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THE INVERSE FUETER MAPPING THEOREM IN INTEGRAL FORM USING SPHERICAL MONOGENICS

BY

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ABSTRACT

In this paper we prove an integral representation formula for the inverse Fueter mapping theorem for monogenic functions defined on axially symmetric open sets $U \subseteq \mathbb{R}^{n+1}$, i.e. on open sets U invariant under the action of $SO(n)$, where n is an odd number. Every monogenic function on such an open set U can be written as a series of axially monogenic functions of degree k, i.e. functions of type $\breve{f}_k(x) := [A(x_0,\rho)+\underline{\omega}B(x_0,\rho)]\mathcal{P}_k(\underline{x})$, where $A(x_0,\rho)$ and $B(x_0,\rho)$ satisfy a suitable Vekua-type system and $\mathcal{P}_k(\underline{x})$ is a homogeneous monogenic polynomial of degree k. The Fueter mapping theorem says that given a holomorphic function f of a paravector variable defined on U, then the function $\breve{f}(x)\mathcal{P}_k(\underline{x})$ given by

$$\Delta^{k+\frac{n-1}{2}}(f(x)\mathcal{P}_k(\underline{x})) = \breve{f}(x)\mathcal{P}_k(\underline{x})$$

is a monogenic function. The aim of this paper is to invert the Fueter mapping theorem by determining a holomorphic function f of a paravector variable in terms of $\breve{f}(x)\mathcal{P}_k(\underline{x})$. This result allows one to invert the Fueter mapping theorem for any monogenic function defined on an axially symmetric open set.

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1. Introduction and notations

The Fueter mapping theorem (see [14]) is an ingenious tool to generate Cauchy–Fueter regular functions from holomorphic functions defined on open sets in the upper complex plane. Such a theorem has been extended in order to obtain monogenic functions with values in a Clifford algebra $\mathbb{R}_n$ (see [1], [4], [16], [18]), by Sce [26] for n odd and by Qian [23] in the general case. Later on, Fueter's theorem has been generalized to the case in which a function f as above is multiplied by a monogenic homogeneous polynomial of degree k (see [20], [21], [22], [27]) and to the case in which the function f is defined on an open set U not necessarily chosen in the upper complex plane (see [23], [24], [25]).

The setting in which we work is the real Clifford algebra $\mathbb{R}_n$ over n imaginary units $e_1,\dots,e_n$ satisfying the relations $e_ie_j+e_je_i=-2\delta_{ij}$. An element in the Clifford algebra will be denoted by $\sum_A e_Ax_A$ where $A=i_1\cdots i_r$, $i_\ell\in\{1,2,\dots,n\}$, $i_1<\cdots<i_r$, is a multi-index, $e_A=e_{i_1}e_{i_2}\dots e_{i_r}$ and $e_\emptyset=1$. As it is well-known, $\mathbb{R}_1$ is the algebra of complex numbers $\mathbb{C}$ (the only case in which the Clifford algebra is commutative), while for $n=2$ we obtain the division algebra of real quaternions $\mathbb{H}$. For $n>2$, the Clifford algebras $\mathbb{R}_n$ have zero divisors. In $\mathbb{R}_n$, we can identify some specific elements with the vectors in the Euclidean space $\mathbb{R}^n$: an element $(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ can be identified with a so-called 1-vector in the Clifford algebra through the map $(x_1,x_2,\dots,x_n)\mapsto \underline{x}=x_1e_1+\cdots+x_ne_n$.

An element $(x_0,x_1,\dots,x_n)\in\mathbb{R}^{n+1}$ will be identified with the element $x=x_0+\underline{x}$ called, in short, a paravector. The norm of $x\in\mathbb{R}^{n+1}$ is defined as $|x|^2=x_0^2+x_1^2+\cdots+x_n^2$. The real part x_0 of x will be also denoted by $\mathrm{Re}[x]$. A function $f:\ U\subseteq\mathbb{R}^{n+1}\to\mathbb{R}_n$ is seen as a function $f(x)$ of the paravector x. We denote by $\mathbb{S}^{n-1}$ the sphere of unit 1-vectors in $\mathbb{R}^{n+1}$, i.e.

$$\mathbb{S}^{n-1}=\{\underline{x}=e_1x_1+\cdots+e_nx_n\ :\ x_1^2+\cdots+x_n^2=1\}.$$

Any element $\underline{\omega}\in\mathbb{S}^{n-1}$ is such that $\underline{\omega}^2=-1$. We will denote by ∂_x the Dirac operator $\partial_x=\partial_{x_0}+e_1\partial_{x_1}+\cdots+e_n\partial_{x_n}$ and we say that a smooth function is left monogenic on the open set U of $\mathbb{R}^{n+1}$ if it satisfies $\partial_xf(x)=0$ on U. In the sequel, we will denote by $\mathcal{M}(U)$ the right $\mathbb{R}_n$-module of (left) monogenic functions on the open set U and by $\mathcal{AM}(U)$ the $\mathbb{R}_n$-submodule of axially monogenic functions, i.e. monogenic functions of the form

$A(x_0,\rho)+\underline{\omega}B(x_0,\rho)$ with A,B satisfying a Vekua-type system, $x=x_0+\underline{\omega}\rho$ and $\underline{\omega}\in\mathbb{S}^{n-1}$.

