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EXTRINSIC ISOPERIMETRY AND COMPACTIFICATION OF MINIMAL SURFACES IN EUCLIDEAN AND HYPERBOLIC SPACES

BY

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ABSTRACT

We study the topology of (properly) immersed complete minimal surfaces P^2 in Hyperbolic and Euclidean spaces which have finite total extrinsic curvature, using some isoperimetric inequalities satisfied by the extrinsic balls in these surfaces (see [10]). We present an alternative and unified proof of the Chern–Osserman inequality satisfied by these minimal surfaces (in $\mathbb{R}^n$ and in $\mathbb{H}^n(b)$), based in the isoperimetric analysis mentioned above. Finally, we show a Chern–Osserman-type equality attained by complete minimal surfaces in the Hyperbolic space with finite total extrinsic curvature.

1. Introduction

Let P^2 be a complete and minimal surface immersed in $\mathbb{R}^n$ and with finite total curvature $\int_P K^P d\sigma < \infty$, K^P being the Gauss curvature of the surface.

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Then we have the following equality (resp. inequality), known as the **Chern–Osserman formula** (see [1], [3] and [6]):

(1.1)
$$-\chi(P) = \frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma - \mathrm{Sup}_r \frac{\mathrm{Vol}(P^2 \cap B_r^{0,n})}{\mathrm{Vol}(B_r^{0,2})} \leq \frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma - k(P),$$

where $\chi(P)$ is the Euler characterisitic of P, k is its number of ends, B^P is the second fundamental form of P in $\mathbb{R}^n$ and $B_r^{b,n}$ denotes the geodesic r-ball in the simply connected real space form $\mathbb{K}^n(b)$.

To have finite total scalar (extrinsic) curvature $\int_P \|B^P\|^2 d\sigma < \infty$ is equivalent to the finiteness of the total Gaussian curvature (the original assumption in [3]) when the surface is minimal and immersed in $\mathbb{R}^n$. From this point of view, it is natural to expect that it is possible to establish a Chern–Osserman inequality (or equality) for complete minimal surfaces with finite total extrinsic curvature (properly) immersed in the hyperbolic space. This question has been addressed by Qing and Yi in the papers [12] and [13]. They proved, for a complete minimal surface P^2 (properly) immersed in $\mathbb{H}^n(b)$ and such that $\int_P \|B^P\| d\sigma < \infty$, that

$$\mathrm{Sup}_r \frac{\mathrm{Vol}(P^2 \cap B_r^{-1,n})}{\mathrm{Vol}(B_r^{-1,2})} < \infty$$

and the following version of the Chern–Osserman Inequality, in terms of the volume growth of the extrinsic balls:

$$-\chi(P) \leq \frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma - \mathrm{Sup}_r \frac{\mathrm{Vol}(P^2 \cap B_r^{-1,n})}{\mathrm{Vol}(B_r^{-1,2})}. \tag{1.2}$$

The proofs given by these authors are different from those for the Euclidean case, and rely heavily on the properties of the hyperbolic functions.

We present in this paper a unified proof of the Chern–Osserman inequality (in terms of the volume growth) for complete minimal surfaces with finite total extrinsic curvature immersed in Euclidean or Hyperbolic spaces. This unification is based on obtaining estimates for the Euler characteristic of the extrinsic balls (given in Lemma 3.1 and Proposition 3.2) and on the isoperimetric inequality for the extrinsic balls given in Theorem 1.1 in [10]. These results are based on the divergence theorem and the Hessian and Laplacian comparison theory of restricted distance functions (see [4], [5] and [11]) which involves bounds on the mean curvature of the submanifold.

We have proved the following Chern–Osserman inequality, which encompasses inequalities (1.1) and (1.2):

THEOREM A: *Let P^2 be a complete minimal surface immersed in a simply connected real space form $\mathbb{K}^n(b)$ with constant sectional curvature $b \leq 0$. Let us suppose that $\int_P \|B^P\|^2 d\sigma < \infty$. Then:*

(1) *P has finite topological type.*
(2) $\mathrm{Sup}_{t>0}(\mathrm{Vol}(D_t)/\mathrm{Vol}(B_t^{b,2})) < \infty$, *where $D_t = P \cap B_t^{b,n}$ is the extrinsic t-ball in P (see Definition 2.1 in Section 2).*
(3)
$$-\chi(P) \leq \frac{\int_P \|B^P\|^2}{4\pi} - \mathrm{Sup}_{t>0} \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{b,2})},$$

where $\chi(P)$ is the Euler characteristic of P.

