ISRAEL JOURNAL OF MATHEMATICS **196** (2013), 1–31
DOI: 10.1007/s11856-012-0139-4

PRESENTATIONS OF SCHÜTZENBERGER GROUPS OF MINIMAL SUBSHIFTS*

BY

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* Research funded by the European Regional Development Fund, through the programme COMPETE, by the Portuguese Government through Centro de Matemática da Universidade do Porto, Centre for Mathematics of the University of Coimbra, and FCT – Fundação para a Ciência e a Tecnologia, under the projects PEst-C/MAT/UI0144/2011 and PEst-C/MAT/UI0324/2011, and by the FCT project PTDC/MAT/65481/2006, within the framework of the programmes COMPETE and FEDER.
Received January 9, 2010 and in revised form January 31, 2012

ABSTRACT

In previous work, the first author established a natural bijection between minimal subshifts and maximal regular $\mathcal{J}$-classes of free profinite semigroups. In this paper, the Schützenberger groups of such $\mathcal{J}$-classes are investigated, in particular in respect to a conjecture proposed by the first author concerning their profinite presentation. The conjecture is established for all non-periodic minimal subshifts associated with substitutions. It entails that it is decidable whether a finite group is a quotient of such a profinite group. As a further application, the Schützenberger group of the $\mathcal{J}$-class corresponding to the Prouhet–Thue–Morse subshift is shown to admit a somewhat simpler presentation, from which it follows that it has rank three, and that it is non-free relatively to any pseudovariety of groups.

1. Introduction

In recent years, several results on closed subgroups of free profinite semigroups have appeared in the literature [3, 4, 6, 25, 28, 9]. The first author explored a link between symbolic dynamics and free profinite semigroups that allowed him to show, for several classes of maximal subgroups of free profinite semigroups, all associated with minimal subshifts [3, 4], that they are free profinite groups. Rhodes and Steinberg [25] proved that the closed subgroups of free profinite semigroups are precisely the projective profinite groups. Without using ideas from symbolic dynamics, Steinberg proved that the Schützenberger group of the minimal ideal of the free profinite semigroup over a finite alphabet with at least two letters is a free profinite group with infinite countable rank [28]. The same result holds for the Schützenberger group of the regular $\mathcal{J}$-class associated to a non-periodic irreducible sofic subshift [9]; the proof is based on the techniques of [28] and on the conjugacy invariance of the group for arbitrary subshifts [8].

In this paper, we investigate the minimal subshift associated with the iteration of a substitution φ over a finite alphabet A and the Schützenberger group $G(\varphi)$ of the corresponding $\mathcal{J}$-class, $J(\varphi)$, of the free profinite semigroup on A. A minimal subshift can be naturally associated with the substitution φ if and only if φ is weakly primitive [3, Theorem 3.7]. Since weakly primitive substitutions are primitive on the subalphabet consisting of the letters that do not eventually disappear under iteration of the substitution, we will stick in this paper to the more familiar setting of primitive substitutions [13].

A primitive substitution always admits a so-called **connection**, which is a special two-letter block ba of the subshift. Provided φ is an encoding of bounded delay, from the set X of return words for ba, which constitute a finite set, it is shown in [3] that one can then obtain a generating set for a certain maximal subgroup H of $J(\varphi)$ by cancelling the prefix b, adding the same letter as a suffix, and applying the idempotent (profinite) iterate φ^ω. In a lecture given at the *Fields Workshop on Profinite Groups and Applications* (Carleton University, August 2005), the first author proposed, as a problem, a natural profinite presentation for $G(\varphi)$, namely

$$\langle X \mid \Phi^\omega(x) = x \ (x \in X)\rangle, \tag{1.1}$$

where Φ is a continuous endomorphism of the free profinite group on a suitable finite alphabet X that encodes the action of a finite power of φ which acts on the semigroup freely generated by X.

By a result of Lubotzky and Kovács [20], every finitely generated projective profinite group has a finite presentation as a profinite group, and indeed a presentation of the form (1.1) for some continuous endomorphism Φ of the profinite group freely generated by X. Hence, by the previously mentioned result of Rhodes and Steinberg, every finitely generated closed subgroup of a free profinite semigroup has such a presentation. But, to be able to use a presentation of the form (1.1), for instance to determine whether a given finite group is a (continuous) homomorphic image of the profinite group so presented, one needs to be able to verify the relations in a finite group, which imposes some computability requirements on Φ. The problem proposed by the first author in 2005 already addressed this concern, proposing a suitable choice for Φ.

In this paper, we establish the conjecture in full generality, that is without any further restrictions on the (weakly) primitive substitution φ other than being non-periodic (Theorem 6.2, which is our main theorem), thereby showing that it entails the decidability of whether a finite group is a continuous homomorphic image of $G(\varphi)$ (Corollary 3.3).[1] The proof of the conjecture depends on a key result from symbolic dynamics due to Mossé [22, 23] (see [13, Subsection 7.2.1] for its significance and history). Its need had been previously avoided in [3] using the bounded delay encoding condition, which is fulfilled in the case of substitutions that induce automorphisms of the free group.

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[1] It is worth noting that it is decidable whether a given primitive substitution generates a periodic subshift [24, 16].

The case of a **proper** substitution, such that the images of all letters start with the same letter and end with the same letter, has played a special role both in symbolic dynamics [12] and in the connections with free profinite semigroups [6, 3]. The former reference shows that every subshift generated by a primitive substitution is conjugate to a subshift generated by a proper primitive substitution, which can be effectively computed. Since conjugate minimal subshifts have isomorphic Schützenberger groups [8], it is worth considering the special case of subshifts generated by proper primitive substitutions, whose Schützenberger group we show to admit the presentation (1.1) with X the original alphabet and Φ the original substitution, provided the subshift is non-periodic (Theorem 6.4). This gives an alternative approach for the main theorem and its decidability consequences.

The Prouhet–Thue–Morse infinite word and the corresponding subshift are among the most studied in the literature [13]. They are generated by the substitution $\tau(a) = ab$, $\tau(b) = ba$. From the main theorem, we deduce that the profinite group $G(\tau)$ admits a related profinite presentation with three generators and three relations (Theorem 7.4). We deduce that $G(\tau)$ cannot be relatively free with respect to any pseudovariety of groups (Theorem 7.6). This answers in a very strong sense the question raised by the first author as to whether this profinite group is free [3]. In the same paper there is already an argument to reduce the proof of this fact to showing that the Schützenberger group $G(\tau)$ has rank three. From the same simpler presentation, we do prove that this group has rank three (Theorem 7.7).

We also consider the only other type of example in the literature of a non-free Schützenberger group $G(\varphi)$ of a subshift defined by a substitution, illustrated by the substitution $\varphi(a) = ab$, $\varphi(b) = a^3b$ [3, Example 7.2], which is proper. For this group, again we prove that it is not free relatively to any pseudovariety of groups (Theorem 7.2).

The paper is organized as follows. Section 2 discusses presentations of profinite semigroups. Section 3 shows how certain presentations can be used to obtain decidability results, which is our main motivation for considering profinite presentations. Section 4 introduces the necessary background and terminology on symbolic dynamics. The result of B. Mossé and its consequence that a power of any non-periodic primitive substitution φ induces an automorphism of a suitable maximal subgroup of $J(\varphi)$ (Theorem 5.6) are presented in Section 5.

Section 6 contains the main theorem and its version for proper primitive substitutions, as well as the connections between the two. Section 7 is dedicated to applications of the main theorems and Section 8 concludes with some open problems suggested by this work.

We indicate [4, 26] as supporting references on pseudovarieties and free profinite semigroups, and [18, 13] for symbolic dynamics.

2. Presentations of pro-V semigroups

For a homomorphism $\psi : S \to U$ between semigroups, we denote by $\operatorname{Ker}\psi$ the set of all pairs (s_1, s_2) of elements of S such that $\psi(s_1) = \psi(s_2)$.

It can be easily checked that an equivalence relation on a compact space is open (respectively, closed, clopen) if so are its classes. In particular, an equivalence relation on such a space is open if and only if it is clopen. A congruence on a profinite semigroup S is said to be **admissible** if it is the intersection of open congruences. In other words, a congruence ρ is admissible if and only if it is closed and the quotient S/ρ is profinite. Thus, the admissible congruences are the kernels of continuous homomorphisms into profinite semigroups. Since the intersection of admissible congruences is admissible, for every relation $R \subseteq S \times S$ there is a smallest admissible congruence containing R, which we call the **admissible congruence generated** by R. In the case of a profinite group, it turns out that a congruence is admissible if and only if it is closed [27, Proposition 2.2.1(a)]. See [26, Section 3.1] for further details, although we prefer not to call profinite an admissible congruence on a profinite semigroup S since every closed congruence is a profinite subsemigroup of the product $S \times S$, but not every closed congruence is admissible.

Throughout this section, we let V be a pseudovariety of semigroups. Consider a set X and a binary relation R on the pro-V semigroup $\overline{\Omega}_X\mathsf{V}$ freely generated by X [4]. The quotient of $\overline{\Omega}_X\mathsf{V}$ by the admissible congruence generated by R is a pro-V semigroup [4, Proposition 3.7] which is said to admit the **V-presentation** $\langle X \mid R\rangle_{\mathsf{V}}$. In this paper, we are interested in the cases where V is either G, the pseudovariety of all finite groups, or S, the pseudovariety of all finite semigroups, the latter serving sometimes as a convenient way to deal with the former.

We recall that the monoid $\operatorname{End} S$ of continuous endomorphisms of a finitely generated profinite semigroup S is profinite for the pointwise convergence topology, which coincides with the compact-open topology [17, Proposition 1]. For this reason, for the remainder of the paper we only consider finite generating sets. Thus, for φ in the profinite monoid $\operatorname{End}\overline{\Omega}_X\mathsf{S}$, we may consider the idempotent continuous endomorphism φ^ω.

Consider a pro-V semigroup T and an onto continuous homomorphism π from $\overline{\Omega}_X\mathsf{V}$ onto T, where X is an arbitrary set. Let φ be a continuous endomorphism of T. By the universal property of $\overline{\Omega}_X\mathsf{V}$, there is at least one continuous endomorphism Φ of $\overline{\Omega}_X\mathsf{V}$ such that Diagram (2.1) commutes. Call such an endomorphism a **lifting of φ via π.**

$$\begin{array}{ccc} \overline{\Omega}_X\mathsf{V} & \xrightarrow{\Phi} & \overline{\Omega}_X\mathsf{V} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ T & \xrightarrow{\varphi} & T \end{array} \tag{2.1}$$

Remark 2.1: If φ is an automorphism of T then $\pi\circ\Phi^\omega=\pi$.