Let us now recall the classical Fueter theorem in order to construct monogenic functions. Let f be a holomorphic function in an open set $\mathcal{U}$ of the upper half complex plane and let

$$f(u+\iota v)=\alpha(u,v)+\iota\beta(u,v),\quad u+\iota v\in\mathcal{U}\subseteq\mathbb{C}^+,$$

where α and β are differentiable functions with values in $\mathbb{R}$. Let us consider the axially symmetric open set

$$U=\{x=x_0+\underline{x}\in\mathbb{R}^{n+1}\mid x_0+\iota|\underline{x}|\in\mathcal{U}\},$$

which is called the open set induced by $\mathcal{U}$. Finally, let us replace u by x_0, v by $|\underline{x}|$ and ι by $\underline{x}/|\underline{x}|$. The Fueter mapping theorem states that the function

$$\Delta_x^{\frac{n-1}{2}}\big(\alpha(x_0,|\underline{x}|)+\frac{\underline{x}}{|\underline{x}|}\beta(x_0,|\underline{x}|)\big),$$

where Δ_x is the Laplace operator in dimension $n+1$, is a monogenic function on U. For the sake of simplicity, we will use the notation of cylindrical coordinates, i.e. we will write $x=x_0+\underline{\omega}\rho$ where $\rho=|\underline{x}|$, $\underline{\omega}=\underline{x}/|\underline{x}|$. In the sequel, taking two open sets $\mathcal{U}$ and U as above, it will be useful to consider the following set of functions:

$$\begin{aligned}\mathcal{N}(U)=\{f:U\subseteq\mathbb{R}^{n+1}\to\mathbb{R}_n, f(x)=f(x_0+\underline{\omega}|\underline{x}|)=\alpha(x_0,|\underline{x}|)+\underline{\omega}\beta(x_0,|\underline{x}|)|\\ \alpha(u,v)+\iota\beta(u,v)\text{ is a }\mathbb{C}\text{-valued holomorphic function in }u+\iota v\in\mathcal{U}\}.\end{aligned}$$

Note that when considering functions in $\mathcal{N}(U)$ (sometimes called holomorphic functions of a paravector variable (see [27]) or radially holomorphic functions (see [18])) α and β are $\mathbb{R}$-valued functions. However, one can consider a more general class of functions: let $f:U\subset\mathbb{R}^{n+1}\to\mathbb{R}_n$ be of the form $f(x)=f(x_0+\underline{\omega}|\underline{x}|)=\alpha(x_0,|\underline{x}|)+\underline{\omega}\beta(x_0,|\underline{x}|)$, where $\underline{\omega}\in\mathbb{S}^{n-1}$, α and β are $\mathbb{R}_n$-valued functions satisfying the Cauchy–Riemann system and suitable additional conditions. Note that such a function f is slice monogenic in the sense of [7] (for more detail on slice monogenic functions and some of their applications see [2], [3], [5], [8], [9], [10], [11]) or slice regular in the sense of [15]. In the recent paper [6] we have proved an integral representation formula for $f(x_0,\rho)$ in terms of the function $\breve{f}(x_0,\rho)$, where $\breve{f}(x_0,\rho)$ and $f(x_0,\rho)$ are related by

$$\breve{f}(x_0,\rho)=\Delta_x^{(n-1)/2}f(x_0,\rho). \tag{1}$$

The function $\breve{f}(x_0,\rho)$ is axially monogenic (see [12]) and we have proved that $\Delta^{(n-1)/2}$ is surjective onto $\mathcal{AM}(U)$; see [6]. Moreover, given $\breve{f}\in\mathcal{AM}(U)$ we constructed a slice monogenic function f satisfying (1), thus inverting the Fueter mapping theorem.

Remark 1.1: Let U be an axially symmetric open set U. Then, by Theorem 1 in [28], but see also [13], every left monogenic function $\breve{f}$ can be written in the form $\breve{f}(x)=\sum_{k=0}^{\infty}\breve{f}_k(x)$ where $\breve{f}_k(x)$ are axially monogenic functions of degree k , i.e. $\breve{f}_k(x)$ are functions of the form

$$\breve{f}_k(x)=A_k(x_0,\rho,\underline{\omega})+\underline{\omega}B_k(x_0,\rho,\underline{\omega})$$

where $A_k(x_0,\rho,\underline{\omega})$ and $B_k(x_0,\rho,\underline{\omega})$ satisfy the Vekua-type system:

$$\begin{cases}\partial_{x_0}A_k-\partial_\rho B_k=\frac{k+n-1}{\rho}B_k,\\ \partial_{x_0}B_k+\partial_\rho A_k=\frac{k}{\rho}A_k.\end{cases}\tag{2}$$

Even though the following definition is very well-known, we recall it for the sake of completeness.