Although with this approach we are not able to state equality (1.1) in the Euclidean setting, we shall prove in Theorem B the following Chern–Osserman-type equality for complete and minimal immersed (cmi for short) surfaces in the Hyperbolic space:

THEOREM B: *Let P^2 be a complete immersed minimal surface in $\mathbb{H}^n(b)$. Let us suppose that $\int_P \|B^P\|^2 d\sigma < \infty$. Then*

$$-\chi(P) = \frac{1}{4\pi} \int_P \|B^P\|^2 d\sigma - \mathrm{Sup}_{t>0} \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{b,2})} - \frac{1}{2\pi} G_b(P), \tag{1.3}$$

where $G_b(P)$ is a non-negative and finite quantity which does not depend on the exhaustion by extrinsic balls $\{D_t\}_{t>0}$ of P and is given by

$$G_b(P) := \lim_{t\to\infty} \Big(h_b(t)\, \mathrm{Vol}(B_t^{b,2}) \Big(\frac{(\mathrm{Vol}(D_t))}{\mathrm{Vol}(B_t^{b,2})} \Big)' + \int_{\partial D_t} \Big\langle B^P(e,e), \frac{\nabla^\perp r}{\| \nabla^P r\|} \Big\rangle d\sigma_t \Big). \tag{1.4}$$

1.1. OUTLINE. The outline of the paper is the following. In Section 2 we present the basic facts about the Hessian comparison theory of the restricted distance function we are going to use, obtaining as a corollary the compactification of cmi surfaces in $\mathbb{K}^n(b)$ with finite total extrinsic curvature (Corollary 2.3). Section 3 is devoted to the unified proof of the Chern–Osserman inequality for complete minimal surfaces with finite total extrinsic curvature immersed in Euclidean and Hyperbolic spaces (Theorem A), and in Section 4 a Chern–Osserman-type equality satisfied by the cmi surfaces in $\mathbb{H}^n(b)$ is proved (Theorem B).

2. Preliminaries

2.1. THE EXTRINSIC DISTANCE. We assume throughout the paper that P^2 is a complete, non-compact, immersed, 2-dimensional submanifold in a simply connected real space form of non-positive constant sectional curvature $\mathbb{K}^n(b)$ ($\mathbb{K}^n(b) = \mathbb{R}^n$ when $b = 0$ and $\mathbb{K}^n(b) = \mathbb{H}^n(b)$ when $b < 0$). All the points in these manifolds are poles. Recall that a pole is a point o such that the exponential map

$$\exp_o\colon T_oN^n \to N^n$$

is a diffeomorphism. For every $x \in N^n \setminus \{o\}$ we define $r_o(x) = \operatorname{dist}_N(o, x)$, and this distance is realized by the length of a unique geodesic from o to x, which is the **radial geodesic from** o. We also denote by r the restriction $r_o|_P : P \to \mathbb{R}_+\cup\{0\}$. This restriction is called the **extrinsic distance function** from o in P^m. The gradients of r in N and P are denoted by $\nabla^N r$ and $\nabla^P r$, respectively. Let us remark that $\nabla^P r(x)$ is just the tangential component in P of $\nabla^N r(x)$, for all $x \in S$. Then we have the following basic relation:

$$\nabla^N r = \nabla^P r + (\nabla^N r)^\perp, \tag{2.1}$$

where $(\nabla^N r)^\perp(x) = \nabla^\perp r(x)$ is perpendicular to T_xP for all $x \in P$.

On the other hand, we should recall that all immersed surfaces P in the real space forms of non-positive constant sectional curvature $N^n = \mathbb{K}^n(b)$ which satisfy $\int_P \|B^P\|^2 d\sigma < \infty$ are properly immersed (see [1], [8] and [9]). Therefore, we can omit the hypothesis about the properness of the immersion when we assume that $\int_P \|B^P\|^2 d\sigma < \infty$.

Definition 2.1: Given a connected and complete surface P^2 properly immersed in a manifold N^n with a pole $o \in N$, we denote the **extrinsic metric balls** of radius $t > 0$ and center $o \in N$ by $D_t(o)$. They are defined as the intersection

$$D_t(o) = B^N_t(o) \cap P = \{x \in P\colon r(x) < t\},$$

where $B^N_t(o)$ denotes the open geodesic ball of radius R centered at the pole o in N^n.

Remark a: We want to point out that the extrinsic domains $D_t(o)$ are precompact sets (because we assume in the definition above that the submanifold P is properly immersed), with boundary $\partial D_t(o)$ being an immersed curve in P for $t > 0$ almost everywhere. The generic smoothness of $\partial D_t(o)$ follows from the following considerations: the distance function r is smooth in $\mathbb{K}^n(b) \setminus \{o\}$ for $\mathbb{K}^n(b)$ to possess a pole $o \in \mathbb{K}^n(b)$ $(b \leq 0)$. Hence the restriction $r|_P$ is smooth in P, and consequently the radii t that produce smooth boundaries $\partial D_t(o)$ are dense in $\mathbb{R}$ by Sard's theorem and the Regular Level Set Theorem.

