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ON THE EXPONENTIAL GROWTH OF GRADED CAPELLI POLYNOMIALS

BY

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ABSTRACT

In a free superalgebra over a field of characteristic zero we consider the graded Capelli polynomials $Cap_{M+1}[Y,X]$ and $Cap_{L+1}[Z,X]$ alternating on $M+1$ even variables and $L+1$ odd variables, respectively. Here we compute the superexponent of the variety of superalgebras determinated by $Cap_{M+1}[Y,X]$ and $Cap_{L+1}[Z,X]$. An essential tool in our computation is the generalized-six-square theorem proved in [3].

1. Introduction

Let F be a field of characteristic zero, $X = \{x_1, x_2, \ldots\}$ a countable set and $F\langle X\rangle = F\langle x_1, x_2, \ldots\rangle$ the free associative algebra on X over F. Recall that an algebra A is a superalgebra (or $\mathbb{Z}_2$-graded algebra) with grading $(A^{(0)}, A^{(1)})$ if $A = A^{(0)} \oplus A^{(1)}$, where $A^{(0)}, A^{(1)}$ are subspaces of A satisfying

$$A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)} \quad \text{and} \quad A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}.$$

If we write $X = Y \cup Z$ as the disjoint union of two sets, then $F\langle X\rangle = F\langle Y \cup Z\rangle$ has a natural structure of free superalgebra if we require that the variables from Y have degree zero and the variables from Z have degree one.

Recall that an element $f(y_1, \ldots, y_n, z_1, \ldots, z_m)$ of $F\langle Y \cup Z\rangle$ is a graded identity or superidentity for A if $f(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$, for all $a_1, \ldots, a_n \in A^{(0)}$ and $b_1, \ldots, b_m \in A^{(1)}$. The set $Id^{sup}(A)$ of all graded identities of A is

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a T_2-ideal of $F\langle Y \cup Z\rangle$, i.e., an ideal invariant under all endomorphisms of $F\langle Y \cup Z\rangle$ preserving the grading. Moreover, every T_2-ideal Γ of $F\langle Y \cup Z\rangle$ is the ideal of graded identities of some superalgebra $A = A^{(0)} \oplus A^{(1)}$, $\Gamma = Id^{sup}(A)$.

For $\Gamma = Id^{sup}(A)$ a T_2-ideal of $F\langle Y \cup Z\rangle$, we denote by supvar(Γ) or supvar(A) the supervariety of all superalgebra having the elements of Γ as graded identities.

In case char$F$=0, it is well known that $Id^{sup}(A)$ is completely determined by its multilinear polynomials and let V_n^{sup} denote the space of multilinear polynomials in the variables $y_1, z_1, \dots, y_n, z_n$ (i.e., y_i or z_i appears in each monomial at degree 1). Then the sequence of spaces $\{V_n^{sup} \cap Id^{sup}(A)\}_{n\geq 1}$ determines $Id^{sup}(A)$ and

$$c_n^{sup}(A) = \dim_F \left(\frac{V_n^{sup}}{V_n^{sup} \cap Id^{sup}(A)} \right)$$

is called the n-th graded codimension of A.

The asymptotic behaviour of the graded codimensions plays an important role in the PI-theory of graded algebras. It was shown in [5] that the sequence $\{c_n^{sup}(A)\}_{n\geq 1}$ is exponentially bounded if and only if A satisfies an ordinary polynomial identity.

In [2] it was proved that if A is a finitely generated superalgebra satisfying a polynomial identity, then $\lim_{n\to\infty} \sqrt[n]{c_n^{sup}(A)}$ exists and is a non-negative integer. It is called the superexponent (or $\mathbb{Z}_2$-exponent) of A and is denoted by

$$supexp(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{sup}(A)}.$$

We remark that in [1] and [4] the existence of the G-exponent has been proved for an arbitrary PI-algebra graded by a finite abelian group G.

Now, if $f = f(y_1, \dots, y_n, z_1, \dots, z_m) \in F\langle Y \cup Z\rangle$ we denote by $\langle f\rangle_{T_2}$ the T_2-ideal generated by f. Also, for a set of polynomials $V \subset F\langle Y \cup Z\rangle$ we write $\langle V\rangle_{T_2}$ to indicate the T_2-ideal generated by V.

In PI-theory a prominent role is played by the Capelli polynomial. If S_m is the symmetric group on $\{1, \dots, m\}$, the polynomial

$$\begin{aligned} Cap_m[T, X] =& Cap_m[t_1, \dots, t_m; x_1, \dots, x_{m-1}] \\ =& \sum_{\sigma\in S_m} (\mathrm{sgn}\sigma) t_{\sigma(1)} x_1 t_{\sigma(2)} \cdots t_{\sigma(m-1)} x_{m-1} t_{\sigma(m)} \end{aligned}$$

is the m-th graded Capelli polynomial in the homogeneous variables $t_1, \dots, t_m$ ($x_1, \dots, x_{m-1}$ are arbitrary variables). In particular, $Cap_m[Y, X]$ and $Cap_m[Z, X]$

denote the m-th graded Capelli polynomials in the alternating variables of homogeneous degree zero $y_1, \dots, y_m$ and of homogeneous degree one $z_1, \dots, z_m$, respectively.

