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THE LIST-CHROMATIC NUMBER OF INFINITE GRAPHS

BY

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ABSTRACT

We investigate the list-chromatic number of infinite graphs. It is easy to see that $\mathrm{Chr}(X) \le \mathrm{List}(X) \le \mathrm{Col}(X)$ for each graph X. It is consistent that $\mathrm{List}(X) = \mathrm{Col}(X)$ holds for every graph with $\mathrm{Col}(X)$ infinite. It is also consistent that for graphs of cardinality $\aleph_1$, $\mathrm{List}(X)$ is countable iff $\mathrm{Chr}(X)$ is countable.

The paper [6] of Erdős and Hajnal initiated the systematic research of the chromatic number of infinite graphs. They discovered that some of their proofs (for example, that every graph with uncountable chromatic number contains the 4-circuit) work for a broader class of graphs, the graphs with uncountable coloring number. The coloring number thus invented was a new graph invariant close to the chromatic number, but with better properties. It satisfies Shelah's Singular Cardinal Compactness Theorem. This, and an easy lemma make the coloring number quite easy to control: it can be described by the stationarity of certain sets of ordinals, defined from the graph.

In this paper we consider the list-chromatic number of infinite graphs, the following modification of the chromatic number. The list-chromatic number of a graph X, $\mathrm{List}(X)$, is the least cardinality κ such that X has a good coloring f that for every vertex v its color, $f(v)$, is an element chosen from a κ-element

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set $F(v)$ preassigned to v. Each vertex has, therefore, its specific set of possible colors.

The list-chromatic number, introduced by Vizing [20], and Erdős–Rubin–Taylor [7], independently, is easily seen to be between the chromatic number and the coloring number of X; $\mathrm{Chr}(X) \leq \mathrm{List}(X) \leq \mathrm{Col}(X)$. In general, $\mathrm{Chr}(X)$ and $\mathrm{Col}(X)$ can be far apart; there are bipartite graphs with large coloring number. Namely, $\mathrm{Col}(K(\kappa,\kappa)) = \kappa$ and $\mathrm{Col}(K(\kappa,\lambda)) = \kappa^+$ when $\kappa < \lambda$ are infinite. We show that the list-chromatic number can consistently be equal to the coloring number (at least for graphs of arbitrary cardinality with infinite coloring number). Further, for graphs of cardinality $\aleph_1$, the list chromatic number can be equal to the chromatic number (if the latter number is infinite). More precisely, there is a model of set theory, where the Continuum Hypothesis (CH) fails, and $\mathrm{List}(X)$ is countable iff $\mathrm{Chr}(X)$ is countable.

In recent work, M. Kojman considered these notions, and showed that although $\mathrm{List}(X) \leq \mathrm{Col}(X)$ always holds, it is consistent that $\mathrm{List}(X)$ and $\mathrm{Col}(X)$ are not far from each other. Specifically, GCH implies $\mathrm{Col}(X) \leq \mathrm{List}(X)^{++}$ holds generally. We improve this to $\mathrm{Col}(X) \leq \mathrm{List}(X)^+$. Kojman also obtained some non-GCH results using Shelah's results ([13]).

We also consider a modification, $\mathrm{List}^*(X)$ of the list-chromatic number. In this case, the sets $F(v)$ assigned to the vertices are not arbitrary sets of some cardinality κ, but κ-element subsets of κ. This definition, which is halfway between the chromatic number and the list-chromatic number, is meaningful only for κ infinite. We show, under GCH, that $\mathrm{Col}(X) \leq \mathrm{List}^*(X)^+$, if $\mathrm{List}^*(X)$ is regular, and $\mathrm{Col}(X) \leq \mathrm{List}^*(X)^{++}$ if $\mathrm{List}^*(X)$ is arbitrary. We further give forcing models, which show that, at least consistently, $\mathrm{List}^*(X)$ differs both from $\mathrm{Chr}(X)$ and $\mathrm{List}(X)$.

The paper is organized as follows. After the necessary definitions we first make some remarks on the fact that the definition of the list-chromatic number obviously uses the axiom of choice. We show that if the AC fails then there is a graph which is not λ-choosable for arbitrarily large λ (Theorem 1). It is also consistent that some graph is not λ-choosable for any cardinal λ (Theorem 2). Then we give the obvious inequalities between the various notions (Lemma 3). Theorem 4 and Lemma 5 are technical statements giving a complete characterization of graphs with coloring number μ for a given infinite cardinal μ. Then, we calculate List^* and List for some bipartite graphs (Lemmas 6–9). Theorem 10 shows that if X is a countably chromatic graph of cardinality κ and MA_κ holds, then the list-chromatic number of X is also countable. It is, therefore, consistent that for graphs X of cardinality $\aleph_1$, $\mathrm{List}(X)$ is countable iff $\mathrm{Chr}(X)$ is countable. Theorem 11 is a consistency result in the opposite direction; it is consistent that $\aleph_1 < c$ and there is a bipartite graph X of cardinality $\aleph_1$ with $\mathrm{List}^*(X) > \omega$. Theorem 12 shows, under GCH, that $\mathrm{Col}(X) \leq \mathrm{List}(X)^+$ and $\mathrm{Col}(X) \leq \mathrm{List}^*(X)^{++}$. Lemma 14 is a technical lemma describing the effect of adding a Cohen real on the list-chromatic number of a graph in the ground model. In Theorem 15 we give the consistency of $\mathrm{Col}(X) = \mathrm{List}(X)$ for every graph with $\mathrm{Col}(X)$ infinite. In Theorem 16 we deduce from the axiom of constructibility the existence of a bipartite type 2 graph X such that $\mathrm{Col}(X) = \mathrm{List}^*(X) = \omega_1$. Theorems 17 and 19 give the consistency of GCH and the existence of graphs with $\omega_1 = |X| = \mathrm{Col}(X) > \mathrm{List}(X)$ and $\omega_1 = |X| = \mathrm{List}(X) > \mathrm{List}^*(X)$, respectively. We notice that the consistency of the statement "GCH plus each HM graph on ω_1 has countable list-chromatic number" can be established using Shelah's $\mathbb{D}$-completeness theory (Theorem 18). Finally, we prove a result promised in [15] (Theorem 20).

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Definitions: We apply (what we believe are) the current axiomatic set theory conventions. If f is a function, A a set, then $f[A] = \{f(x) : x \in A\}$. Each ordinal is identified with the set of smaller ordinals, and each cardinal with the least ordinal of that cardinality. If $\kappa > \tau$ are uncountable, regular cardinals, then $S^\kappa_\tau = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \tau\}$.

If S is a set, κ a cardinal, then we define $[S]^\kappa = \{X \subseteq S : |X| = \kappa\}$ and $[S]^{<\kappa} = \{X \subseteq S : |X| < \kappa\}$.

A **graph** (V, X) is any pair where V is a set (the vertices) and $X \subseteq [V]^2$ (the edges). In some cases we denote a graph by its edge set only. In a graph (V, X) for any vertex $v \in V$, $\Gamma_X(v)$ or just $\Gamma(v)$ is **the neighborhood of** v, that is, $\Gamma_X(v) = \{w \in V : \{v, w\} \in X\}$. If, further, V is an ordered set (most of our graphs are on some ordinal), then we let $\Gamma^-_X(v) = \{w < v : \{w, v\} \in X\}$, $\Gamma^+_X(v) = \{w > v : \{w, v\} \in X\}$. A graph (V, X) is **bipartite** if V is the union of the disjoint sets A and B (the **bipartition classes**) and all edges go between A and B. A bipartite graph X with bipartition classes A, B is a **graph of type** (κ, λ, μ) if $|A| = \kappa$, $|B| = \lambda$, and $|\Gamma(x)| \geq \mu$ $(x \in A)$. We sometimes call a graph of type $(\lambda^+, \lambda, \mu)$ a **graph of type 1**.

$K(\kappa,\lambda)$ denotes the complete bipartite graph with bipartition classes of cardinalities κ, λ, respectively. A **good coloring** of a graph (V,X) is any function on V such that $f(x)\neq f(y)$ holds whenever $\{x,y\}\in X$. The **chromatic number** $\mathrm{Chr}(V,X)$ or $\mathrm{Chr}(X)$ of a graph (V,X) is the least cardinal κ such that there is a good coloring $f:V\to\kappa$. If κ is a cardinal, the graph (V,X) is **κ-choosable**, if for every function F from V with $|F(v)|=\kappa$ $(v\in V)$ (we call these funcions **assignments**) there is a choice function $f(v)\in F(v)$ $(v\in V)$ which is a good coloring of (V,X). The **list-chromatic number** $\mathrm{List}(V,X)$ or $\mathrm{List}(X)$ of a graph (V,X) is the least cardinal κ such that (V,X) is κ-choosable. The **restricted list-chromatic number**, $\mathrm{List}^*(V,X)$ or $\mathrm{List}^*(X)$, is the smallest cardinal $\kappa\geq\omega$ such that the following holds: for every assignment $F:V\to[\kappa]^\kappa$ there is a choice function $f(x)\in F(x)$ which is a good coloring of (V,X).

The **coloring number** $\mathrm{Col}(V,X)$ or $\mathrm{Col}(X)$ of a graph (V,X) is the smallest cardinal μ such that there is a well ordering $<$ of the vertex set V such that $|\Gamma^-(x)|<\mu$ holds for every $x\in V$.

A Hajnal–Máté graph (HM graph for short) is a graph X on some $\delta\leq\omega_1$ such that for every $\alpha<\delta$ the set $\Gamma^-(\alpha)$ is either finite or it is an ω-sequence converging to α (and so α is limit). This notion was introduced and first investigated in [9]. We let $S(X)$ be the set of those α for which the latter option holds.

A graph X on some regular cardinal $\kappa>\omega$ is a **graph of type** (κ,μ) if there is a stationary set $S=S(X)\subseteq S^\kappa_{\mathrm{cf}(\mu)}$ such that for each $\alpha\in S$ the set $\Gamma^-(\alpha)$ is a set of ordinal μ, cofinal in α. Sometimes we call graphs of this kind **graphs of type 2**.

If $\mathcal{H}\subseteq[S]^\omega$ is a set system, then we call $\mathcal{H}$ **2-chromatic**, if there is a coloring $f:S\to\{0,1\}$ such that for each $H\in\mathcal{H}$ neither $H\subseteq f^{-1}(0)$ nor $H\subseteq f^{-1}(1)$ holds.

$\mathrm{Add}(\mu,\kappa)$ denotes the Cohen notion of forcing that adds κ subsets to μ. $\mathrm{Col}(\kappa,\lambda)$ is the Levy-collapse making $\lambda=\kappa^+$.

For the statement and applications of Martin's axiom, see [8].

If χ is a regular cardinal, then $\mathcal{H}(\chi)$ denotes the set of those sets whose transitive closure has cardinality less than χ. We usually consider the model $(\mathcal{H}(\chi),\in,<_w)$, where $<_w$ is a well order of $\mathcal{H}(\chi)$.

We first make some remarks showing that in the absence of the axiom of choice, the list-chromatic number may not exist.