Definition 1.2: A left monogenic polynomial $\mathcal{P}_k$ in $\mathbb{R}^{n+1}$ (resp. $\mathbb{R}^n$) is called an inner left spherical monogenic polynomial of degree $k\in\mathbb{N}_0$ if it is homogeneous of degree k, that is $\mathcal{P}_k(x/|x|)|x|^k$ (resp. $\mathcal{P}_k(\underline{x}/|\underline{x}|)|\underline{x}|^k$) and it satisfies $\partial_x\mathcal{P}_k(x)=0$ (resp. $\partial_{\underline{x}}\mathcal{P}_k(\underline{x})=0$).

Remark 1.3: Any axially monogenic function of degree k defined on an axially symmetric open set U can be written in the form

$$(A(x_0,\rho)+\underline{\omega}B(x_0,\rho))\mathcal{P}_k(\underline{x})\tag{3}$$

where $A(x_0,\rho)$, $B(x_0,\rho)$ are real valued; see [13] and also [22]. Note that the assumption that A, B are real valued is not a restriction; in fact, in view of the finite dimensionality of the space of inner spherical monogenics we can always decompose any axially monogenic function of degree k of the form $A_k(x_0,\rho,\underline{\omega})+\underline{\omega}B_k(x_0,\rho,\underline{\omega})$ into a finite sum of functions of the form (3). The two functions A, B satisfy a Vekua-type system which is a variation of (2); see [22].

The preceding discussion leads to the following result (see [13]):

THEOREM 1.4: *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set. Then every monogenic function $\breve{f} : U \to \mathbb{R}_n$ can be written in the form $\breve{f}(x) = \sum_{k=0}^{\infty} \breve{f}_k(x)$ with*

$$\breve{f}_k(x) = \sum_{j=1}^{m_k} [A_{k,j}(x_0,\rho) + \underline{\omega} B_{k,j}(x_0,\rho)]\mathcal{P}_{k,j}(\underline{x}), \tag{4}$$

where $\mathcal{P}_{k,j}$ form a basis for the space of spherical monogenics of degree k which has dimension m_k and $A_{k,j}$, $B_{k,j}$ are suitable real-valued functions.

The main aim of this paper is to find the inversion of the Fueter mapping theorem in the case of monogenic functions of type $(A_{k,j}(x_0,\rho)+\underline{\omega}B_{k,j}(x_0,\rho))\mathcal{P}_{k,j}(\underline{x})$ by providing their so-called Fueter primitive.

Problem 1.5: Suppose that $U \subseteq \mathbb{R}^{n+1}$ is an axially symmetric domain, where n is an odd number. Given the axially monogenic function of degree $k \in \mathbb{N}_0$

$$\breve{f}(x)\mathcal{P}_k(\underline{x}) = (A(x_0,\rho) + \underline{\omega}B(x_0,\rho))\mathcal{P}_k(\underline{x}),$$

where $\mathcal{P}_k(\underline{x})$ is a spherical monogenic polynomial of degree k, determine a function $f(x_0,\rho) = \alpha(x_0,\rho) + \underline{\omega}\beta(x_0,\rho) \in \mathcal{N}(U)$ such that

$$\Delta_x^{k+\frac{n-1}{2}}(f(x)\mathcal{P}_k(\underline{x})) = \breve{f}(x)\mathcal{P}_k(\underline{x}) \quad \text{on } U, \tag{5}$$

where Δ_x is the Laplace operator in dimension $n+1$.

The solution of Problem 1.5 is given by the following integral representation formula for $f(x)\mathcal{P}_k(\underline{x})$:

$$\begin{aligned} f(x)&\mathcal{P}_k(\underline{x}) \\ &= \int_\Gamma \mathcal{W}_{k,n}^{-}\Big(\frac{x-y_0}{\rho}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{\rho}\Big)\rho^{2k+n-2}[dy_0\, A(y_0,\rho) - d\rho\, B(y_0,\rho)] \\ &\quad - \int_\Gamma \mathcal{W}_{k,n}^{+}\Big(\frac{x-y_0}{\rho}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{\rho}\Big)\rho^{2k+n-2}[dy_0 B(y_0,\rho) + d\rho A(y_0,\rho)], \end{aligned} \tag{6}$$

where $f(x)\mathcal{P}_k(\underline{x})$ is the solution to equation (5) and Γ is a suitable regular curve. Here $\mathcal{W}_{k,n}^{+}$ and $\mathcal{W}_{k,n}^{-}$ are explicit kernels that are determined in Section 4.