Let Cap_m^0 denote the set of 2^{m-1} polynomials obtained from $Cap_m[Y, X]$ by deleting any subset of variables x_i (by evaluating the variables x_i to 1 in every possible way). Similarly, we define by Cap_m^1 the set of 2^{m-1} polynomials obtained from $Cap_m[Z, X]$ by deleting any subset of variables x_i.

If L and M are two natural numbers, we denote the T_2-ideal generated by the polynomials Cap^0_{M+1}, Cap^1_{L+1} by $\Gamma_{M+1,L+1} = \langle Cap^0_{M+1}, Cap^1_{L+1}\rangle_{T_2}$. We also write $\mathcal{U}^{sup}_{M+1,L+1} = \mathrm{supvar}(\Gamma_{M+1,L+1})$.

In this paper we calculate the superexponent of the supervariety $\mathcal{U}^{sup}_{M+1,L+1}$. We shall prove that the superexponent of $\mathcal{U}^{sup}_{M+1,L+1}$ can be explicitly computed from the basic invariants of an upper block triangular matrix algebra and we shall show that

$$(M + L) - 10 \leq \mathrm{supexp}(\mathcal{U}^{\mathrm{sup}}_{M+1,L+1}) \leq (M + L).$$

This paper was inspired by the ordinary case (see [8]) where Mishchenko, Regev and Zaicev proved that, depending on m, the variety corresponding to the Capelli polynomial of rank m has exponent m or $m - 1$ or $m - 2$ or $m - 3$. Moreover, we should mention that a basic tool for proving the results of this paper is a result of Cohen and Regev concerning the generalized-six-square theorem [3] (see also Appendix A of [6]).

2. Computing the superexponent of the supervariety $\mathcal{U}^{sup}_{M+1,L+1}$

Throughout the paper we will denote by F a field of characteristic zero. Recall that, if F is an algebraically closed field, then a simple finite-dimensional superalgebra B_i over F is isomorphic to one of the following algebras (see [7]):

(1) $M_n(F)$ with trivial grading $(M_n(F), 0)$;

(2) $M_{k,l}(F)$ with grading $\left(\begin{pmatrix} F_{11} & 0 \\ 0 & F_{22}\end{pmatrix}, \begin{pmatrix} & F_{12} \\ F_{21} & 0\end{pmatrix}\right)$, where F_{11}, F_{12}, F_{21}, F_{22} are $k \times k$, $k \times l$, $l \times k$ and $l \times l$ matrices respectively, $k \geq 1$ and $l \geq 1$;

(3) $M_n(F \oplus cF)$ with grading $(M_n(F), cM_n(F))$, where $c^2 = 1$.

Also recall that by the Wedderburn–Malcev theorem (see [6, Theorem 3.4.3]), if A is a finite dimensional superalgebra, then $A = B + J$ where B is a maximal semisimple subalgebra and J is the Jacobson radical of A. Moreover, we may assume (see [7]) that B is a superalgebra and B can be decomposed as $B = B_1 \oplus \cdots \oplus B_t$, where $B_1, \ldots, B_t$ are simple superalgebras.

Lemma 1: *Let A be a finite-dimensional superalgebra with Jacobson radical $J = J(A)$ and let $B = B_1 \oplus \cdots \oplus B_k$ be a subsuperalgebra of A, where $B_1, \ldots, B_k$ are finite-dimensional simple superalgebras over F. Let $B_i \cong M_{n_i}(F)$ for $i \in \{i_1, \ldots, i_r\}$, $B_i \cong M_{k_i,l_i}(F)$ for $i \in \{j_1, \ldots, j_s\}$ and $B_i \cong M_{n_i}(F + cF)$ for $i \in \{h_1, \ldots, h_t\}$, where $r, s, t \geq 0$ and $\{i_1, \ldots, i_r\}$, $\{j_1, \ldots, j_s\}$ and $\{h_1, \ldots, h_t\}$ are disjoint subsets of $\{1, \ldots, k\}$ whose union is $\{1, \ldots, k\}$. For each i, $B_i = B_i^{(0)} \oplus B_i^{(1)}$. Let $B^{(j)} = \bigoplus_i B_i^{(j)}$, $j = 0, 1$, so $B = B^{(0)} \oplus B^{(1)}$. Denote $d^{(j)} = dim_F B^{(j)}$. If we suppose that*

$$B_1 J B_2 J \cdots J B_k \neq 0 \tag{1}$$

in A, then the following two properties hold:

- *If $r \geq 1$, then A does not satisfy the graded Capelli identities*

$$Cap_{d^0+s+t+r_0}[Y; X] \quad \text{and} \quad Cap_{d^1+s+t+r_1}[Z; X],$$

  for some r_0, $r_1 \geq 0$ such that $r_0 + r_1 = r - 1$.
- *If $r = 0$, then A does not satisfy the graded Capelli identities*

$$Cap_{d^0+s+t-1}[Y; X] \quad \text{and} \quad Cap_{d^1+s+t-1}[Z; X].$$

Proof. Suppose that $r \geq 1$. It is well known that $J = J^{(0)} \oplus J^{(1)}$ is a homogeneous ideal (see [7]). The inequality (1) implies that there exist $c_1, \ldots, c_{k-1} \in J^{(0)} \cup J^{(1)}$ and $e_i \in B_i$, $i = 1, \ldots, k$, such that

$$e_1 c_1 e_2 c_2 \cdots e_{k-1} c_{k-1} e_k \neq 0, \tag{2}$$

where each $e_i \in B_i^{(0)} \cup B_i^{(1)}$ is a matrix unit of B_i. Moreover, for every $i = 1, \ldots, k$, we have

$$1_i c_i = c_i 1_{i+1} = c_i, \tag{3}$$

where $1_i \in B_i^{(0)}$ is the identity element of B_i.