THEOREM 1: *If the axiom of choice fails, then there is a graph X such that for every cardinal κ there is a cardinal $\lambda \geq \kappa$ such that X is not λ-choosable.*

Proof. As the axiom of choice fails, there is a set $\{A_i : i \in I\}$ of nonempty sets with no choice functions. Set $\mu = \sum\{|A_i| : i \in I\}$, i.e., the cardinality of $B = \bigcup\{\{i\} \times A_i : i \in I\}$. If $\kappa > 0$ is an arbitrary cardinal, $|K| = \kappa$, set $C = B \times K$, $L = {}^{\omega}C$, let $\lambda = |L|$. Then $\lambda \geq \kappa$ and as

$$\lambda \leq |A_i \times L| \leq |B \times L| \leq |C \times L| = |{}^{1+\omega}C| = |{}^{\omega}C| = \lambda,$$

we have $|A_i \times L| = \lambda$ for each $i \in I$. There is no choice function for the system $\{A_i \times L : i \in I\}$, as it would project to a choice function of $\{A_i : i \in I\}$. Therefore, if X is the complete graph on I, then X is not λ-choosable. ∎

THEOREM 2: *It is consistent with the negation of the axiom of choice, that there is a graph which is not κ-choosable for any cardinal κ.*

Proof. We use a model of Halpern and Howard (cf. [10], [11]), in which $2\kappa = \kappa$ holds for any infinite cardinal κ and there is a system $\{A_i : i \in I\}$ of sets which does not have a choice function and $|A_i| = 2$ holds for every $i \in I$. Our graph X is the complete graph on I. If κ is an infinite cardinal, let K be a set with $|K| = \kappa$. If F is the assignment $F(i) = A_i \times K$ for X, then $|A_i \times K| = \kappa$ by the hypothesis that for each infinite cardinal κ, $2\kappa = \kappa$ holds, and there is no choice function for $\{F(i) : i \in I\}$, as it would project to a choice function of $\{A_i : i \in I\}$. We obtained, therefore, that X is not κ-choosable for any infinite cardinal κ. As X is an infinite complete graph, this holds for κ finite, as well. ∎

From now on we assume the axiom of choice.

LEMMA 3: *For every graph (V, X) we have*

$$\mathrm{Chr}(V, X) \leq \mathrm{List}^*(V, X) \leq \mathrm{List}(V, X) \leq \mathrm{Col}(V, X) \leq |V|.$$

Proof. The first inequality follows from the fact that the assigment $F(x) = \kappa$ is of the kind considered in the definition of $\mathrm{List}^*(V, X)$. The second inequality holds as the definition of $\mathrm{List}^*(V, X)$ is the same as that of the list-chromatic number, only some specific assignments are used.

For the third inequality assume that $\mathrm{Col}(V, X) \leq \kappa$ for some cardinal κ. This means that there is a well ordering $<$ of the vertex set V such that $|\Gamma^-(x)| < \kappa$ holds for every $x \in V$. If F is a function with $|F(x)| = \kappa$ for every $x \in V$, then we select the good coloring $f(x) \in F(x)$ by transfinite recursion on $<$: let $f(x)$ be an arbitrary element of the set $F(x) - \{f(y) : y \in \Gamma^-(x)\}$. The latter set is nonempty as $|F(x)| = \kappa$ and we subtract a smaller set. It is clear that f is, indeed, a good coloring.

For proving the last inequality it suffices to consider any well ordering into the order type $|V|$. Recall that in axiomatic set theory, $|V|$, the cardinality of V, is identified with the smallest ordinal which is equicardinal to V, so a well ordering into order type $|V|$ is one where each element is preceded by less than $|V|$ elements. ∎

Next we quote an important and powerful result of Shelah.

THEOREM 4 (Shelah [18]): *If X is a graph on a set V of cardinality λ, where λ is singular, and $\mathrm{Col}(X|V') \leq \mu$ for every $V' \subseteq V$, $|V'| < \lambda$, then $\mathrm{Col}(X) \leq \mu$.* ∎

LEMMA 5: (a) *If X is a graph of type $(\lambda^+, \lambda, \mu)$ or (κ, μ), then $\mathrm{Col}(X) > \mu$.*
(b) *If $\mathrm{Col}(X) > \mu$, then X contains a type 1 or type 2 subgraph.*

Proof. We use the following characterization.

CLAIM: *Let X be a graph on the regular cardinal κ such that if $A \in [\kappa]^{<\kappa}$ then $\mathrm{Col}(X|A) \leq \mu$. Define*

$$S = \left\{\alpha < \kappa : (\exists \beta(\alpha) \geq \alpha) \left|\Gamma_X^-(\beta(\alpha)) \cap \alpha\right| \geq \mu\right\}.$$

Then X has $\mathrm{Col}(X) \geq \mu^+$ if and only if S is stationary.

Proof. See, e.g., in [14]. ∎

The Claim immediately gives part (a).

For part (b), assume that X is a graph on some κ with $\mathrm{Col}(X) \geq \mu^+$. By passing to a subgraph of minimal cardinality with coloring number at least μ^+, we can assume that each $X' \subseteq X$ with $|X'| < \kappa$ has $\mathrm{Col}(X') \leq \mu$; κ is regular by Theorem 4. By the Claim, the set

$$S = \{\alpha < \kappa : (\exists \beta(\alpha) \geq \alpha)|\Gamma_X^-(\beta(\alpha)) \cap \alpha| \geq \mu\}$$

is stationary. For $\alpha \in S$ let $f(\alpha) \leq \alpha$ be the supremum of the first μ elements of $\Gamma_X^-(\beta) \cap \alpha$.

Assume first that the set $S' = \{\alpha \in S : f(\alpha) < \alpha\}$ is stationary. Then by Fodor's theorem there is a stationary $S'' \subseteq S'$ such that $f(\alpha) = \gamma$ for $\alpha \in S''$.

We now have a graph of type 1; set $A = \gamma$, $B = \{\beta(\alpha) : \alpha \in S''\}$ and the edges of X between A and B.

Assume finally that S' is nonstationary. Let Y be the subgraph of X obtained by removing all vertices in $\bigcup\{[\alpha, \beta(\alpha)) : \alpha \in S - S'\}$ and all edges $\{\xi, \alpha\} \in X$ with $\xi < \alpha \in S'$.

Then Y is a graph of type (κ, μ). ∎

Lemma 6: *If X is a bipartite graph with bipartition classes A and B, $|A| = \omega$, $|B| = c$, and $|\Gamma(x)| = \omega$ for each vertex $x \in B$, then* $\mathrm{List}^*(X) = \omega_1$.

Proof. We first show that $\mathrm{List}^*(X) \le \omega_1$. Let $F : A \cup B \to [\omega_1]^{\omega_1}$ be an assignment of X. We define the coloring $f(x) \in F(x)$ $(x \in A \cup B)$ as follows. Let $f(x) \in F(x)$ be arbitrary for $x \in A$. Then pick an element $f(y) \in F(y) - \{f(x) : x \in A\}$ for each $y \in B$. This is possible, as the latter set is nonempty.

For the other direction we are going to construct an assignment $F : A \cup B \to [\omega]^\omega$ such that no good coloring can be chosen from F. Let $\{F(x) : x \in A\}$ be arbitrary pairwise disjoint elements of $[\omega]^\omega$. For each $g \in \prod\{F(x) : x \in A\}$ choose an element $y_g \in B$ and make $F(y_g) = \{g(x) : \{x, y_g\} \in X\}$. The latter set is plainly infinite, and we can make the choice as $|B| = c$ by assumption. Assume that $f \in \prod F$ is a good coloring of X. Set $g = f|A$. Then no color $f(y_g)$ can be chosen from $F(y_g)$. ∎

A similar argument gives the following.

Lemma 7 (GCH): *Let X be a bipartite graph on the bipartition classes A and B with $|A| = \lambda^+$, $|B| = \lambda$, and $|\Gamma(x)| \ge \mu$ $(x \in A)$. Then:*

(a) $\mathrm{List}(X) > \mu$,

(b) *if $\lambda = \mu$ or μ is regular, then* $\mathrm{List}^*(X) > \mu$.

Proof. (a) We can as well assume that $|\Gamma(x)| = \mu$ holds for every $x \in A$ (by removing edges). Let $\{F(y) : y \in B\}$ be pairwise disjoint sets of cardinality μ. For each $g \in \prod\{F(y) : y \in B\}$ let x_g be an element of A such that $x_g \ne x_{g'}$ whenever $g \ne g'$. Define F by $F(x_g) = \{g(y) : y \in \Gamma(x_g)\}$ for these points; for the other elements of A, set $F(x) = \mu$. Now, there is no good coloring $f \in \prod F$.

(b) The case when $\lambda = \mu$ follows from the preceding argument, as then $|\bigcup\{F(y) : y \in B\}| = \mu$.

Assume now that μ is regular and X is a bipartite graph with bipartition classes A and B with $|A| = \lambda^+$, $|B| = \lambda$. Let λ be minimal that X contains a subgraph of this kind. We assume that $B = \lambda$.

If $\tau < \lambda$ then $|\Gamma(x) \cap \tau| = \mu$ can hold only for at most λ $x \in A$, as otherwise we could decrease the value of λ. Removing these elements of A, we obtain that for each $x \in A$, $\Gamma(x)$ is a cofinal set of λ, which is of type μ, specifically, $\mathrm{cf}(\lambda) = \mu$. Let $\{\lambda_\alpha : \alpha < \mu\}$ be a continuous sequence of cardinals, converging to λ, with $\lambda_0 = 0$. Let $\{A_\alpha : \alpha < \mu\}$ be pairwise disjoint elements of $[\mu]^\mu$ and set $F(\xi) = A_\alpha$ for $\lambda_\alpha \leq \xi < \lambda_{\alpha+1}$. For each $g \in \prod\{F(\xi) : \xi < \lambda\}$ pick an element $x(g) \in A$ with $x(g) \neq x(g')$ for $g \neq g'$. This is possible, as there are $\mu^\lambda = \lambda^+$ such functions g. If we now let $F(x(g)) = \{F(y) : y \in \Gamma(x(g))\}$ then there is no good coloring f of X with $f \in \prod F$, and we have therefore established that $\mathrm{List}^*(X) > \mu$. ∎

The above argument specifically gives the following, without assuming GCH.

Lemma 8: *If κ is an infinite cardinal then* $\mathrm{List}^*(K_{\kappa,2^\kappa}) = \kappa^+$. ∎

Lemma 9: *If X is a bipartite graph with bipartition classes A and B, $|A| = \omega$, $|B| < c$, then* $\mathrm{List}(X) \leq \omega$.

Proof. Let $A = \{a_n : n < \omega\}$, $B = \{b_\alpha : \alpha < \kappa\}$ where $\kappa < c$. Let F be an assignment on $A \cup B$ with $|F(x)| = \omega$ for $x \in A \cup B$. Choose $x_s \in F(a_n)$ for every function $s : n \to 2$ ($n < \omega$) such that $\{x_s : s : n \to 2, n < \omega\}$ are all distinct. For $h : \omega \to 2$ set $K_h = \{x_{h|n} : n < \omega\}$.

If $h : \omega \to 2$ then $|K_h| = \omega$, and for $h \neq h'$, $|K_h \cap K_{h'}| < \omega$. Therefore for each $\alpha < \kappa$ there can be only one h such that $F(b_\alpha) \subseteq K_h$ holds. We can choose a function $h : \omega \to 2$ such that for no $\alpha < \kappa$ does $F(b_\alpha) \subseteq K_h$ hold.

Given h as above, set $f(a_n) = x_{h|n}$, $f(b_\alpha) \in F(b_\alpha) - K_h$, a good coloring of X. ∎

Theorem 10 (MA_κ): *If (V, X) is a graph with $|V| \leq \kappa$ and $\mathrm{Chr}(X) \leq \omega$, then* $\mathrm{List}(X) \leq \omega$.

Proof. Let $V = \bigcup\{V_i : i < \omega\}$ be a partition witnessing that $\mathrm{Chr}(X) \leq \omega$, i.e., if $\{x, y\} \in X$, $x \in V_i$, $y \in V_j$, then $i \neq j$. Assume we are given an assignment $F : V \to [\kappa]^\omega$.

Define the poset $(P, \leq)$ as follows. Set $p = (s, g) \in P$ if

(a) $s \in [V]^{<\omega}$,
(b) g is a function on s, $g(x) \in F(x)$,
(c) $g(x) \neq g(y)$ for $x \in V_i$, $y \in V_j$, $i \neq j$.