Proof. The facts that Diagram (2.1) commutes and π is continuous entail the equality $\pi\circ\Phi^\omega=\varphi^\omega\circ\pi$. On the other hand, φ^ω is the identity on T because φ is an automorphism of T. ∎

Suppose now that φ is an automorphism of the pro-V semigroup T. Put

$$R=\{(\Phi^\omega(x),x):x\in X\}$$

and let ρ be the admissible congruence on $\overline{\Omega}_X\mathsf{V}$ generated by R. From Remark 2.1 it follows that $R\subseteq\operatorname{Ker}\pi$, which yields $\rho\subseteq\operatorname{Ker}\pi$. If $\rho=\operatorname{Ker}\pi$, then

$$\langle X\mid\Phi^\omega(x)=x\ (x\in X)\rangle_{\mathsf{V}} \tag{2.2}$$

is a presentation of T. Note also that $u\ \rho\ \Phi^\omega(u)$ for every $u\in\overline{\Omega}_X\mathsf{S}$ since ρ is a closed congruence containing R. It follows that $\operatorname{Ker}\Phi^\omega\subseteq\rho$. Conversely, we have $R\subseteq\operatorname{Ker}\Phi^\omega$ since Φ^ω is idempotent, which entails $\rho\subseteq\operatorname{Ker}\Phi^\omega$. We have thus shown that $\rho=\operatorname{Ker}\Phi^\omega$.

LEMMA 2.2: *Let T be a pro-V semigroup and suppose that there is a commutative diagram (2.1) of continuous homomorphisms, where π is onto and φ is an automorphism of T. If $\operatorname{Ker}\pi\subseteq\operatorname{Ker}\Phi^\omega$, then T admits the presentation $\langle X\mid R\rangle_{\mathsf{V}}$.*

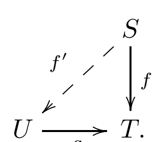
Proof. By Remark 2.1, we have $\operatorname{Ker}\pi \supseteq \operatorname{Ker}\Phi^\omega$. Hence, if $\operatorname{Ker}\pi \subseteq \operatorname{Ker}\Phi^\omega$, then $T \simeq \overline{\Omega}_X\mathsf{V}/\operatorname{Ker}\Phi^\omega = \overline{\Omega}_X\mathsf{S}/\rho = \langle X \mid R\rangle_{\mathsf{V}}$. ∎

The group analogue of Lemma 2.2 involving the group kernel, which is just a translation in the language of profinite group theory of the lemma, also follows from a result of Lubotzky [20, Proposition 1.1], who presents a proof attributed to L. Kovács. The same proof can also be found in the second edition of [27], namely by combining Lemma C.1.5 and Example C.1.6.

Let W be a subpseudovariety of V. For a pro-W semigroup, there is a simple relationship between V-presentations and W-presentations. If $\lambda : \overline{\Omega}_X\mathsf{V} \to S$ is a continuous homomorphism onto a pro-W semigroup, then $\lambda = \lambda' \circ q$, where $q : \overline{\Omega}_X\mathsf{V} \to \overline{\Omega}_X\mathsf{W}$ is the canonical homomorphism and $\lambda' : \overline{\Omega}_X\mathsf{W} \to S$ is a continuous homomorphism. Let $u, v \in \overline{\Omega}_X\mathsf{V}$. It is routine to check that if $\operatorname{Ker}\lambda$ is the admissible congruence on $\overline{\Omega}_X\mathsf{V}$ generated by $R \subseteq \overline{\Omega}_X\mathsf{V} \times \overline{\Omega}_X\mathsf{V}$, then $\operatorname{Ker}\lambda'$ is the admissible congruence on $\overline{\Omega}_X\mathsf{W}$ generated by $(q \times q)(R)$. We thus have the following simple observation, which we record here for later reference.

LEMMA 2.3: *Let W be a subpseudovariety of V. If the pro-W semigroup S admits the presentation $\langle X \mid R\rangle_{\mathsf{V}}$, then it also admits the presentation $\langle X \mid (q \times q)(R)\rangle_{\mathsf{W}}$.* ∎

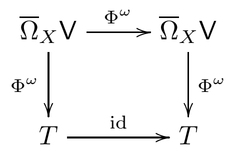
We say that a pro-V semigroup S is **V-projective** if, whenever T and U are pro-V semigroups and $f : S \to T$ and $g : U \to T$ are continuous homomorphisms with g onto, there is some continuous homomorphism $f' : S \to U$ such that the following diagram commutes:



PROPOSITION 2.4: *The following are equivalent for a pro-V semigroup S and a finite set X:*

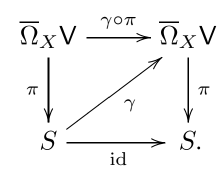
(1) *S admits a presentation of the form* (2.2) *for some continuous endomorphism Φ of $\overline{\Omega}_X\mathsf{V}$;*
(2) *S is V-projective and X-generated;*
(3) *S is a retract of $\overline{\Omega}_X\mathsf{V}$.*

Proof. (1) ⇒ (3) Let Φ be a continuous endomorphism of $\overline{\Omega}_X\mathsf{V}$ and denote by T the image of the retraction Φ^ω. It suffices to establish that T admits the presentation (2.2). For this purpose, we apply the general setting of this section to the following commutative diagram:



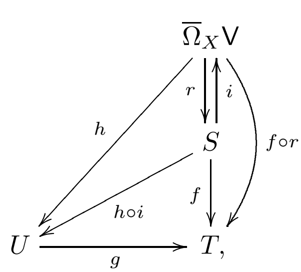
From Lemma 2.2 we deduce that indeed T admits the presentation (2.2).

(2) ⇒ (1) Let $\pi : \overline{\Omega}_X\mathsf{V} \to S$ be an onto continuous homomorphism. Since S is V-projective, there is a continuous homomorphism $\gamma : S \to \overline{\Omega}_X\mathsf{V}$ such that $\pi \circ \gamma$ is the identity on S. Consider the diagram



Since $\gamma \circ \pi$ is idempotent and $\operatorname{Ker}\pi \subseteq \operatorname{Ker}(\gamma\circ\pi)$, Lemma 2.2 yields that S admits the presentation $\langle X \mid (\gamma\circ\pi)^\omega(x) = x\ (x \in X)\rangle_{\mathsf{V}}$.

(3) ⇒ (2) Suppose that the continuous homomorphism $r : \overline{\Omega}_X\mathsf{V} \to S$ is a retraction. Let T and U be pro-V semigroups and let $f : S \to T$ and $g : U \to T$ be continuous homomorphisms with g onto. Denote by i the inclusion mapping $S \to \overline{\Omega}_X\mathsf{V}$. Then we have the following diagram



where the existence of the continuous homomorphism h such that the outer triangle commutes follows from the universal property of the free pro-V semigroup $\overline{\Omega}_X\mathsf{V}$. Since $r \circ i$ is the identity on S, we deduce that $g \circ h \circ i = f \circ r \circ i = f$, and so we may take $f' = h \circ i$. ∎

Combining Proposition 2.4 with the fact that closed subgroups of a free profinite semigroup are G-projective [25], we obtain the following result.

Corollary 2.5: *Every finitely generated closed subgroup of a free profinite semigroup admits a presentation of the form*

$$\langle X \mid \Phi^\omega(x) = x \ (x \in X)\rangle_{\mathsf{G}} \tag{2.3}$$

for some continuous endomorphism Φ *of* $\overline{\Omega}_X\mathsf{G}$. ∎

The next result provides a method to drop relations in such presentations corresponding to superfluous generators.

For a profinite semigroup S and a subset X, the notation $\overline{\langle X\rangle}$ stands for the closed subsemigroup of S generated by X.

Proposition 2.6: *Let* Φ *be a continuous endomorphism of* $\overline{\Omega}_X\mathsf{V}$*,* x_0 *an element of the finite set* X*, and* $Y = X \setminus \{x_0\}$*. Suppose that* $w \in \overline{\langle Y\rangle}$ *is such that* $\Phi^\omega(x_0) = \Phi^\omega(w)$ *and let* r *be the unique continuous endomorphism of* $\overline{\Omega}_X\mathsf{V}$ *that fixes each* $y \in Y$ *and maps* x_0 *to* w*. Then the pro-*V *semigroup presented by*

$$\langle X \mid w = x_0,\, \Phi^\omega(x) = x \ (x \in X)\rangle_{\mathsf{V}} \tag{2.4}$$

also admits the presentation

$$\langle Y \mid \Psi^\omega(y) = y \ (y \in Y)\rangle_{\mathsf{V}}, \tag{2.5}$$

where $\Psi = r \circ \Phi$*.*

Proof. Note that we may add in the presentation (2.5) the generator x_0 and the relation $w = x_0$ without changing the pro-V semigroup thus presented.

Let θ be the admissible congruence on $\overline{\Omega}_X\mathsf{V}$ generated by the relation $w = x_0$. Since $r(u)\ \theta\ u$ for every $u \in \overline{\Omega}_X\mathsf{V}$, we conclude that $\Psi(v) = r(\Phi(v))\ \theta\ \Phi(v)$ whenever $v \in \overline{\Omega}_X\mathsf{V}$.

Let ρ and σ be the admissible congruences on $\overline{\Omega}_X\mathsf{V}$ generated by the relation $w = x_0$ together with, respectively, the relations $\Phi^\omega(x) = x$ $(x \in X)$ and $\Psi^\omega(y) = y$ $(y \in Y)$. To complete the proof, it suffices to show that $\rho = \sigma$. For this purpose, in view of the preceding paragraph, it remains to show that $\Phi^\omega(x_0)\ \sigma\ x_0$. Indeed, we have

$$\Phi^\omega(x_0) = \Phi^\omega(w)\ \theta\ \Psi^\omega(w)\ \sigma\ w\ \sigma\ x_0,$$

which gives the desired relation since $\theta \subseteq \sigma$. ∎

Note that, with the same proof, we could relax the hypothesis $\Phi^\omega(x_0) = \Phi^\omega(w)$ to the relation $\Phi^\omega(x_0)\ \theta\ \Phi^\omega(w)$.

3. Decidability

For a set X, denote by $\mathcal{T}(X)$ the semigroup of all full transformations of X. The following lemma will be useful. As has been observed by the referee, it can be seen as an application of Yoneda's Lemma but we prefer to give an elementary proof.

LEMMA 3.1: *Let A be a finite set, V be a pseudovariety of semigroups, $\varphi \in \operatorname{End}\overline{\Omega}_A\mathsf{V}$, and S a semigroup from V. Consider the transformation $\varphi_S \in \mathcal{T}(S^A)$ defined by $\varphi_S(f) = \hat{f}\circ\varphi|_A$, where $\hat{f}$ is the unique extension of $f \in S^A$ to a continuous homomorphism $\overline{\Omega}_A\mathsf{V} \to S$. Then the correspondence*

$$\operatorname{End}\overline{\Omega}_A\mathsf{V} \to \mathcal{T}(S^A)$$
$$\varphi \mapsto \varphi_S$$

is a continuous anti-homomorphism. In particular, we have $(\varphi^\omega)_S = (\varphi_S)^\omega$.