Notice that if $B_i \cong M_{n_i}(F + cF)$, then there exist $\overline{c}_i$, $\overline{c}_{i-1} \in J^{(0)}$ such that $e_i c_i = e_i \overline{c}_i$ or $e_i c_i = c e_i \overline{c}_i$ and $c_{i-1} e_i = \overline{c}_{i-1} e_i$ or $c_{i-1} e_i = c \overline{c}_{i-1} e_i$. Also, if $B_i \cong M_{k_i,l_i}(F)$, then there exist $\widetilde{e}_i$, $\widehat{e}_i \in B_i^{(0)} \cup B_i^{(1)}$ and $\overline{c}_i$, $\overline{c}_{i-1} \in J^{(0)}$ such that $e_ic_i = \widetilde{e}_i\overline{c}_i$ and $c_{i-1}e_i = \overline{c}_{i-1}\widehat{e}_i$. Thus, by (2), we have that there exist $\overline{c}_1, \ldots, \overline{c}_{k-1} \in J^{(0)} \cup J^{(1)}$ and $\overline{e}_i \in B_i^{(0)} \cup B_i^{(1)}$, $i = 1, \ldots, k$, such that

$$\overline{e}_1\overline{c}_1\overline{e}_2\overline{c}_2 \cdots \overline{e}_{k-1}\overline{c}_{k-1}\overline{e}_k \neq 0, \tag{4}$$

and the number of $\overline{c}_i$ in $J^{(0)}$ is $s + t + r_0$, the number of $\overline{c}_i$ in $J^{(1)}$ is r_1, where r_0, r_1 are fixed non-negative integers such that $r_0 + r_1 = r - 1$.

Now, we consider one of the simple superalgebras $B_i = B_i^{(0)} \oplus B_i^{(1)}$ and we fix a basis of its even component $B_i^{(0)}$ made of matrix units

$$u_1^i, \ldots, u_{d_i^0}^i.$$

We can choose matrix units

$$a_1^i, \ldots, a_{d_i^0+1}^i \in B_i = B_i^{(0)} \oplus B_i^{(1)}$$

such that

$$f_i^0 = f_i^0(u_1^i, \ldots, u_{d_i^0}^i; a_1^i, \ldots, a_{d_i^0+1}^i) = a_1^iu_1^ia_2^i \cdots u_{d_i^0}^ia_{d_i^0+1}^i = b^i \neq 0 \tag{5}$$

and

$$f_i^0(u_{\sigma(1)}^i, \ldots, u_{\sigma(d_i^0)}^i; a_1^i, \ldots, a_{d_i^0+1}^i) = 0, \tag{6}$$

for any non-trivial permutation $\sigma \in S_{d_i^0}$. Since b^i is a matrix unit, there exist b_1^i and b_2^i in $B_i^{(0)} \cup B_i^{(1)}$ such that

$$b_1^ib^ib_2^i = \overline{e}_i.$$

Moreover, since $B_iB_j = 0$ for $i \neq j$, if $a \in B_1 \oplus \cdots \oplus B_k$ or $a = c_j$ with $j \neq i$, by (3) we have

$$b_2^iab_1^{i+1} = 0 \tag{7}$$

and, if $a = c_j$ or $a \in B_h$ with $j \neq i$ and $h \neq i$, we obtain

$$b_1^iab_2^i = 0. \tag{8}$$

Now, we construct a polynomial which is alternating on $d^0+s+t+r_0$ variables of homogeneous degree zero and has a non-zero value on A. Let

$$\begin{aligned} g =& g(y_1^1, \ldots, y_{d_1^0}^1, y_1^2, \ldots, y_{d_2^0}^2, \ldots, y_1^k, \ldots, y_{d_k^0}^k, x_1^1, \ldots, x_{d_1^0}^1, \\ & x_1^2, \ldots, x_{d_2^0}^2, \ldots, x_1^k, \ldots, x_{d_k^0}^k, t_1^{(j)}, \ldots, t_{k-1}^{(j)}, w_1, \ldots, w_{2k}) \\ =& w_1f_1^0w_2t_1^{(j)}w_3f_2^0w_4t_2^{(j)}w_5 \cdots w_{2k-2}t_{k-1}^{(j)}w_{2k-1}f_k^0w_{2k}, \end{aligned}$$

where f_i^0 is the monomial constructed earlier for $B_i^{(0)}$(see (6)), $j \in \{0,1\}$, $t_i^{(0)}$ is a variable of even degree and $t_i^{(1)}$ is a variable of odd degree. Moreover, the number of variables $t_i^{(0)}$ is $s+t+r_0$ and the number of variables $t_i^{(1)}$ is r_1. Then we define

