizations. In each step, none of the natural parameters of the body is being preserved, but the ones which are of interest remain under control. The MM^*-estimate is again crucial for the proof.

Our approach is based on properties of the isotropic position of a convex body and combines a very simple one-step isomorphic symmetrization argument (which is reminiscent of [20]) with the method of convex perturbations that Klartag invented in [12] for his solution to the isomorphic slicing problem. Aside from the use of the latter, the approach is elementary, in the sense that it uses only standard tools from convex geometry, namely, some classical consequences of the Brunn–Minkowski inequality. Recall that a convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered and its inertia matrix is a multiple of the identity: there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{1.6}$$

for every $\theta \in S^{n-1}$. It is relatively easy to show that every convex body has an isotropic position and that this position is well-defined (by this we mean unique up to orthogonal transformations): if K is a centered convex body, then any linear image $\tilde{K}$ of K which has volume 1 and satisfies

$$\int_{\tilde{K}} \|x\|_2^2 \, dx = \inf\Big\{ \int_{T(\tilde{K})} \|x\|_2^2 \, dx : T \text{ is linear and volume-preserving}\Big\} \tag{1.7}$$

is an isotropic image of K. This also implies that any isotropic image of K has the same isotropic constant, and thus L_K can be defined for the entire affine class of K. One of the main problems in the asymptotic theory of convex bodies is the hyperplane conjecture, which, in an equivalent formulation, says that there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leq C. \tag{1.8}$$

A classical reference on the subject is the paper of Milman and Pajor [21] (see also [7]). The problem remains open: Bourgain [4] has obtained the upper bound $L_K \leq c\sqrt[4]{n}\log n$, and Klartag [12] has improved that to $L_K \leq c\sqrt[4]{n}$—see also [13]. However, in this paper we only need a few basic results from the theory of isotropic convex bodies and, more generally, of isotropic log-concave probability measures. All this background information is given in Section 2; there we also list a few more necessary tools from the general asymptotic theory of convex bodies.

In Section 3 we prove the reverse Santaló inequality in two stages. First, using elementary covering estimates, we prove a version of it which involves the isotropic constant L_K of K.

THEOREM 1.1: *Let K be a convex body in $\mathbb{R}^n$ which contains 0 in its interior. Then $4ns(K)^{1/n} \geq ns(K-K)^{1/n} \geq c_1/L_K$, where $c_1 > 0$ is an absolute constant.*

Then, we use Klartag's ideas from [12] to show that every symmetric convex body K is "close" to a convex body T with isotropic constant L_T bounded by $1/\sqrt{ns(K)^{1/n}}$.

THEOREM 1.2: *Let K be a symmetric convex body in $\mathbb{R}^n$. There exists a convex body T in $\mathbb{R}^n$ such that (i) $c_2K \subseteq T-T \subseteq c_3K$ and (ii) $L_T \leq c_4/\sqrt{ns(K)^{1/n}}$, where $c_2, c_3, c_4 > 0$ are absolute constants.*

Since K and $T-T$ have bounded geometric distance, we easily check that $s(K)^{1/n} \simeq s(T-T)^{1/n}$. Then we can use Theorem 1.1 for T to obtain the lower bound $L_T \geq c_5/\big(ns(K)^{1/n}\big)$. Combining this estimate with Theorem 1.2(ii), we immediately get the reverse Santaló inequality for symmetric bodies, and hence for all bodies.

THEOREM 1.3: *Let K be a symmetric convex body in $\mathbb{R}^n$. Then $s(K)^{1/n} \geq c_6/n$, where $c_6 > 0$ is an absolute constant.*

In Section 4 we briefly indicate how one can use Theorem 1.3 in order to establish the existence of M-ellipsoids and the reverse Brunn–Minkowski inequality. The procedure is rather standard.

The existence of an "M-ellipsoid" associated with any centered convex body K in $\mathbb{R}^n$ was proved by Milman in [19] (see also [20]): there exists an absolute constant $c > 0$ such that for any centered convex body K in $\mathbb{R}^n$ we can find an origin symmetric ellipsoid $\mathcal{E}_K$ satisfying $|K| = |\mathcal{E}_K|$ and

$$\begin{aligned} &\frac{1}{c}|\mathcal{E}_K + T|^{1/n} \leq |K+T|^{1/n} \leq c|\mathcal{E}_K + T|^{1/n}, \\ &\frac{1}{c}|\mathcal{E}_K^{\circ} + T|^{1/n} \leq |K^{\circ}+T|^{1/n} \leq c|\mathcal{E}_K^{\circ} + T|^{1/n}, \end{aligned} \tag{1.9}$$

for every convex body T in $\mathbb{R}^n$. The existence of M-ellipsoids can be equivalently established by introducing the M-position of a convex body. To any given centered convex body K in $\mathbb{R}^n$ we can apply a linear transformation and find a position $\tilde{K} = u_K(K)$ of volume $|\tilde{K}| = |K|$ such that (1.9) is satisfied with $\mathcal{E}_K$ a multiple of B_2^n. This is the so-called M-position of K. It follows then that for every pair of convex bodies K_1 and K_2 in $\mathbb{R}^n$ and for all $t_1, t_2 > 0$,

$$|t_1\tilde{K_1} + t_2\tilde{K_2}|^{1/n} \leq c' \left(t_1|\tilde{K_1}|^{1/n} + t_2|\tilde{K_2}|^{1/n} \right), \tag{1.10}$$

where $c' > 0$ is an absolute constant, and that (1.10) remains true if we replace $\tilde{K_1}$ or $\tilde{K_2}$ (or both) by their polars. This statement is Milman's reverse Brunn–Minkowski inequality.

Another way to define the M-position of a convex body is through covering numbers. Recall that the covering number $N(A,B)$ of a body A by a second body B is the least integer N for which there exist N translates of B whose union covers A. Then, as Milman proved, there exists an absolute constant $\beta > 0$ such that every centered convex body K in $\mathbb{R}^n$ has a linear image $\tilde{K}$ which satisfies $|\tilde{K}| = |B_2^n|$ and

$$\max\{N(\tilde{K}, B_2^n), N(B_2^n, \tilde{K}), N(\tilde{K}^\circ, B_2^n), N(B_2^n, \tilde{K}^\circ)\} \leq \exp(\beta n). \tag{1.11}$$

We say that a convex body K which satisfies (1.11) is in M-position with constant β. If K_1 and K_2 are two such convex bodies, there is a standard way to show that they and their polar bodies satisfy the reverse Brunn–Minkowski inequality (1.10). Note that M-ellipsoids and the M-position of a convex body are not uniquely defined; see [2] for a recent description in terms of isotropic restricted Gaussian measures.