From (4) and the integral representation formula (6) we show that we can find a Fueter primitive for **any** monogenic function on an axially symmetric open set; see Corollary 5.4.

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2. A restriction result for the Fueter primitive

Definition 2.1 (Fueter's Primitive): Let n be an odd number and let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain. Let

$$\breve{f}(x)\mathcal{P}_k(\underline{x}) = (A(x_0,\rho) + \underline{\omega}B(x_0,\rho))\mathcal{P}_k(\underline{x})$$

be an axially monogenic function of degree $k \in \mathbb{N}_0$. We say that a function $f(x)\mathcal{P}_k(\underline{x})$, $f \in \mathcal{N}(U)$ is a Fueter primitive of $\breve{f}(x)\mathcal{P}_k(\underline{x})$ if

$$\Delta_x^{k+\frac{n-1}{2}}(f(x)\mathcal{P}_k(\underline{x})) = \breve{f}(x)\mathcal{P}_k(\underline{x}) \quad \text{on } U,$$

where Δ_x is the Laplace operator in dimension $n+1$.

Given any paravector $x = x_0 + \underline{\omega}\rho$, where $\rho \neq 0$, it is obvious that x belongs to the complex plane $\mathbb{C}_{\underline{\omega}}$ and that any paravector on the real axis belongs to $\mathbb{C}_{\underline{\omega}}$ for all $\underline{\omega} \in \mathbb{S}^{n-1}$.

Definition 2.2: Let $U \subseteq \mathbb{R}^{n+1}$ be a domain. We say that U is a slice domain (s-domain for short) if $U \cap \mathbb{R}$ is non-empty and if $U \cap \mathbb{C}_{\underline{\omega}}$ is a domain in $\mathbb{C}_{\underline{\omega}}$ for all $\underline{\omega} \in \mathbb{S}^{n-1}$.

Remark 2.3: Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain. Suppose that $W \in \mathcal{N}(U)$. So function W admits the power series expansion

$$W(x) = \sum_{\ell \geq 0} \frac{1}{\ell!}\,\underline{x}^{\ell}\, V^{(\ell)}(x_0), \tag{7}$$

where $V(x_0) := W(x)|_{\underline{x}=0}$. For functions $W \in \mathcal{N}(U)$ the terms $V^{(\ell)}(x_0)$ are real numbers. The convergence is in a suitable ball $B(x_0, r)$ centered at $x_0 \in U \cap \mathbb{R}$ and radius $r > 0$. Finally, observe that the product $W(x)\mathcal{P}_k(\underline{x})$ is well defined and we can write

$$W(x)\mathcal{P}_k(\underline{x}) = \sum_{\ell \geq 0} \frac{1}{\ell!}\,\underline{x}^{\ell}\, V^{(\ell)}(x_0)\mathcal{P}_k(\underline{x}).$$

We have the following results which will be crucial in the sequel. We reason in the ball $B(x_0, r)$ of convergence, and for the Identity Principle for slice monogenic functions (see [7]) the result can be extended to the whole axially symmetric s-domain U.

Proposition 2.4: *Let U be an axially symmetric s-domain in $\mathbb{R}^{n+1}$, n be an odd number, and suppose that $W \in \mathcal{N}(U)$. Let $\mathcal{P}_k(\underline{x})$ be an inner left spherical* *monogenic polynomial of degree $k \in \mathbb{N}_0$. Let $x_0 \in U \cap \mathbb{R}$ and suppose that (7) is the power series expansion of W in $B(x_0, r)$. Then there exists a positive constant $\mathcal{H}_{k,n}$, independent of x_0, such that, for $\underline{x} \to \underline{0}$,*

$$\Delta_x^{k+\frac{(n-1)}{2}}(W(x)\mathcal{P}_k(\underline{x})) = \mathcal{H}_{n,k} V^{(2k+n-1)}(x_0)\mathcal{P}_k(\underline{x}) + \mathcal{R}(x_0, \underline{x})\mathcal{P}_k(\underline{x}),$$

where

$$\mathcal{H}_{k,n} := \sum_{j=0}^{k+\frac{n-1}{2}} \binom{k+\frac{n-1}{2}}{j} (-1)^j 2^{2j} j! \, \frac{1}{(2j)!} \, \frac{\Gamma\left(\frac{2k+n}{2}+j\right)}{\Gamma\left(\frac{2k+n}{2}\right)}, \tag{8}$$

and

$$\lim_{\underline{x}\to\underline{0}} \mathcal{R}(x_0, \underline{x}) = 0.$$

Proof. We set for simplicity $m = k+(n-1)/2$ and we calculate $\Delta_x^m(W(x)\mathcal{P}_k(\underline{x}))$, keeping in mind that we have to take the limit $\underline{x} \to \underline{0}$. Since $\Delta_x = \partial_{x_0}^2 + \Delta_{\underline{x}}$, we can write