Proof. Let $\varphi,\psi \in \operatorname{End}\overline{\Omega}_A\mathsf{V}$ and $f \in S^A$. Since $\widehat{\hat{f}\circ\varphi|_A} = \hat{f}\circ\varphi$, we obtain the following chain of equalities:

$$(\varphi\circ\psi)_S(f) = \hat{f}\circ\varphi\circ\psi|_A = \widehat{\hat{f}\circ\varphi|_A}\circ\psi|_A = \psi_S(\hat{f}\circ\varphi|_A) = \psi_S\circ\varphi_S(f),$$

which proves that our mapping is an anti-homomorphism. To prove that it is continuous, consider a net limit $\varphi = \lim\varphi_i$ in $\operatorname{End}\overline{\Omega}_A\mathsf{V}$. Then, for every $f \in \mathcal{T}(S^A)$ and every $a \in A$, we may perform the following computation:

$$\begin{aligned}\varphi_S(f)(a) = \hat{f}(\varphi(a)) = \hat{f}\big((\lim\varphi_i)(a)\big) &= \hat{f}(\lim\varphi_i(a))\\ &= \lim\hat{f}(\varphi_i(a)) = \lim(\varphi_i)_S(f)(a),\end{aligned}$$

which yields the desired equality $\varphi_S = \lim(\varphi_i)_S$. ∎

The following result will be useful to draw structural and computational information about presentations of the form (2.2). To state it, we require some further terminology. For a semigroup S, we say that a mapping $f \in S^A$ is a **generating mapping** if $f(A)$ generates S. Given a pseudovariety of semigroups V, a subpseudovariety W, and an endomorphism φ of $\overline{\Omega}_A\mathsf{V}$, let φ_{W} be the unique continuous endomorphism of $\overline{\Omega}_A\mathsf{W}$ such that $\varphi_{\mathsf{W}}\circ p = p\circ\varphi$, where

$p : \overline{\Omega}_A \mathsf{V} \to \overline{\Omega}_A \mathsf{W}$ is the canonical projection. In particular, if $\varphi \in \operatorname{End} \overline{\Omega}_A \mathsf{W}$, then $\varphi_{\mathsf{W}} = \varphi$.

Proposition 3.2: *Let* V *and* W *be pseudovarieties of semigroups such that* $\mathsf{W} \subseteq \mathsf{V}$. *Let* A *be a finite alphabet and let* φ *be a continuous endomorphism of* $\overline{\Omega}_A \mathsf{V}$. *The following are equivalent for an arbitrary semigroup* S *from* W*:*

(1) *S is a continuous homomorphic image of the semigroup presented by*

$$\langle A \mid \varphi_{\mathsf{W}}^{\omega}(a) = a \ (a \in A) \rangle_{\mathsf{W}}; \tag{3.1}$$

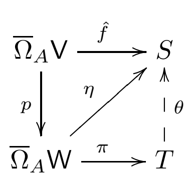
(2) *there is some generating mapping* $f : A \to S$ *and some integer* n *such that* $1 \leq n \leq |S^A|$ *and* $\varphi_S^n(f) = f$;

(3) *there is some generating mapping* $f : A \to S$ *and some integer* n *such that* $\varphi_S^n(f) = f$.

Proof. Let T be the profinite semigroup defined by the presentation (3.1) and consider the natural homomorphisms $p : \overline{\Omega}_A \mathsf{V} \to \overline{\Omega}_A \mathsf{W}$ and $\pi : \overline{\Omega}_A \mathsf{W} \to T$.

We begin by proving (1) $\Rightarrow$ (2). Suppose that $\theta : T \to S$ is an onto continuous homomorphism. Consider the mapping $f = \theta \circ \pi \circ p|_A \in S^A$, whose unique continuous homomorphic extension $\hat{f} : \overline{\Omega}_A \mathsf{V} \to S$ is the mapping $\theta \circ \pi \circ p$. Since $\varphi_{\mathsf{W}} \circ p = p \circ \varphi$, we deduce that $\varphi_S^k(f) = \theta \circ \pi \circ \varphi_{\mathsf{W}}^k \circ p|_A$ for every $k \geq 0$, where we write φ^0 and φ_{W}^0 for the identity mappings on $\overline{\Omega}_X \mathsf{V}$ and $\overline{\Omega}_X \mathsf{W}$, respectively. Hence, for every $a \in A$, the following equalities hold: $\varphi_S^{\omega}(f)(a) = \theta \circ \pi \circ \varphi_{\mathsf{W}}^{\omega}(a) = \theta \circ \pi \circ \varphi_{\mathsf{W}}^0(a) = \varphi_S^0(f)(a) = f(a)$. We have thus proved that $\varphi_S^{\omega}(f) = f$. As φ_S is a transformation of the set S^A, the successive iterates $f, \varphi_S(f), \varphi_S^2(f), \ldots, \varphi_S^{|S^A|}(f)$ cannot all be distinct and $\varphi_S^{\omega}(f)$ must be found in the sequence on the first repeated point or between it and its first repetition. Hence, the equality $\varphi_S^{\omega}(f) = f$ implies that $\varphi_S^n(f) = f$ for some integer n such that $1 \leq n \leq |S^A|$.

The implication (2) $\Rightarrow$ (3) being trivial, it remains to prove the implication (3) $\Rightarrow$ (1). It suffices to show that $\hat{f}$ factors through $\pi \circ p$. Since $S \in \mathsf{W}$, $\hat{f}$ factors through p, and we have the following commutative diagram, where the existence of the dashed arrow θ is yet to be established:



Thus, it is enough to verify that, for every $a \in A$, $\eta(\varphi_{\mathsf{W}}^{\omega}(a)) = \eta(a)$. Taking into account the definition of p, the desired equality is equivalent to $\eta(\varphi_{\mathsf{W}}^{\omega}(p(a))) = \eta(p(a))$. In view of $\varphi_{\mathsf{W}} \circ p = p \circ \varphi$ and $\eta \circ p = \hat{f}$, this translates into the equality $\hat{f}(\varphi^{\omega}(a)) = \hat{f}(a)$. Indeed, by hypothesis, we have $\varphi_S^n(f) = f$ for some n, hence f is fixed by all powers of φ_S^n and, therefore, also by $\varphi_S^{\omega} = (\varphi_S^n)^{\omega}$. ∎

We say that a profinite semigroup S is **decidable** if there is an algorithm to determine, for a given finite semigroup T, whether there is a continuous homomorphism from S onto T. For instance, if V is a pseudovariety of semigroups and A is a finite set, then $\overline{\Omega}_A\mathsf{V}$, the pro-$\mathsf{V}$ semigroup freely generated by A, is decidable if and only if it is decidable whether a finite A-generated semigroup belongs to V. Thus, the pseudovariety V has a decidable membership problem if and only if all finitely generated free pro-V semigroups are decidable.

The following immediate application of Proposition 3.2 could be stated, and essentially proved in the same way, for much more general presentations. To avoid introducing further notation, we stick here to the type of presentations in which we are mostly interested.

Corollary 3.3: *Let φ be an endomorphism of the free group $FG(A)$ on a finite set A and let $\hat{\varphi}$ be its unique extension to a continuous endomorphism of $\overline{\Omega}_A\mathsf{G}$. Then the profinite group presented by $\langle A \mid \hat{\varphi}^{\omega}(a) = a \ (a \in A)\rangle_{\mathsf{G}}$ is decidable.* ∎

4. Preliminaries on symbolic dynamics

Let A be a finite alphabet. We denote by A^+ the free semigroup on A. A **code** is a nonempty subset of A^+ that generates a free subsemigroup.

The subsemigroup of $\overline{\Omega}_A\mathsf{S}$ generated by A is a free semigroup, and so we identify it with A^+. The elements of $\overline{\Omega}_A\mathsf{S} \setminus A^+$ are said to be **infinite**, while those of A^+, which are isolated elements of $\overline{\Omega}_A\mathsf{S}$, are said to be **finite**.

We may represent an element x of $A^{\mathbb{Z}}$ as the biinfinite word

$$\cdots x(-2)x(-1) \cdot x(0)x(1)x(2) \cdots .$$

For $x \in A^{\mathbb{Z}}$ and integers k, ℓ with $k \leq \ell$, we denote by $x_{[k,\ell]}$ the word $x(k)x(k+1) \cdots x(\ell)$; a word of this form is called a **finite block** of x.

A **symbolic dynamical system** $\mathcal{X}$ of $A^{\mathbb{Z}}$, also called **subshift** or **shift space** of $A^{\mathbb{Z}}$, is a nonempty closed subset of $A^{\mathbb{Z}}$ invariant under the shift operation and its inverse [18]. We denote by $L(\mathcal{X})$ the set of all finite blocks of elements of $\mathcal{X}$.

A subshift $\mathcal{X}$ is **minimal** if it does not contain proper subshifts. There is another useful characterization of minimal subshifts, with a combinatorial flavor. An element $x \in A^{\mathbb{Z}}$ is **uniformly recurrent** if for every finite block w of x, there is a positive integer N such that w is a factor of every finite block of x with length N. It turns out that a subshift is minimal if and only if it is generated by a uniformly recurrent biinfinite sequence [13, Proposition 5.1.13].

A trivial example of minimal subshift is that of a minimal finite subshift, generated by a periodic biinfinite word. Such a subshift is said to be **periodic**.

Given a subshift $\mathcal{X}$ and $u \in L(\mathcal{X})$, say that a nonempty word v is a **return word of** u **in** $\mathcal{X}$ if $vu \in L(\mathcal{X})$, u is a prefix of vu and u occurs in vu only as a prefix and a suffix. The set of all return words of u is denoted $R(u)$. See [7] for a recent account on return words. A subshift generated by an element of $A^{\mathbb{Z}}$ is minimal if and only if each of its finite blocks has a finite set of return words.

The following discussion summarizes results that can be found in [3, Section 2] and [5, Section 6]. If the subshift $\mathcal{X}$ is minimal, then the topological closure of $L(\mathcal{X})$ in $\overline{\Omega}_A\mathsf{S}$ is the disjoint union of $L(\mathcal{X})$ and a $\mathcal{J}$-class $J(\mathcal{X})$ of maximal regular elements of $\overline{\Omega}_A\mathsf{S}$. The correspondence $\mathcal{X} \mapsto J(\mathcal{X})$ is a bijection between the set of minimal subshifts of $A^{\mathbb{Z}}$ and the set of maximal regular $\mathcal{J}$-classes of $\overline{\Omega}_A\mathsf{S}$. Moreover, an infinite element w of $\overline{\Omega}_A\mathsf{S}$ belongs to $J(\mathcal{X})$ if and only if all its finite factors lie in $L(\mathcal{X})$.

It is natural to ask what is the structure of the (isomorphic) maximal subgroups of $J(\mathcal{X})$, denoted $G(\mathcal{X})$. Since the expression "maximal subgroup of $J(\mathcal{X})$" refers to a concrete subgroup of the free profinite semigroup and we wish to investigate its structure as an abstract profinite group, we prefer to call $G(\mathcal{X})$ the **Schützenberger group of** $\mathcal{X}$. This is in accordance with the literature in semigroup theory in which, more generally, one associates an abstract group with every $\mathcal{D}$-class of a semigroup, which is known as its **Schützenberger group**.