Pisier (see [26] and [27, Chapter 7]) has proposed a different approach to these results, which allows one to find a whole family of special M-ellipsoids satisfying stronger entropy estimates. The precise statement is as follows. For every $0 < \alpha < 2$ and every symmetric convex body K in $\mathbb{R}^n$, there exists a linear image $\tilde{K}$ of K which satisfies $|\tilde{K}| = |B_2^n|$ and

$$\max\{N(\tilde{K}, tB_2^n), N(B_2^n, t\tilde{K}), N(\tilde{K}^\circ, tB_2^n), N(B_2^n, t\tilde{K}^\circ)\} \leq \exp\left(\frac{c(\alpha)n}{t^\alpha}\right) \tag{1.12}$$

for every $t \geq 1$, where $c(\alpha)$ is a constant depending only on α, with $c(\alpha) = O\big((2-\alpha)^{-1}\big)$ as $\alpha \to 2$. We then say that $\tilde{K}$ is in M-position of order α (or α-regular M-position). It is an interesting question to give an elementary proof of the existence of, say, a 1-regular M-position. Another interesting question is to check if the isotropic position is α-regular for some $\alpha \geq 1$ (assuming that $L_K \simeq 1$).

2. Tools from asymptotic convex geometry

2.1. Basic notation. As mentioned at the beginning of the Introduction, we denote the Euclidean norm on $\mathbb{R}^n$ by $\|\cdot\|_2$. More generally, if B is any symmetric convex body in $\mathbb{R}^n$, we write $\|\cdot\|_B$ for the norm induced on $\mathbb{R}^n$ by B. For every $q \geq 1$, every convex body K and every symmetric convex body B, we define

$$I_q(K,B) := \left(\frac{1}{|K|^{1+\frac{q}{n}}}\int_K \|x\|_B^q\, dx\right)^{1/q}. \tag{2.1}$$

If B is the Euclidean ball B_2^n and K is an isotropic convex body in $\mathbb{R}^n$, then from (1.6) we see that

$$I_2^2(K,B_2^n) = \int_K \|x\|_2^2\, dx = \int_K \Big(\sum_{i=1}^n \langle x, e_i\rangle^2\Big)\, dx = nL_K^2, \tag{2.2}$$

so $L_K = I_2(K,B_2^n)/\sqrt{n}$. More generally, as was explained in the Introduction, if K is an arbitrary convex body in $\mathbb{R}^n$, and we write $\tilde{K}$ for the translate of K which is centered, $\tilde{K} = K - \mathrm{bar}(K)$, then the isotropic constant L_K of K can be defined by

$$L_K := \frac{1}{\sqrt{n}} \inf\big\{I_2\big(T(\tilde{K}), B_2^n\big) : T \text{ is an invertible linear transformation}\big\}. \tag{2.3}$$

In the sequel, we write $\overline{B}$ for the homothetic image of volume 1 of a convex body $B \subset \mathbb{R}^n$, i.e. $\overline{B} := \frac{B}{|B|^{1/n}}$.

As a generalization to convex bodies, we also consider logarithmically concave (or log-concave) measures on $\mathbb{R}^n$. This more general approach is justified by a well-known and very fruitful idea of K. Ball from [1] which allows one to transfer results from the setting of convex bodies to the broader setting of log-concave measures and vice versa. We write $\mathcal{P}_{[n]}$ for the class of all Borel probability measures on $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure. The density of $\mu \in \mathcal{P}_{[n]}$ is denoted by f_μ. A probability measure $\mu \in \mathcal{P}_{[n]}$ is called symmetric if f_μ is an even function on $\mathbb{R}^n$. We say that $\mu \in \mathcal{P}_{[n]}$ is centered if for all $\theta \in S^{n-1}$,

$$\int_{\mathbb{R}^n} \langle x, \theta\rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x, \theta\rangle f_\mu(x) dx = 0. \tag{2.4}$$

A measure μ on $\mathbb{R}^n$ is called log-concave if for any Borel subsets A and B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$, $\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$. A function $f : \mathbb{R}^n \to [0,\infty)$ is called log-concave if $\log f$ is concave on its support $\{f > 0\}$.

It is known that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H, then $\mu \in \mathcal{P}_{[n]}$ and its density f_μ is log-concave (see [3]). Note that if K is a convex body in $\mathbb{R}^n$, then the Brunn–Minkowski inequality implies that $\mathbf{1}_K$ is the density of a log-concave measure.

There is also a way to generalize the notion of the isotropic constant of a convex body in the setting of log-concave measures. Set

$$\|\mu\|_\infty = \sup_{x\in\mathbb{R}^n} f_\mu(x). \tag{2.5}$$

The isotropic constant of μ is defined by

$$L_\mu := \left(\frac{\|\mu\|_\infty}{\int_{\mathbb{R}^n} f_\mu(x)dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}}, \tag{2.6}$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}\frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx} \tag{2.7}$$

(in the case that μ is a centered probability measure, we can write more simply $\mathrm{Cov}(\mu)_{ij} := \int_{\mathbb{R}^n} x_i x_j f_\mu(x)\,dx$). It is straightforward to see that this definition coincides with the original definition of the isotropic constant when f_μ is the characteristic function of a convex body. In addition, any bounds that we have for the isotropic constants of convex bodies continue to hold essentially in this more general setting. This can be seen through the following construction: let $\mu \in \mathcal{P}_{[n]}$ and assume that $0 \in \mathrm{supp}(\mu)$. For every $p > 0$, we define a set $K_p(\mu)$ as follows:

$$K_p(\mu) := \left\{x \in \mathbb{R}^n : p\int_0^\infty f_\mu(rx)r^{p-1}dr \geq f_\mu(0)\right\}. \tag{2.8}$$

The sets $K_p(\mu)$ were introduced in [1] and allow us to study log-concave measures using convex bodies. K. Ball proved that if μ is log-concave, then $K_p(\mu)$ is a convex body. Moreover, if μ is centered, then $K_{n+1}(\mu)$ is also centered, and we can prove that

$$c_1 L_{K_{n+1}(\mu)} \leq L_\mu \leq c_2 L_{K_{n+1}(\mu)} \tag{2.9}$$

for some constants $c_1, c_2 > 0$ independent of n.