$$\mathrm{Alt}(g)$$

as the polynomial obtained from g by alternating on $y_1^1, \ldots, y_{d_1^0}^1, \ldots, y_1^k, \ldots, y_{d_k^0}^k$ and on $t_i^{(0)}$. Now, we consider the substitution φ such that

$$\varphi(y_j^i) = u_j^i, \quad \varphi(x_j^i) = a_j^i,$$

$$\varphi(t_i^{(j)}) = \overline{c}_i,$$

$$\varphi(w_{2i-1}) = b_1^i, \quad \varphi(w_{2i}) = b_2^i.$$

Hence, from (5), (6), (7), (8) and (3), (4) we get

$$\varphi(\mathrm{Alt}(g)) = \varphi(g) = \overline{e}_1\overline{c}_1\overline{e}_2\overline{c}_2 \cdots \overline{e}_{k-1}\overline{c}_{k-1}\overline{e}_k \neq 0.$$

This last inequality proves that A does not satisfy the graded Capelli identity $Cap_{d^0+s+t+r_0}[Y;X]$.

In a similar way we can construct a polynomial which is alternating on $d^1+s+t+r_1$ variables of odd degree and has a non-zero value on A. Then the first statement of Lemma 1 is proved.

Now let $r=0$. The proof of the second part of this lemma is very similar to that of the first part. Notice that in (4) the number of $\overline{c}_i$ in $J^{(0)}$ is $s+t-1$ and the number of $\overline{c}_i$ in $J^{(1)}$ is 0. Moreover, in the polynomial g the number of variables $t_i^{(0)}$ is $s+t-1$ and the number of variables $t_i^{(1)}$ is 0. Thus the polynomial $\mathrm{Alt}(g)$ is alternating on $d^0+s+t-1$ variables of even degree and has a non-zero value on A. In a similar way we obtain a polynomial which is alternating on $d^1+s+t-1$ variables of odd degree and has a non-zero value on A. Thus the lemma is proved. ∎

Lemma 2: *For any $r, s, t \geq 0$ and for any integers $a_1, \ldots, a_r$, $k_1, l_1, \ldots, k_s, l_s$, $b_1, \ldots, b_t \geq 0$, there exists a finite-dimensional superalgebra $A = B + J$ with maximal semisimple subsuperalgebra B and Jacobson radical $J = J(A)$ such that:*

(1) $B = B_1 \oplus \cdots \oplus B_{r+s+t}$

(2) *All B_i's are simple superalgebras. In particular,*

$$\begin{aligned} B_i \cong & M_{a_i}(F), && \forall i = 1, \dots, r, \\ B_i \cong & M_{k_i, l_i}(F), && \forall i = r+1, \dots, r+s, \\ B_i \cong & M_{b_i}(F + cF), && \forall i = r+s+1, \dots, r+s+t. \end{aligned}$$

(3)

$$\begin{aligned} \text{supexp}(A) = & \dim B = \sum_{i=1}^{r} a_i^2 + \sum_{i=r+1}^{r+s} (k_i + l_i)^2 + 2 \sum_{i=r+s+1}^{r+s+t} b_i^2 \\ = & \dim B^{(0)} + \dim B^{(1)} = d^0 + d^1 \\ = & a_1^2 + \cdots + a_r^2 + (k_{r+1}^2 + l_{r+1}^2) + \cdots + (k_{r+s}^2 + l_{r+s}^2) \\ & + b_{r+s+1}^2 + \cdots + b_{r+s+t}^2 + 2k_{r+1} l_{r+1} + \cdots + 2k_{r+s} l_{r+s} \\ & + b_{r+s+1}^2 + \cdots + b_{r+s+t}^2. \end{aligned}$$

(4)
- *If $r \geq 1$, then $A \in \mathcal{U}^{\text{sup}}_{d^0+s+t+r_0+1, d^1+s+t+r_1+1}$, for some r_0, r_1 such that $r_0 + r_1 = r - 1$.*
- *If $r = 0$, then $A \in \mathcal{U}^{\text{sup}}_{d^0+s+t, d^1+s+t}$.*

Proof. We construct A as a block upper-triangular matrix algebra

$$A = \begin{pmatrix} B_1 & * & & & & & & & * \\ 0 & \ddots & & & & & & & \\ & & B_r & & & & & & \\ & & & B_{r+1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & B_{r+s} & & & \\ & & & & & & B_{r+s+1} & & \\ & & & & & & & \ddots & * \\ 0 & & & & & & & 0 & B_{r+s+t} \end{pmatrix},$$

where the B_i's are the following simple superalgebras:

- $B_i \cong M_{a_i}(F), \quad \forall i = 1, \dots, r,$
- $B_i \cong M_{k_i, l_i}(F), \quad \forall i = r+1, \dots, r+s,$
- $B_i \cong M_{b_i}(F + cF), \quad \forall i = r+s+1, \dots, r+s+t.$

Hence $B = B_1 \oplus \cdots \oplus B_{r+s+t}$ is a semisimple subsuperalgebra of A and the Jacobson radical $J = J(A)$ consists of all block strictly uppertriangular matrices. Then we have immediately (1), (2).