For instance, it is proved in [3] that, if $\mathcal{X}$ is an Arnoux–Rauzy subshift of degree k, of which the case $k = 2$ is that of the extensively studied Sturmian subshifts [19, 13], then $G(\mathcal{X})$ is a free profinite group of rank k. An example of a minimal subshift $\mathcal{X}$ such that $G(\mathcal{X})$ is not freely generated, with rank two, is also given in the same paper [3, Example 7.2].

A **right** (respectively **left**) **infinite word** is an element of $A^{\mathbb{N}}$ (resp. of $A^{\mathbb{Z}^-}$). Given $w \in \overline{\Omega}_A \mathsf{S}$, we denote by $\overrightarrow{w}$ (resp. $\overleftarrow{w}$) the right (resp. left) infinite word whose finite prefixes (resp. suffixes) are those of w.

Lemma 4.1 ([5, Lemma 6.6]): *For a minimal subshift $\mathcal{X}$, two elements $u, v \in J(\mathcal{X})$ are $\mathcal{R}$-equivalent if and only if $\overrightarrow{u} = \overrightarrow{v}$ and $\mathcal{L}$-equivalent if and only if $\overleftarrow{u} = \overleftarrow{v}$.*

Taking into account [6, Lemma 8.2], we deduce that $w \in J(\mathcal{X})$ lies in a subgroup if and only if the doubly infinite word $\overleftarrow{w} \cdot \overrightarrow{w}$ belongs to $\mathcal{X}$. Indeed, $w \in J(\mathcal{X})$ lies in a subgroup if and only if w^2 stays in the same $\mathcal{J}$-class, that is it has the same finite factors as w. Now, by [6, Lemma 8.2], the finite factors of w^2 are those of w together with the products of the form uv, where u is a finite suffix of w and v is a finite prefix of w. Thus, altogether, the finite factors of w^2 are the finite factors of $\overleftarrow{w} \cdot \overrightarrow{w}$.

The maximal subgroups H of $J(\mathcal{X})$ are thus in bijection with the elements of $\mathcal{X}$ via the mapping that sends H to $\overleftarrow{w} \cdot \overrightarrow{w}$, where w is any element of H. For $x \in \mathcal{X}$, we denote by H_x the maximal subgroup corresponding to x.

By a **substitution over a finite alphabet** A we mean an endomorphism of the free semigroup A^+. The substitution φ over the alphabet A is **primitive** if there is a positive integer n such that, for all $a, b \in A$, a occurs in $\varphi^n(b)$ and $\lim |\varphi^n(b)| = \infty$, where $|u|$ denotes the length of the word u. It is well known that to each primitive substitution φ over a finite alphabet A, we can associate a minimal subshift $\mathcal{X}_\varphi$. In terms of biinfinite words, there are some such words that are periodic for the action of φ given by

$$x \mapsto \cdots \varphi(x(-2))\varphi(x(-1)) \cdot \varphi(x(0))\varphi(x(1))\varphi(x(2)) \cdots$$

(cf. [13, Exercise 1.2.1]) and the subshift $\mathcal{X}_\varphi$ is generated by it. A finite word belongs to the language $L(\mathcal{X}_\varphi)$ if and only if it is a factor of $\varphi^k(a)$ for all $a \in A$ and all sufficiently large $k \geq 1$. Note that $\varphi(L(\mathcal{X}_\varphi)) \subseteq L(\mathcal{X}_\varphi)$ (see, for instance, [3, Lemma 4.1(a)]). We say that φ is **periodic** in case so is $\mathcal{X}_\varphi$.

We shall denote $J(\mathcal{X}_\varphi)$ and $G(\mathcal{X}_\varphi)$ respectively by $J(\varphi)$ and $G(\varphi)$: this notation is more synthetic and emphasizes the exclusive dependence of these structures on φ, which in turn is a mathematical object completely determined by a finite amount of data, namely the images (in A^+) of letters by φ. Naturally, we also call $G(\varphi)$ the **Schützenberger group** of the primitive substitution φ.

The unique continuous endomorphism of $\overline{\Omega}_A\mathsf{S}$ extending φ will also be denoted by φ. A **connection for** φ is a word ba, with $b, a \in A$, such that $ba \in L(\mathcal{X}_\varphi)$, the first letter of $\varphi^\omega(a)$ is a, and the last letter of $\varphi^\omega(b)$ is b. Every primitive substitution has a connection [3, Corollary 4.12]. In terms of the subshift $\mathcal{X}_\varphi$, a connection is simply a word of the form $x(-1)x(0)$ for some periodic point x of the action of φ on biinfinite words. For a connection ba, the intersection H_{ba} of the $\mathcal{R}$-class containing $\varphi^\omega(a)$ with the $\mathcal{L}$-class containing $\varphi^\omega(b)$ is a maximal subgroup of $J(\varphi)$. There is a finite power $\tilde{\varphi}$ of φ such that the first letter of $\tilde{\varphi}(a)$ is a and the last letter of $\tilde{\varphi}(b)$ is b. We call $\tilde{\varphi}$ a **connective power** of φ (with respect to the connection ba).

We let $X_\varphi(a, b) = b^{-1}(R(ba)b)$. To avoid overloaded notation, $X_\varphi(a, b)$ will be usually denoted by X. The set $R(ba)$ is easily recognized to be a code and so is $X = b^{-1}(R(ba)b)$. Let i be the unique homomorphism from the semigroup freely generated by X into the semigroup freely generated by A such that $i(x) = x$ for all $x \in X$. Then i is injective, because X is a code. If $x \in X$ then $\tilde{\varphi}(x)$ belongs to the subsemigroup of A^+ generated by X. Therefore, we can consider the word $w_x = i^{-1}(\tilde{\varphi}(x))$, the unique decomposition of $\tilde{\varphi}(x)$ in the elements of X. The homomorphism i has a unique extension to a continuous homomorphism $\overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$, which we also denote by i, and which we call the **encoding associated with** the connection ba.

Theorem 4.2 ([21, Corollary 2.2]): *The mapping i is injective.*

Let q be the canonical projection $\overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_X\mathsf{G}$, namely the unique continuous homomorphism from $\overline{\Omega}_X\mathsf{S}$ into $\overline{\Omega}_X\mathsf{G}$ that is the identity on the generators. Then there are unique continuous endomorphisms $\tilde{\varphi}_X$ and $\tilde{\varphi}_{X,\mathsf{G}}$ such that Diagram (4.1) commutes. More explicitly, for each $x \in X$ we have $\tilde{\varphi}_X(x) = w_x$ and $\tilde{\varphi}_{X,\mathsf{G}}(x) = w_x$, where we regard w_x as a semigroup word and a group word, respectively.

$$
\begin{array}{ccccc}
\overline{\Omega}_A\mathsf{S} & \xleftarrow{\ i\ } & \overline{\Omega}_X\mathsf{S} & \xrightarrow{\ q\ } & \overline{\Omega}_X\mathsf{G} \\
\downarrow{\scriptstyle \tilde{\varphi}} & & \downarrow{\scriptstyle \tilde{\varphi}_X} & & \downarrow{\scriptstyle \tilde{\varphi}_{X,\mathsf{G}}} \\
\overline{\Omega}_A\mathsf{S} & \xleftarrow{\ i\ } & \overline{\Omega}_X\mathsf{S} & \xrightarrow{\ q\ } & \overline{\Omega}_X\mathsf{G}
\end{array}
\tag{4.1}
$$

5. Maximal subgroups fixed by powers of primitive substitutions

Let A be a finite alphabet and let $\mathcal{X} \subseteq A^{\mathbb{Z}}$ be a subshift. Given a word $u \in L(\mathcal{X})$, let n be a nonnegative integer less than or equal to the length of u. Let u_1 and u_2 be words such that $u = u_1 u_2$ and $|u_1| = n$. An **n-delayed return word of u in $\mathcal{X}$** is a word v such that $u_1 v u_2 \in L(\mathcal{X})$ and $u_1 v \in R(u)u_1$ (see [12, Definition 11]). The set of n-delayed return words of u in $\mathcal{X}$ shall be denoted by $R(n,u)$ or $R(u_1,u_2)$. Note that

$$R(u_1,u_2) = u_1^{-1}(R(u)u_1),$$

thus $R(u_1,u_2)$ and $R(u)$ have the same cardinality.

Lemma 5.1: *Let $\mathcal{X}$ be a minimal subshift of $A^{\mathbb{Z}}$. Let $v \in \overline{\Omega}_A\mathsf{S}$ be an element of a maximal subgroup of $J(\mathcal{X})$. If u_1 and u_2 are words such that u_1 is a suffix and u_2 is a prefix of v, then v belongs to $\overline{\langle R(u_1,u_2)\rangle}$.*

Proof. Since v lies in a subgroup, v^3 also belongs to $J(\mathcal{X})$, whence so does $u_1 v u_2$. The set $\overline{L(\mathcal{X})}$ is closed under taking factors by [5, Proposition 2.4], and so there is a sequence (w_n) of words in $L(\mathcal{X})$ that converges to $u_1 v u_2$. We may as well assume that $u_1 u_2$ is a prefix and a suffix of each w_n. Hence, $w_n(u_1u_2)^{-1}$ is a product of words in $R(u_1u_2)$ and, therefore, $u_1^{-1} w_n u_2^{-1}$ belongs to the subsemigroup generated by $R(u_1,u_2)$, from which the lemma follows by [1, Exercise 10.2.10]. ∎

We recall that the evaluation mapping

$$\begin{aligned} \overline{\Omega}_M\mathsf{S} \times (\overline{\Omega}_A\mathsf{S})^M &\to \overline{\Omega}_A\mathsf{S} \\ (w, v_1, \ldots, v_M) &\mapsto w(v_1,\ldots,v_M) \end{aligned} \tag{5.1}$$

is continuous for every positive integer M [2, Subsection 2.3].

Proposition 5.2: *Let $\mathcal{X}$ be a minimal non-periodic subshift of $A^{\mathbb{Z}}$ and let $x \in \mathcal{X}$. Suppose there are strictly increasing sequences of positive integers $(p_n)_n$ and $(q_n)_n$ such that $R(p_n, x_{[-p_n,q_n]})$ has exactly M elements $r_{n,1},\ldots,r_{n,M}$, for every n. Let $(r_1,\ldots,r_M)$ be an arbitrary accumulation point of the sequence $(r_{n,1},\ldots,r_{n,M})_n$ in $(\overline{\Omega}_A\mathsf{S})^M$. Then $\overline{\langle r_1,\ldots,r_M\rangle}$ is the maximal subgroup H_x of $J(\mathcal{X})$.*

Proof. Clearly the proof needs only to deal with the case where the sequence $(r_{n,1},\ldots,r_{n,M})_n$ converges to $(r_1,\ldots,r_M)$. Since $r_{n,i} \in L(\mathcal{X})$ for all n, we know that $r_i \in \overline{L(\mathcal{X})}$. Let

$$p_{(n,i)} = \min\{p_n, |r_{n,i}|\} \quad \text{and} \quad q_{(n,i)} = \min\{q_n + 1, |r_{n,i}|\}.$$

By [10, Lemma 3.2],

$$\lim \min\{|v| : v \in R(p_n, x_{[-p_n,q_n]})\} = \infty.$$

Hence, $r_i \in J(\mathcal{X})$ and $\lim p_{(n,i)} = \lim q_{(n,i)} = \infty$. Since for all n the word $x_{[0,q_{(n,i)}]}$ is a prefix of $r_{n,i}$ and $x_{[-p_{(n,i)},-1]}$ is a suffix of $r_{(n,i)}$, we obtain $r_i \in H_x$ by definition of H_x.