For basic facts from the Brunn–Minkowski theory and the asymptotic theory of finite-dimensional normed spaces, we refer to the books [31], [24] and [27].

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$ for two quantities a, b

associated with convex bodies or measures on $\mathbb{R}^n$, we mean that we can find positive constants c_1, c_2, independent of the dimension n, such that $c_1 a \leq b \leq c_2 a$. Also, if $K, L \subseteq \mathbb{R}^n$, we will write $K \simeq L$ if there exist absolute positive constants c_1, c_2 such that $c_1 K \subseteq L \subseteq c_2 K$.

In the rest of the Section, we collect several tools and results from the asymptotic theory of convex bodies which will be used in Section 3.

2.2. Some lemmas on covering numbers. Let K, B be convex bodies in $\mathbb{R}^n$ with B symmetric. One can give an estimate for the covering numbers $N(K, tB)$, $t > 0$, in terms of the quantity

$$I_1(K,B) = \frac{1}{|K|^{1+\frac{1}{n}}} \int_K \|x\|_B \, dx. \tag{2.10}$$

Lemma 2.1: *Let K be a convex body of volume 1 in $\mathbb{R}^n$ containing 0 as an interior point. For any symmetric convex body B in $\mathbb{R}^n$ and any $t > 0$, one has*

$$\log N(K, tB) \leq \frac{c_1 n I_1(K,B)}{t} + \log 2, \tag{2.11}$$

where $c_1 > 0$ is an absolute constant.

Remark 2.1: (i) In the case that B is the Euclidean ball B_2^n and K is an isotropic convex body, we have that $I_1(K,B) \leq \sqrt{n} L_K$ and therefore

$$\log N(K, tB_2^n) \leq \frac{c_1' n^{3/2} L_K}{t} \tag{2.12}$$

for any $t > 0$ (for very large t the estimate is trivially true, since every isotropic body K satisfies the inclusion $K \subseteq cnL_K B_2^n$ for some absolute constant $c > 0$; see, e.g., [7, Theorem 1.2.4]). Given (1.7), this is essentially the best way we can apply Lemma 2.1 when $B = B_2^n$. This version of the lemma appeared in the Ph.D. Thesis of Hartzoulaki [11] (see [7, Theorem 1.6.4]), but the same argument yields Lemma 2.1 too (the key idea of that argument comes from Talagrand's proof of the dual Sudakov inequality). The parameter $I_1(K, B_2^n)$ has been used again in entropy estimates for isotropic convex bodies [22], and also in a proof of the low M^*-estimate in the case of quasi-convex bodies [17] .

(ii) Knowing that we have for any set S,

$$N(S - S, 2B_2^n) = N(S - S, B_2^n - B_2^n) \leq N(S, B_2^n)^2, \tag{2.13}$$

we can use (2.12) to also get an upper bound for the covering numbers of the difference body of an isotropic convex body K by the Euclidean ball:

$$\log N(K-K,tB_2^n) \leq \frac{2c_1' n^{3/2} L_K}{t}. \tag{2.14}$$

(iii) Lemma 2.1 is also related to the problem of estimating the mean width of an isotropic convex body K, namely the parameter $w(K) := \int_{S^{n-1}} h_K(\theta)d\sigma(\theta)$ where h_K is the support function of K and σ is the uniform probability measure on S^{n-1}. The best upper bound we have is $w(K) \leq cn^{3/4}L_K$ (there are several arguments leading to this estimate; see [9] and the references therein). It is known (see, e.g., [8, Theorem 5.6]) that an improvement of the form

$$\log N(K,tB_2^n) \leq \frac{c_1' n^{3/2} L_K}{t^{1+\delta}} \tag{2.15}$$

(for some $\delta > 0$) in (2.12) would immediately imply a better bound for $w(K)$ in the isotropic case.

The next lemma allows us to bound the dual covering numbers $N(B_2^n, tK^\circ)$ (the proof, which we include for the reader's convenience, uses a well-known idea from [32]; see also [16, Section 3.3]).

Lemma 2.2: *Let K be a convex body in $\mathbb{R}^n$ which contains 0 in its interior. For every $t>0$ we set $A(t) := t\log N(K,tB_2^n)$ and $B(t) := t\log N(B_2^n, tK^\circ)$. Then, one has*

$$\sup_{t>0} B(t) \leq 16 \sup_{t>0} A(t). \tag{2.16}$$

In particular, if K is isotropic (or a translate of an isotropic convex body which still contains 0 in its interior), then (2.12) and (2.14) imply that

$$\log N(B_2^n, tK^\circ) \leq \log N\big(B_2^n, t(K-K)^\circ\big) \leq \frac{c_2 n^{3/2} L_K}{t}, \tag{2.17}$$

where $c_2>0$ is an absolute constant.

Proof. For any $t>0$ we have $(t^2K^\circ) \cap (4K) \subseteq 2tB_2^n$. Passing to the polar bodies we see that

$$B_2^n \subseteq \mathrm{conv}\left(\frac{t}{2}K^\circ, \frac{2}{t}K\right) \subseteq \frac{t}{2}K^\circ + \frac{2}{t}K. \tag{2.18}$$

We write

$$\begin{aligned} N(B_2^n, tK^\circ) &\leq N\left(\frac{t}{2}K^\circ + \frac{2}{t}K, tK^\circ\right) = N\left(\frac{2}{t}K, \frac{t}{2}K^\circ\right) \\ &\leq N\left(\frac{2}{t}K, \frac{1}{4}B_2^n\right) N\left(\frac{1}{4}B_2^n, \frac{t}{2}K^\circ\right) \\ &= N\left(K, \frac{t}{8}B_2^n\right) N(B_2^n, 2tK^\circ). \end{aligned} \tag{2.19}$$

Taking logarithms we get $B(t) \leq 8A(t/8) + \frac{1}{2}B(2t)$, for all $t > 0$. This implies that $B := \sup_{t>0} B(t) \leq 16A$, and the result follows. ∎

The last covering lemma contains some standard entropy estimates which are valid for arbitrary convex bodies in $\mathbb{R}^n$.