Since $B_1 J B_2 J \cdots J B_{r+s+t} \neq 0$ by [2] we have that the superexponent of A is $\dim B = \dim B^{(0)} + \dim B^{(1)}$. Thus (3) holds.

Now, let $r \geq 1$ and let r_0, r_1 be fixed non-negative integers such that $r_0 + r_1 = r - 1$. We consider the grading of A obtained by requiring the following conditions on elementary matrices:

$$e_{1,a_1+1},\, e_{a_1+1,a_1+a_2+1},\, \ldots,\, e_{a_1+\cdots+a_{r_0-1}+1,a_1+\cdots+a_{r_0}+1} \in J^{(0)},$$

$$e_{a_1+\cdots+a_{r_0}+1,a_1+\cdots+a_{r_0+1}+1},\, \ldots,\, e_{a_1+\cdots+a_{r_0+r_1-1}+1,a_1+\cdots+a_{r_0+r_1}+1} \in J^{(1)}.$$

It follows that, in such grading, any monomial of elements of A containing at least $s + t + r_0 + 1$ elements of $J^{(0)}$ must be zero. Similarly, any monomial of elements of A containing at least $s+t+r_1+1$ elements of $J^{(1)}$ must vanish. We claim that any multilinear polynomial $f^0 = f^0(y_1, \ldots, y_{d^0+s+t+r_0+1}, x_1, x_2, \ldots)$ alternating on $d^0+s+t+r_0+1$ variables of homogeneous degree zero must vanish in A. In fact, by multilinearity we can consider only substitutions $\varphi : y_i \to \overline{y}_i$, $\varphi : x_j \to \overline{x}_j$, such that $\overline{y}_i \in B^{(0)} \cup J^{(0)}$, for $1 \leq i \leq d^0 + s + t + r_0 + 1$. However, since $\dim B^{(0)} = d^0$, if we substitute at least $d^0 + 1$ alternating variables in elements of $B^{(0)}$, the polynomial vanishes. On the other hand, from the above, if we substitute at least $s + t + r_0 + 1$ elements of $J^{(0)}$, we also get that f^0 vanishes in A. The outcome of this is that A satisfies $Cap_{d^0+s+t+r_0+1}[Y; X]$. With similar arguments it is easily seen that any multilinear polynomial $f^1 = f^1(z_1, \ldots, z_{d^1+s+t+r_1+1}, x_1, x_2, \ldots)$ alternating on $d^1 + s + t + r_1 + 1$ variables of degree one must vanish in A. Thus A satisfies $Cap_{d^1+s+t+r_1+1}[Z; X]$ and $A \in \mathcal{U}^{\sup}_{d^0+s+t+r_0+1,d^1+s+t+r_1+1}$, where $r_0 + r_1 = r - 1$.

In case $r = 0$, clearly any monomial of elements of A containing at least $s+t$ elements of $J^{(0)}$ (or $J^{(1)}$, respectively) must be zero, since in this case $J^{s+t} = (0)$. Then, similarly to the previous case, we obtain that $A \in \mathcal{U}^{\sup}_{d^0+s+t,d^1+s+t}$. Thus (4) holds and the lemma is proved. ∎

Let L, M be two natural numbers. Let $A = A^{(0)} \oplus A^{(1)}$ be a generating superalgebra of $\mathcal{U}^{\sup}_{M+1,L+1}$.Then, since $A^{(0)}$ and $A^{(1)}$ satisfy Capelli identities, by [6, Lemma 11.4.1], A satisfies a Capelli identity. Then, by [6, Theorem 11.4.3], we may assume that A is a finitely generated superalgebra and by a theorem of Kemer [7, Theorem 2.2] we may assume that A is a finite-dimensional superalgebra. Since any polynomial alternating on $M+1$ variables of degree zero vanishes in A, we get that $\dim A^{(0)} \leq M$. Similarly, we get that $\dim A^{(1)} \leq L$ and $\operatorname{supexp}(A) \leq \dim A \leq M + L$ (see also [2]). Thus we have the following:

Lemma 3: $\operatorname{supexp}(\mathcal{U}^{\sup}_{M+1,L+1}) \leq M + L$. ∎

Let M and L be fixed. Then, for any integers s, $t \geq 0$, $r \geq 1$ such that $r - 1 = r_0 + r_1$ for some non-negative integers r_0, r_1, we define the set

$$\begin{aligned}\overline{A}_{r,s,t;r_0,r_1} = \{a_1, \ldots, a_r, k_1, l_1, \ldots, k_s, l_s, b_1, \ldots, b_t \in \mathbb{Z}^+ \mid \\ a_1^2 + \cdots + a_r^2 + (k_1^2 + l_1^2) + \cdots (k_s^2 + l_s^2) \\ + b_1^2 + \cdots + b_t^2 + r_0 + s + t \leq M, \\ \text{and } 2k_1 l_1 + \cdots + 2k_s l_s + b_1^2 + \cdots + b_t^2 + r_1 + s + t \leq L\}.\end{aligned}$$