$p' = (s', g') \leq p = (s, g)$ if $s' \supseteq s$, $g' \supseteq g$, i.e., g' extends g.

Notice that (c) implies that g is good coloring of X on s.

Claim 1: *If $x \in V$, then $D_x = \{(s, g) : x \in s\}$ is dense in $(P, \leq)$.*

Proof. Given $x \in V$ and $(s, g) \in P$ with $x \notin s$, extend (s, g) to (s', g') where $s' = s \cup \{x\}$ and $g' \supseteq g$ with $g'(x) \in F(x) - \{g(y) : y \in s\}$. ∎

Claim 2: *$(P, \leq)$ is ccc.*

Proof. Assume that $p_\alpha \in P$ for $\alpha < \omega_1$. By the Δ-system lemma we can assume that $p_\alpha = (s \cup s_\alpha, g_\alpha)$ where the members of the family $\{s\} \cup \{s_\alpha : \alpha < \omega_1\}$ are pairwise disjoint. With a similar argument we assume that the range of g_α is $t \cup t_\alpha$ with the members of the family $\{t\} \cup \{t_\alpha : \alpha < \omega_1\}$ pairwise disjoint. As $\prod\{F(x) : x \in s\}$ is countable, we can assume that $g_\alpha | s = g$ for $\alpha < \omega_1$.

If $y \in t$, $\alpha < \omega_1$, there is a unique $i_\alpha(y) < \omega$ such that if $g_\alpha(x) = y$ then $x \in V_{i_\alpha(y)}$. Shrinking the family again, we can assume that $i_\alpha(y) = i(y)$ for $y \in t$. We claim that now (s', g') with $s' = s \cup s_\alpha \cup s_\beta$, $g' = g_\alpha \cup g_\beta$ is a condition extending p_α, p_β for any $\alpha \neq \beta$.

First, g' is a function as $g_\alpha | s = g = g_\beta | s$. Obviously, $g'(x) \in F(x)$ for $x \in s'$.

In order to show (c) for g', assume that $g'(x) = g'(x') = y$. As p_α, p_β are conditions, $x \in s \cup s_\alpha$, $x' \in s \cup s_\beta$ (or vice versa), and $y \in t$. Therefore $x \in V_{i_\alpha(y)}$, $x' \in V_{i_\beta(y)}$ and by our above arguments $i_\alpha(y) = i_\beta(y)$, that is, x and x' are in the same V_i. ∎

Let $G \subseteq P$ be a filter meeting each D_x $(x \in V)$. Let $f(x)$ be the unique value such that some $(s, g) \in G$ has $x \in s$, $f(x) = g(x)$. Then $f(x) \in F(x)$ for each $x \in V$ and f is a good coloring of X. ∎

A result complementing the above is the following.

Theorem 11: *It is consistent that $\aleph_1 < c$ and there is a bipartite graph X of cardinality $\aleph_1$ with* $\mathrm{List}^*(X) > \omega$.

Proof. We show that if there is a non-2-chromatic system $\mathcal{H} = \{H_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$, then there is a graph as described. For this, let $A = \{a_\alpha : \alpha < \omega_1\}$ and

$B = \{b_\alpha : \alpha < \omega_1\}$ be disjoint sets of cardinality $\aleph_1$ and let

$$X = \{(a_\alpha, b_\beta) : \alpha, \beta < \omega_1\}$$

be the complete bipartite graph on A and B. Define the assignment $F : A \cup B \to [\omega]^\omega$ by $F(a_\alpha) = F(b_\alpha) = H_\alpha$.

Assume that $f(a_\alpha) \in F(a_\alpha)$, $f(b_\alpha) \in F(b_\alpha)$ for $\alpha < \omega_1$ and f is a good coloring of X. Then the sets $U = \{f(a_\alpha) : \alpha < \omega_1\}$ and $V = \{f(b_\alpha) : \alpha < \omega_1\}$ must be disjoint as X is the complete bipartite graph. As $f(a_\alpha) \in H_\alpha \cap U$ and $f(b_\alpha) \in H_\alpha \cap V$, the sets U, V give a good 2-coloring of $\mathcal{H}$, a contradiction.

The consistency of $\aleph_1 < c$ with the existence of a non-2-chromatic set system of cardinality $\aleph_1$ consisting of countable sets was first proved by Kunen (unpublished; see, however, [4]). Kunen's argument was the following. Let V be a model of $\aleph_1 < c$. Force with the finite support iteration $\{P_\alpha : \alpha \leq \omega_1\}$ where the factor Q_α is defined as follows. Let D_α be a nonprincipal ultrafilter on ω in V^{P_α}. Set $q \in Q_\alpha$ if $q = (x, A)$, $x \in [\omega]^{<\omega}$, $A \in D$, $x \cap A = \emptyset$; $q' = (x', A') \leq q = (x, A)$ if $x' \supseteq x$, $x' - x \subseteq A$, $A' \subseteq A$. It is immediate that $P = P_{\omega_1}$ is ccc. Density arguments give that if $G_\alpha \subseteq Q_\alpha$ is generic, then $A_\alpha = \bigcup\{x : (x, A) \in G_\alpha\}$ is a set $A_\alpha \in [\omega]^\omega$ such that for every $B \in V^{P_\alpha}$, $B \subseteq \omega$, either $B \cap A_\alpha$ or $(\omega - B) \cap A_\alpha$ is finite. This implies that if

$$\mathcal{H} = \left\{A_\alpha - s : \alpha < \omega_1, s \in [\omega]^{<\omega}\right\}$$

then $\mathcal{H}$ is not 2-chromatic in V^P. ∎

THEOREM 12 (GCH): *Assume that* $\mathrm{List}(X)$ *is infinite. Then* $\mathrm{Col}(X) \leq \mathrm{List}(X)^+$, $\mathrm{Col}(X) \leq \mathrm{List}^*(X)^{++}$, *and if* $\mathrm{List}^*(X)$ *is regular, then* $\mathrm{Col}(X) \leq \mathrm{List}^*(X)^+$.

Proof. In order to prove the first statement, it suffices to show the following. If μ is an infinite cardinal, and $\mathrm{Col}(X) > \mu^+$, then $\mathrm{List}(X) > \mu$.

For this, assume that $\mathrm{Col}(X) > \mu^+$. We can assume that $\kappa = |X|$ is minimal with respect to this condition and, for simplicity, that X is on κ. Then, by the Claim in Lemma 5, κ is regular, and there is a stationary set $S \subseteq \kappa$ such that for each $\alpha \in S$ there is some $\beta(\alpha)$ with $\alpha \leq \beta(\alpha) < \kappa$ that $|\Gamma^-(\beta(\alpha))| \geq \mu^+$. By thinning out S, we can obtain that the mapping $\alpha \mapsto \beta(\alpha)$ is injective. For each $\alpha \in S$ let $f(\alpha)$ denote the supremum of the first μ elements of $\Gamma^-(\beta(\alpha))$. Then obviously $f(\alpha) < \alpha$, and so we can apply Fodor's lemma and get $\gamma < \kappa$ and a stationary $S' \subseteq S$ such that $f(\alpha) = \gamma$ for $\alpha \in S'$.

If we now set $\lambda = |\gamma|$, then we have a subgraph of X of the following type: a bipartite graph with bipartition classes A and B, $|A| = \lambda^+$, $|B| = \lambda$, for each $x \in A$, $|\Gamma(x)| \geq \mu$. We can apply Lemma 7(a) to obtain $\mathrm{List}(X) > \mu$.

In order to prove the other two claims, it suffices to show that if μ is regular and $\mathrm{List}^*(X) \leq \mu$, then $\mathrm{Col}(X) \leq \mu^+$. This follows as above, using Lemma 7(b). ∎

Before proceeding to the next result, we notice the following.

Lemma 13: *If* $\mathrm{Col}(X) = \omega$, *then* $\mathrm{List}(X) = \omega$.

Proof. It suffices to show that X contains finite subgraphs with arbitrarily large finite list-chromatic number. Theorem 9.1 of Erdős and Hajnal in [6] states that if $2 \leq k < \omega$ and $\mathrm{Col}(X) \geq 2k-2$, then X contains a finite subgraph Y with $\mathrm{Col}(Y) \geq k$. Alon in [2] gives a function $f(n)$ such that if the finite graph Y has $\mathrm{Col}(Y) \geq f(n)$ then $\mathrm{List}(Y) \geq n$. As by the above result X contains finite subgraphs with arbitrary large coloring number, $\mathrm{List}(X) = \omega$ follows. ∎

Lemma 14 (GCH): *Let* $\kappa > \omega$ *be regular, and, if* $\kappa = \lambda^+$, λ *singular, assume* $\square_\lambda$. *Let* X *be a* (κ, μ)*-graph with some* $\mu < \kappa$. *If* $P = \mathrm{Add}(\kappa, 1)$, *then* $\mathrm{List}(X) > \mu$ *holds in* V^P.

We notice that the hypothesis $\square_\lambda$ is used only in Case 6 below.

Proof. We assume, therefore, that X is a graph on κ, $S \subseteq \kappa$ is stationary, and, for each $\alpha \in S$, $\Gamma^-(\alpha)$ is a cofinal subset of α of order type μ. Notice that $\mathrm{cf}(\alpha) = \mathrm{cf}(\mu)$ for $\alpha \in S$.

We can assume that P is the following poset: $p \in P$ if $p : \alpha \to [\kappa]^\mu$ for some $\alpha < \kappa$; $p' \leq p$ if p' end-extends p. If G is a V-P generic filter, then $F = \bigcup G$ is an assignment $F : \kappa \to [\kappa]^\mu$. We are going to prove that, in $V[G]$, there is no good coloring $f \in \prod F$ for X. Assume that $p \in P$ forces that $\underline{f} \in \prod F$ is a good coloring of X.

We consider six cases.

Case 1. κ is inaccessible and μ is regular.

Let N be an elementary submodel with

$$\{X, P, p, \underline{f}\} \subseteq N \prec (\mathcal{H}(\chi), \in, <_w)$$

for some large regular χ with $[N]^{<\mu} \subseteq N$ and $\delta = N \cap \kappa \in S$.

Set $H = \Gamma^-(\delta)$, the set of points below δ which are joined to δ. By assumption, $\mathrm{tp}(H) = \mu$ and $\sup(H) = \delta$.

Construct the decreasing sequence $\{p_\xi : \xi < \mu\}$ of conditions in N as follows. Let $p_0 = p$. If $\xi < \mu$ is limit, let p_ξ be $\bigcup\{p_\eta : \eta < \xi\}$, the largest lower bound of $\{p_\eta : \eta < \xi\}$. If $p_\xi \in N$ is given, set $\alpha_\xi = \mathrm{Dom}(p_\xi) < \delta$, pick $x_\xi \in H - \alpha_\xi$, let $p'_\xi \in N$ be an extension of p_ξ such that $p'_\xi(x_\xi)$ is disjoint from $p_\xi(\beta)$ for $\beta < \alpha_\xi$, and let $p_{\xi+1} \in N$ be an extension of p'_ξ with $p_{\xi+1} \Vdash \underline{f}(x_\xi) = y_\xi$ for some y_ξ. Finally, let $p' = \bigcup\{p_\xi : \xi < \mu\}$. Clearly, $\mathrm{Dom}(p') = \delta$. Extend p' to p'' where $p''(\delta) = \{y_\xi : \xi < \mu\}$. Notice that $y_\xi \neq y_\eta$ for $\eta < \xi < \mu$, so the latter set is indeed of cardinality μ. If now $p^* \leq p''$ determines $\underline{f}(\delta)$, say $p^* \Vdash \underline{f}(\delta) = y_\xi$, then $p^* \Vdash \underline{f}(\delta) = \underline{f}(x_\xi)$, while $\{x_\xi, \delta\} \in X$, so p^* forces a contradiction.