Lemma 2.3: *Let K and L be convex bodies in $\mathbb{R}^n$. If L is symmetric, then*

$$N(K, L) \leq \frac{|K + L/2|}{|L/2|} \leq 2^n \frac{|K + L|}{|L|}, \tag{2.20}$$

whereas in the general case

$$N(K, L) \leq 4^n \frac{|K + L|}{|L|}. \tag{2.21}$$

Moreover,

$$\frac{|K + L|}{|L|} \leq 2^n N(K, L). \tag{2.22}$$

Proof. Both (2.22) and (2.20) are direct consequences of the definitions (recall that if N is a maximal subset of K with respect to the property "if $x, y \in N$ and $x \neq y$, then $\|x-y\|_L \geq 1$", we obviously have that every two sets $x+L/2, y+L/2$ with $x, y \in N, x \neq y$, have disjoint interiors, while $K \subseteq \bigcup_{x\in N}(x+L)$). To prove (2.21), we note that $N(K + x, L + y) = N(K, L)$ for every $x, y \in \mathbb{R}^n$, and also that the ratio $|K+L|/|L|$ remains unaltered if we translate K or L. This means that we can assume L is centered, in which case it follows from [23, Corollary 3] that $|L \cap (-L)| \geq 2^{-n}|L|$. But then, from (2.20) we get that

$$N(K, L) \leq N(K, L \cap (-L)) \leq 2^n \frac{|K + (L \cap (-L))|}{|L \cap (-L)|} \leq 4^n \frac{|K + L|}{|L|}, \tag{2.23}$$

and we have (2.21). ∎

COROLLARY 2.4: *Let K and L be two convex bodies in $\mathbb{R}^n$. Then*

$$N(K,L)^{1/n} \simeq \frac{|K+L|^{1/n}}{|L|^{1/n}}. \tag{2.24}$$

It also follows that if K and L have the same volume, then

$$N(K,L)^{1/n} \leq 8N(L,K)^{1/n}. \tag{2.25}$$

2.3. THE METHOD OF CONVEX PERTURBATIONS. In [12] Klartag gave an affirmative answer to the following question: even if we don't know that every convex body in $\mathbb{R}^n$ has bounded isotropic constant, given a body K can we find a second body T "geometrically close" to K with isotropic constant $L_T \simeq 1$? Here when we say that K and T are "geometrically close", we will mean that there exists an absolute constant $c > 0$ such that for some $x, y \in \mathbb{R}^n$,

$$\frac{1}{c}(T-x) \subseteq K - y \subseteq c(T-x). \tag{2.26}$$

The method Klartag used is based on two key observations. The first one is that in order to find a body T close to K which has bounded isotropic constant, it suffices to define a positive log-concave function on K (vanishing everywhere else) with bounded isotropic constant and the extra property that its range is not too large.

PROPOSITION 2.5: *Let K be a convex body in $\mathbb{R}^n$ and let $f : K \to (0,\infty)$ be a log-concave function such that*

$$\sup_{x\in K} f(x) \leq m^n \inf_{x\in K} f(x) \tag{2.27}$$

for some $m > 1$. Let x_0 be the barycenter of f, i.e.,

$$x_0 = \frac{\int_{\mathbb{R}^n} xf(x)\,dx}{\int_{\mathbb{R}^n} f(x)\,dx},$$

and set $g(x) = f(x+x_0)$. Then, for the centered convex body $T := K_{n+1}(g)$, defined as in (2.8), we have that $L_f \simeq L_T$ and

$$\frac{1}{m}T \subseteq K - x_0 \subseteq mT. \tag{2.28}$$

The second observation is that a family of suitable candidates for the function f we need so as to apply Proposition 2.5 can be found through the logarithmic

Laplace transform on K. In general, the logarithmic Laplace transform of a finite Borel measure μ on $\mathbb{R}^n$ is defined by

$$\Lambda_\mu(\xi) := \log\left(\int_{\mathbb{R}^n} e^{\langle \xi, x\rangle} \frac{d\mu(x)}{\mu(\mathbb{R}^n)}\right). \tag{2.29}$$

In [12], Klartag makes use of the following properties of Λ_μ:

Proposition 2.6: *Let $\mu = \mu_K$ denote the Lebesgue measure on some convex body K in $\mathbb{R}^n$. Then*

$$\big(\nabla\Lambda_\mu\big)(\mathbb{R}^n) = \mathrm{int}(K) \tag{2.30}$$

(*actually, for the arguments in* [12] *and for our proof here, it suffices to know that* $\big(\nabla\Lambda_\mu\big)(\mathbb{R}^n) \subseteq K$). *If μ_ξ is the probability measure on $\mathbb{R}^n$ with density proportional to the function $e^{\langle \xi, x\rangle}\mathbf{1}_K(x)$, then*

$$\mathrm{bar}(\mu_\xi) = \nabla\Lambda_\mu(\xi) \quad \text{and} \quad \mathrm{Hess}\,(\Lambda_\mu)(\xi) = \mathrm{Cov}(\mu_\xi). \tag{2.31}$$

Moreover, the map $\nabla\Lambda_\mu$, which is one-to-one, transports the measure ν with density $\det\mathrm{Hess}\,(\Lambda_\mu)$ *to μ. In other words, for every continuous non-negative function $\phi : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_K \phi(x)\,dx = \int_{\mathbb{R}^n} \phi(\nabla\Lambda_\mu(\xi))\det\mathrm{Hess}(\Lambda_\mu)(\xi)\,d\xi = \int_{\mathbb{R}^n} \phi(\nabla\Lambda_\mu(\xi))d\nu(\xi). \tag{2.32}$$

Klartag's approach has been recently applied in [6] where Dadush, Peikert and Vempala provide an algorithm for enumerating lattice points in a convex body, a result which has further applications to integer programming and other problems about lattice points. In particular, they use the arguments from [12] in order to give an expected $2^{O(n)}$-time algorithm for computing an M-ellipsoid for any convex body in $\mathbb{R}^n$.

3. Proof of the reverse Santaló inequality

We now prove the reverse Santaló inequality using the results that were described in Section 2. The proof consists of three steps which roughly are the following: (i) we obtain a lower bound for the volume product $s(K)$ which is optimal up to the value of the isotropic constant L_K of K, (ii) by adapting Klartag's main argument from [12] we show that every symmetric convex body K has bounded geometric distance (in the sense defined in (2.26)) from a second convex body T whose isotropic constant L_T can be expressed in terms of $s(K)$, and (iii) we use the lower bound for $s(T)$ in terms of L_T, and the fact that $s(K)$ and $s(T)$ are comparable, to get a lower bound for $s(K)$ in which L_K does not appear anymore.