Let g be an element of H_x. By Lemma 5.1, there is an element $w_n \in \overline{\Omega}_M\mathsf{S}$ such that $g = w_n(r_{n,1}, \ldots, r_{n,M})$. Let w be an accumulation point of $(w_n)_n$ in $\overline{\Omega}_M\mathsf{S}$. Since the evaluation mapping (5.1) is continuous, it follows that $g = w(r_1, \ldots, r_M)$. This proves that $H_x = \overline{\langle r_1, \ldots, r_M \rangle}$. ∎

The following result shows that a primitive substitution φ induces natural actions on certain maximal subgroups of $J(\varphi)$.

Lemma 5.3: *Let φ be a primitive substitution and let ba be a connection for φ. If $\tilde{\varphi}$ is a connective power of φ, then $\tilde{\varphi}(H_{ba}) \subseteq H_{ba}$.*

Proof. Let u be a word from $R(b,a)$. Then $\varphi^n(u)$ belongs to $L(\mathcal{X}_\varphi)$ for every n. Hence, $\varphi^\omega(u)$ belongs to $J(\varphi)$. Since u starts with a and ends with b, it follows that $\varphi^\omega(u) \in H_{ba}$. Let $K = \tilde{\varphi}(H_{ba})$. Since, by [3, Proposition 4.2], φ maps $J(\varphi)$ to itself, K is a subgroup of $\overline{\Omega}_A\mathsf{S}$ contained in $J(\varphi)$. Thus, since H_{ba} is a maximal subgroup of $\overline{\Omega}_A\mathsf{S}$, to show that $\tilde{\varphi}(H_{ba}) \subseteq H_{ba}$, it suffices to show that $K \cap H_{ba}$ is nonempty. Indeed, $\tilde{\varphi}\varphi^\omega(u) = \varphi^\omega\tilde{\varphi}(u)$ belongs to K, by definition of K, and to H_{ba}, since $\tilde{\varphi}$ is a connective power of φ. ∎

Let φ be a primitive substitution over A. A **biinfinite fixed point of** φ is an element x of $A^{\mathbb{Z}}$ such that $x_{[0,n]}$ is a prefix of $\varphi(x_{[0,n]})$ and $x_{[-n,-1]}$ is a suffix of $\varphi(x_{[-n,-1]})$, for every positive integer n.

Given a biinfinite word $x \in A^{\mathbb{Z}}$ and a positive integer ℓ, let $\sim_\ell$ be the equivalence relation on $\mathbb{Z}$ defined by $i \sim_\ell j$ if $x_{[i-\ell,i+\ell]} = x_{[j-\ell,j+\ell]}$. Note that, for $k > \ell$, $\sim_k$ refines $\sim_\ell$.

Suppose additionally that x is a biinfinite fixed point of φ. Following the notation of [13],[2] let

$$E_1(\varphi) = \{0\} \cup \bigcup_{n\geq 1} \big\{-|\varphi(x_{[-n,-1]})|, |\varphi(x_{[0,n-1]})|\big\}.$$

[2] The minus sign in $-|\varphi(x_{[-n,-1]})|$ in the formula for $E_1(\varphi)$ is missing in [13, Section 7.2.1]. The correct formulation can be found in [12, Section 2.4].

The substitution φ is said to be **bilaterally recognizable** if there exists $\ell > 0$ such that E_1 is a union of $\sim_\ell$-classes. Denote by $\ell(\varphi)$ the least possible value of ℓ.

The following result of Mossé [22, 23], stated in [13, Theorem 7.2.2], will be crucial in the sequel.

THEOREM 5.4: *Every non-periodic primitive substitution with a biinfinite fixed point is bilaterally recognizable.*

In the case of non-periodic primitive substitutions, the following consequence of Theorem 5.4 provides the key tool to prove the reverse inclusion of that given by Lemma 5.3.

PROPOSITION 5.5: *Let φ be a non-periodic primitive substitution and let ba be a connection for φ. If $\tilde{\varphi}$ is a connective power of φ, then $H_{ba} \subseteq \operatorname{Im}\tilde{\varphi}$.*

Proof. Let x be the unique element of the subshift $\mathcal{X}_\varphi$ such that $H_x = H_{ba}$. By Lemma 5.3, x is also the biinfinite word

$$\cdots \tilde{\varphi}(x(-2))\tilde{\varphi}(x(-1)) \cdot \tilde{\varphi}(x(0))\tilde{\varphi}(x(1))\tilde{\varphi}(x(2)) \cdots .$$

Therefore, x is a biinfinite fixed point of $\tilde{\varphi}$. By Theorem 5.4, $\tilde{\varphi}$ is bilaterally recognizable.

By [12, Proposition 25 and Theorem 24], the sequence $|R(n, x_{[-n,n]})|$ is bounded. Hence, there is a strictly increasing sequence (p_n) for which $|R(p_n, x_{[-p_n,p_n]})|$ is a constant M, and such that $p_n > \ell(\tilde{\varphi})$ for all n.

Let $R(p_n, x_{[-p_n,p_n]}) = \{r_{n,1}, \ldots, r_{n,M}\}$. Let $k \in \{1, \ldots, M\}$. Because x is uniformly recurrent, there are $i > 0$ and $j > i$ such that

$$x_{[-p_n,p_n]} = x_{[i-p_n,i+p_n]} = x_{[j-p_n,j+p_n]} \tag{5.2}$$

and $r_{n,k} = x_{[i,j-1]}$. Since $0 \in E_1(\tilde{\varphi})$ and $p_n > \ell(\tilde{\varphi})$, it follows from (5.2) that $i, j \in E_1(\tilde{\varphi})$. As $r_{n,k} = x_{[i,j-1]}$, we conclude that $r_{n,k}$ belongs to $\operatorname{Im}\tilde{\varphi}$.

Let $(r_1, \ldots, r_M)$ be an accumulation point of the sequence $(r_{n,1}, \ldots, r_{n,M})_n$. Since $\operatorname{Im}\tilde{\varphi}$ is closed in $\overline{\Omega}_A\mathsf{S}$, we have $r_k \in \operatorname{Im}\tilde{\varphi}$ for all k. It then follows from Proposition 5.2 that $H_{ba} \subseteq \operatorname{Im}\tilde{\varphi}$. ∎

We can now establish the announced reverse inclusion of that given by Lemma 5.3 in the case of non-periodic primitive substitutions.

Theorem 5.6: *Let φ be a non-periodic primitive substitution. Consider a connection ba for φ and a connective power $\tilde{\varphi}$. Then $H_{ba} = \tilde{\varphi}(H_{ba}) = \varphi^\omega(H_{ba})$.*

Proof. Let k be a positive integer. Then $\tilde{\varphi}^k$ is also a connective power of φ relatively to the connection ba. Therefore, by Proposition 5.5, we obtain the inclusion $H_{ba} \subseteq \operatorname{Im} \tilde{\varphi}^k$. Hence, given $g \in H_{ba}$, for each positive integer k, there is $u_k \in \overline{\Omega}_A \mathsf{S}$ such that $g = (\tilde{\varphi})^{k!}(u_k)$. Since the evaluation mapping on continuous endomorphisms of finitely generated profinite semigroups is continuous (cf. [17, Proposition 1]), there is an accumulation point u of the sequence (u_k) such that $g = \tilde{\varphi}^\omega(u) = \varphi^\omega(u)$. Hence, we have $\varphi^\omega(g) = g$. This proves the equality $H_{ba} = \varphi^\omega(H_{ba})$.

By Lemma 5.3, the inclusion $\tilde{\varphi}(H_{ba}) \subseteq H_{ba}$ holds. Hence, $\tilde{\varphi}^{k!-1}(H_{ba}) \subseteq H_{ba}$ holds for all $k \geq 1$, which shows that $\tilde{\varphi}^{\omega-1}(H_{ba}) \subseteq H_{ba}$ because H_{ba} is closed. We then have

$$H_{ba} = \varphi^\omega(H_{ba}) = \tilde{\varphi}(\tilde{\varphi}^{\omega-1}(H_{ba})) \subseteq \tilde{\varphi}(H_{ba}),$$

which, together with Lemma 5.3, establishes the equality $H_{ba} = \tilde{\varphi}(H_{ba})$. ∎

The first author [3, Theorem 4.13] managed to avoid using Mossé's Theorem 5.4 to obtain the equality $H_{ba} = \varphi^\omega(H_{ba})$ by adding the extra synchronization hypothesis that φ is an "encoding of bounded delay with respect to the finite factors of $J(\varphi)$" (cf. [3]). This restriction turns out not to be significant in case φ induces an automorphism of the free group $FG(A)$, because then the extra hypothesis always holds [3, Corollary 5.6].

Theorem 5.6 fails if φ is periodic. For example, consider the periodic primitive substitution defined by $\varphi(a) = aba$ and $\varphi(b) = bab$. Then ba is a connection for φ, and $X_\varphi(a,b) = \{ab\}$. Note that $\varphi^n(ab) = (ab)^{3^n}$, for every positive integer n. By the definition of H_{ba}, we know that $K = \overline{\langle \varphi^\omega(ab) \rangle}$ is a closed subgroup of H_{ba}. Note that $(ab)^{\omega+1}$ is $\mathcal{H}$-equivalent to $\varphi^\omega(ab)$, that is $(ab)^{\omega+1} \in H_{ba}$. Note also that $(ab)^{\omega+1} \notin K$. If we had $H_{ba} \subseteq \operatorname{Im} \varphi^\omega$, then we would have $(ab)^{\omega+1} = \varphi^\omega(ab)^{\omega+1} \in K$, a contradiction. This shows the necessity of the non-periodicity hypothesis in Theorem 5.6.

6. Presentations of Schützenberger groups of primitive substitutions

By Corollary 2.5, every finitely generated maximal subgroup of $\overline{\Omega}_X \mathsf{S}$ admits a finite presentation of the form (2.3). However, to be able to apply the decidability results of Section 3, one needs computability properties of the continuous endomorphism Φ of $\overline{\Omega}_X\mathsf{G}$. In this section, we show that this is always possible for the Schützenberger group of an arbitrary primitive substitution over a finite alphabet.