Remark b: When the submanifold considered is totally geodesic, namely, when P is a Hyperbolic or an Euclidean subspace $\mathbb{K}^m(b)$ of the ambient real space form $\mathbb{K}^n(b)$, the extrinsic t-balls become geodesic t-balls, and its boundary is the distance sphere of radius t in $\mathbb{K}^m(b)$. We recall here that the mean curvature of the geodesic sphere of radius t in the real space form $\mathbb{K}^n(b)$ (denoted as $S_t^{b,n-1}$) 'pointing inward' is (see [10])

$$h_b(t) = \begin{cases} \sqrt{b}\cot\sqrt{b}t & \text{if } \ b > 0, \\ 1/t & \text{if } \ b = 0, \\ \sqrt{-b}\coth\sqrt{-b}t & \text{if } \ b < 0. \end{cases}$$

2.2. Hessian comparison analysis of the extrinsic distance. Let us now consider D_t an extrinsic ball in a complete and properly immersed minimal surface P in the real space form $\mathbb{K}^n(b)$ with $b \leq 0$. We are going to apply the Gauss–Bonnet formula to the curve ∂D_t. To do that, we need to compute its geodesic curvature in the following

Proposition 2.2: *When ∂D_t is a smooth curve, its geodesic curvature $k_g^{\partial D_t}$ is given by*

$$k_g^{\partial D_t} = \frac{h_b(t)}{\|\nabla^P r\|} + \left\langle B^P(e,e), \frac{\nabla^\perp r}{\|\nabla^P r\|}\right\rangle. \tag{2.2}$$

Proof. Let $\{e,\nu\} \subset TP$ be an orthonormal frame along the curve ∂D_t, where e is the unit tangent vector to ∂D_t and $\nu = \nabla^P r/\|\nabla^P r\|$ is the unit normal to ∂D_t in P, pointing outward.

From the definition of the geodesic curvature of the extrinsic boundaries ∂D_t, we have

$$k_g^t = -\left\langle \nabla_e^P e, \frac{\nabla^P r}{\|\nabla^P r\|}\right\rangle \tag{2.3}$$

Then, taking into account the definition of the Hessian

$$\operatorname{Hess}^P r(e,e) = \langle \nabla^P \nabla^P r, e\rangle$$

and the fact that $\nabla^P r$ and e are orthogonal,

$$k_g^t = \frac{1}{\|\nabla^P r\|}\mathrm{Hess}^P r(e,e). \tag{2.4}$$

But given $X \in T_q P$ unitary (see [5] and [11] for detailed computations),

$$\mathrm{Hess}^P(r)(X,X) = h_b(r)\big(1 - \langle X, \nabla^{\mathbb{K}^n(b)} r\rangle^2\big) + \langle \nabla^{\mathbb{K}^n(b)} r, B^P(X,X)\rangle, \tag{2.5}$$

where B^P is the second fundamental form of P in N. Applying at this point equation (2.5) we have

$$k_g^t = \frac{1}{\|\nabla^P r\|}\{h_b(r) + \langle \nabla^\perp r, B^P(e,e)\rangle\}. \tag{2.6}$$ ∎

Now we consider $\{D_t\}_{t>0}$ an exhaustion of P by extrinsic balls. Recall than an exhaustion of the submanifold P is a sequence of subsets $\{D_t \subseteq P\}_{t>0}$ such that

- $D_t \subseteq D_s$ when $s \geq t$,
- $\bigcup_{t>0} D_t = P$.

Using the equality (2.2) for the geodesic curvature of the extrinsic curves we have the following result:

Theorem 2.3: *Let P^2 be a complete minimal surface immersed in a simply connected real space form $\mathbb{K}^n(b)$ with constant sectional curvature $b \leq 0$. Let us suppose that $\int_P \|B^P\|^2 d\sigma < \infty$. Then:*

(i) *P is diffeomorphic to a compact surface P^* punctured at a finite number of points.*

(ii) *For all sufficiently large $t > R_0 > 0$, $\chi(P) = \chi(D_t)$ and hence, given $\{D_t\}_{t>0}$ an exhaustion of P by extrinsic balls,*

$$\chi(P) = \lim_{t\to\infty} \chi(D_t).$$

Proof. Let us consider $\{D_t\}_{t>0}$ an exhaustion of P by extrinsic balls, centered at the pole $o \in \mathbb{K}^n(b)$. We apply Lemma 2.2 to the smooth curves ∂D_t: As

$$-\|B^P\| \leq \langle B^P(e,e), \nabla^\perp r\rangle \leq \|B^P\|,$$

we have, on the points of the curve $q \in \partial D_t$,

$$\begin{aligned}\|\nabla^P r\|(q)\cdot k_g^{\partial D_t}(q) &= h_b(r_o(q)) + \langle B^P(e,e), \nabla^\perp r\rangle(q)\\ &\geq h_b(r_o(q)) - \|B^P\|(q).\end{aligned} \tag{2.7}$$