3.1. Lower bound involving the isotropic constant. Our first step will be to prove the following lower bound for $s(K)$.

Proposition 3.1: *Let K be a convex body in $\mathbb{R}^n$ which contains 0 in its interior. Then*

$$4|K|^{1/n}|nK^\circ|^{1/n} \geq |K-K|^{1/n}|n(K-K)^\circ|^{1/n} \geq \frac{c_1}{L_K}, \tag{3.1}$$

where $c_1 > 0$ is an absolute constant.

Proof. We may assume that $|K|=1$. From the Brunn–Minkowski inequality and the classical Rogers–Shephard inequality (see [29]), we have $2 \leq |K-K|^{1/n} \leq 4$. Since $(K-K)^\circ \subseteq K^\circ$, we immediately see that

$$|K|^{1/n}|nK^\circ|^{1/n} \geq \frac{1}{4}|K-K|^{1/n}|n(K-K)^\circ|^{1/n}, \tag{3.2}$$

so it remains to prove the second inequality. Since

$$\big|A(K)-A(K)\big|\big|\big(A(K)-A(K)\big)^\circ\big| = |K-K||(K-K)^\circ| \tag{3.3}$$

for any invertible affine transformation A of K, we may assume for the rest of the proof that K is isotropic. We define

$$K_1 := \frac{K-K}{L_K} \cap \overline{B}_2^n \tag{3.4}$$

and observe that the inclusion $K_1 \subseteq \overline{B}_2^n$ implies that $\overline{B}_2^n \subseteq c_1 n K_1^\circ$ for some absolute constant $c_1 > 0$. Moreover,

$$nK_1^\circ \simeq \mathrm{conv}\{nL_K(K-K)^\circ, \overline{B}_2^n\} \subseteq nL_K(K-K)^\circ + \overline{B}_2^n, \tag{3.5}$$

therefore we can use (2.22) from Lemma 2.3 to bound $|nK_1^\circ|$ from above; recalling (2.17) from Lemma 2.2 as well (with $t \simeq \sqrt{n}L_K$), we see that

$$\begin{aligned} c_1^{-n} \leq |nK_1^\circ| &\leq c_2^n|\mathrm{conv}\{nL_K(K-K)^\circ, \overline{B}_2^n\}| \\ &\leq (2c_2)^n|nL_K(K-K)^\circ|\, N\Big(\overline{B}_2^n, nL_K(K-K)^\circ\Big) \\ &\leq (2c_2)^n|nL_K(K-K)^\circ|\, N\big(B_2^n, c_3\sqrt{n}L_K(K-K)^\circ\big) \\ &\leq e^{c_4 n}|nL_K(K-K)^\circ|. \end{aligned} \tag{3.6}$$

This shows that there exists an absolute constant $c_1' > 0$ so that

$$|nL_K(K-K)^\circ|^{1/n} \geq c_1', \tag{3.7}$$

and since $|K-K|^{1/n} \geq 2$, we have proven that

$$|K-K|^{1/n}|(K-K)^\circ|^{1/n} \geq \frac{2c_1'}{nL_K}. \tag{3.8}$$ ∎

3.2. A variant of Klartag's argument. Our second step will be to show that every convex body K in $\mathbb{R}^n$ has bounded geometric distance from a second convex body T whose isotropic constant L_T can be bounded in terms of $s(K–K)$.

Proposition 3.2: *Let K be a convex body in $\mathbb{R}^n$. For every $\varepsilon \in (0,1)$ there exist a centered convex body $T \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that*

$$\frac{1}{1+\varepsilon}T \subseteq K + x \subseteq (1+\varepsilon)T \tag{3.9}$$

and

$$L_T \leq \frac{c_2}{\sqrt{\varepsilon n s(K-K)^{1/n}}}, \tag{3.10}$$

where $c_2 > 0$ is an absolute constant.

Proof. We may assume that K is centered and that $|K-K| = 1$. Indeed, once we prove the proposition for $\tilde{K} := (K-\mathrm{bar}(K))/|K-K|^{1/n}$ and some $\varepsilon \in (0,1)$, and find a convex body T which satisfies (3.9) and (3.10) with $\tilde{K}$ instead of K, it will immediately hold that the pair $(K, |K-K|^{1/n}T)$ also satisfies these properties, because L_T and $s(K-K)$ are affine invariants.

Recall from Proposition 2.6 that if $\mu = \mu_K$ is the Lebesgue measure restricted on K, then the function $\nabla\Lambda_\mu$ transports the measure ν with density

$$\frac{d\nu}{d\xi} = \det \mathrm{Hess}\,(\Lambda_\mu)(\xi) \equiv \det \mathrm{Cov}(\mu_\xi) \tag{3.11}$$

to μ. This implies that

$$\nu(\mathbb{R}^n) = \int_{\mathbb{R}^n} \mathbf{1} \det \mathrm{Hess}\,(\Lambda_\mu)(\xi)\, d\xi = \int_K \mathbf{1}\, dx = |K| \leq |K-K| = 1. \tag{3.12}$$

Thus, for every $\varepsilon > 0$ we may write

$$\begin{aligned} |\varepsilon n(K-K)^\circ| &\min_{\xi \in \varepsilon n(K-K)^\circ} \det \mathrm{Cov}(\mu_\xi) \\ &\leq \int_{\varepsilon n(K-K)^\circ} \det \mathrm{Cov}(\mu_\xi)\, d\xi = \nu(\varepsilon n(K-K)^\circ) \leq 1, \end{aligned} \tag{3.13}$$

which means that there exists $\xi \in \varepsilon n(K-K)^\circ$ such that

(3.14)
$$\det \mathrm{Cov}(\mu_\xi) = \min_{\xi' \in \varepsilon n(K-K)^\circ} \det \mathrm{Cov}(\mu_{\xi'}) \leq |\varepsilon n(K-K)^\circ|^{-1} = \big(\varepsilon n s(K-K)^{1/n}\big)^{-n}$$

(where the last equality holds because $|K-K|=1$). Now, from the definition of μ_ξ and (2.6) we have that

$$L_{\mu_\xi} = \left(\frac{\sup_{x\in K} e^{\langle \xi, x\rangle}}{\int_K e^{\langle \xi, x\rangle} dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu_\xi)]^{\frac{1}{2n}}. \tag{3.15}$$