Case 2. κ is inaccessible and μ is singular.

Fix a sequence $\{\mu_\alpha : \alpha < \mathrm{cf}(\mu)\}$ of cardinals converging to μ.

Let χ be a large regular cardinal and choose an elementary submodel N such that

$$\{p, X, \underline{f}\} \subseteq N \prec (\mathcal{H}(\chi), \in, <_w)$$

such that $[N]^{<\mathrm{cf}(\mu)} \subseteq N$, $\delta = N \cap \kappa \in S$ is a cardinal and

$$\bigcup\{\mathcal{P}(\alpha) : \alpha < \delta\} \subseteq N.$$

Set $H = \Gamma^-(\delta)$, the set of points less than δ, joined to δ. For $\tau < \mathrm{cf}(\mu)$ let H_τ denote the set of the first μ_τ elements of H.

We are going to construct a sequence $\{p_\xi : \xi < \mathrm{cf}(\mu)\}$ of conditions as follows. Set $p_0 = p$. If $\xi < \mathrm{cf}(\mu)$ is limit, let $p_\xi = \bigcup\{p_\eta : \eta < \xi\}$. Given $p_\xi \in N$, we construct $p_{\xi+1}$ as follows. Set $\alpha_\xi = \mathrm{Dom}(p_\xi)$; let τ_ξ be the minimal $\tau < \mathrm{cf}(\mu)$ such that $H_\tau \not\subseteq \alpha_\xi$. Extend p_ξ to a $p'_{\xi+1} \in N$ such that the sets $\{p'_{\xi+1}(\beta) : \beta \in H_{\tau_\xi} - \alpha_\xi\}$ are disjoint from $\bigcup p_\xi[\alpha_\xi]$ and each other. Next select $p_{\xi+1} \in N$ such that $p_{\xi+1} \leq p'_{\xi+1}$ and $p_{\xi+1}$ determines the values of $\underline{f}$ on $H_{\tau_\xi} - \alpha_\xi$. Finally, set $p' = \bigcup\{p_\xi : \xi < \mathrm{cf}(\mu)\}$. Then $\mathrm{Dom}(p') = \delta$ and we can define an extension p^* of p' which forces that

$$F(\delta) = \left\{\underline{f}(\beta) : \beta \in \bigcup\{H_{\tau_\xi} - \alpha_\xi : \xi < \mathrm{cf}(\mu)\}\right\}.$$

As the latter set is of cardinality μ, this is possible. We are done, as p^* forces that $\underline{f}(\delta) = \underline{f}(\beta)$ for some $\beta \in H$.

Case 3. $\kappa = \lambda^+$ and μ is regular with $\mu \leq \mathrm{cf}(\lambda)$.

Pick N with

$$\{p, X, \underline{f}\} \subseteq N \prec (\mathcal{H}(\chi), \in, <_w)$$

such that $[N]^{<\mu} \subseteq N$ and $\delta = N \cap \lambda^+ \in S$.

Set $H = \Gamma^-(\delta)$, the set of points less than δ, which are joined to δ.

Construct the decreasing sequence $\{p_\xi : \xi < \mu\}$ of conditions with $p_\xi \in N$ as follows. Set $p_0 = p$. If $\xi < \mu$ is limit, let $p_\xi = \bigcup\{p_\eta : \eta < \xi\}$. If $p_\xi \in N$ is given, set $\alpha_\xi = \mathrm{Dom}(p_\xi)$, $\beta_\xi = \min(H - \alpha_\xi)$. Let $p'_\xi \in N$ be a condition extending p_ξ such that $p'_\xi(\beta_\xi)$ is disjoint from $p_\xi[H \cap \alpha_\xi]$. Then, extend p'_ξ to some condition $p_{\xi+1} \in N$ which determines the value of $\underline{f}(\beta_\xi)$. Set $p' = \bigcup\{p_\xi : \xi < \mu\}$, a condition with $\mathrm{Dom}(p) = \delta$. Next, extend p' to p^* forcing $F(\delta) = \{\underline{f}(\beta_\xi) : \xi < \mu\}$. Notice that this latter set is of cardinality μ. As no element of $F(\delta)$ can be chosen as $\underline{f}(\delta)$, we are done.

Case 4. $\kappa = \lambda^+$, $\mu < \mathrm{cf}(\lambda)$ is singular.

Fix a sequence $\{\mu_\alpha : \alpha < \mathrm{cf}(\mu)\}$ of cardinals converging to μ.

Pick N with

$$\{p, X, \underline{f}\} \subseteq N \prec (\mathcal{H}(\chi), \in, <_w)$$

such that $[N]^{<\mathrm{cf}(\mu)} \subseteq N$, $\bigcup\{[\alpha]^\mu : \alpha < \delta\} \subseteq N$, and $\delta = N \cap \lambda^+ \in S$.

Set $H = \Gamma^-(\delta)$, the set of points less than δ, joined to δ. For $\tau < \mathrm{cf}(\mu)$ let H_τ denote the set of the first μ_τ elements of H.

We are going to construct a sequence $\{p_\xi : \xi < \mathrm{cf}(\mu)\}$ of conditions in N as follows. Set $p_0 = p$. If $\xi < \mathrm{cf}(\mu)$ is limit, let $p_\xi = \bigcup\{p_\eta : \eta < \xi\}$. Given $p_\xi \in N$, we construct $p_{\xi+1}$ as follows. Set $\alpha_\xi = \mathrm{Dom}(p_\xi)$; let τ_ξ be the minimal $\tau < \mathrm{cf}(\mu)$ such that $H_\tau \not\subseteq \alpha_\xi$. Extend p_ξ to a $p'_{\xi+1} \in N$ such that the sets $\{p'_{\xi+1}(\beta) : \beta \in H_{\tau_\xi} - \alpha_\xi\}$ are disjoint from $\bigcup p_\xi[\alpha_\xi]$ and each other. Next select $p_{\xi+1} \in N$ such that $p_{\xi+1} \leq p'_{\xi+1}$ and $p_{\xi+1}$ determines the values of $\underline{f}$ on $H_{\tau_\xi} - \alpha_\xi$. Finally, set $p' = \bigcup\{p_\xi : \xi < \mathrm{cf}(\mu)\}$. Then $\mathrm{Dom}(p') = \delta$ and we can define an extension p^* of p' which forces that

$$F(\delta) = \left\{\underline{f}(\beta) : \beta \in \bigcup\{H_{\tau_\xi} - \alpha_\xi : \xi < \mathrm{cf}(\mu)\}\right\}.$$

As the latter set is of cardinality μ, this is possible. We can now conclude as in Case 2.

Case 5. $\kappa = \lambda^+$, μ is singular with $\mathrm{cf}(\mu) \leq \mathrm{cf}(\lambda) < \mu$.

Fix a sequence $\overline{\varphi} = \langle \varphi_\alpha : \alpha < \lambda^+ \rangle$ where $\varphi_\alpha : \alpha \to \lambda$ is injective. Fix also an increasing sequence $\{\mu_\tau : \tau < \mathrm{cf}(\mu)\}$ of regular cardinals converging to μ with $\mathrm{cf}(\lambda) < \mu_0$.

Pick N such that

$$\{p, X, \underline{f}, \overline{\varphi}\} \cup \bigcup\{\mathcal{P}(\alpha) : \alpha < \lambda\} \subseteq N \prec (\mathcal{H}(\chi), \in, <_w)$$

with $[N]^{<\mathrm{cf}(\mu)} \subseteq N$ and $\delta = N \cap \lambda^+ \in S$.

CLAIM 1: *If $T \subseteq \delta$, $\sup(T) < \delta$, $\tau = |T|$ is regular, $\mathrm{cf}(\lambda) < \tau < \lambda$, then there is a subset $T' \subseteq T$, $|T'| = \tau$, $T' \in N$.*

Proof. Pick α with $\sup(T) < \alpha < \delta$. The set $Z = \varphi_\alpha[T] \subseteq \lambda$ and has cardinality τ, so there is a subset $Z' \subseteq Z$ which is bounded below λ and $|Z'| = \tau$. By our construction, $Z' \in N$, and so $T' = \varphi_\alpha^{-1}[Z'] \subseteq T$ and $T' \in N$. ∎

Set $H = \Gamma^-(\delta)$, the set of points less than δ, which are joined to δ. Let H_τ denote the set of the first μ_τ elements of H ($\tau < \mathrm{cf}(\mu)$).

We are going to define a decreasing sequence $\{p_\xi : \xi < \mathrm{cf}(\mu)\}$ of conditions from N. Set $p_0 = p$. If $\xi < \mathrm{cf}(\mu)$ is limit, let $p_\xi = \bigcup\{p_\eta : \eta < \xi\}$. Given p_ξ, we define $p_{\xi+1}$ as follows. Set $\alpha_\xi = \mathrm{Dom}(p_\xi)$. Let $\tau(\xi) < \mathrm{cf}(\mu)$ be such that $H_{\tau(\xi)} \not\subseteq \alpha_\xi$. Now $H_{\tau(\xi)} - \alpha_\xi$ has ordinal $\mu_{\tau(\xi)}$, a regular cardinal with $\mathrm{cf}(\lambda) < \mu_{\tau(\xi)}$. We can apply Claim 1, and obtain a set $T_\xi \subseteq H_{\tau(\xi)} - \alpha_\xi$ with $|T_\xi| = \mu_{\tau(\xi)}$ and $T_\xi \in N$. Let $p'_\xi \in N$ be an extension of p_ξ such that $\mathrm{Dom}(p'_\xi) = \sup(T_\xi)$ and the sets $\{p'_\xi(\beta) : \beta \in T_\xi\}$ are disjoint from $p_\xi[\alpha_\xi]$ and each other. Let $p_{\xi+1}$ be an extension of p'_ξ which determines the values of $\underline{f}(\beta)$ for $\beta \in T_\xi$. Set $p' = \bigcup\{p_\xi : \xi < \mathrm{cf}(\mu)\}$. Extend p' to a p^* which forces $F(\delta) = \{\underline{f}(\beta) : \beta \in T\}$ where $T = \bigcup\{T_\xi : \xi < \mathrm{cf}(\mu)\}$. Now p^* forces that $\underline{f}$ is not a good coloring.

CASE 6. $\kappa = \lambda^+$ for some λ with $\mathrm{cf}(\lambda) < \mathrm{cf}(\mu)$.

This is the only case that uses the hypothesis $\Box$.

Notice that we have $\mathrm{cf}(\mu) > \omega$.

By assumption, there is a $\Box$-sequence $\overline{C} = \{C_\alpha : \alpha < \lambda^+,\ \text{limit}\}$. Let the increasing enumeration of C_α be $C_\alpha = \{\gamma^\alpha_\xi : \xi < \mathrm{tp}(C_\alpha)\}$, where we assume $\gamma^\alpha_0 = 0$.

Fix the strictly increasing sequence $\{\lambda_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ of cardinals converging to λ. Let $\{\mu_\tau : \tau < \mathrm{cf}(\mu)\}$ be a sequence as follows. If μ is regular, then set $\mu_\tau = \tau$ for $\tau < \mu$. If μ is singular, then let $\{\mu_\tau : \tau < \mathrm{cf}(\mu)\}$ be a sequence of regular cardinals converging to μ.

Fix also a sequence $\overline{\varphi} = \{\varphi_{\alpha,\beta} : \alpha < \beta < \lambda^+\}$ where $\varphi_{\alpha,\beta} : [\alpha, \beta) \to \lambda$ is an injection.