Using now Proposition 2.2 in [1], when P^2 is a cmi in $\mathbb{R}^n$, or Lemma 3.1 in [9], when P^2 is a cmi in $\mathbb{H}^n(b)$, we know that $\|B^P\|(q)$ goes uniformly to 0 as $t = r_o(q) \to \infty$. Hence, for all the points $q \in \partial D_t$ and for sufficiently large t,

$$\| \nabla^P r\|(q) \cdot k_g^{\partial D_t}(q) \; > 0. \tag{2.8}$$

We are going to examine this last assertion in more detail: when the ambient space is the Hyperbolic space $\mathbb{H}^n(b)$ $(b < 0)$, then we have that $h_b(r_o(q)) - \|B^P\|(q) > 0$ on ∂D_t and for all sufficiently large t because, applying Lemma 3.1 in [9], $\|B^P\|(q)$ goes uniformly to 0 as $t = r_o(q) \to \infty$ and we have in this case that $h_b(r) = \sqrt{-b}\coth\sqrt{-b}r \geq \sqrt{-b} > 0 \quad \forall r \geq 0$.

When the ambient space is the Euclidean space we need a more careful analysis: applying Proposition 2.2 in [1] we have that, for all sufficiently large t, $\operatorname{Sup}_{q\in\partial D_t} \|B^P\| \leq \epsilon(t)/t$, where $\epsilon(t)$ is a non-negative function such that $\epsilon(t) \to 0$ when $t \to \infty$. Hence we have, for all sufficiently large t and $\forall q \in \partial D_t$, that $h_0(r_o(q)) - \|B^P\|(q) \geq (1 - \epsilon(t))/t > 0$.

Hence $\| \nabla^P r\| > 0$ in ∂D_t, for all sufficiently large t (in the Euclidean case, we have moreover Lemma 2.4 in [1] where it is proved that, when P is a cmi surface in $\mathbb{R}^n$ with finite total extrinsic curvature, then $\| \nabla^P r\| > \frac{1}{2}$ outside a compact in P). Fixing a sufficiently large radius R_0, we can conclude that the extrinsic distance r_o has no critical points in $P \setminus \mathrm{D}_{R_0}$.

The above inequality implies that for this sufficiently large fixed radius R_0, there is a diffeomorphism

$$\Phi : P \setminus \mathrm{D}_{R_0} \to \partial D_{R_0} \times [0, \infty[.$$

In particular, P has only finitely many ends, each one of finite topological type.

To prove this we apply Theorem 3.1 in [7], concluding that, as the extrinsic annuli $A_{R_0,R}(o) = D_R(o) \setminus D_{R_0}(o)$ contain no critical points of the extrinsic distance function $r_o : P \longrightarrow \mathbb{R}^+$ because of inequality (2.8), then $D_R(o)$ is diffeomorphic to $D_{R_0}(o)$ for all $R \geq R_0$.

The above diffeomorphism implies that we can construct P from D_{R_0} (R_0 large enough) attaching annuli, and that $\chi(P \setminus D_t) = 0$ when $t \geq R_0$. Then, for all $t > R_0$,

$$\chi(P) = \chi(D_t \cup (P \setminus D_t)) = \chi(D_t). \qquad \blacksquare$$

3. Proof of Theorem A

We begin with the following results which are the common ingredients of the proof, for both the Euclidean and Hyperbolic cases:

Lemma 3.1: *Let $P^2 \subset \mathbb{K}^n(b)$ be a surface properly immersed in a real space form with curvature $b \leq 0$, let D_t be an extrinsic disc in P of radius $t > 0$ and let ∂D_t be the extrinsic circle. Then*

$$\int_{\partial D_t} \frac{\|\nabla^\perp r\|^2}{\|\nabla^P r\|} d\sigma_t \leq \int_{\partial D_t} \frac{d\sigma_t}{\|\nabla^P r\|} - h_b(t)\operatorname{Vol}(D_t). \tag{3.1}$$

Proof. Tracing equality (2.5) we obtain the following expression for the Laplacian of the extrinsic distance in this context:

$$\Delta^P(r) = (2 - \|\nabla^P r\|^2) h_b(r) + 2\langle \nabla^N r, H_P \rangle, \tag{3.2}$$

where H_P denotes the mean curvature vector of P in N and $h_b(r)$ is the mean curvature of the geodesic r-spheres in $\mathbb{K}^n(b)$. Applying the divergence theorem we have

$$\begin{aligned}
\int_{\partial D_t} \frac{\|\nabla^\perp r\|^2}{\|\nabla^P r\|} d\sigma_t &= \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t - \int_{\partial D_t} \|\nabla^P r\| d\sigma_t \\
&= \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t - \int_{D_t} \Delta^P r d\sigma \\
&= \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t - \int_{D_t} (2 - \|\nabla^P r\|^2) h_b(r) d\sigma \\
&\leq \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t - \int_{D_t} h_b(r) d\sigma \\
&\leq \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t - h_b(t)\operatorname{Vol}(D_t). \qquad \blacksquare
\end{aligned} \tag{3.3}$$