Since $\xi \in \varepsilon n(K-K)^\circ$ and $K \cup (-K) \subset K-K$, we know that $|\langle \xi, x\rangle| \leq \varepsilon n$ for all $x \in K$, therefore $\sup_{x\in K} e^{\langle \xi, x\rangle} \leq \exp(\varepsilon n)$. On the other hand, since K is centered, from Jensen's inequality we have that

$$\frac{1}{|K|}\int_K e^{\langle \xi, x\rangle} dx \geq \exp\left(\frac{1}{|K|}\int_K \langle \xi, x\rangle\, dx\right) = 1, \tag{3.16}$$

which means that $\int_K e^{\langle \xi, x\rangle} dx \geq |K| \geq 4^{-n}|K-K|$ by the Rogers–Shephard inequality. Combining all these we get

$$L_{\mu_\xi} \leq \frac{4e^\varepsilon}{\sqrt{\varepsilon n s(K-K)^{1/n}}}. \tag{3.17}$$

Finally, we note that the function $f_\xi(x) = e^{\langle \xi, x\rangle}\mathbf{1}_K(x)$ (which is proportional to the density of μ_ξ) is obviously log-concave and satisfies

$$\sup_{x\in \mathrm{supp}(f_\xi)} f_\xi(x) \leq e^{2\varepsilon n} \inf_{x\in \mathrm{supp}(f_\xi)} f_\xi(x) \tag{3.18}$$

(since $|\langle \xi, x\rangle| \leq \varepsilon n$ for all $x \in K$). Therefore, applying Proposition 2.5, we can find a centered convex body T_ξ in $\mathbb{R}^n$ such that

$$L_{T_\xi} \simeq L_{f_\xi} = L_{\mu_\xi} \leq \frac{4e^\varepsilon}{\sqrt{\varepsilon n s(K-K)^{1/n}}} \tag{3.19}$$

and

$$\frac{1}{e^{2\varepsilon}} T_\xi \subseteq K - b_\xi \subseteq e^{2\varepsilon} T_\xi \tag{3.20}$$

where b_ξ is the barycenter of f_ξ. Since $e^{2\varepsilon} \leq 1 + c\varepsilon$ when $\varepsilon \in (0,1)$, the result follows. ∎

3.3. Removing the isotropic constant. Combining the previous two results we can remove the isotropic constant L_K from the lower bound for $s(K)^{1/n}$.

Theorem 3.3: *Let K be a convex body in $\mathbb{R}^n$ which contains 0 in its interior. Then*

$$|K|^{1/n}|nK^\circ|^{1/n} \geq c_3, \tag{3.21}$$

where $c_3 > 0$ is an absolute constant.

Proof. Since $|K|^{1/n}|nK^\circ|^{1/n} \geq \frac{1}{4}|K-K|^{1/n}|n(K-K)^\circ|^{1/n}$, we may assume for the rest of the proof that K is symmetric. Using Proposition 3.2 with $\varepsilon = 1/2$, we find a convex body $T \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that

$$\frac{2}{3}T \subseteq K + x \subseteq \frac{3}{2}T \tag{3.22}$$

and $L_T \leq c_0/\sqrt{ns(K)^{1/n}}$ for some absolute constant $c_0 > 0$. Proposition 3.1 shows that

$$|T-T|^{1/n}|n(T-T)^\circ|^{1/n} \geq \frac{c_1}{L_T}, \tag{3.23}$$

where $c_1 > 0$ is an absolute constant too. Observe that $\frac{2}{3}(T-T) \subseteq K-K = 2K \subseteq \frac{3}{2}(T-T)$, and thus $K^\circ \supseteq \frac{4}{3}(T-T)^\circ$. Therefore, combining the above, we get

$$\begin{aligned} ns(K)^{1/n} &= |nK^\circ|^{1/n}|K|^{1/n} \geq \frac{4}{9}|n(T-T)^\circ|^{1/n}|T-T|^{1/n} \\ &\geq \frac{c_1'}{L_T} \geq c_2\sqrt{ns(K)^{1/n}}, \end{aligned} \tag{3.24}$$

and so it follows that

$$s(K)^{1/n} \geq \frac{c_3}{n} \tag{3.25}$$

with $c_3 = c_2^2$. This completes the proof. ∎

Remark 3.1: Having proved the reverse Santaló inequality, one can go back to Proposition 3.2 and insert the lower bound for $s(K-K)$, exactly as in Klartag's solution of the isomorphic slicing problem. We see that if K is a convex body in $\mathbb{R}^n$, then for every $\varepsilon \in (0,1)$ there exist a centered convex body $T \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that

$$\frac{1}{1+\varepsilon}T \subseteq K + x \subseteq (1+\varepsilon)T \tag{3.26}$$

and

$$L_T \leq \frac{c_4}{\sqrt{\varepsilon}}, \tag{3.27}$$

where $c_4 > 0$ is an absolute constant.

4. M-ellipsoids and the reverse Brunn–Minkowski inequality

Pisier notes in [27] that the asymptotic form of the Santaló inequality and its inverse and the existence of an M-position for any convex body are interconnected results. In fact, there is a standard way to prove the reverse Santaló inequality if we know that every centered convex body has an M-ellipsoid (see the final comment of this paper). In this last Section we briefly discuss how one can establish the inverse implication and, as a consequence, get the reverse Brunn–Minkowski inequality too, using simple "geometric" arguments.

4.1. Existence of M-ellipsoids. Let K be a centered convex body in $\mathbb{R}^n$. We will give a proof of the existence of an M-ellipsoid for K. The next Proposition is the first step.

Proposition 4.1: *Let K be a centered convex body in $\mathbb{R}^n$. Then there exists an ellipsoid $\mathcal{E}_K$ such that $|K| = |\mathcal{E}_K|$ and*

$$\max\{\log N(K, t\mathcal{E}_K), \log N(\mathcal{E}_K^\circ, tK^\circ)\} \leq \frac{cn}{t} \tag{4.1}$$

for all $t > 0$, where $c > 0$ is an absolute constant.