$$\begin{aligned}\Delta_x^m(W(x)\mathcal{P}_k(\underline{x})) &= \sum_{j=0}^{m}\sum_{\ell\geq 0}\binom{m}{j}\frac{1}{\ell!}\,\partial_{x_0}^{2(m-j)}\Delta_{\underline{x}}^j\left(\underline{x}^\ell\,\mathcal{P}_k(\underline{x})V^{(\ell)}(x_0)\right)\\ &= \sum_{j=0}^{m}\sum_{\ell\geq 0}\binom{m}{j}\frac{1}{\ell!}\,\Delta_{\underline{x}}^j\left(\underline{x}^\ell\,\mathcal{P}_k(\underline{x})\right)\partial_{x_0}^{2(m-j)}V^{(\ell)}(x_0)\end{aligned}$$

and we observe that

$$\Delta_{\underline{x}}^j\left(\underline{x}^\ell\mathcal{P}_k(\underline{x})\right) = \begin{cases} 0 & \text{if } 2j > \ell,\\ C_\ell\mathcal{P}_k(\underline{x}) & \text{if } 2j = \ell,\\ \mathcal{E}(\underline{x})\mathcal{P}_k(\underline{x}) & \text{if } 2j < \ell,\end{cases}$$

where C_ℓ are constants depending on ℓ, and $\mathcal{E}$ is a continuous function such that $\mathcal{E}(\underline{x}) \to 0$ for $\underline{x} \to \underline{0}$. Moreover, it is easy to see that all the terms corresponding to $2j = \ell$ contain

$$\partial_{x_0}^{2(m-j)}V^{(2j)}(x_0) = V^{(2m)}(x_0).$$

So we have

$$\begin{aligned}\Delta_x^m(W(x)\mathcal{P}_k(\underline{x})) &= \sum_{j=0}^{m}\sum_{\ell\geq 0}\binom{m}{j}\frac{1}{\ell!}\,\Delta_{\underline{x}}^j\left(\underline{x}^\ell\,\mathcal{P}_k(\underline{x})\right)\partial_{x_0}^{2(m-j)}V^{(\ell)}(x_0)\\ &= \sum_{j=0}^{m}\frac{1}{(2j)!}\binom{m}{j}\Delta_{\underline{x}}^j\left(\underline{x}^{2j}\mathcal{P}_k(\underline{x})\right)\;\; V^{(2m)}(x_0) + \mathcal{R}(x_0,\underline{x})\mathcal{P}_k(\underline{x}).\end{aligned}$$

Let us set

$$\mathcal{H}_{n,k} := \sum_{j=0}^{m} \frac{1}{(2j)!} \binom{m}{j} \Delta_{\underline{x}}^{j}\Big(\underline{x}^{2j}\mathcal{P}_k(\underline{x})\Big), \quad \text{where } m = k + \frac{(n-1)}{2}.$$

Now recall that $\Delta_{\underline{x}} = -\partial_{\underline{x}}^2$, where $\partial_{\underline{x}}$ is the Dirac operator in dimension n, and the well-known relations

$$\partial_{\underline{x}}\Big(\underline{x}^{2s}\mathcal{P}_k(\underline{x})\Big) = -2s\ \underline{x}^{2s-1}\mathcal{P}_k(\underline{x}),$$

$$\partial_{\underline{x}}\Big(\underline{x}^{2s+1}\mathcal{P}_k(\underline{x})\Big) = -(2s+2k+n)\ \underline{x}^{2s}\mathcal{P}_k(\underline{x}).$$

Observe that $\Delta_{\underline{x}}^0\big(\underline{x}^0\mathcal{P}_k(\underline{x})\big) = 1$ and consider the terms

$$\begin{aligned}\Delta^j\left(\underline{x}^{2j}\mathcal{P}_k(\underline{x})\right) &= -\Delta^{j-1}\partial_{\underline{x}}^2\big(\underline{x}^{2j}\mathcal{P}_k(\underline{x})\big)\\ &= \Delta^{j-1}\partial_{\underline{x}}\big(2j\ \underline{x}^{2j-1}\mathcal{P}_k(\underline{x})\big)\\ &= -2j(2j-2+2k+n)\ \Delta^{j-1}\big(\underline{x}^{2(j-1)}\mathcal{P}_k(\underline{x})\big).\end{aligned}$$

So we get

$$\Delta\Big(\underline{x}^2\mathcal{P}_k(\underline{x})\Big) = -2(2k+n)\mathcal{P}_k(\underline{x}),$$

$$\Delta^2\Big(\underline{x}^4\mathcal{P}_k(\underline{x})\Big) = 2\cdot 4(2+2k+n)(2k+n)\mathcal{P}_k(\underline{x}),$$

and by induction we have

$$\Delta^j\Big(\underline{x}^{2j}\mathcal{P}_k(\underline{x})\Big) = (-1)^j 2^{2j} j!\ \frac{\Gamma\Big(\frac{2k+n}{2}+j\Big)}{\Gamma\Big(\frac{2k+n}{2}\Big)}\mathcal{P}_k(\underline{x}),$$

and we finally obtain that

$$\mathcal{H}_{k,n} := \sum_{j=0}^{m} \frac{1}{(2j)!}\binom{m}{j}(-1)^j 2^{2j} j!\ \frac{\Gamma\Big(\frac{2k+n}{2}+j\Big)}{\Gamma\Big(\frac{2k+n}{2}\Big)}.$$