Proposition 3.2: *Let $P^2 \subset \mathbb{K}^n(b)$ be a complete minimal surface properly immersed in a real space form with curvature $b \leq 0$, let D_t be an extrinsic disc in P of radius $t > 0$ and let ∂D_t be its boundary. Then*

$$\begin{aligned}
&-2\pi\chi(D_t) + \left(b + \frac{f_{b,\alpha}^2(t) h_b(t)}{2}\right)\operatorname{Vol}(D_t) \\
&\quad + \left(h_b(t) - \frac{f_{b,\alpha}^2(t)}{2}\right) \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t \leq \frac{1}{2} R(t) + \frac{1}{2 f_{b,\alpha}^2(t)} R'(t),
\end{aligned} \tag{3.4}$$

where $R(t) = \int_{D_t} \|B^P\|^2 d\sigma$, $\|B^P\|$ *is the norm of the second fundamental form of* P *in* $\mathbb{K}^n(b)$, $\chi(D_t)$ *is the Euler's characterisc of* D_t *and, given any fixed number* $\alpha \in]0,2[$,

$$f_{b,\alpha}^2(t) = \alpha h_b(t).$$

Proof. Integrating equation (2.2) along ∂D_t and using the Gauss–Bonnet theorem, we obtain

$$\begin{aligned}(3.5)\quad 2\pi\chi(D_t) - \int_{D_t} K^P d\sigma \\ = h_b(t)\int_{\partial D_t} \frac{1}{\| \nabla^P r\|} d\sigma_t + \int_{\partial D_t} \Big\langle B^P(e,e), \frac{\nabla^\perp r}{\| \nabla^P r\|}\Big\rangle d\sigma_t,\end{aligned}$$

where we denote by K^P the Gauss curvature of P.

But on ∂D_t,

$$-\|B^P\| \frac{\| \nabla^\perp r\|}{\| \nabla^P r\|} \leq \Big\langle B^P(e,e), \frac{\nabla^\perp r}{\| \nabla^P r\|}\Big\rangle \leq \|B^P\| \frac{\| \nabla^\perp r\|}{\| \nabla^P r\|},$$

so as $f_{b,\alpha}(t) \geq 0 \forall t > 0$, taking into account the arithmetic–geometric inequality and applying the co-area formula, we have

$$\begin{aligned}
&2\pi\chi(D_t) - \int_{D_t} K^P d\sigma \\
&= h_b(t)\int_{\partial D_t} \frac{1}{\| \nabla^P r\|} d\sigma_t + \int_{\partial D_t} \Big\langle B^P(e,e), \frac{\nabla^\perp r}{\| \nabla^P r\|}\Big\rangle d\sigma_t \\
(3.6)\quad &\geq h_b(t)\int_{\partial D_t} \frac{1}{\| \nabla^P r\|} d\sigma_t \\
&\quad - \frac{1}{2}\int_{\partial D_t} \frac{\|B^P\|^2}{f_{b,\alpha}^2(r)\| \nabla^P r\|} d\sigma_t - \frac{1}{2}\int_{\partial D_t} \frac{f_{b,\alpha}^2(r)\| \nabla^\perp r\|^2}{\| \nabla^P r\|} d\sigma_t \\
&\geq h_b(t)\int_{\partial D_t} \frac{1}{\| \nabla^P r\|} d\sigma_t - \frac{1}{2f_{b,\alpha}^2(t)} R'(t) - \frac{f_{b,\alpha}^2(t)}{2}\int_{\partial D_t} \frac{\| \nabla^\perp r\|^2}{\| \nabla^P r\|} d\sigma_t.
\end{aligned}$$

Then, using inequality (3.1) of Lemma 3.1 in the last member of the inequalities (3.6) and applying the Gauss equation for minimal surfaces in the real space forms $\mathbb{K}^n(b)$, we have

$$2\pi\chi(D_t) - b\,\mathrm{Vol}(D_t) + \frac{1}{2}R(t) \geq \Big(h_b(t) - \frac{f_{b,\alpha}^2(t)}{2}\Big)\int_{\partial D_t} \frac{1}{\|\nabla^P r\|}d\sigma_t - \frac{1}{2f_{b,\alpha}^2(t)}R'(t) + \frac{f_{b,\alpha}^2(t)h_b(t)}{2}\mathrm{Vol}(D_t), \tag{3.7}$$

and hence

$$-2\pi\chi(D_t) + \Big(b + \frac{f_{b,\alpha}^2(t)h_b(t)}{2}\Big)\mathrm{Vol}(D_t) + \Big(h_b(t) - \frac{f_{b,\alpha}^2(t)}{2}\Big)\int_{\partial D_t}\frac{1}{\|\nabla^P r\|} \leq \frac{1}{2}R(t) + \frac{1}{2f_{b,\alpha}^2(t)}R'(t). \tag{3.8}$$

∎

We are going to divide the proof into two cases: Case I, where the ambient space is the Hyperbolic space $\mathbb{H}^n(b)$, and Case II where the ambient space is the Euclidean space $\mathbb{R}^n$.