Proof. As explained in Remark 3.1, we can combine Theorem 3.3 with Proposition 3.2 to find a centered convex body T with isotropic constant $L_T \leq C$ such that

$$\frac{2}{3}T \subseteq K + x \subseteq \frac{3}{2}T \tag{4.2}$$

for some $x \in \mathbb{R}^n$. Let $Q(T)$ be an isotropic position of T. From Remark 2.1(ii) and Lemma 2.2 we know that

$$\max\{\log N\big(Q(T) - Q(T), t\sqrt{n}B_2^n\big), \log N\big(B_2^n, t\sqrt{n}(Q(T) - Q(T))^\circ\big)\} \leq \frac{cn}{t} \tag{4.3}$$

for every $t > 0$. Since

$$\frac{2}{3}(Q(T) - Q(T)) \subseteq Q(K) - Q(K) \subseteq \frac{3}{2}(Q(T) - Q(T)) \tag{4.4}$$

and $Q(K) \subseteq Q(K) - Q(K)$, $(Q(K) - Q(K))^\circ \subseteq (Q(K))^\circ$, from (4.3) it follows that

$$\max\{\log N\big(Q(K), t\sqrt{n}B_2^n\big), \log N\big(B_2^n, t\sqrt{n}(Q(K))^\circ\big)\} \leq \frac{c'n}{t} \tag{4.5}$$

for every $t > 0$. We define $\mathcal{E}_K := Q^{-1}(a\sqrt{n}B_2^n)$ where a is chosen so that $|Q(K)| = |a\sqrt{n}B_2^n|$ (equivalently, so that $|\mathcal{E}_K| = |K|$), and from (4.5) we get that

$$\max\{\log N(K, t\mathcal{E}_K), \log N(\mathcal{E}_K^\circ, tK^\circ)\} \leq \frac{c'an}{t} \tag{4.6}$$

for all $t > 0$. It remains to observe that

$$|\sqrt{n}B_2^n|^{1/n} \simeq 1 = |Q(T)|^{1/n} \simeq |Q(K+x)|^{1/n} = |Q(K)|^{1/n}, \tag{4.7}$$

whence it follows that $a \simeq 1$. ∎

Combining Proposition 4.1 with the classical Santaló inequality and Corollary 2.4, we can now prove the existence of M-ellipsoids for any centered convex body in $\mathbb{R}^n$.

Theorem 4.2: *Let K be a centered convex body in $\mathbb{R}^n$. There exists an ellipsoid $\mathcal{E}_K$ such that $|K| = |\mathcal{E}_K|$ and*

$$\max\Big\{\log N(K, \mathcal{E}_K), \log N(\mathcal{E}_K, K), \log N(K^\circ, \mathcal{E}_K^\circ), \log N(\mathcal{E}_K^\circ, K^\circ)\Big\} \leq cn, \tag{4.8}$$

where $c > 0$ is an absolute constant.

Proof. Let $\mathcal{E}_K$ be the ellipsoid defined in Proposition 4.1. It immediately follows that

$$\max\Big\{N(K, \mathcal{E}_K), N(\mathcal{E}_K^\circ, K^\circ)\Big\} \leq \exp(cn). \tag{4.9}$$

For the other two covering numbers we use Lemma 2.3: $N(\mathcal{E}_K, K) \leq 8^n N(K, \mathcal{E}_K)$, which means that $\log N(\mathcal{E}_K, K) \leq (\log 8)n + \log N(K, \mathcal{E}_K)$. Similarly,

$$N(K^\circ, \mathcal{E}_K^\circ) \leq 2^n \frac{|K^\circ + \mathcal{E}_K^\circ|}{|\mathcal{E}_K^\circ|} \leq 2^n \frac{|K^\circ + \mathcal{E}_K^\circ|}{|K^\circ|} \leq 4^n N(\mathcal{E}_K^\circ, K^\circ), \tag{4.10}$$

where we have also used the fact that $|K| = |\mathcal{E}_K| \Rightarrow |K^\circ| \leq |\mathcal{E}_K^\circ|$ from the classical Santaló inequality. This completes the proof. ∎

4.2. Reverse Brunn–Minkowski inequality. As a consequence of Theorem 4.2 and Corollary 2.4, we get the "reverse" Brunn–Minkowski inequality.

Theorem 4.3: *Let K be a centered convex body in $\mathbb{R}^n$. There exists an ellipsoid $\mathcal{E}_K$ such that $|K| = |\mathcal{E}_K|$ and for every convex body T in $\mathbb{R}^n$,*

$$e^{-(c+\log 8)}\,|\mathcal{E}_K+T|^{1/n} \leq |K+T|^{1/n} \leq e^{c+\log 8}\,|\mathcal{E}_K+T|^{1/n}, \tag{4.11}$$

$$e^{-(c+\log 8)}\,|\mathcal{E}_K^{\circ}+T|^{1/n} \leq |K^{\circ}+T|^{1/n} \leq e^{c+\log 8}\,|\mathcal{E}_K^{\circ}+T|^{1/n}, \tag{4.12}$$

where c is the constant we found in Theorem 4.2.

Proof. Let $\mathcal{E}_K$ be the ellipsoid defined in Proposition 4.1. Using Lemma 2.3, we can write

$$\begin{aligned}|\mathcal{E}_K+T|^{1/n} &\leq 2|T|^{1/n}N(\mathcal{E}_K,T)^{1/n} \leq 2|T|^{1/n}N(\mathcal{E}_K,K)^{1/n}N(K,T)^{1/n}\\ &\leq 2e^c|T|^{1/n}N(K,T)^{1/n} \leq 8e^c|K+T|^{1/n}.\end{aligned} \tag{4.13}$$

The same reasoning gives us the second part of (4.11) and (4.12). ∎

To conclude this discussion, let us finally explain why, once we know that for every centered convex body K there exists an ellipsoid $\mathcal{E}_K$ such that

$$\max\big\{\log N(K,\mathcal{E}_K), \log N(\mathcal{E}_K,K), \log N(K^{\circ},\mathcal{E}_K^{\circ}), \log N(\mathcal{E}_K^{\circ},K^{\circ})\big\} \leq cn \tag{4.14}$$

for some absolute constant $c>0$, it is easy to prove that

$$e^{-2(c+\log 8)}s(B_2^n) \leq s(K) \leq e^{2(c+\log 8)}s(B_2^n) \tag{4.15}$$

for all centered bodies K. Indeed, if $\mathcal{E}_K$ is an M-ellipsoid for K as above, then from Lemma 2.3,

$$\frac{|\mathcal{E}_K+K|^{1/n}}{|K|^{1/n}} \leq 2N(\mathcal{E}_K,K)^{1/n} \leq 2e^c \leq 2e^cN(K,\mathcal{E}_K)^{1/n} \leq 8e^c\frac{|\mathcal{E}_K+K|^{1/n}}{|\mathcal{E}_K|^{1/n}},$$

so $|\mathcal{E}_K|^{1/n} \leq 8e^c|K|^{1/n}$, and in the same manner

$$\max\Big\{\frac{|K|^{1/n}}{|\mathcal{E}_K|^{1/n}}, \frac{|\mathcal{E}_K^{\circ}|^{1/n}}{|K^{\circ}|^{1/n}}, \frac{|K^{\circ}|^{1/n}}{|\mathcal{E}_K^{\circ}|^{1/n}}\Big\} \leq 8e^c. \tag{4.16}$$

(4.15) now follows.