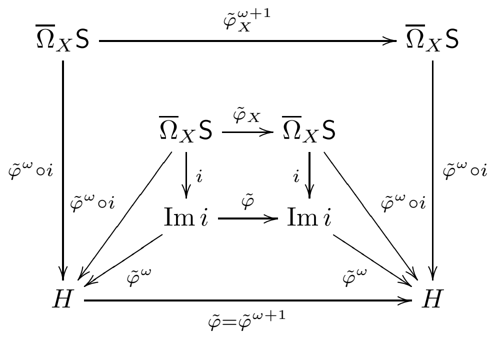
We separate into two subsections the general case, which involves return words, and a special case, in which the idempotent iterate φ^ω of the substitution φ maps all letters to the same $\mathcal{H}$-class. In the special case, the presentation can be expressed more directly in terms of the given substitution. In the third subsection, we show that one can actually obtain the general case from the special one.

6.1. The general case. We first apply the simple remarks of Subsection 2 to obtain a semigroup presentation for a profinite subgroup associated with a primitive substitution φ and a connection of φ.

Proposition 6.1: *Let φ be a primitive substitution over the alphabet A, ba be a connection for φ, and $\tilde\varphi$ be a connective power of φ. Put $X = X_\varphi(a,b)$ and $H = \operatorname{Im}(\varphi^\omega \circ i)$, where i is the encoding associated with ba. Then $\operatorname{Ker}(\varphi^\omega \circ i) \subseteq \operatorname{Ker}\tilde\varphi^\omega_X$ and so H admits the presentation*

$$\langle X \mid \tilde\varphi^\omega_X(x) = x \ (x \in X)\rangle_{\mathsf{S}}. \tag{6.1}$$

Proof. Note that $\varphi^\omega(i(X))$ is contained in H_{ba} [3, Proposition 4.8(1)], whence H is a subgroup of $\overline{\Omega}_A\mathsf{S}$. Moreover, $\tilde\varphi$ acts as an automorphism on H and as an endomorphism of $\operatorname{Im} i$. We obtain the following commutative diagram, where the commutativity of the outer rectangle follows from that of the largest trapezoid.



Let $(u,v) \in \operatorname{Ker}(\tilde\varphi^\omega \circ i)$. We claim that $(u,v) \in \operatorname{Ker}\tilde\varphi^\omega_X = \operatorname{Ker}\tilde\varphi^{\omega+1}_X$. Indeed, since i is injective by Theorem 4.2, it suffices to show that u and v have the same image under $i \circ \tilde{\varphi}^{\omega}_X$. Now, by the commutativity of the diagram, the following holds for an arbitrary $w \in \overline{\Omega}_X\mathsf{S}$: $i \circ \tilde{\varphi}^{\omega}_X(w) = \tilde{\varphi}^{\omega} \circ i(w)$. Combining with the hypothesis that $\tilde{\varphi}^{\omega} \circ i(u) = \tilde{\varphi}^{\omega} \circ i(v)$, we deduce that $i \circ \tilde{\varphi}^{\omega}_X(u) = i \circ \tilde{\varphi}^{\omega}_X(v)$. We have thus shown that $\operatorname{Ker}(\tilde{\varphi}^{\omega} \circ i) \subseteq \operatorname{Ker}\tilde{\varphi}^{\omega}_X$. To conclude the proof, it suffices to invoke Lemma 2.2. ∎

We are now ready for the main theorem of this paper.

Theorem 6.2: *Let φ be a non-periodic primitive substitution over the alphabet A. Let ba be a connection of φ and let $X = X_{\varphi}(a,b)$. Then $G(\varphi)$ admits the presentation*

$$\langle X \mid \tilde{\varphi}^{\omega}_{X,\mathsf{G}}(x) = x \ (x \in X)\rangle_{\mathsf{G}}, \tag{6.2}$$

where $\tilde{\varphi}$ is a connective power of φ.

Proof. Let $H = \operatorname{Im}(\varphi^{\omega} \circ i)$. As in the proof of Proposition 6.1, we know that H is contained in H_{ba}. On the other hand, by Lemma 5.1, H_{ba} is contained in $\operatorname{Im} i$, whence $\varphi^{\omega}(H_{ba}) \subseteq \operatorname{Im}(\varphi^{\omega} \circ i) = H$. By Theorem 5.6, it follows that $H = H_{ba}$. Hence, H is the Schützenberger group $G(\varphi)$. According to Proposition 6.1, H admits the profinite semigroup presentation (6.1). Lemma 2.3 yields that H admits the presentation

$$\langle X \mid q(\tilde{\varphi}^{\omega}_X(x)) = q(x) \ (x \in X)\rangle_{\mathsf{G}},$$

where $q : \overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_X\mathsf{G}$ is the canonical projection. In view of commutativity of Diagram (4.1), and noting also that $q(x) = x$ for each $x \in X$, it remains to observe that the above presentation is just a reformulation of (6.2). ∎

6.2. The case of proper substitutions. This subsection is dedicated to a special case in which the Schützenberger group of the primitive substitution is realized as a retract of the free profinite semigroup under the ω-power of the substitution. This leads to a somewhat simpler presentation of the form (1.1).

We say that a substitution φ over a finite alphabet A is **proper** if there are letters $a, b \in A$ such that, for every $d \in A$, the word $\varphi(d)$ starts with a and ends with b.

Lemma 6.3: *Let φ be a non-periodic proper substitution over a finite alphabet A. Then $\operatorname{Im}\varphi^{\omega}$ is a maximal subgroup of $\overline{\Omega}_A\mathsf{S}$ contained in $J(\varphi)$.*

Proof. By [6, Proposition 5.3], all the elements of $\overline{\Omega}_A\mathsf{S}$ of the form $\varphi^\omega(a)$ $(a \in A)$ lie in the same maximal subgroup H. By Theorem 5.6, the image of φ^ω is H. ∎

The special case of the following result where φ is an encoding of bounded delay with respect to the finite factors of $J(\varphi)$ was announced in a lecture by the first author at the *Fields Workshop on Profinite Groups and Applications* (Carleton University, August 2005). Its proof appears here for the first time, and furthermore does not depend on that hypothesis.

THEOREM 6.4: *Let φ be a non-periodic proper primitive substitution over a finite alphabet A. Then $G(\varphi)$ admits the presentation*

$$\langle A \mid \varphi_{\mathsf{G}}^\omega(a) = a \ (a \in A)\rangle_{\mathsf{G}}. \tag{6.3}$$

Proof. By Lemma 6.3, $H = \operatorname{Im}\varphi^\omega$ is a maximal subgroup of $\overline{\Omega}_A\mathsf{S}$ and it is also the Schützenberger group of φ. In particular, φ acts on H as an automorphism.

Consider the following commutative diagram

$$\begin{array}{ccc} \overline{\Omega}_A\mathsf{S} & \xrightarrow{\varphi^{\omega+1}} & \overline{\Omega}_A\mathsf{S} \\ {\scriptstyle\varphi^\omega}\downarrow & & \downarrow{\scriptstyle\varphi^\omega} \\ H & \xrightarrow{\varphi} & H. \end{array}$$

Since $\operatorname{Ker}\varphi^\omega \subseteq \operatorname{Ker}\varphi^{\omega+1}$, it follows from Lemma 2.2 that H admits the presentation $\langle A \mid \varphi^\omega(a) = a \ (a \in A)\rangle_{\mathsf{S}}$. Applying Lemma 2.3, we deduce that H can also be presented as $\langle A \mid p(\varphi^\omega(a)) = p(a) \ (a \in A)\rangle_{\mathsf{G}}$. Finally, since $p \circ \varphi = \varphi_{\mathsf{G}} \circ p$ and $p(a) = a$ for every $a \in A$, the latter presentation coincides with (6.3). ∎

6.3. A REDUCTION TO THE PROPER CASE. In this subsection, we show how to reduce the case of a general primitive substitution to that of a proper primitive substitution, thus providing an alternative proof of Theorem 6.2 based on Theorem 6.4. The first ingredient is the following lemma, which can be extracted from [12, Lemma 21], noting that the definition of proper substitution adopted in that paper translates in the language of the present paper as a substitution which admits a power which is proper.

LEMMA 6.5: *Let φ be a primitive substitution over a finite alphabet A and let ba be a connection for φ. Let $X = X_\varphi(a,b)$ and suppose that $\tilde\varphi$ is a connective power of φ such that* $\min\{|\tilde\varphi(a)|, |\tilde\varphi(b)|\} > \max\{|x| : x \in X\}$. *Then $\tilde\varphi_X$ is a* *proper primitive substitution over the alphabet X. The subshift $\mathcal{X}_\varphi \subseteq A^{\mathbb{Z}}$ is periodic if and only if so is $\mathcal{X}_{\tilde{\varphi}_X} \subseteq X^{\mathbb{Z}}$.*

Provided φ is non-periodic, for a choice of $\tilde{\varphi}$ as in Lemma 6.5, we may apply Lemma 6.3 to conclude that $\operatorname{Im}\tilde{\varphi}_X^\omega = \operatorname{Im}\varphi_X^\omega$ is a maximal subgroup of $J(\varphi_X)$, which we denote by K.

On the other hand, Theorem 5.6 shows that, for a connection ba of φ, the maximal subgroup H_{ba} is such that $H_{ba} = \varphi^\omega(H_{ba})$. Hence, for the encoding i associated with ba, we have

$$H_{ba} = \varphi^\omega(H_{ba}) \subseteq \varphi^\omega(i(\overline{\Omega}_X \mathsf{S})) = i(\varphi_X^\omega(\overline{\Omega}_X \mathsf{S})) = i(K).$$

Since K is a subgroup and H_{ba} is a maximal subgroup, we have $i(K) = H_{ba}$. As i is injective by Theorem 4.2, we obtain the following result.

THEOREM 6.6: *Let φ be a non-periodic primitive substitution over a finite alphabet and let $\tilde{\varphi}_X$ be as in Lemma 6.5. Then the encoding i associated with the connection ba defines an isomorphism between a maximal subgroup of $J(\mathcal{X}_{\tilde{\varphi}_X})$ and H_{ba}.* ∎

Combining Theorems 6.4 and 6.6, it is now immediate to obtain an alternative way to deduce Theorem 6.2.

7. Applications

This section is devoted to applications of the main results of Section 6.

7.1. DECIDABILITY OF SCHÜTZENBERGER GROUPS OF PRIMITIVE SUBSTITUTIONS. The following result is our main motivation for obtaining presentations of Schützenberger groups. The periodic case of the following theorem follows from the fact that the corresponding Schützenberger group is a free procyclic group [6, Theorem 7.5]. For this reason, the two alternative proofs presented below handle only the non-periodic case.

THEOREM 7.1: *Let φ be a primitive substitution over a finite alphabet. Then the profinite group $G(\varphi)$ is decidable.*

First proof. By [3, Lemmas 3.3 and 4.5], one may effectively compute a connection ba for φ and the set $X = X_\varphi(a, b)$. Thus, to conclude the proof, it suffices to invoke Corollary 3.3 taking into account Theorem 6.2. ∎

Second proof. Assuming that the substitution φ is proper, Corollary 3.3 combined with Theorem 6.4, yields that the group $G(\varphi)$ is decidable. To obtain the general case, we invoke a result from symbolic dynamics which states that, for every primitive substitution φ, one can effectively compute a proper primitive substitution ψ such that the subshifts $\mathcal{X}_\varphi$ and $\mathcal{X}_\psi$ are conjugate [12] (see also [11, Proposition 31]). Since the profinite groups $G(\varphi)$ and $G(\psi)$ are isomorphic by [8, Theorem 3.11], and the latter is decidable, so is the former. ∎

Note that, while the first proof depends less on results on symbolic dynamics, the second proof does not depend on the injectivity of the encoding i associated with the connection (Theorem 4.2). By using also the injectivity of i, one may modify the second proof by applying instead Theorem 6.6 to obtain the isomorphism of $G(\varphi)$ with a decidable profinite group.