Choose an elementary submodel

$$\bigcup\{\mathcal{P}(\tau) : \tau < \lambda\} \cup \{p, \overline{C}, \overline{\varphi}, X, \underline{f}\} \subseteq N \prec (\mathcal{H}(\chi), \in, <_w)$$

where χ is a large regular cardinal, such that $|N| = \lambda$, $\delta = N \cap \lambda^+ \in S$. Notice that $\mathrm{tp}(C_\delta) \leq \lambda$ as $\overline{C}$ is a $\square$-sequence, and by $\mathrm{cf}(\lambda) < \mathrm{cf}(\mu)$ we cannot have equality here, so $\mathrm{tp}(C_\delta) < \lambda$.

Set $H = \Gamma^-(\delta)$, the set of points below δ which are joined to δ. By assumption, $\mathrm{tp}(H) = \mu$ and $\sup(H) = \delta$.

Set, for $\eta < \mathrm{cf}(\lambda)$,

$$A_\eta = \left\{\langle \xi, x\rangle : \xi < \mathrm{tp}(C_\delta), \varphi^{-1}_{\gamma^\delta_\xi, \gamma^\delta_{\xi+1}}(x) \in H, x < \lambda_\eta\right\}.$$

Clearly, $\{A_\eta : \eta < \mathrm{cf}(\lambda)\}$ is increasing in η and, as $A_\eta \subseteq \mathrm{tp}(C_\delta) \times \lambda_\eta$, $A_\eta \in N$. Further, $H = \bigcup\{B_\eta : \eta < \mathrm{cf}(\lambda)\}$ where

$$B_\eta = \{\varphi^{-1}_{\gamma^\delta_\xi, \gamma^\delta_{\xi+1}}(x) : \langle \xi, x\rangle \in A_\eta\}.$$

As $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\mu)$, there is an $\eta < \mathrm{cf}(\lambda)$ such that $|B_\eta| = \mu$. If we write $\overline{H} = B_\eta$, then $\overline{H} \subseteq H$ and $|\overline{H}| = \mu$.

Claim 2: *If $\alpha < \delta$ then $\overline{H} \cap \alpha \in N$.*

Proof. Assume first that $\gamma < \delta$ is a limit point of C_δ. Then there is a limit $\zeta < \mathrm{tp}(C_\delta)$ such that $\gamma = \gamma^\delta_\zeta$. Then $C_\gamma = C_\delta \cap \gamma$ and

$$\overline{H} \cap \gamma = \left\{\varphi^{-1}_{\gamma^\gamma_\xi, \gamma^\gamma_{\xi+1}}(x) : \langle \xi, x\rangle \in A_\eta, \xi < \zeta\right\}$$

is in N, as $A_\eta \in N$.

If $\alpha < \delta$ is arbitrary, as $\mathrm{cf}(\delta) = \mathrm{cf}(\mu) > \omega$, we can let γ be a limit point of C_δ such that $\alpha < \gamma < \delta$ and then $\overline{H} \cap \alpha = (\overline{H} \cap \gamma) \cap \alpha$ is in N. ∎

Recall that we have fixed a sequence $\langle \mu_\tau : \tau < \mathrm{cf}(\mu)\rangle$ of regular cardinals converging to μ. Let $\overline{H}_\tau$ denote the set of the first μ_τ members of $\overline{H}$.

We construct a decreasing sequence $\{p_\xi : \xi < \mathrm{cf}(\mu)\}$ of conditions, as follows. Set $p'_0 = p$. For $0 < \xi < \mathrm{cf}(\mu)$ let $p'_\xi = \bigcup\{p_\eta : \eta < \xi\}$, i.e., $p_{\xi-1}$, if ξ is successor, and the largest lower bound of $\{p_\eta : \eta < \xi\}$ if ξ is limit. Set $\alpha_\xi = \mathrm{Dom}(p'_\xi)$. Let $\tau_\xi < \mathrm{cf}(\mu)$ be the least τ such that $\overline{H}_{\tau_\xi} \not\subseteq \alpha_\xi$. Let p''_ξ be the $<_w$-minimal extension of p'_ξ such that $\mathrm{Dom}(p''_\xi) = \sup(\overline{H}_{\tau_\xi})$ and the sets $\{p''_\xi(\beta) : \alpha_\xi \leq \beta \in \overline{H}_{\tau_\xi}\}$ are pairwise disjoint and disjoint from $\bigcup p'_\xi[\alpha_\xi]$. Let p_ξ be the $<_w$-least extension of p''_ξ which determines the values of f for each element of $\overline{H}_{\tau_\xi} - \alpha_\xi$. Finally, set $p' = \bigcup\{p_\xi : \xi < \mathrm{cf}(\mu)\}$.

CLAIM 3: $p_\xi \in N$ $(\xi < \mathrm{cf}(\mu))$.

Proof. As for each $\xi < \mathrm{cf}(\mu)$ the sequence $\langle p_\eta : \eta \leq \xi\rangle$ can be defined from $\overline{H} \cap \alpha$ for $\alpha < \delta$ sufficiently large. ∎

From Claim 3 we obtain that all p_ξ are defined and $\mathrm{Dom}(p') = \delta$. Set

$$Z = \bigcup\{\overline{H}_{\tau_\xi} - \alpha_\xi : \xi < \mathrm{cf}(\mu)\}.$$

Notice that $|Z| = \mu$. Now p' forces that the sets $\{F(\beta) : \beta \in Z\}$ are disjoint and p' determines $f(\beta) \in F(\beta)$ for these values of β. Let $p^* \leq p'$ be a condition with $p^*(\delta) = \{f(\beta) : \beta \in Z\}$. Then obviously p^* forces that $\underset{\sim}{f}$ is not a good coloring of X. ∎

THEOREM 15: *It is consistent that GCH holds, and if X is a graph with* $\mathrm{Col}(X) \geq \omega$ *then* $\mathrm{List}(X) = \mathrm{Col}(X)$.

Proof. Let V be a model of ZFC+GCH plus $\square_\lambda$ for each singular cardinal λ. This is consistent as it can be obtained by iterated forcing; it also follows from the axiom of constructibility, V=L. We force with $(P, \leq)$, the Easton support iteration of $\mathrm{Add}(\kappa, \kappa^+)$ for each uncountable regular κ. Forcing with $(P, \leq)$ does not collapse cardinals and does not change cofinalities; see Lemma 2.5 of [12]. Let X be a graph in $V[G]$ such that $\mathrm{Col}(X) \geq \omega$; we have to show that $\mathrm{List}(X) \geq \mathrm{Col}(X)$. By Lemma 14 we can assume that $\mathrm{Col}(X) > \omega$. As each uncountable cardinal is either a successor cardinal, or a supremum of successor cardinals, it suffices to prove that if $\mathrm{Col}(X) \geq \mu^+$ then $\mathrm{List}(X) \geq \mu^+$. Lemma 5 further reduces this to the cases when X is of type 1 or 2. The case when X is of type 1 is settled by Lemma 7.

We can therefore assume that X is a graph on some regular cardinal $\kappa > \mu$ and

$$S = \left\{\alpha < \kappa : \mathrm{tp}(\Gamma^-(\alpha)) = \mu, \sup(\Gamma^-(\alpha)) = \alpha\right\}$$

is stationary.

Split P as follows: $P = P_\kappa * Q * P_\infty^{\kappa+1}$ where P_κ is the iteration up to and excluding κ, $Q = \mathrm{Add}(\kappa, \kappa^+)$, and $P_\infty^{\kappa+1}$ is the rest of the iteration. As we consider Easton supports, $P_\infty^{\kappa+1}$ does not introduce any new κ-sequence of ordinals, therefore X is in $V^{P_\kappa * Q}$ and it suffices to show that $\mathrm{List}(X) \geq \mu^+$ holds in this model.

Set $W = V^{P_\kappa}$. If $\kappa = \lambda^+$ for some singular cardinal λ, then λ^+ remains cardinal and successor of λ in W, hence $\square_\lambda$ still holds in W.

As Q is κ^+-c.c., there is some $\alpha < \kappa^+$ such that X is in $W' = W^{\mathrm{Add}(\kappa,\alpha)}$. By moving the addition of the α-th Cohen subset of κ to the end of the forcing, we can arrange that $V^{P_\kappa * Q} = (W'')^{\mathrm{Add}(\kappa,1)}$ for some appropriate W'' containing X. We finally apply Lemma 13 and obtain an assignment F witnessing that $\mathrm{List}(X) \geq \mu^+$ in $V^{P_\kappa, Q}$, and therefore in V^P by our remark above. ∎

Theorem 16 (V=L): *There is a type 2 graph X with* $\mathrm{Chr}(X) = 2$, $\mathrm{List}^*(X) = \omega_1$, $\mathrm{Col}(X) = \omega_1$.

Proof. Let $S, T \subseteq \omega_1$ be disjoint stationary sets. As V=L is assumed, $\diamondsuit_S$ and $\diamondsuit_T$ both hold and so there exist sequences $\{f_\alpha : \alpha \in S\}$ and $\{f_\alpha : \alpha \in T\}$ such that if $f : \omega_1 \to \omega$, then both $\{\alpha \in S : f_\alpha = f|\alpha\}$ and $\{\alpha \in T : f_\alpha = f|\alpha\}$ are stationary.

Our X will be bipartite with bipartition classes S and T, with $\Gamma^-(\alpha) = \emptyset$ ($\alpha \in T$), and $\Gamma^-(\alpha)$ is either $\emptyset$ or an ω-sequence converging to α for $\alpha \in S$. Also, we are going to determine an assignment $F : S \cup T \to [\omega]^\omega$ that will eventually witness $\mathrm{List}^*(X) > \omega$.

For $\alpha \in S \cup T$, set

$$I_\alpha = \left\{ i < \omega : \sup(f^{-1}(i) \cap T) = \alpha \right\}.$$

If $\alpha \in T$ and I_α is finite, then we let $F(\alpha) = \omega - I_\alpha$.

If $\alpha \in S$ and I_α is infinite, enumerate I_α as $I_\alpha = \{i_n : n < \omega\}$, and set $\Gamma^-(\alpha) = \{\beta_n : n < \omega\}$ where the element $\beta_n \in f^{-1}(i_n) \cap T$ is chosen so that $\beta_n \to \alpha$. Further, let $F(\alpha) = I_\alpha$. Otherwise, set $\Gamma^-(\alpha) = \emptyset$ and $F(\alpha) = \omega$.

Claim: *There is no good coloring f of X such that $f(\alpha) \in F(\alpha)$ for $\alpha \in S \cup T$.*

Proof. Assume on the contrary that $f : S \cup T \to \omega$ is a good coloring of X and $f(\alpha) \in F(\alpha)$ for $\alpha \in S \cup T$. First, we claim that

$$I = \left\{ i < \omega : \sup(f^{-1}(i) \cap T) = \omega_1 \right\}$$

is infinite. Indeed, assume I is finite, and define

$$\gamma = \sup\{f^{-1}(i) : i \in \omega - I\}.$$

The set

$$E = \{\alpha > \gamma : i \in I \longrightarrow \sup(f^{-1}(i) \cap \alpha) = \alpha\}$$

is a closed, unbounded set in ω_1. Pick $\alpha \in E \cap T$ such that $f_\alpha = f|\alpha$. Then $I_\alpha = I$ and $f(\alpha) \in F(\alpha) = \omega - I_\alpha = \omega - I$ which is a contradiction, as $\alpha > \gamma$.

We obtained, therefore, that I is infinite. Again, E is closed, unbounded, where $\alpha \in E$ if the following hold: if $i \in \omega - I$ then $\sup(f^{-1}(i)) < \alpha$; if $i \in I$ then $\sup(f^{-1}(i) \cap \alpha) = \alpha$. Pick $\alpha \in S \cap E$ with $f_\alpha = f|\alpha$. Now $I_\alpha = I$, $f(\alpha) \neq i$ for $i \notin I$, and for each $i \in I$ there is a $\beta \in \Gamma^-(\alpha)$ with $f(\beta) = i$, so, as f is supposed to be a good coloring, $f(\alpha) \neq i$, either. ∎

With the Claim we have proved the Theorem. ∎

Theorem 17: *GCH is consistent with the existence of a graph with* $|X| = \omega_1$, $\mathrm{Col}(X) = \omega_1$, $\mathrm{List}(X) = \omega$.