Case I. Let us consider P (properly) immersed in $\mathbb{H}^n(b)$. Let $\{D_t\}_{t>0}$ be an exhaustion of P by extrinsic balls. Using the co-area formula, we know that, for $t > 0$ almost everywhere,

$$\frac{d}{dt}\mathrm{Vol}(D_t) = \int_{\partial D_t}\frac{1}{\|\nabla^P r\|}d\sigma_t. \tag{3.9}$$

Hence, applying Proposition 3.2 we have

$$-2\pi\chi(D_t) + \Big(b + \frac{f_{b,\alpha}^2(t)h_b(t)}{2}\Big)\mathrm{Vol}(D_t) + \Big(h_b(t) - \frac{f_{b,\alpha}^2(t)}{2}\Big)\frac{d}{dt}\mathrm{Vol}(D_t) \leq \frac{1}{2}R(t) + \frac{1}{2f_{b,\alpha}^2(t)}R'(t). \tag{3.10}$$

On the other hand, from (3.9), $\frac{d}{dt}\mathrm{Vol}(D_t) \geq \mathrm{Vol}(\partial D_t)$. Therefore, using inequality (3.10) we obtain

$$-2\pi\chi(D_t) + \mathrm{Vol}(D_t)\Big[\Big(b+\frac{f_{b,\alpha}^2(t)h_b(t)}{2}\Big)+\Big(h_b(t)-\frac{f_{b,\alpha}^2(t)}{2}\Big)\frac{\mathrm{Vol}(\partial D_t)}{\mathrm{Vol}(D_t)}\Big] \leq \frac{1}{2}R(t)+\frac{1}{2f_{b,\alpha}^2(t)}R'(t). \tag{3.11}$$

Applying the isoperimetric inequality in [10] (Theorem 1.1), we have

$$
\begin{aligned}
(3.12)\quad & -2\pi\chi(D_t) \\
& + \operatorname{Vol}(D_t)\Big[\Big(b+\frac{f_{b,\alpha}^2(t)h_b(t)}{2}\Big)+\Big(h_b(t)-\frac{f_{b,\alpha}^2(t)}{2}\Big)\frac{\operatorname{Vol}(S_t^{b,1})}{\operatorname{Vol}(B_t^{b,2})}\Big] \\
& \qquad \leq \frac{1}{2}R(t)+\frac{1}{2f_{b,\alpha}^2(t)}R'(t),
\end{aligned}
$$

where $S_t^{b,1}$ and $B_t^{b,2}$ denote, respectively, the geodesic t-sphere and t-ball in $\mathbb{K}^2(b)$. Hence, using the fact that

$$b\operatorname{Vol}(B_t^{b,2})+h_b(t)\operatorname{Vol}(S_t^{b,1})=2\pi \quad \forall t>0,$$

we obtain, with some computations,

$$
\begin{aligned}
(3.13)\quad & -2\pi\chi(D_t)+\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})}\Big[2\pi-2\pi\frac{f_{b,\alpha}^2(t)}{2}\frac{\operatorname{Vol}(B_t^{b,2})}{\operatorname{Vol}(S_t^{b,1})}\Big] \\
& \qquad \leq \frac{1}{2}R(t)+\frac{1}{2f_{b,\alpha}^2(t)}R'(t).
\end{aligned}
$$

Therefore, for all $t>0$,

$$(3.14)\quad \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})}\Big(1-\frac{\alpha h_b(t)}{2}\frac{\operatorname{Vol}(B_t^{b,2})}{\operatorname{Vol}(S_t^{b,1})}\Big)-\chi(D_t)\leq\frac{R(t)}{4\pi}+\frac{R'(t)}{4\pi\alpha h_b(t)}.$$

As

$$\frac{\|B^P\|^2}{h_b(t)}\leq\frac{1}{\sqrt{-b}}\|B^P\|^2,$$

then $\int_P\|B^P\|^2d\sigma<\infty$ implies

$$\int_P\frac{\|B^P\|^2}{h_b(t)}d\sigma<\infty.$$

Hence, by the co-area formula,

$$(3.15)\quad \int_0^\infty\Big(\int_{\partial D_t}\frac{\|B^P\|^2}{\|\nabla^P r\|h_b(r)}\Big)dt=\int_0^\infty\Big(\frac{R'(t)}{h_b(t)}\Big)dt<\infty.$$

Therefore, there is a monotone increasing (sub)sequence $\{t_i\}_{i=1}^\infty$ tending to infinity (namely, $t_i\to\infty$ when $i\to\infty$) such that $R'(t_i)/h_b(t_i)\to 0$ when $i\to\infty$.