7.2. A FIRST NON-RELATIVELY FREE EXAMPLE. We give an example to illustrate how to apply Theorem 6.4 to prove that the Schützenberger group of a primitive substitution is not relatively free. Let $A = \{a, b\}$ and define a substitution φ by $\varphi(a) = ab$ and $\varphi(b) = a^3b$, which is non-periodic and proper primitive. Hence, by Theorem 6.4, the group $G(\varphi)$ admits the presentation

$$\langle a, b \mid \varphi_{\mathsf{G}}^{\omega}(a) = a,\ \varphi_{\mathsf{G}}^{\omega}(b) = b\rangle_{\mathsf{G}}.$$

It is shown in [3, Example 7.2] that $G(\varphi)$ is not a free profinite group. We proceed to improve this result by showing that it is not relatively free, that is, not of the form $\overline{\Omega}_X\mathsf{V}$, although in fact we do not know whether the pseudovariety generated by all its finite continuous homomorphic images is a proper subclass of G.

THEOREM 7.2: *Let φ be the substitution given by $\varphi(a) = ab$ and $\varphi(b) = a^3b$. Then $G(\varphi)$ is not a relatively free profinite group.*

Proof. By Lemma 6.3, the closed subsemigroup $H = \overline{\langle\varphi^{\omega}(a), \varphi^{\omega}(b)\rangle}$ is a maximal subgroup isomorphic to $G(\varphi)$. The argument in [3, Example 7.2] shows that H cannot be relatively free with respect to any pseudovariety containing the two-element group. Hence, it suffices to show that the pseudovariety generated by the finite continuous homomorphic images of H contains the two-element group, i.e., that H has a continuous homomorphic image of finite even order. We claim, more specifically, that the alternating group A_5 is a continuous homomorphic image of H.

Let $A = \{a, b\}$ and let $h : \overline{\Omega}_A\mathsf{S} \to A_5$ be the unique continuous homomorphism such that $h(a) = (1\,2\,3)$ and $h(b) = (3\,4\,5)$. Note that h is onto. To establish the claim, in view of Proposition 3.2 it is enough to check that $h \circ \varphi^{12}|_A = h_A$. Although the length of the word $\varphi^n(a)$ depends exponentially on n, the verification can be done easily by applying Lemma 3.1 since $h \circ \varphi^{12}|_A = (\varphi_{A_5})^{12}(h|_A)$. The computation of $(\varphi_{A_5})^{12}(h|_A)$ can be carried out either by hand or by using a computer algebra system like GAP [14] and it confirms that indeed $(\varphi_{A_5})^{12}$ fixes $h|_A$, thereby proving the theorem. ∎

7.3. The case of the Prouhet–Thue–Morse substitution. Let A be the two-letter alphabet $\{a, b\}$. The **Prouhet–Thue–Morse substitution** is the non-periodic primitive substitution τ over A given by $\tau(a) = ab$ and $\tau(b) = ba$ [13]. Note that no power of τ is proper. The word aa is a connection for τ and $\tilde{\tau} = \tau^2$ is a connective power of τ. The four elements of $X = X_\tau(a, a)$ are $x = abba$, $y = ababba$, $z = abbaba$ and $t = ababbaba$, cf. [7, Section 3.2]. By Theorem 6.2, the $\mathcal{H}$-class $H = H_{aa}$ of $\tau^\omega(a)$, which is generated by $\tau^\omega(X)$, admits the following presentation:

$$\langle X \mid \tilde{\tau}^{\omega}_{X,\mathsf{G}}(u) = u \ (u \in X)\rangle_{\mathsf{G}}. \tag{7.1}$$

More precisely, the kernel of the continuous homomorphism $\overline{\Omega}_X\mathsf{G} \to H$ that maps each $u \in X$ to $\tau^\omega(u)$ is the closed congruence generated by the relations in the presentation (7.1). Let $\alpha = \tau^\omega(x)$, $\beta = \tau^\omega(y)$, $\gamma = \tau^\omega(z)$, and $\delta = \tau^\omega(t)$.

Remark 7.3: Let ζ be a continuous semigroup homomorphism from $\overline{\Omega}_A\mathsf{S}$ into a profinite semigroup S. Suppose that $\zeta(x)$ belongs to a subgroup of S. Then $\zeta(y) \cdot \zeta(x)^{\omega-1} \cdot \zeta(z) = \zeta(t)$.

Proof. We have $\zeta(ababba) \cdot \zeta(abba)^{\omega-1} \cdot \zeta(abbaba) = \zeta(ab) \cdot \zeta(abba)^{\omega+1} \cdot \zeta(ba)$, and $\zeta(abba)^{\omega+1} = \zeta(abba)$, because $\zeta(abba)$ is a group element of S. ∎

Applying Remark 7.3 to the continuous homomorphism τ^ω, we conclude that $\beta\alpha^{-1}\gamma = \delta$ in H, so that the profinite group H is generated by $\{\alpha, \beta, \gamma\}$ and so the relation $yx^{-1}z = t$ turns out to be a consequence of the relations in the presentation (7.1).

A routine calculation shows that the images of letters of X by $\tilde{\tau}_X$ are given by

$$\tilde{\tau}_X(x) = zxy, \quad \tilde{\tau}_X(y) = ztxy, \quad \tilde{\tau}_X(z) = zxty, \quad \tilde{\tau}_X(t) = ztxty.$$

We proceed to give an alternative presentation of $G(\tau)$ as a profinite group.

THEOREM 7.4: *The group $G(\tau)$ admits the following presentation:*

$$\langle x, y, z \mid \Psi^{\omega}(x) = x,\ \Psi^{\omega}(y) = y,\ \Psi^{\omega}(z) = z\rangle_{\mathsf{G}}, \tag{7.2}$$

where Ψ is the unique continuous endomorphism of $\overline{\Omega}_{\{x,y,z\}}\mathsf{G}$ such that $\Psi(x) = zxy$, $\Psi(y) = zyx^{-1}zxy$, and $\Psi(z) = zxyx^{-1}zy$.

Proof. Let $X = \{x, y, z, t\}$ and $Y = X \setminus \{t\}$. Consider the continuous endomorphism r of $\overline{\Omega}_X\mathsf{G}$ which fixes the elements of Y and maps t to $yx^{-1}z$. Let $\Phi = \tilde{\tau}_{X,\mathsf{G}}$. Note that Ψ has been defined so that $r \circ \Phi$ coincides with Ψ on Y. We extend Ψ to $\overline{\Omega}_X\mathsf{G}$ by putting $\Psi(t) = r(\Phi(t))$, which yields the equality $\Psi = r \circ \Phi$. On the other hand, we have $\Phi(t) = ztxty = ztxy \cdot (zxy)^{-1} \cdot zxty = \Phi(yx^{-1}z)$. As argued above, from Theorem 6.2 it follows that $G(\tau)$ admits the presentation

$$\langle X \mid yx^{-1}z = t,\ \Phi^{\omega}(u) = u\ (u \in X)\rangle_{\mathsf{G}}.$$

To finish the proof, it now suffices to invoke Proposition 2.6. ∎

For a profinite group G and a pseudovariety of groups V, denote by G_{V} the largest pro-V factor group of G. For a prime p, let Ab_p denote the pseudovariety of all elementary Abelian p-groups. The following result is well known (cf. [27, Proposition 3.4.2 and Lemma 3.3.5]).

LEMMA 7.5: *Let V and W be pseudovarieties of groups such that $\mathsf{V} \subseteq \mathsf{W}$ and suppose that G is a finitely generated free pro-W group. Then G_{V} is a free pro-V group and, if V contains some nontrivial group, then the two groups have the same rank.*

In [3, Example 7.3] it was proved that the profinite group $G(\tau)$ is not free on three generators: although the computation starts from an incorrect set of return words, the same argument goes through with the correct set. We may now adopt a different approach to establish the following improvement.

THEOREM 7.6: *The profinite group $G(\tau)$ is not relatively free.*

Proof. We first note that, in view of Theorem 7.4, the following presentation defines a finite quotient group K_p of $G(\tau)$:

$$\begin{aligned}\langle x, y, z \mid{}& \Psi^{\omega}(x) = x,\ \Psi^{\omega}(y) = y,\ \Psi^{\omega}(z) = z,\\ & xy = yx,\ yz = zy,\ zx = xz,\ x^p = y^p = z^p = 1\rangle_{\mathsf{G}}.\end{aligned}$$

Let $Y = \{x, y, z\}$. By Lemma 2.3, the group K_p also admits the presentation

$$\langle x, y, z \mid \Lambda^\omega(x) = x,\, \Lambda^\omega(y) = y,\, \Lambda^\omega(z) = z\rangle_{\mathsf{Ab}_p},$$

where Λ is the continuous endomorphism of $\overline{\Omega}_Y\mathsf{Ab}_p$ induced by Ψ, which is given by $\Lambda(x) = zxy$ and $\Lambda(y) = \Lambda(z) = y^2z^2$. In the group K_p, we have $y = \Lambda^\omega(y) = \Lambda^\omega(z) = z$. Moreover, identifying each function f from Y to K_p with the triple $(f(x), f(y), f(z))$ and applying iteratively the transformation $\Lambda_{K_p} \in \mathcal{T}(K_p^Y)$, one obtains inductively $\Lambda^n_{K_p}(x, y, y) = (xy^{2(4^n-1)/3}, y^{4^n}, y^{4^n})$. By Lemma 3.1, it follows that, in K_p and for $n = m!$ sufficiently large, the equalities $xy^{2(4^n-1)/3} = x$ and $y^{4^n} = y$ hold. In particular, for $p = 2$, we get $y = 1$, which shows that K_2 is a cyclic group of order 2. On the other hand, for a prime $p > 2$, the previous calculations show that K_p is an elementary Abelian p-group of rank two.

Suppose that $G(\tau)$ were a free pro-V group for some pseudovariety of groups V. By the above, V contains Ab_p for every prime p. Moreover, since $G(\tau)_{\mathsf{Ab}_p} = K_p$, the above calculations imply that $G(\tau)_{\mathsf{Ab}_p}$ has rank 1 for $p = 2$ and rank 2 for an odd prime p, which contradicts Lemma 7.5. Hence, $G(\tau)$ cannot be a relatively free profinite group. ∎

The proof of Theorem 7.6 shows that the rank of $G(\tau)$ is either two or three. The following result settles the precise value of the rank. It is a further application of the presentation of $G(\tau)$ given by Theorem 7.4.