Proof. We show that GCH is consistent with the existence of a type 2 graph X with $\mathrm{List}(X) = \omega$.

Let V be a model of GCH. We are going to construct an iterated forcing $\{P_\alpha : \alpha \leq \omega_2\}$ with countable supports. The first factor, Q_0, adds a type 2 graph X with initial segment forcing. That is, $q \in Q_0$ if $q = (\delta + 1, x)$ where $\delta < \omega_1$ and x is an HM graph on $\delta + 1$; $q' = (\delta' + 1, x') \leq q = (\delta + 1, x)$ if $\delta' \geq \delta$ and $x = [\delta + 1]^2 \cap x'$. The **height** of q, $h(q)$, is the first coordinate of q. If G_0 is V–Q_0-generic, we let X be the graph added by Q_0, that is, $X = \bigcup\{x : (\delta + 1, x) \in G_0\}$.

We set $S = S(X) = \{\beta < \omega_1 : |\Gamma_X^-(\beta)| = \omega\}$.

If P_α is specified, let $F_\alpha : \omega_1 \to [\omega_1]^{\aleph_0}$ be an assignment in V^{P_α}. We let $q \in Q_\alpha$ if either $q = \emptyset$ or $q : \delta \to \omega_1$ is a good coloring of X for some $\delta \notin S$ with $q(\gamma) \in F_\alpha(\gamma)$ for $\gamma < \delta$; $q' \leq q$ if q' end-extends q. Here the **height** of q, $h(q)$, is the domain of of q.

As we have already indicated, we iterate with countable supports, that is, $|\mathrm{supp}(p)| \leq \omega$ for $p \in P_\alpha$, where $\mathrm{supp}(p) = \{\beta < \alpha : p(\beta) \neq 1\}$.

Claim 1: *If* $\beta < \alpha < \omega_2$, $p \in P_\alpha$, $q = p|\beta$, $r = p|[\beta, \alpha)$, $q' \leq q$, *then there is a unique* $p' \in P_\alpha$ *such that* $p'|\beta = q'$, $p'|[\beta, \alpha) = r$.

Proof. Straightforward. ∎

In what follows, if $p \in P_\alpha$, $\beta < \alpha$, $p|\beta = q$, $p|[\beta, \alpha) = r$ then we use the notation $p = q + r$.

Claim 2: *In* V^{P_α} $(\alpha < \omega_2)$, *each condition in* Q_α *has extensions of arbitrarily large height.*

Proof. The statement is obvious for $\alpha = 0$; each condition in Q_0 has extensions of arbitrarily large height.

Assume that $\alpha > 0$. In V^{P_α}, we are given the graph X on ω_1, the assignment $F_\alpha : \omega_1 \to [\omega_1]^{\aleph_0}$, a condition $q : \delta \to \omega_1$ for some $\delta \notin S$ and we want to extend it to $q' \leq q$ where $q' : \delta' \notin S$. Let $\{Y_\xi : \delta \leq \xi < \delta'\}$ be a system of disjoint sets with $Y_\xi \subseteq F_\alpha(\xi)$, $|Y_\xi| = \omega$. These sets can be chosen as $\{F_\alpha(\xi) : \delta \leq \xi < \delta'\}$ is a family of countably many infinite sets. Let

$$Y'_\xi = Y_\xi - \{\tau < \delta : \{\tau, \xi\} \in X\}.$$

As X is an HM graph, we subtract finitely many elements, therefore Y'_ξ is infinite. Finally, pick the distinct elements $q'(\xi) \in Y'_\xi$ for $\delta \leq \xi < \delta'$. ∎

Claim 3: *If $\alpha \leq \omega_2$, $p \in P_\alpha$, $\xi < \omega_1$, then there is a condition $p' \in P_\alpha$, $p' \leq p$, such that $h(p'(\beta)) \geq \xi$ for every $\beta \in \operatorname{supp}(p')$.*

Proof. Immediate by transfinite induction, using Claim 2. ∎

For $\alpha \leq \omega_2$ we let D_α be the set of conditions p such that for each $\beta \in \operatorname{supp}(p)$, $p|\beta$ fully determines $p(\beta)$ and there is some limit ordinal δ such that $h(p(0)) = \delta + 1$, $p(0)$ forces that $\delta \notin S$, and $h(p(\beta)) = \delta$ for each $\beta \in \operatorname{supp}(p)$, $\beta \neq 0$. We call δ the **height of** p and denote it by $h(p)$, although the height of $p(0)$ is $\delta + 1$.

Claim 4: *Let $\alpha \leq \omega_2$. If $p_0 \geq p_1 \geq p_2 \geq \cdots$ are in D_α then there is a $p \in D_\alpha$ such that, for each n, $p \leq p_n$.*

Proof. Let δ_n be the height of p_n. If $\delta_k = \delta_{k+1} = \cdots$ for some $k < \omega$, then we may let p be simply the coordinatewise union of the p_n's.

In the other case we can assume, by possibly thinning out the sequence, that $\delta_0 < \delta_1 < \cdots$. Set $\delta = \sup\{\delta_n : n < \omega\}$. Let p be the following condition. If $p_n(0) = (\delta_n + 1, x_n)$, then $x = \bigcup\{x_n : n < \omega\}$, $p(0) = (\delta + 1, x)$ (this makes sure that $\delta \notin S$), $p(\xi) = \bigcup\{p_n(\xi) : n < \omega\}$ for the other coordinates. ∎

We call the condition p constructed above the **canonical limit of** $\{p_n : n < \omega\}$.

Claim 5: (a) *D_α is dense in P_α ($\alpha \leq \omega_2$).*
(b) *Forcing with P_α does not introduce new reals ($\alpha \leq \omega_2$).*

Proof. Notice that if we have (a) for some α, then we have (b) for α by Claim 4.

We prove (a) by induction on α. Assume we have (a) for α and want to show it for $\alpha+1$. Let $p+q$ be an element of $P_{\alpha+1}$ where $p \in P_\alpha$, $q \in Q_\alpha$.

Extend p to a $p_0 \in D_\alpha$ of height δ_0 which determines $q_0 = q$ and forces that $h(q_0) < \delta_0$. For $n = 0, 1, \ldots$ inductively extend p_n to a $p_{n+1} \in D_\alpha$ of height $\delta_{n+1} > \delta_n$ which determines a $q_{n+1} \leq q_n$ with $h(q_{n+1}) = \delta_n$. Finally let $\overline{p}$ be the canonical limit of $\{p_n : n < \omega\}$ and $\overline{q} = \bigcup\{q_n : n < \omega\}$. Clearly $\overline{p}+\overline{q} \leq p+q$ and $\overline{p} + \overline{q} \in D_{\alpha+1}$.

Assume now that α is a limit ordinal. The case $\mathrm{cf}(\alpha) \geq \omega_1$ is obvious as we are forcing with countable supports. Assume that $\mathrm{cf}(\alpha) = \omega$ and let $\{\alpha_n : n < \omega\}$ be a sequence converging to α. Let $p \in P_\alpha$ be arbitrary. Define by induction on $n < \omega$ the conditions $p_n \in D_{\alpha_n}$ such that

(1) $p_{n+1}|\alpha_n \leq p_n$,
(2) $p_{n+1}|[\alpha_n, \alpha_{n+1}) \leq p|[\alpha_n, \alpha_{n+1})$,
(3) $h(p_n) < h(p_{n+1})$.

Then the canonical limit of $\{p_n : n < \omega\}$ will be in D_α and below p. ∎

Claim 6: *P_α is $\aleph_2$-c.c. ($\alpha \leq \omega_2$).*

Proof. Assume that $p_\gamma \in P_\alpha$ for $\gamma < \omega_2$. We can assume that $p_\gamma \in D_\alpha$, by CH, that $\mathrm{supp}(p_\gamma) = A \cup B_\gamma$ where the sets $\{A, B_\gamma : \gamma < \omega_2\}$ are disjoint and $h(p_\gamma) = \delta$ for $\gamma < \omega_2$. Further, again by CH, we can assume that $p_\gamma(0)$, and, in general, that $p_\gamma(\xi)$ is the same for all $\gamma < \omega_2$, for all $\xi \in A$. But now for any $\gamma < \gamma'$ the coordinatewise union of the conditions $p_\gamma, p_{\gamma'}$ is a common extension of p_γ and $p_{\gamma'}$. ∎

Claim 7: *S remains stationary when forcing with P_{ω_2}.*

Proof. Assume that 1 forces that $E \subseteq \omega_1$ is a closed, unbounded set. Pick $E \in N \prec (\mathcal{H}(\theta); \in, <_w)$, where θ is a large regular cardinal, $<_w$ is a well order of $\mathcal{H}(\theta)$, $|N| = \omega$. Set $\delta = N \cap \omega_1$. Fix a sequence $\{\delta_n : n < \omega\}$ converging to δ.

Set $U = (N \cap \omega_2) - \{0\}$. Enumerate U as $U = \{\alpha_n : n < \omega\}$.

Subclaim 1: *There are ordinals $\xi_n < \delta$, conditions $p_s \in P_\alpha \cap N$, and ordinals $\nu(s)$ $(s : \{\alpha_0, \ldots, \alpha_{n-1}\} \to 2)$ $(n < \omega)$, such that*

(1) *$\delta_n < \xi_n$ $(n < \omega)$;*
(2) *$p_s \Vdash \nu(s) \in E$;*
(3) *$\xi_{n-1} < \nu(s) < \xi_n$ $(s : \{\alpha_0, \ldots, \alpha_{n-1}\} \to 2)$;*
(4) *$h(p_s) < \xi_n$ $(s : \{\alpha_0, \ldots, \alpha_{n-1}\} \to 2)$;*

(5) $p_s|\alpha_i \Vdash F_{\alpha_i}(\xi_n) = X(\alpha_i, s, \xi_n)$ *($i < n$);*
(6) $p_s(\alpha_i)(\xi_n) = \tau(\alpha_i, s(i), \xi_n)$ *where* $\tau(\alpha_i, 0, \xi_n), \tau(\alpha_i, 1, \xi_n) \in X(\alpha_i, s, \xi_n)$*, they differ from each other and from each* $\tau(\alpha_i, \varepsilon, \xi_m)$ *($\varepsilon < 2$, $m < n$);*
(7) $p_s \le p_{s'}$ *where* $s' = s|\{\alpha_0, \dots, \alpha_{n-2}\}$*;*
(8) *if* $s|\alpha = s'|\alpha$*, then* $p_s|\alpha = p_{s'}|\alpha$ *($\alpha < \omega_2$).*

Proof. We construct these objects by induction on $n < \omega$. For $n = 0$ we choose a condition $p_\emptyset \in N \cap P$ such that $p_\emptyset \Vdash \nu(\emptyset) \in E$ for some $\nu(\emptyset) < \delta$ and then pick ξ_0 such that $\max(\delta_0, \nu(\emptyset)) < \xi_0 < \delta$.

For the inductive step assume that we have already determined ξ_n and the system of conditions $\{p_s : s : \{\alpha_0, \dots, \alpha_{n-1}\} \to 2\}$. Choose the ordinal $\delta_{n+1} < \xi_{n+1} < \delta$ such that $\xi_{n+1} > h(p_s), \nu(s)$ holds for every s as above. Next we define p_s for every $s : \{\alpha_0, \dots, \alpha_n\} \to 2$ by $p_s = p_{s^*}$ where $s^* = s|\{\alpha_0, \dots, \alpha_{n-1}\}$. Reorder $\{\alpha_0, \dots, \alpha_n\}$ as $\{\beta_0, \dots, \beta_n\}$ with $\beta_0 < \beta_1 < \cdots < \beta_n$.