Also, given integers s, $t \geq 0$ $(r = 0)$, we define the set

$$\begin{aligned}\widetilde{A}_{s,t} = \{k_1, l_1, \ldots, k_s, l_s, b_1, \ldots, b_t \in \mathbb{Z}^+ \mid \\ (k_1^2 + l_1^2) + \cdots + (k_s^2 + l_s^2) + b_1^2 + \cdots + b_t^2 + s + t \leq M + 1, \\ \text{and } 2k_1 l_1 + \cdots + 2k_s l_s + b_1^2 + \cdots + b_t^2 + s + t \leq L + 1\}.\end{aligned}$$

Moreover, let

$$\overline{a}_{r,s,t;r_0,r_1} = \max_{a_i,k_i,l_i,b_i \in \overline{A}_{r,s,t;r_0,r_1}} \{a_1^2 + \cdots + a_r^2 + (k_1 + l_1)^2 + \cdots + (k_s + l_s)^2 + 2b_1^2 + \cdots + 2b_t^2\}$$

and

$$\widetilde{a}_{s,t} = \max_{k_i,l_i,b_i \in \widetilde{A}_{s,t}} \{(k_1 + l_1)^2 + \cdots + (k_s + l_s)^2 + 2b_1^2 + \cdots + 2b_t^2\},$$

then we define

$$a_0 = \max\{\overline{a}_{r,s,t;r_0,r_1}, \widetilde{a}_{s,t} \mid r + s + t \leq 11\}.$$

The main result of the paper is the following:

Theorem 4: *If $M \geq L \geq 0$, then*

(1) $\operatorname{supexp}(\mathcal{U}^{\sup}_{M+1,L+1}) = a_0$,

and

(2) $(M + L) - 10 \leq \operatorname{supexp}(\mathcal{U}^{\sup}_{M+1,L+1}) \leq (M + L)$.

Proof. By Lemma 3,

$$\mathrm{supexp}(\mathcal{U}^{\sup}_{M+1,L+1}) \leq M+L.$$

Let $r \geq 1$. Take $a_1,\ldots,a_r,k_1,l_1,\ldots,k_s,l_s,b_1,\ldots,b_t$ in $\overline{A}_{r,s,t;r_0,r_1}$. Then, by Lemma 2, there exists a finite-dimensional superalgebra $A = B + J = B_1 \oplus \cdots \oplus B_{r+s+t} + J$ such that

$$\begin{aligned}\mathrm{supexp}(A) =&\dim B = \dim B^{(0)} + \dim B^{(1)} = d^0 + d^1\\ =&a_1^2 + \cdots + a_r^2 + (k_1^2 + l_1^2) + \cdots + (k_s^2 + l_s^2) + b_1^2 + \cdots + b_t^2\\ &+ 2k_1l_1 + \cdots + 2k_sl_s + b_1^2 + \cdots + b_t^2.\end{aligned}$$

Moreover, $A \in \mathcal{U}^{\sup}_{d^0+s+t+r_0+1,d^1+s+t+r_1+1}$. Since $d^0 + s + t + r_0 \leq M$ and $d^1+s+t+r_1 \leq L$, we have that $d^0+s+t+r_0+1 \leq M+1$, $d^1+s+t+r_1+1 \leq L+1$ and $A \in \mathcal{U}^{\sup}_{M+1,L+1}$. Hence

$$\mathrm{supexp}(\mathcal{U}^{\sup}_{M+1,L+1}) \geq \mathrm{supexp}(A)$$

and, by definition of $\overline{a}_{r,s,t;r_0,r_1}$, we also have that

$$\mathrm{supexp}(\mathcal{U}^{\sup}_{M+1,L+1}) \geq \overline{a}_{r,s,t;r_0,r_1}. \tag{9}$$

Now let $r = 0$. Similarly, we can apply Lemma 2 to a generic element of $\widetilde{A}_{s,t}$ and we get that there exists a finite-dimensional superalgebra $A = B + J = B_1 \oplus \cdots \oplus B_{s+t} + J$ such that

$$\begin{aligned}\mathrm{supexp}(A) =&\dim B = \dim B^{(0)} + \dim B^{(1)} = d^0 + d^1\\ =&(k_1^2 + l_1^2) + \cdots + (k_s^2 + l_s^2) + b_1^2 + \cdots + b_t^2\\ &+ 2k_1l_1 + \cdots + 2k_sl_s + b_1^2 + \cdots + b_t^2.\end{aligned}$$

Moreover, $A \in \mathcal{U}^{\sup}_{d^0+s+t,d^1+s+t}$. Since $d^0+s+t \leq M+1$ and $d^1+s+t \leq L+1$ as above, we get that $A \in \mathcal{U}^{\sup}_{M+1,L+1}$. Hence

$$\mathrm{supexp}(\mathcal{U}^{\sup}_{M+1,L+1}) \geq \mathrm{supexp}(A)$$

and

$$\mathrm{supexp}(\mathcal{U}^{\sup}_{M+1,L+1}) \geq \widetilde{a}_{s,t}. \tag{10}$$