This concludes the proof recalling that $m = k + (n-1)/2$. ∎

3. The kernels $\mathcal{F}_{k,n}^{+}(x)$ and $\mathcal{F}_{k,n}^{-}(x)$ and their factorization

Using the monogenic Cauchy kernel and the inner left spherical monogenic polynomials $\mathcal{P}_k(\underline{x})$ we define two important kernels that we will use in the sequel. We start by recalling the following:

Definition 3.1 (The monogenic Cauchy kernel): We denote by $\mathcal{G}$ the monogenic Cauchy kernel

$$\mathcal{G}(x) = \frac{1}{A_{n+1}}\,\frac{\overline{x}}{|x|^{n+1}}, \quad x \in \mathbb{R}^{n+1} \setminus \{0\}, \tag{9}$$

where A_{n+1} is the area of the unit sphere:

$$A_{n+1} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}.$$

Definition 3.2 (The kernels $\mathcal{F}^{+}_{k,n}(x)$ and $\mathcal{F}^{-}_{k,n}(x)$): Let $\mathcal{G}(x-\underline{y})$ be the monogenic Cauchy kernel defined in (9) with $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$, and for $\underline{y} = r\underline{\omega} \in \mathbb{R}^n$ we assume $r = 1$ and $\underline{\omega} \in \mathbb{S}^{n-1}$. Let $\mathcal{P}_k(\underline{x})$ be an inner left spherical monogenic polynomial of degree $k \in \mathbb{N}_0$. We define the kernels

$$\begin{aligned} \mathcal{F}^{+}_{k,n}(x) &= \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\,\mathcal{P}_k(\underline{\omega})\,dS(\underline{\omega}), \\ \mathcal{F}^{-}_{k,n}(x) &= \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\,\underline{\omega}\mathcal{P}_k(\underline{\omega})dS(\underline{\omega}), \end{aligned} \tag{10}$$

where $dS(\underline{\omega})$ is the scalar element of surface area of $\mathbb{S}^{n-1}$.

Before we prove the main result of this section, that is the factorization property of the kernels $\mathcal{F}^{+}_{k,n}(x)$ and $\mathcal{F}^{-}_{k,n}(x)$, we recall some results that we will use in the sequel.

THEOREM 3.3 (Funk–Hecke (see [19])): *Denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. Let ξ and η be two unit vectors in $\mathbb{R}^n$. Let ψ be a real-valued function whose domain contains $[-1,1]$ and let $\mathcal{P}_k(\xi)$ be spherical harmonics of degree k. Then we have*

$$\int_{\mathbb{S}^{n-1}} \psi(\langle \xi,\eta\rangle)\mathcal{P}_k(\eta)\,dS(\eta) = A_{n-1}\mathcal{P}_k(\xi)\int_{-1}^{1} \psi(t)P_{k,n}(t)(1-t^2)^{(n-3)/2}\,dt,$$

where $dS(\eta)$ is the scalar element of surface area on $\mathbb{S}^{n-1}$, $\langle \xi,\eta\rangle$ is the scalar product of ξ, η, $P_{k,n}(t)$ are the Legendre polynomials and

$$A_{n-1} = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}.$$

Remark 3.4: By the Rodriguez formula the Legendre polynomials $P_{k,n}(t)$ can be expressed by

$$P_{k,n}(t) = \Big(-\frac{1}{2}\Big)^k \frac{\Gamma((n-1)/2)}{\Gamma(k+(n-1)/2)}(1-t^2)^{(3-n)/2}\frac{d^n}{dt^n}(1-t^2)^{k+(n-3)/2}.$$

We recall the following formula (see [19] p. 188).

Proposition 3.5: *Let $f : [-1,1] \to \mathbb{R}$ be a continuous function with its n derivatives; then the following formula holds:*

$$\int_{-1}^{1} f(t)P_{k,n}(t)(1-t^2)^{(n-3)/2}dt = \Big(\frac{1}{2}\Big)^k \frac{\Gamma((n-1)/2)}{\Gamma(k+(n-1)/2)} \int_{-1}^{1} (1-t^2)^{k+(n-3)/2} f^{(k)}(t)\,dt, \tag{11}$$

where $P_{k,n}$ are the Legendre polynomials, n is the dimension and k is the degree of $P_{n,k}$.