Split each p_s as

$$p_s = q_\emptyset + q_{i_0} + \cdots + q_{i_0 i_1 \cdots i_n}$$

where $i_j = s(\beta_j)$, $q_{i_0 \cdots i_{k-1}} \in P^{\beta_{k-1}}_{\beta_k}$ (with $\beta_{n+1} = \omega_2$). The indexing, that is, that $q_{i_0 i_1 \cdots i_k} = p_s|[\beta_k, \beta_{k+1})$ depends only on the sequence $i_0 i_1 \cdots i_k$, is justified by (8).

Enumerate $\{s : \{\beta_0, \dots, \beta_{k-1}\} \to 2, k \le n+1\}$ as $\{s_j : j \le m\}$ with $m = 2^{n+2} - 1$ such that shorter sequences precede longer ones. Set $q^0_t = q_t$ for each sequence t. We are going to construct the decreasing sequence $q^0_t \ge \cdots \ge q^j_t \ge \cdots \ge q^m_t$. At step j we set $q^{j+1}_t = q^j_t$ unless t is a segment of s_j. If it is, we proceed as follows. Let

$$q^{j+1}_\emptyset + q^{j+1}_{i_0} + \cdots + q^{j+1}_{i_0 i_1 \cdots i_{k-1}} \le q^j_\emptyset + q^j_{i_0} + \cdots + q^j_{i_0 i_1 \cdots i_{k-1}}$$

force that $F_{\beta_k}(\xi_{n+1}) = X(\beta_k, i_0 i_1 \cdots i_{k-1}, \xi_{n+1})$ and

$$q^{j+1}_{i_0 i_1 \cdots i_{k-1}}(\beta_{k-1})(\xi_{n+1}) = \tau(\beta_{k-1}, i_{k-1}, \xi_{n+1}),$$

where $\tau(\beta_{k-1}, 0, \xi_{n+1}), \tau(\beta_{k-1}, 1, \xi_{n+1}) \in X(\beta_{k-1}, i_0 i_1 \cdots i_{k-2}, \xi_{n+1})$ are different from each other and from each $\tau(\beta_{k-l}, \varepsilon, \xi_m)$ ($\varepsilon < 2$, $m \le k$). Further, if $k = n+1$ then we also make $q^{j+1}_\emptyset + q^{j+1}_{i_0} + \cdots + q^{j+1}_{i_0 i_1 \cdots i_{k-1}}$ force that $\nu(s_j) \in E$ for some $\nu(s_j) > \xi_{n+1}$.

Eventually we obtain the conditions

$$\overline{p}_s = q^m_\emptyset + q^m_{i_0} + \cdots + q^m_{i_0 i_1 \cdots i_n},$$

and $s(\beta_0)=i_0$, $s(\beta_1)=i_1,\ldots,s(\beta_n)=i_n$. The system $\{\overline{p}_s : s : \{\beta_0,\ldots,\beta_n\} \to 2\}$ satisfies (5–8). ∎

Let $\{p_s : s : \{\alpha_0,\ldots,\alpha_{n-1}\} \to 2, n < \omega\}$ be a system as in Subclaim 1. For $g : U \to 2$ set $p_g = \bigcup\{p_{s_n} : n < \omega\}$ where $s_n = g|\{\alpha_0,\ldots,\alpha_{n-1}\}$.

Notice that p_g is not a condition as the height of $p_g(0)$ is not a successor ordinal.

Subclaim 2: *If $g|\alpha = g'|\alpha$ then $p_g|\alpha = p_{g'}|\alpha$ ($\alpha < \omega_2$).*

Proof. By property (8) in Subclaim 1, $p_{s_n}|\alpha = p_{s'_n}|\alpha$ where

$$s_n = g|\{\alpha_0,\ldots,\alpha_{n-1}\}, \quad \text{and} \quad s'_n = g'|\{\alpha_0,\ldots,\alpha_{n-1}\}$$

for $n < \omega$. ∎

Enumerate U in increasing order as $U = \{\alpha_\xi : \xi < \theta\}$. We are going to construct $p_\xi \in P_{\alpha_\xi}$, $g_\xi = g|\alpha_\xi$, $g_\xi : U \cap \alpha_\xi \to 2$, such that

(1) $h(p_\xi(\tau)) \geq \delta + 1$ ($\tau \in \mathrm{supp}(p_\xi)$);
(2) $p_\xi|\alpha_\eta \leq p_\eta$ ($\eta < \xi$);
(3) p_ξ determines that $F_{\alpha_\xi}(\delta) = X(\alpha_\xi, \delta)$;
(4) $p_\xi \leq p_{g|\alpha_\xi}$.

To start, we let p_0 be $p_g(0)$ (any g) which is a graph on δ, extend it to a graph on $\delta + 1$, by joining δ to each ξ_n ($n < \omega$).

At step α_ξ set $T_k = \{\tau(\alpha_\xi, k, \xi_n) : i < n < \omega\}$ for $k < 2$, where i is the index of α_ξ in the ω-enumeration of U. As $T_0 \cap T_1 = \emptyset$, we can choose $k < 2$ such that it is not true that all but finitely many elements of $X(\alpha_\xi, \delta)$ are in T_k. Let $g(\alpha_\xi)$ be this k. Then, $X(\alpha_\xi, \delta) \not\subseteq \{p_g(\alpha_\xi)(\xi_n) : n < \omega\}$, as the latter set is T_k with finitely many elements added. Choose an

$$r \in X(\alpha_\xi, \delta) - \{p_g(\alpha_\xi)(\xi_n) : n < \omega\}$$

and let $p_g(\alpha_\xi)(\delta) = r$.

The condition $p = p_{\alpha_\theta}$ finally obtained forces $\delta \in S$ and $p \leq p_s$ for $s = g|\{\alpha_0,\ldots,\alpha_{n-1}\}$ for $n < \omega$, therefore $p \Vdash \delta \in E$, i.e., $p \Vdash \delta \in S \cap E$, a contradiction. ∎

As Claim 7 implies $\mathrm{Col}(X) = \omega_1$, and the generic functions for the factors Q_α establish $\mathrm{List}(X) = \omega$, the proof of the Theorem is concluded. ∎

With the powerful method of Shelah's $\mathbb{D}$-completeness systems one can prove a much stronger result.

THEOREM 18: *It is consistent with GCH that every HM graph X on ω_1 has* $\mathrm{List}(X) \le \omega$.

Proof. If X is an HM-graph on ω_1 and $F : \omega_1 \to [\omega_1]^{\aleph_0}$ is an assignment, let $Q_{X,F}$ be the following notion of forcing: $q \in Q_{X,F}$ if either $q = \emptyset$ or $q : \alpha+1 \to \omega_1$ is a good coloring of $X|\alpha+1$ with $q(\beta) \in F(\beta)$ for $\beta \le \alpha$; $q' \le q$ if q' extends q as a function; $(Q_{X,F}, \le)$ is $\mathbb{D}$-complete, as shown in [1] and [18] for the case when $F(\alpha) = \omega$ $(\alpha < \omega_1)$. The theory of $\mathbb{D}$-completeness gives the result. ∎

THEOREM 19: *It is consistent with GCH that there exists a graph X on ω_1 for which* $\mathrm{List}(X) = \omega_1$ *and* $\mathrm{List}^*(X) = \omega$ *hold.*

Proof. Let V model GCH. We are going to force with a countable support iteration of length ω_2.

The first iterand, Q_0, will add the graph X on ω_1 and the assignment witnessing $\mathrm{List}(X) = \omega_1$. We let $q \in Q_0$ if $q = (\delta+1, x, g)$ where $\delta < \omega_1$, x is an HM graph on $\delta+1$, g is a function on $S(x)$, $g(\xi) \in [\xi]^\omega$. We call δ **the height of** q and denote it by $h(q)$.

We order Q_0 as follows: $q' = (\delta'+1, x', g') \le q = (\delta+1, x, g)$ if $\delta' \ge \delta$, $x = x' \cap [\delta+1]^2$, and $g' \supseteq g$, that is, $g'(\xi) = g(\xi)$ holds for every $\xi \in S(x)$.

CLAIM 1: *Every $q \in Q_0$ has extensions of arbitrarily large height.*

Proof. If $q = (\delta+1, x, g)$ and $\delta' \ge \delta$ then $q' = (\delta'+1, x, g)$ extends q. ∎

CLAIM 2: *If $q_0 \ge q_1 \ge \cdots$ where $q_n = (\delta_n+1, x_n, g_n)$, $\delta_0 < \delta_1 < \cdots$, then $q \le q_n$ for all n, where $\delta = \sup\{\delta_n : n < \omega\}$, $x = \bigcup\{x_n : n < \omega\}$, $g = \bigcup\{g_n : n < \omega\}$, $q = (\delta+1, g, x)$.*

Proof. Obvious. ∎

We call the above condition q **the no-edge limit of** $\{q_n : n < \omega\}$.

CLAIM 3: *If $q = (\delta+1, x, g) \in Q_0$, then there exists an extension $q' = (\delta'+1, x', g') \le q$ with some element $\beta \in S(x')$ such that $\delta < \min(g(\beta))$ and $\Gamma^-_{x'}(\beta) \cap (\delta+1) = \emptyset$.*

Proof. Straightforward. ∎

If $G_0 \subseteq Q_0$ is generic then we let $X = \bigcup\{x : (\delta+1, x, g) \in G_0\}$, the HM graph added by Q_0, and $S = S(X)$. Moreover, let $g^*(\beta) = g(\beta)$ for any $(\delta+1, x, g) \in G_0$ where g is defined at β, and so $g^* : S \to [\omega_1]^\omega$ is the assignment added by Q_0.

If $\alpha < \omega_2$ and we have constructed P_α then let $F_\alpha : \omega_1 \to [\omega]^\omega$ be an assignment, and set $q \in Q_\alpha$ if for some $\delta \notin S$ we have $q : \delta \to \omega$, $q(\xi) \in F_\alpha(\xi)$ $(\xi < \delta)$ and q is a good coloring of X on δ; $h(q) = \delta$ is **the height of** q. As we will prove that GCH still holds in the intermediate models, it is possible to enumerate all possible assignments as $\{F_\alpha : \alpha < \omega_2\}$. Our forcing will be $P = P_{\omega_2}$.

CLAIM 4: *In V^{P_α}, every $q \in Q_\alpha$ has extensions of arbitrarily large height $(1 \le \alpha < \omega_2)$.*

Proof. Assume that $q \in Q_\alpha$, $h(q) = \delta$ and we are given $\delta' > \delta$, $\delta' \notin S$; for example, δ' can be a successor ordinal. Choose the distinct elements $\{q'(\xi) : \delta \le \xi < \delta'\}$ such that

$$q'(\xi) \in F_\alpha(\xi) - \{q(\eta) : \{\eta, \xi\} \in X\}$$

for $\delta \le \xi < \delta'$. This is possible, as one has to choose distinct elements from countably many infinite sets. ∎

For $p \in P_\alpha$ set $p \in D_\alpha$ if for every $\beta \in \operatorname{supp}(p)$ the condition $p|\beta$ fully determines $p(\beta)$, and there is an ordinal δ such that $p(0) = (\delta+1, x, g)$, $\delta \notin S(x)$, and $h(p(\beta)) = \delta$ for all $0 < \beta \in \operatorname{supp}(p)$. The ordinal δ, denoted by $h(p)$, is called **the height of** p.

CLAIM 5: *D_α is $< \omega_1$-closed.*

Proof. Assume that $p_0 \ge p_1 \ge \cdots$ are elements of D_α. Set $\delta_n = h(p_n)$. We can assume that either $\delta_0 = \delta_1 = \cdots$ or $\delta_0 < \delta_1 < \cdots$. In the former case we simply take the following union of the conditions: $\operatorname{supp}(p) = \bigcup\{\operatorname{supp}(p_n) : n < \omega\}$, and if $\beta \in \operatorname{supp}(p)$ then $p(\beta) = p_n(\beta)$ where $n < \omega$ is arbitrary such that $\beta \in \operatorname{supp}(p_n)$.