Putting together (9) and (10), we obtain that

$$\mathrm{supexp}(\mathcal{U}^{\sup}_{M+1,L+1}) \geq a_0. \tag{11}$$

By a result of Cohen and Regev [3] (see also [6, Theorem 9.5.4]), we can decompose the pair $(M-5, L-5)$ as a sum of at most six generalized squares in the following sense. There exists $k \leq 6$ such that

$$M-5 = a_1 + \cdots + a_k, \quad L-5 = b_1 + \cdots + b_k, \tag{12}$$

where $(a_i, b_i) = (u_i^2, u_i^2)$ or $(a_i, b_i) = (p_i^2 + q_i^2, 2p_iq_i)$, $i = 1, \ldots, k$. Thus

$$a_1 + \cdots + a_k + k \leq a_1 + \cdots + a_k + 6 = M - 5 + 6 = M + 1, \tag{13}$$

$$b_1 + \cdots + b_k + k \leq b_1 + \cdots + b_k + 6 = L - 5 + 6 = L + 1. \tag{14}$$

Write $k = \overline{r} + \overline{s} + \overline{t}$, where $\overline{r}$ denotes the number of pairs (a_i, b_i) of type $(v_i^2, 0)$, $\overline{s}$ denotes the number of (a_i, b_i) of type $(p_i^2 + q_i^2, 2p_iq_i)$ and $\overline{t}$ the number of (a_i, b_i) of type (u_i^2, u_i^2). If $\overline{r} \geq 1$, we claim that $v_1, \ldots, v_{\overline{r}}, p_1, q_1, \ldots, p_{\overline{s}}, q_{\overline{s}}, u_1, \ldots, u_{\overline{t}} \in \overline{A}_{\overline{r},\overline{s},\overline{t};r_0,r_1}$, with r_0 and r_1 non-negative integers such that $r_0 + r_1 = \overline{r} - 1$. In fact, by (13) and (14),

$$\begin{aligned} v_1^2 + \cdots + v_{\overline{r}}^2 + (p_1^2 + q_1^2) + \cdots + (p_{\overline{s}}^2 + q_{\overline{s}}^2) &+ u_1^2 + \cdots + u_{\overline{t}}^2 + \overline{s} + \overline{t} + r_0 \\ &= a_1 + \cdots + a_k + \overline{s} + \overline{t} + r_0 \\ &\leq a_1 + \cdots + a_k + \overline{s} + \overline{t} + \overline{r} - 1 \\ &= a_1 + \cdots + a_k + k - 1 \leq M, \end{aligned} \tag{15}$$

and

$$\begin{aligned} 2p_1q_1 + \cdots + 2p_{\overline{s}}q_{\overline{s}} + u_1^2 + \cdots + u_{\overline{t}}^2 + \overline{s} + \overline{t} + r_1 & \\ &= b_1 + \cdots + b_k + \overline{s} + \overline{t} + r_1 \\ &\leq b_1 + \cdots + b_k + \overline{s} + \overline{t} + \overline{r} - 1 \\ &= b_1 + \cdots + b_k + k - 1 \leq L. \end{aligned} \tag{16}$$

Moreover, by (12),

$$\begin{aligned} M + L - 10 =& (M-5) + (L-5) \\ =& a_1 + \cdots + a_k + b_1 + \cdots + b_k \\ =& v_1^2 + \cdots + v_{\overline{r}}^2 + (p_1^2 + q_1^2) + \cdots + (p_{\overline{s}}^2 + q_{\overline{s}}^2) + u_1^2 + \cdots + u_{\overline{t}}^2 \\ & + 2p_1q_1 + \cdots 2p_{\overline{s}}q_{\overline{s}} + u_1^2 + \cdots + u_{\overline{t}}^2. \end{aligned} \tag{17}$$

Therefore, from (15), (16) and (17) we obtain that

$$M + L - 10 \leq \overline{a}_{\overline{r},\overline{s},\overline{t};r_0,r_1}. \tag{18}$$

In case $\overline{r}=0$, then $p_1, q_1, \ldots, p_{\overline{s}}, q_{\overline{s}}, u_1, \ldots, u_{\overline{t}} \in \widetilde{A}_{\overline{s},\overline{t}}$, since, by (13) and (14),

(19)
$$\begin{aligned}(p_1^2+q_1^2)+\cdots+(p_{\overline{s}}^2+q_{\overline{s}}^2)+u_1^2+\cdots+u_{\overline{t}}^2+\overline{s}+\overline{t} &= a_1+\cdots+a_k+\overline{s}+\overline{t}\\ &= a_1+\cdots+a_k+k\\ &\leq M+1\end{aligned}$$

and

(20)
$$\begin{aligned}2p_1q_1+\cdots+2p_{\overline{s}}q_{\overline{s}}+u_1^2+\cdots+u_{\overline{t}}^2+\overline{s}+\overline{t}+r_1 =& b_1+\cdots+b_k+\overline{s}+\overline{t}\\ =& b_1+\cdots+b_k+k\leq L+1.\end{aligned}$$

Moreover, by (12),

$$\begin{aligned}M+L-10 =& (M-5)+(L-5)\\ =& a_1+\cdots+a_k+b_1+\cdots+b_k\\ =& (p_1^2+q_1^2)+\cdots+(p_{\overline{s}}^2+q_{\overline{s}}^2)+u_1^2+\cdots+u_{\overline{t}}^2\\ &+2p_1q_1+\cdots+2p_{\overline{s}}q_{\overline{s}}+u_1^2+\cdots+u_{\overline{t}}^2.\end{aligned} \tag{21}$$

Therefore, from (19), (20) and (21) we get

$$M+L-10\leq \widetilde{a}_{\overline{s},\overline{t}}. \tag{22}$$

Since $\overline{r}+\overline{s}+\overline{t}=k\leq 6$, putting together (18) and (22) we obtain that

$$M+L-10\leq a_0. \tag{23}$$

Hence, from (11) and (23) we have that

$$\operatorname{supexp}(\mathcal{U}_{M+1,L+1}^{\sup})\geq a_0\geq M+L-10. \tag{24}$$

Putting together Lemma 3 and (24) we get the proof of the second part of the theorem.