Acknowledgements. The second-named author wishes to thank the A. Sloan Foundation and the US National Science Foundation for support through the grant DMS-0906150. The third-named author is supported by a scholarship of the University of Athens.

References

[1] K. M. Ball, *Logarithmically concave functions and sections of convex sets in* $\mathbb{R}^n$, Studia Mathematica **88** (1988), 69–84.

[2] S. G. Bobkov, *On Milman's ellipsoids and M-position of convex bodies*, in *Concentration, Functional Inequalities and Isoperimetry*, Contemporary Mathematics, Vol. 545, American Mathematical Society, Providence, RI, 2011, pp. 23–33.

[3] C. Borell, *Convex set functions in d-space*, Periodica Mathematica Hungarica **6** (1975), 111–136.

[4] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, in *Geometric Aspects of Functional Analysis* (J. Lindenstrauss and V. D. Milman, eds.), Lecture Notes in Mathematics, Vol. 1469, Springer-Verlag, Berlin, 1991, pp. 127–137.

[5] J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in* $\mathbb{R}^n$, Inventiones Mathematicae **88** (1987), 319–340.

[6] D. Dadush, C. Peikert and S. Vempala, *Enumerative lattice algorithms in any norm via M-ellipsoid coverings*, in *Proceedings of the IEEE 52nd Annual Symposium on Foundations of Computer Science*, IEEE Computer Society, 2011, pp. 580–589.

[7] A. Giannopoulos, *Notes on isotropic convex bodies*, Lecture Notes, Warsaw 2003, available at `http://users.uoa.gr/~apgiannop/`.

[8] A. Giannopoulos and V. D. Milman, *Mean width and diameter of proportional sections of a symmetric convex body*, Journal für die Reine und Angewandte Mathematik **497** (1998), 113–139.

[9] A. Giannopoulos, G. Paouris and P. Valettas, *On the distribution of the* ψ_2*-norm of linear functionals on isotropic convex bodies*, in *Geometric Aspects of Functional Analysis* (B. Klartag, S. Mendelson and V. D. Milman, eds.), Lecture Notes in Mathematics, Vol. 2050, Springer-Verlag, Berlin, 2012, pp. 227–253.

[10] Y. Gordon, M. Meyer and S. Reisner, *Zonoids with minimal volume product—a new proof*, Proceedings of the American Mathematical Society **104** (1988), 273–276.

[11] M. Hartzoulaki, *Probabilistic methods in the theory of convex bodies*, Ph.D. Thesis, University of Crete, March 2003.

[12] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geometric and Functional Analysis **16** (2006), 1274–1290.

[13] B. Klartag and E. Milman, *Centroid bodies and the logarithmic laplace transform—A unified approach*, Journal of Functional Analysis **262** (2012), 10–34.

[14] G. Kuperberg, *A low-technology estimate in convex geometry*, International Mathematics Research Notices **9** (1992), 181–183.

[15] G. Kuperberg, *From the Mahler conjecture to Gauss linking integrals*, Geometric and Functional Analysis **18** (2008), 870–892.

[16] M. Ledoux and M. Talagrand, *Probability in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 23, Springer, Berlin, 1991.

[17] A. E. Litvak, V. D. Milman and A. Pajor, *The covering numbers and low* M^**-estimate for quasi-convex bodies*, Proceedings of the American Mathematical Society **127** (1999), 1499–1507.

[18] M. Meyer, *Une characterisation volumique de certains espaces normés*, Israel Journal of Mathematics **55** (1986), 317–326.

[19] V. D. Milman, *Inegalité de Brunn–Minkowski inverse et applications à la théorie locale des espaces normés*, Comptes Rendus de l'Académie des Sciences. Série I. Mathématique **302** (1986), 25–28.

[20] V. D. Milman, *Isomorphic symmetrization and geometric inequalities*, in *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics, Vol. 1317, Springer-Verlag, Berlin, 1988, pp. 107–131.

[21] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, in *Geometric Aspects of Functional Analysis* (J. Lindenstrauss and V. D. Milman, eds.), Lecture Notes in Mathematics, Vol. 1376, Springer-Verlag, Berlin, 1989, pp. 64–104.

[22] V. D. Milman and A. Pajor, *Cas limites dans les inégalités du type de Khinchine et applications géometriques*, Comptes Rendus de l'Académie des Sciences. Série I. Mathématique **308** (1989), 91–96.

[23] V. D. Milman and A. Pajor, *Entropy and asymptotic geometry of non-symmetric convex bodies*, Advances in Mathematics **152** (2000), 314–335.

[24] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Mathematics, Vol. 1200, Springer, Berlin, 1986.

[25] F. Nazarov, *The Hörmander proof of the Bourgain–Milman theorem*, in *Geometric Aspects of Functional Analysis* (B. Klartag, S. Mendelson and V. D. Milman, eds.), Lecture Notes in Mathematics, Vol. 2050, Springer-Verlag, Berlin, 2012, pp. 335–343.

[26] G. Pisier, *A new approach to several results of V. Milman*, Journal für die Reine und Angewandte Mathematik **393** (1989), 115–131.

[27] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Tracts in Mathematics, Vol. 94, Cambridge University Press, Cambridge, 1989.

[28] S. Reisner, *Zonoids with minimal volume product*, Mathematische Zeitschrift **192** (1986), 339–346.

[29] C. A. Rogers and G. C. Shephard, *Convex bodies associated with a given convex body*, Journal of the London Mathematical Society **33** (1958), 270–281.

[30] J. Saint-Raymond, *Sur le volume des corps convexes symétriques*, in *Initiation Seminar on Analysis: G. Choquet-M. Rogalski-J. Saint-Raymond, 20th Year: 1980/1981*, Publications Mathématiques de l'Université Pierre et Marie Curie, Vol. 46, Université Paris VI, Paris, 1981.

[31] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, Encyclopedia of Mathematics and its Applications, Vol. 44, Cambridge University Press, Cambridge, 1993.

[32] N. Tomczak-Jaegermann, *Dualité des nombres d'entropie pour des opérateurs a valeurs dans un espace de Hilbert*, Comptes Rendus de l'Académie des Sciences. Série I. Mathématique **305** (1987), 299–301.