Theorem 3.6 (Factorization property of the kernels $\mathcal{F}^+_{k,n}(x)$ and $\mathcal{F}^-_{k,n}(x)$): *Let n be an odd number. Let $\mathcal{P}_k(\underline{x})$ be an inner left spherical monogenic polynomial of degree $k \in \mathbb{N}_0$. Let $\mathcal{F}^+_{k,n}(x)$ and $\mathcal{F}^-_{k,n}(x)$ be the kernels defined in* (10). *Then there exist two functions $\mathcal{S}^+_{k,n}(x)$ and $\mathcal{S}^-_{k,n}(x)$ independent of $\mathcal{P}_k(\underline{x})$, such that*

$$\mathcal{F}^+_{k,n}(x) = \mathcal{S}^+_{k,n}(x)\mathcal{P}_k(\underline{x}), \quad \mathcal{F}^-_{k,n}(x) = \mathcal{S}^-_{k,n}(x)\mathcal{P}_k(\underline{x}) \tag{12}$$

and

(13)

$$\lim_{\underline{x}\to\underline{0}} \mathcal{S}^+_{k,n}(x) = \mathcal{C}_{k,n}\frac{x_0}{(x_0^2+1)^{k+(n+1)/2}}, \quad \lim_{\underline{x}\to\underline{0}} \mathcal{S}^-_{k,n}(x) = -\mathcal{C}_{k,n}\frac{1}{(x_0^2+1)^{k+(n+1)/2}},$$

where

$$\mathcal{C}_{k,n} := \frac{(-1)^k}{\sqrt{\pi}}\frac{\Gamma(k+\frac{n+1}{2})}{\Gamma(k+\frac{n}{2})}. \tag{14}$$

Proof. Let us consider first the functions $\mathcal{F}^+_{k,n}(x)$, for all odd numbers n. Recalling (9), we can write $\mathcal{F}^+_{k,n}(x)$ as

$$\mathcal{F}^+_{k,n}(x) = \frac{1}{A_{n+1}}\int_{\mathbb{S}^{n-1}} \frac{x_0-\underline{x}+\underline{\omega}}{(x_0^2+\langle \underline{x}-\underline{\omega},\underline{x}-\underline{\omega}\rangle)^{(n+1)/2}}\,\mathcal{P}_k(\underline{\omega})\,dS(\underline{\omega}).$$

By setting $r=|\underline{x}|$, $I=\underline{x}/r$, we split it as

$$\mathcal{F}^+_{k,n}(x) = (x_0-\underline{x})\mathcal{J}_{k,n}(x_0,r) + \mathcal{L}_{k,n}(x_0,r),$$

where we have set

$$\mathcal{J}_{k,n}(x_0,r) := \frac{1}{A_{n+1}}\int_{\mathbb{S}^{n-1}} \psi_r(t)\mathcal{P}_k(\underline{\omega})\,dS(\underline{\omega}),$$

$$\mathcal{L}_{k,n}(x_0,r) := \frac{1}{A_{n+1}}\int_{\mathbb{S}^{n-1}} \psi_r(t)\;\underline{\omega}\;\mathcal{P}_k(\underline{\omega})\,dS(\underline{\omega})$$

and

$$\psi_r(t) := \frac{1}{(x_0^2 + 1 + r^2 - 2rt)^{(n+1)/2}}, \quad t := \langle I, \underline{\omega} \rangle. \tag{15}$$

To compute $\mathcal{F}^+_{k,n}(x)$ we proceed by steps. First we calculate $\mathcal{J}_{k,n}(x_0, r)$ and $\mathcal{L}_{k,n}(x_0, r)$ using Theorem 3.3 (Funk–Hecke). We have

$$\mathcal{J}_{k,n}(x_0, r) = \frac{A_{n-1}}{A_{n+1}} \mathcal{P}_k(I) \int_{-1}^{1} \psi_r(t) P_{k,n}(t)(1-t^2)^{(n-3)/2} dt \tag{16}$$

and

$$\mathcal{L}_{k,n}(x_0, r) = \frac{A_{n-1}}{A_{n+1}} I\mathcal{P}_k(I) \int_{-1}^{1} \psi_r(t) P_{k+1,n}(t)(1-t^2)^{(n-3)/2}\, dt. \tag{17}$$

If we set

$$\mathcal{Q}_{k,n}(x_0, r) := \frac{A_{n-1}}{A_{n+1}} \int_{-1}^{1} \psi_r(t) P_{k,n}(t)(1-t^2)^{(n-3)/2}\, dt, \tag{18}$$

then we can write $\mathcal{J}_{k,n}(x_0, r)$ and $\mathcal{L}_{k,n}(x_0, r)$ as

$$\mathcal{J}_{k,n}(x_0, r) = \mathcal{Q}_{k,n}(x_0, r)\mathcal{P}_k(I) \tag{19}$$

and

$$\mathcal{L}_{k,n}(x_0, r) = \mathcal{Q}_{k+1,n}(x_0, r) I\mathcal{P}_k(I), \tag{20}$$

so we obtain

$$\mathcal{F}^+_{k,n}(x) = \Big(\frac{(x_0 - \underline{x})}{r^k} \mathcal{Q}_{k,n}(x_0, r) + \frac{\underline{x}}{r^{k+1}} \mathcal{Q}_{k+1,n}(x_0, r) \Big) \mathcal{P}_k(\underline{x}) = \mathcal{S}^+_{k,n}(x)\mathcal{P}_k(\underline{x}).$$

These computations prove the factorization of $\mathcal{F}^+_{k,n}(x)$ given in (12).