Let us consider the exhaustion of P by these extrinsic balls, namely, $\{D_{t_i}\}_{i=1}^{\infty}$. Then we have, replacing t by t_i and taking limits when $i \to \infty$ in inequality (3.14) and applying Theorem 2.3 (ii),

$$\text{(3.16)} \quad \mathrm{Sup}_i \frac{\mathrm{Vol}(D_{t_i})}{\mathrm{Vol}(B^{b,2}_{t_i})}\Big(1-\frac{\alpha}{2}\Big)-\chi(P) \leq \lim_{i\to\infty}\frac{R(t_i)}{4\pi} = \frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma < \infty$$

for all α such that $0<\alpha<2$.

Hence, as $\mathrm{Vol}(D_t)/\mathrm{Vol}(B^{b,2}_t)$ is a continuous non-decreasing function of t, we can conclude that

$$\mathrm{Sup}_{t>0}\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B^{b,2}_t)}<\infty \quad \text{and} \quad -\chi(P)<\infty.$$

Then, letting α tend to 0 in (3.16), we get, for all $t>0$,

$$\text{(3.17)} \qquad \mathrm{Sup}_{t>0}\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B^{b,2}_t)}-\chi(P)\leq \frac{\int_P \|B^P\|^2}{4\pi}.$$

Case II. Let us consider P immersed in $\mathbb{R}^n$. We consider, as in the proof above, an exhaustion of P by extrinsic balls, $\{D_t\}_{t>0}$, but now, and following [1], these extrinsic balls will be centered at the origin $0\in\mathbb{R}^n$, which we assume, without loss of generality, belongs to the surface P. Applying Proposition 3.2 we have

$$\text{(3.18)} \quad -2\pi\chi(D_t)+\Big(\frac{\alpha}{2t^2}\Big)\mathrm{Vol}(D_t) + \Big(\frac{1}{t}-\frac{\alpha}{2t}\Big)\int_{\partial D_t}\frac{1}{\|\nabla^P r\|} \leq \frac{1}{2}R(t)+\frac{t}{2\alpha}R'(t).$$

Now, as $\int_P ||B^P||^2 d\sigma<\infty$, we can apply Proposition 2.2 in [1], so we have, for $\alpha\in]0,2[$,

$$\text{(3.19)} \quad \frac{t}{2\alpha}R'(t) = \frac{t}{2\alpha}\int_{\partial D_t}\frac{\|B^P\|^2}{\|\nabla^P r\|}d\sigma \leq \frac{\mu(t)}{2\alpha t}\int_{\partial D_t}\frac{1}{\|\nabla^P r\|}d\sigma,$$

$\mu(t)$ being such that $\lim_{t\to\infty}\mu(t)=0$ and therefore, from (3.18),

$$\text{(3.20)} \quad -2\pi\chi(D_t)+\mathrm{Vol}(D_t)\Big(\frac{\alpha}{2t^2}\Big) + \Big(\frac{1}{t}-\frac{\alpha}{2t}-\frac{\mu(t)}{2\alpha t}\Big)\int_{\partial D_t}\frac{1}{\|\nabla^P r\|}d\sigma_t \leq \frac{1}{2}R(t).$$

On the other hand,

$$\frac{1}{t}-\frac{\alpha}{2t}-\frac{\mu(t)}{2\alpha t}\geq 0$$

if and only if $\mu(t) \leq \alpha(2-\alpha)$, which is true for t large enough, namely, for $t > t_\alpha$ because $\lim_{t\to\infty} \mu(t) = 0$. Hence, as

$$\mathrm{Vol}(\partial D_t) \leq \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t,$$

and applying Theorem 1.1 in [10], we have that inequality (3.20) becomes, for all $t > t_\alpha$,

$$-2\pi\chi(D_t) + \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{0,2})}\Big[2\pi(1-\frac{\alpha}{2}-\frac{\mu(t)}{2\alpha}) + \frac{\pi\alpha}{2}\Big] \leq \frac{1}{2}R(t). \tag{3.21}$$

Then, taking limits when $t \to \infty$ in inequality (3.21) and applying Theorem 2.3, we have that $\lim_{t\to\infty} \mu(t) = 0$ and $\chi(P) = \lim_{t\to\infty} \chi(D_t)$, so we obtain, for all α such that $0 < \alpha < 2$,

$$2\pi \,\mathrm{Sup}_t \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{0,2})}\Big(1-\frac{\alpha}{2}+\frac{\pi\alpha}{2}\Big) - 2\pi\chi(P) \leq \frac{\int_P \|B^P\|^2}{2} < \infty. \tag{3.22}$$