Theorem 7.7: *The group $G(\tau)$ has a group of order 18 of rank three as a continuous homomorphic image. Hence, $G(\tau)$ has rank three.*

Proof. Set $Y = \{x, y, z\}$. Let K be the group given by the following presentation:

$$\langle a, b, c \mid a^2 = b^3 = c^3 = 1,\, bc = cb,\, aba = b^2,\, aca = c^2\rangle_{\mathsf{G}}.$$

Note that K is the semidirect product of the subgroup $\langle b, c\rangle$, which is the direct product of two three-element groups, by the two-element subgroup $\langle a\rangle$. Let $h : \overline{\Omega}_Y\mathsf{G} \to K$ be the continuous homomorphism that sends x, y, z respectively to a, b, c.

We first verify that $h(\Psi^2(u)) = h(u)$ for all $u \in Y$. Since the calculations are quite similar, we treat only the case where $u = y$, leaving the other two cases for the reader to check:

$$\begin{aligned} h(\Psi^2(y)) &= caba^{-1}cb \cdot cba^{-1}cab \cdot (cab)^{-1} \cdot caba^{-1}cb \cdot cab \cdot cba^{-1}cab \\ &= c \cdot aba \cdot cbcbac \cdot aba \cdot cbcabcb \cdot aca \cdot b \\ &= c \cdot b^2 \cdot cbcbac \cdot b^2 \cdot cbcabcb \cdot c^2 \cdot b = b = h(y). \end{aligned}$$

From Proposition 3.2, it follows that K is a continuous homomorphic image of $G(\tau)$. Since it is easily checked that K has rank three, it follows that so does $G(\tau)$. ∎

One can also use the same technique as in the proof of Theorem 7.2 to establish that other finite groups are continuous homomorphic images of $G(\tau)$, as in the following observation.

Remark 7.8: The alternating group A_5 is a continuous homomorphic image of $G(\tau)$.

Proof. Let $A = \{a, b\}$ and consider the transformation $\tau_{A_5} \in A_5^A$ associated with the substitution τ according to Lemma 3.1. Identifying here $f \in A_5^A$ with the pair $(f(a), f(b))$, we have $\tau_{A_5}(x, y) = (xy, yx)$. Again, a straightforward calculation shows that $\tau_{A_5}^6$ fixes the pair of 3-cycles $((1\,2\,3), (3\,4\,5))$. Hence, for the continuous homomorphism $h : \overline{\Omega}_A\mathsf{S} \to A_5$ given by $h(a) = (1\,2\,3)$ and $h(b) = (3\,4\,5)$, by Lemma 3.1 we obtain the equalities $h(\tau^\omega(a)) = (1\,2\,3)$ and $h(\tau^\omega(a)) = (3\,4\,5)$, from which it follows that $h(\tau^\omega(abba)) = (1\,3\,2\,5\,4)$ and $h(\tau^\omega(ababba)) = (1\,5\,2)$. This proves the claim since A_5 is generated by the latter two cycles, while $\tau^\omega(abba)$ and $\tau^\omega(ababba)$ belong to $G(\tau)$. ∎

We do not know whether the finite quotients of $G(\tau)$ generate a proper pseudovariety of groups. On the other hand, every finite cyclic group is a continuous homomorphic image of $G(\tau)$. Indeed, adding to the defining relations of the presentation of $G(\tau)$ given by Theorem 7.4, the relations $y = z = 1$, we obtain the free procyclic group.

The following result adds further information about the presentation of Theorem 7.4.

Proposition 7.9: *For each $u \in \{x, y, z\}$, the pseudoidentity $\Psi^\omega(u) = u$ fails in the two-element group C_2. Hence, for each $u \in \{x, y, z\}$, the relation $\Psi^\omega(u) = u$, which holds in $G(\tau)$, is nontrivial.*

Proof. Let a be the nonidentity element of C_2 and let $Y = \{x, y, z\}$. Then, in the notation of Lemma 3.1, one verifies that the transformation Ψ_{C_2} is idempotent. Moreover, if we identify each function $f \in C_2^Y$ with the triple $(f(x), f(y), f(z))$, then $\Psi_{C_2}(f_1, f_2, f_3) = (f_3 f_1 f_2, 1, 1)$. In particular, we obtain $\Psi^\omega_{C_2}(a, a, 1) = \Psi^\omega_{C_2}(a, 1, a) = (1, 1, 1)$. Hence, none of the pseudoidentities $\Psi^\omega(u) = u$, with $u \in Y$, is satisfied by C_2. ∎

8. Open problems

We end with a few open problems.

Problem 8.1: Let φ be a primitive substitution over a finite alphabet. And let $\mathsf{V}(\varphi)$ be the pseudovariety generated by the (continuous) homomorphic images of $G(\varphi)$.

(1) When is $\mathsf{V}(\varphi) = \mathsf{G}$?
(2) "Compute" $\mathsf{V}(\varphi)$.

Problem 8.2: (1) For which (minimal) subshifts $\mathcal{X}$ is the associated Schützenberger group $G(\mathcal{X})$ decidable?
(1) In particular, is there any such group which is undecidable?

It is well known that a free profinite group relatively to an extension-closed pseudovariety is G-projective as a profinite group (cf. [15] and [29, Corollary 11.2.3]) and so, in view of the results of [3] or [25], finitely generated such groups certainly appear as closed subgroups of free profinite semigroups on two generators. P. Zalesskiĭ asked in the *Fields Workshop on Profinite Groups and Applications* (Carleton University, August 2005) and also in the Meeting of the *ESI Programme on Profinite Groups* (Vienna, December 2008) whether in particular free pro-p groups can appear as Schützenberger groups of free profinite semigroups. In our setting, and in view of the results of this section, this suggests the following question.

Problem 8.3: Let Φ be a continuous endomorphism of $\overline{\Omega}_X\mathsf{G}$ and let G be the profinite group presented by $\langle X \mid \Phi^\omega(x) = x \ (x \in X)\rangle$. Under what assumptions on Φ is G a relatively free profinite group?

As has been pointed out by the referee, if H is a pseudovariety of groups that contains every finite group whose Frattini quotient belongs to H, then $\overline{\Omega}_X\mathsf{H}$

admits such a presentation. Indeed every free pro-H group is G-projective [27, Proposition 7.6.7], and every finitely generated G-projective profinite group has such a presentation by Corollary 2.5.

Acknowledgment. We thank the anonymous referee for several suggestions, corrections, and comments that contributed to a significant improvement in the presentation of the paper.

References

[1] J. Almeida, *Finite Semigroups and Universal Algebra*, World Scientific, Singapore, 1995, English translation.

[2] J. Almeida, *Dynamics of implicit operations and tameness of pseudovarieties of groups*, Transactions of the American Mathematical Society **354** (2002), 387–411.

[3] J. Almeida, *Profinite groups associated with weakly primitive substitutions*, Fundamentalnaya i Prikladnaya Matematika (Fundamental and Applied Mathematics) **11** (2005), 13–48, In Russian. English version in Journal of Mathematical Sciences **144** (2007), 3881–3903.

[4] J. Almeida, *Profinite semigroups and applications*, in *Structural Theory of Automata, Semigroups, and Universal Algebra (New York)* (V. B. Kudryavtsev and I. G. Rosenberg, eds.), NATO Science Series II: Mathematics, Physics and Chemistry, Vol. 207, Springer, Berlin, 2005, Proceedings of the NATO Advanced Study Institute on Structural Theory of Automata, Semigroups and Universal Algebra, Montréal, Québec, Canada, 7–18 July, 2003, pp. 1–45.

[5] J. Almeida and A. Costa, *Infinite-vertex free profinite semigroupoids and symbolic dynamics*, Journal of Pure and Applied Algebra **213** (2009), 605–631.

[6] J. Almeida and M. V. Volkov, *Subword complexity of profinite words and subgroups of free profinite semigroups*, International Journal of Algebra and Computation **16** (2006), 221–258.

[7] L. Balková, E. Pelantová and W. Steiner, *Sequences with constant number of return words*, Monatshefte für Mathematik **155** (2008), 251–263.

[8] A. Costa, *Conjugacy invariants of subshifts: an approach from profinite semigroup theory*, International Journal of Algebra and Computation **16** (2006), 629–655.

[9] A. Costa and B. Steinberg, *Profinite groups associated to sofic shifts are free*, Proceedings of the London Mathematical Society **102** (2011), 341–369.

[10] F. Durand, *A characterization of substitutive sequences using return words*, Discrete Mathematics **179** (1998), 89–101.

[11] F. Durand, *Linearly recurrent subshifts have a finite number of non-periodic subshift factors*, Ergodic Theory and Dynamical Systems **20** (2000), 1061–1078.

[12] F. Durand, B. Host and C. Skau, *Substitutional dynamical systems, Bratteli diagrams and dimension groups*, Ergodic Theory and Dynamical Systems **19** (1999), 953–993.

[13] N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lecture Notes in Mathematics, Vol. 1794, Springer-Verlag, Berlin, 2002.

[14] The GAP Group, *GAP—Groups, Algorithms, and Programming, Version 4.4*, 2006 (`http://www.gap-system.org`).

[15] K. W. Gruenberg, *Projective profinite groups*, Journal of the London Mathematical Society **42** (1967), 155–165.

[16] T. Harju and M. Linna, *On the periodicity of morphisms on free monoids*, RAIRO Informatique Théorique et Applications **20** (1986), 47–54.

[17] R. P. Hunter, *Some remarks on subgroups defined by the Bohr compactification*, Semigroup Forum **26** (1983), 125–137.

[18] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, 1996.

[19] M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, Cambridge, 2002.

[20] A. Lubotzky, *Pro-finite presentations*, Journal of Algebra **242** (2001), 672–690.

[21] S. Margolis, M. Sapir and P. Weil, *Irreducibility of certain pseudovarieties*, Communications in Algebra **26** (1998), 779–792.

[22] B. Mossé, *Puissances de mots et reconnaissabilité des points fixes d'une substitution*, Theoretical Computer Science **99** (1992), 327–334.

[23] B. Mossé, *Reconnaissabilité des substitutions et complexité des suites automatiques*, Bulletin de la Société Mathématique de France **124** (1996), 329–346.

[24] J.-J. Pansiot, *Decidability of periodicity for infinite words*, RAIRO Informatique Théorique et Applications **20** (1986), 43–46.

[25] J. Rhodes and B. Steinberg, *Closed subgroups of free profinite monoids are projective profinite groups*, The Bulletin of the London Mathematical Society **40** (2008), 375–383.

[26] J. Rhodes and B. Steinberg, *The q-theory of Finite Semigroups*, Springer Monographs in Mathematics, Springer, Berlin, 2009.

[27] L. Ribes and P. A. Zalesskiĭ, *Profinite groups*, Ergebenisse der Mathematik und ihrer Grenzgebiete, Vol. 40, Springer, Berlin, 2000.

[28] B. Steinberg, *Maximal subgroups of the minimal ideal of a free profinite monoid are free*, Israel Journal of Mathematics **176** (2010), 139–155.

[29] J. Wilson, *Profinite Groups*, London Mathematical Society Monographs, New Series, Vol. 19, Clarendon, Oxford, 1998.