In the latter case, i.e., if $\delta_0 < \delta_1 < \cdots$, we define p as follows. Let $p(0)$ be the no-edge limit of $\{p_n(0) : n < \omega\}$. If $0 < \beta \in \bigcup\{\operatorname{supp}(p_n) : n < \omega\}$ then we let

$$p(\beta) = \bigcup\{p_n(\beta) : k \le n < \omega\},$$

where k is sufficiently large such that $\beta \in \mathrm{supp}(p_k)$. ∎

We call the condition constructed in the previous Claim the **canonical limit of** $\{p_n : n < \omega\}$. Similarly we can define the canonical limit of $\{p_\xi : \xi < \varphi\}$ for any decreasing sequence of some limit length $\varphi < \omega_1$ from D_α.

Claim 6: *D_α is dense in P_α ($\alpha \le \omega_2$).*

Proof. By transfinite induction on α.

Assume that we have the Claim for α and we are given the condition $(p, q) \in P_{\alpha+1} = P_\alpha * Q_\alpha$. We choose by recursion the conditions $p = p_0 \ge p_1 \ge p_2 \ge \cdots$, ordinals $\delta_0 < \delta_1 < \cdots$, names $q = q_0, q_1, \ldots$, such that

(1) $p_{n+1} \in D_\alpha$;
(2) $h(p_n) = \delta_n$ $(1 \le n < \omega)$;
(3) p_{n+1} determines q_n and forces that $h(q_n) = \delta_n$;
(4) $p_{n+1} \Vdash \delta_{n+1} \notin S$;
(5) $p_{n+2} \Vdash q_{n+1} \le q_n$.

Let $\overline{p}$ be the canonical limit of $\{p_n : n < \omega\}$, and $\overline{q}$ be the union of $\{q_n : n < \omega\}$. Notice that $\overline{p} \Vdash \overline{q} \in Q_\alpha$ as $\sup\{\delta_n : n < \omega\} \notin S$. Then clearly $(\overline{p}, \overline{q}) \in D_{\alpha+1}$ and $(\overline{p}, \overline{q}) \le (p, q)$.

Next assume that $\mathrm{cf}(\alpha) = \omega$ and we have the Claim for all ordinals $\beta < \alpha$. Let $p \in P_\alpha$ be arbitrary. Fix a sequence $\{\alpha_n : n < \omega\}$ converging to α with $\alpha_0 = 0$. Choose the conditions p_n and ordinals δ_n for $n < \omega$ such that

(1) $p_n \in D_{\alpha_n}$;
(2) $p_{n+1}|\alpha_n \le p_n$;
(3) $p_{n+1}|[\alpha_n, \alpha_{n+1}) \le p|[\alpha_n, \alpha_{n+1})$;
(4) $\delta_n = h(p_n)$, and $\delta_0 < \delta_1 < \cdots$.

We now let $\overline{p}$ be the canonical limit of $\{p_n : n < \omega\}$. Clearly, $\overline{p} \in D_\alpha$ and $h(\overline{p}) = \delta$ where $\delta = \sup\{\delta_n : n < \omega\}$.

If α is limit with $\mathrm{cf}(\alpha) > \omega$ then we immediately get the result by induction. ∎

Claim 7: *$(P, \le)$ is $\aleph_2$-c.c.*

Proof. Assume that $p_\xi \in P$ for $\xi < \omega_2$. Without loss of generality we can assume that $p_\xi \in D_{\omega_2}$, $h(p_\xi) = \delta$, $\mathrm{supp}(p_\xi) = S \cup S_\xi$ where the sets $\{S, S_\xi : \xi < \omega_2\}$

are pairwise disjoint. As there are at most $\aleph_1$ structures on $(\delta+1)\times S$, there are $\xi<\eta<\omega_2$ such that $p_\xi|S=p_\eta|S$.

If $p\in P$ is the following condition,

$$p(\alpha)=\begin{cases}p_\xi(\alpha) & \alpha\in S\cup S_\xi;\\ p_\eta(\alpha) & \alpha\in S_\eta,\end{cases}$$

then clearly $p\le p_\xi,p_\eta$. ∎

In order to prove the Theorem assume that some $p^*\in D_{\omega_2}$ forces that there is a choice function $H(\xi)\in g^*(\xi)$ for $\xi\in S$ which is a good coloring of X on S. Without loss of generality, $\operatorname{supp}(p^*)$ is infinite.

Claim 8: *There exist conditions $p^*\ge p_0\ge p_1\ge\cdots$, ordinals $i_n\in\operatorname{supp}(p_n)$, $h(p_n)<\alpha_n<\beta_n<\omega_1$, $k_n<\omega$, such that*

(1) $p_n\Vdash H(\beta_n)=\alpha_n$;
(2) $p_n\in D_{\omega_2}$;
(3) $\{i_n:n<\omega\}=\bigcup\{\operatorname{supp}(p_n):n<\omega\}-\{0\}$;
(4) $p_n(i_j)(\beta_n)=k_j$ $(j\le n)$;
(5) *for each $p\le p_n$, there exist $p'\le p$, α, β, such that $h(p)<\alpha<\beta<h(p')$, $p'\Vdash H(\beta)=\alpha$, and $p'(i_j)(\beta)=k_j$ $(j\le n)$.*

Proof. With an obvious bookkeeping we can take care of (3); further, if we can find a condition with (5) then there is a condition satisfying (1) and (4), as well. We can therefore assume that we are given p_n and i_{n+1} and we cannot choose an appropriate $p_{n+1}\le p_n$ satisfying (5).

Choose the decreasing sequence $p_n\ge q_0\ge q_1\ge\cdots$ of conditions, such that $q_t\in D_{\omega_2}$ and there is no $q\le q_t$ in which there are $h(q_t)<\alpha<\beta$ with $q\Vdash H(\beta)=\alpha$ and $q(i_j)(\beta)=k_j$ $(j\le n)$, $q(i_{n+1})(\beta)=t$. Now let $q\le q_t$ $(t<\omega)$, $q\in D_{\omega_2}$. Pick $q'\le q$ with some $h(q)<\alpha<\beta<h(q')$, $q'\Vdash H(\beta)=\alpha$, $q'(i_j)(\beta)=k_j$ $(j\le n)$. This is possible as $q\le p_n$. Then $q'(i_{n+1})(\beta)=t$ for some $t<\omega$ and this contradicts the choice of q_t. ∎

Let $\{p_n,\alpha_n,\beta_n,k_n:n<\omega\}$ be given as in Claim 8. Set $\beta=\sup\{\beta_n:n<\omega\}$. Let $\{i_\xi:\xi<\varphi\}$ be the increasing enumeration of $\bigcup\{\operatorname{supp}(p_n):n<\omega\}-\{0\}$ for some ordinal $\varphi<\omega_1$. We are going to define a descending sequence $\{r_\xi:\xi<\varphi\}$ of conditions such that $r_\xi\in D_{i_\xi}$ and $r_\xi\le p_n|i_\xi$ $(n<\omega)$. Let $r_0\in Q_0$ be the following condition. If $p_n(0) = (\beta_n+1, x_n, g_n)$, then $r_0 = (\beta+1, x, g)$ where $x = \bigcup\{x_n : n < \omega\} \cup \{\{\beta_n, \beta\} : n < \omega\}$, and $g \supseteq g_n$ such that $g(\beta) = \{\alpha_n : n < \omega\}$.

If $\xi < \varphi$ is limit, then we let $r_\xi \in D_{i_\xi}$ be the canonical lower bound of $\{r_\eta : \eta < \xi\}$.

Assume next that r_ξ is given and $\xi+1 < \varphi$. We construct $r_{\xi+1}$ as follows. Let $r_{\xi+1}|i_\xi \in D_{i_\xi}$ be such that it determines $F_{i_\xi}(\beta) \in [\omega]^\omega$. Next, to define $r_{\xi+1}(\beta)$, extend $\bigcup\{p_n(\beta) : n < \omega\}$ by assigining to β some value $r_{\xi+1}(\beta) \in F_{i_\xi}(\beta)$, $r_{\xi+1}(\beta) \notin \{p_n(i_\xi) : n < \omega\}$. This is possible, as the latter set is finite.

With the construction of the sequence $\{r_\xi : \xi < \varphi\}$ finished, let r be r_φ if φ is a successor ordinal, and the canonical limit of $\{r_\xi : \xi < \varphi\}$ if φ is a limit ordinal. Then, r forces that $\Gamma^-_X(\beta) = \{\beta_n : n < \omega\}$, $g^*(\beta) = \{\alpha_n : n < \omega\}$, $H(\beta_n) = \alpha_n$ $(n < \omega)$. Now it is not possible to choose any α_n as $H(\beta)$. ∎

Finally, we pay our debt to [15].

Theorem 20: *It is consistent that $c = \aleph_2$, and if $\{A_\alpha : \alpha < \omega_2\} \subseteq [\omega_2]^{\aleph_1}$ then there is a coloring $g : \omega_2 \to \omega$ such that g assumes every value on every A_α.*

Proof. Let V be a model of GCH. We force with $\mathrm{Add}(\omega, \omega_2) \oplus \mathrm{Add}(\omega_1, \omega_3)$. Classical theory (see, e.g., [16]) gives that the forcing is cardinal preserving.

Set $P = \mathrm{Add}(\omega, \omega_2)$. Let $(Q, \leq)$ be the following notion of forcing: $q \in Q$ if q is a function with $\mathrm{Dom}(q) \in [\omega_2]^{\leq\aleph_0}$, $\mathrm{Ran}(q) \subseteq \omega$. $q' \leq q$ if q' extends q. Let $(Q_\xi, \leq)$ be a copy of $(Q, \leq)$ for $\xi < \omega_3$. Let $Q(X)$ be the countable support product of $\{(Q_\xi, \leq) : \xi \in X\}$. Finally, set $\overline{Q} = Q(\omega_3)$. We force with $P \oplus \overline{Q}$ (which is equivalent to the above said $\mathrm{Add}(\omega, \omega_2) \oplus \mathrm{Add}(\omega_1, \omega_3)$). Let $G \subseteq P \oplus \overline{Q}$ be generic. If $\mathcal{A} = \{A_\alpha : \alpha < \omega_2\} \subseteq [\omega_2]^{\aleph_1}$ has $\mathcal{A} \in V[G]$, then $\mathcal{A} \in P \oplus Q(\xi)$ for some $\xi < \omega_3$, as $P \oplus \overline{Q}$ is $\aleph_3$-c.c. Split $\overline{Q}$ as $Q(\omega_3 - \{\xi\}) \oplus Q(\{\xi\})$; then $P \oplus \overline{Q} = P \oplus Q(\omega_3 - \{\xi\}) \oplus Q(\{\xi\})$ and $\mathcal{A} \in W$ where $W = V[G \cap (P \oplus Q(\omega_3 - \{\xi\}))]$. By the product theorem (see [16]) the final model is obtained from W via forcing with $(Q, \leq)$.

It suffices, therefore, to show that if $\mathcal{A} \in W$, and if we force with $(Q, \leq)$, then the generic function is a function $g : \omega_2 \to \omega$ as required by the Theorem. Indeed, assume that we are given $q \in Q$, $\alpha < \omega_2$, and $i < \omega$. Then choose $q' \leq q$ such that $q'(\eta) = i$ for some $\eta \in A_\alpha$. This is possible, as $\mathrm{Dom}(q)$ is countable and A_α is uncountable. Then q' forces that $g(\eta) = i$, and we are done. ∎

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