Therefore we obtain

$$\mathrm{Sup}_{t>0} \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{0,2})} < \infty \quad \text{and} \quad -\chi(P) < \infty.$$

Then, letting α tend to 0, we obtain, for all $t > 0$,

$$\mathrm{Sup}_{t>0} \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{0,2})} - \chi(P) \leq \frac{\int_P \|B^P\|^2}{4\pi}. \tag{3.23}$$

4. Proof of Theorem B

In Corollary 2.3, we obtained a sufficiently large radius R_0 such that the extrinsic distance r_p has no critical points in $P \setminus \mathrm{D}_{R_0}$.

Hence for this sufficiently large fixed radius R_0, there is a diffeomorphism

$$\Phi : P \setminus \mathrm{D}_{R_0} \to \partial D_{R_0} \times [0, \infty[$$

so, in particular, P has only finitely many ends, each of finite topological type.

The above diffeomorphism implied that we could construct P from D_{R_0} (R_0 large enough) attaching annuli and that $\chi(P \setminus D_t) = 0$ when $t \geq R_0$, and hence for all $t > R_0$, $\chi(P) = \chi(D_t)$.

Let us consider now an exhaustion by extrinsic balls $\{D_t\}_{t>0}$ of P such that the extrinsic distance r_o has no critical points in $P \setminus \mathrm{D}_{R_0}$.

Applying now the Gauss–Bonnet Theorem to the extrinsic balls D_t,

$$2\pi\chi(P) = \int_{D_t} K^P d\sigma + \int_{\partial D_t} k_g d\sigma_t. \tag{4.1}$$

Taking into account equation (2.2) and the Gauss formula, we have, for every sufficiently large radius $t > R_0$,

$$\begin{aligned}2\pi\chi(P) =& -\frac{1}{2}\int_{D_t} \|B^P\|^2 + b\operatorname{Vol}(D_t) + h_b(t)(\operatorname{Vol}(D_t))' \\ &+ \int_{\partial D_t} \Big\langle B^P(e,e), \frac{\nabla^\perp r}{\|\nabla^P r\|}\Big\rangle d\sigma_t \\ =& -\frac{1}{2}\int_{D_t} \|B^P\|^2 d\sigma \\ &+ \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})}\Bigg(b\cdot\operatorname{Vol}(B_t^{b,2}) + h_b(t)(\operatorname{Vol}(D_t))'\frac{\operatorname{Vol}(B_t^{b,2})}{\operatorname{Vol}(D_t)} \\ &+ \frac{\operatorname{Vol}(B_t^{b,2})}{\operatorname{Vol}(D_t)}\int_{\partial D_t}\langle B^P(e,e), \frac{\nabla^\perp r}{\|\nabla^P r\|}\rangle d\sigma_t\Bigg).\end{aligned} \tag{4.2}$$

But $2\pi = b\cdot\operatorname{Vol}(B_t^{b,2}) + h_b(t)\operatorname{Vol}(S_t^{b,1})\ \forall t>0$, so, for every sufficiently large radius $t > R_0$ and after some computations,

(4.3)
$$\begin{aligned}2\pi\chi(P) =& -\frac{1}{2}\int_{D_t}\|B^P\|^2 d\sigma + 2\pi\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})} \\ &+ h_b(t)\operatorname{Vol}(B_t^{b,2})\Big(\frac{(\operatorname{Vol}(D_t))}{\operatorname{Vol}(B_t^{b,2})}\Big)' + \int_{\partial D_t}\Big\langle B^P(e,e), \frac{\nabla^\perp r}{\|\nabla^P r\|}\Big\rangle d\sigma_t.\end{aligned}$$

The above equation is valid for all $t > R_0$, so, taking limits when $t\to\infty$, we can define

$$\begin{aligned}G_b(P) := \lim_{t\to\infty}\Bigg(& h_b(t)\operatorname{Vol}(B_t^{b,2})\Big(\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})}\Big)' \\ &+ \int_{\partial D_t}\Big\langle B^P(e,e), \frac{\nabla^\perp r}{\|\nabla^P r\|}\Big\rangle d\sigma_t\Bigg).\end{aligned} \tag{4.4}$$

Using equalities (4.3), we have that

$$G_b(P) = 2\pi\chi(P) + \frac{1}{2}\int_{D_t}\|B^P\|^2 d\sigma - 2\pi\operatorname{Sup}_t\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})} < \infty, \tag{4.5}$$

and hence $G_b(P)$ does not depend on the exhaustion $\{D_t\}_{t>0}$.

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