By [6, Lemma 11.4.1, Theorem 11.4.3] and [7, Theorem 2.2], $\mathcal{U}_{M+1,L+1}^{\sup}$ is generated by a finite-dimensional superalgebra $A=A^{(0)}\oplus A^{(1)}$ and

$$\operatorname{supexp}(\mathcal{U}_{M+1,L+1}^{\sup})=\operatorname{supexp}(A).$$

Moreover (see [2]), we can decompose $A=B+J$, where $J=J(A)$ is the Jacobson radical of A and $B=B_1\oplus\cdots\oplus B_k$ is a direct sum of simple superalgebras with

$$B_1JB_2J\cdots JB_k\neq 0.$$

We have

$$\mathrm{supexp}(A) = d^0 + d^1,$$

where $d^0 = \dim B^{(0)}$ and $d^1 = \dim B^{(1)}$.

Let r be the number of B_i's isomorphic to $M_{a_i}(F)$, s the number of B_i's isomorphic to $M_{k_i,l_i}(F)$ and t the number of B_i's isomorphic to $M_{b_i}(F + cF)$, $r + s + t = k$.

Suppose first that $r \geq 1$. Then by Lemma 1, A does not satisfy the graded Capelli identities $Cap_{d^0+s+t+r_0}[Y;X]$ and $Cap_{d^1+s+t+r_1}[Z;X]$, where r_0 and r_1 are suitable non-negative integers such that $r_0+r_1 = r-1$. Since $A \in \mathcal{U}^{\sup}_{M+1,L+1}$ it follows that

$$d^0 + s + t + r_0 < M + 1, \quad d^1 + s + t + r_1 < L + 1. \tag{25}$$

Then

$$d^0 + s + t + r_0 \leq M, \quad d^1 + r_1 \leq L.$$

It follows that

$$(d^0 + d^1) + s + t + (r_0 + r_1) \leq M + L,$$

i.e.,

$$\mathrm{supexp}(A) + s + t + r - 1 \leq M + L.$$

Hence

$$\mathrm{supexp}(A) \leq M + L + 1 - (s + t + r).$$

Now, if $k = s + t + r > 11$ then

$$\mathrm{supexp}(A) \leq M + L + 1 - (s + t + r) < M + L + 1 - 11 = M + L - 10,$$

which contradicts the first statement of the theorem.

If $k = s + t + r \leq 11$ then, from (25), it follows that

$$d^0 + s + t + r_0 \leq M, \quad d^1 + s + t + r_1 \leq L,$$

with $r_0 + r_1 = r - 1$. Hence $a_1, \ldots, a_r, k_1, l_1, \ldots k_s, l_s, b_1, \ldots, b_t \in \overline{A}_{r,s,t;r_0,r_1}$. Thus, in case $r \geq 1$,

$$\mathrm{supexp}(A) \leq \overline{a}_{r,s,t;r_0,r_1} \leq a_0. \tag{26}$$

Suppose now that $r = 0$; then by Lemma 1, A does not satisfy the graded Capelli identities $Cap_{d^0+s+t-1}[Y;X]$ and $Cap_{d^1+s+t-1}[Z;X]$. Thus, since $A \in \mathcal{U}^{\sup}_{M+1,L+1}$, it follows that

$$d^0 + s + t - 1 < M + 1, \quad d^1 + s + t - 1 < L + 1. \tag{27}$$

Then

$$d^0 + s + t - 1 \leq M, \quad d^1 \leq L,$$

and we get

$$(d^0 + d^1) + s + t - 1 \leq M + L.$$

It follows that

$$\operatorname{supexp}(A) + s + t - 1 \leq M + L,$$

and so

$$\operatorname{supexp}(A) \leq M + L + 1 - (s + t).$$

Now, if $k = s + t > 11$ then

$$\operatorname{supexp}(A) \leq M + L + 1 - (s + t) < M + L + 1 - 11 = M + L - 10,$$

which contradicts the first statement of the theorem.

If $k = s + t \leq 11$ then, from (27), it follows that

$$d^0 + s + t \leq M + 1, \quad d^1 + s + t \leq L + 1.$$

Hence $k_j,\ l_j,\ b_h \in \widetilde{A}_{s,t}$, for all $j = 1, \ldots, s,\ h = 1, \ldots, t$. Thus

$$\operatorname{supexp}(A) \leq \widetilde{a}_{s,t} \leq a_0. \tag{28}$$

From (26) and (28) we obtain

$$\operatorname{supexp}(A) \leq a_0, \tag{29}$$

and by comparing (11) and (29) we get the second statement of the theorem. ∎

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