Let us now calculate the limit

$$\lim_{r\to 0} \Big(\frac{(x_0 - \underline{x})}{r^k} \mathcal{Q}_{k,n}(x_0, r) + \frac{\underline{x}}{r^{k+1}} \mathcal{Q}_{k+1,n}(x_0, r) \Big),$$

where the non trivial term is only

$$\lim_{r\to 0} \frac{x_0}{r^k} \mathcal{Q}_{k,n}(x_0, r).$$

To this end, we must study the function $\mathcal{Q}_{k,n}(x_0, r)$ defined in (18) for $r \to 0$. First of all, we expand in power series the function $\psi_r(t)$ defined in (15) using the binomial series

$$\psi_r(t) = \frac{1}{(x_0^2 + 1 + r^2)^{(n+1)/2}} \sum_{j=0}^{\infty} \binom{-(n+1)/2}{j} \Big(\frac{2rt}{x_0^2 + 1 + r^2} \Big)^j;$$

using the orthogonality properties of the Legendre polynomials we get

$$
\begin{aligned}
\mathcal{Q}_{k,n}(x_0,r) =& \frac{A_{n-1}}{A_{n+1}}\,\frac{1}{(x_0^2+1+r^2)^{(n+1)/2}}\binom{-(n+1)/2}{k}\Big(\frac{2r}{x_0^2+1+r^2}\Big)^k \\
&\times\int_{-1}^{1} t^k P_{k,n}(t)(1-t^2)^{(n-3)/2}\,dt \\
=&\frac{A_{n-1}}{A_{n+1}}\,2^k\frac{r^k}{(x_0^2+1+r^2)^{k+(n+1)/2}}\binom{-(n+1)/2}{k} \\
&\times\int_{-1}^{1} t^k P_{k,n}(t)(1-t^2)^{(n-3)/2}\,dt.
\end{aligned}
\tag{21}
$$

To explicitly compute the last integral, we use formula (21) and the well-known integral (see [17])

$$\int_{-1}^{1}(1-t^2)^q\,dt=\sqrt{\pi}\;\frac{\Gamma(q+1)}{\Gamma(q+\frac{3}{2})}\quad\text{for } q\in\mathbb{R}^+,$$

so we get

$$
\begin{aligned}
&\int_{-1}^{1} t^k P_{k,n}(t)(1-t^2)^{(n-3)/2}dt \\
&\quad=\frac{1}{2^k}\;k!\sqrt{\pi}\frac{\Gamma((n-1)/2)}{\Gamma(k+(n-1)/2)}\frac{\Gamma(1+k+(n-3)/2)}{\Gamma(3/2+k+(n-2)/2)} \\
&\quad=\frac{1}{2^k}k!\sqrt{\pi}\frac{\Gamma((n-1)/2)}{\Gamma(k+n/2)}.
\end{aligned}
$$

Moreover, we have

$$\binom{-(n+1)/2}{k}=\frac{(-1)^k}{k!}\frac{\Gamma(\frac{n+1}{2}+k)}{\Gamma(\frac{n+1}{2})},$$

which follows from

$$
\begin{aligned}
\binom{-(n+1)/2}{k} &= \frac{1}{k!}\Big(-\frac{n+1}{2}\Big)\Big(-\frac{n+3}{2}\Big)\cdots\Big(-\frac{n+2k-1}{2}\Big) \\
&=\frac{(-1)^k}{k!}\Big(\frac{n+1}{2}\Big)\Big(\frac{n+1}{2}+1\Big)\Big(\frac{n+1}{2}+2\Big)\cdots\Big(\frac{n+1}{2}+k-1\Big) \\
&=\frac{(-1)^k}{k!}\frac{\Gamma(\frac{n+1}{2}+k)}{\Gamma(\frac{n+1}{2})}.
\end{aligned}
$$

Finally, we get

$$\begin{aligned}\mathcal{Q}_{k,n}(x_0,r) =& \frac{A_{n-1}}{A_{n+1}} 2^k \frac{r^k}{(x_0^2+1+r^2)^{k+(n+1)/2}} \binom{-(n+1)/2}{k} \\ & \times \int_{-1}^{1} t^k P_{k,n}(t)(1-t^2)^{(n-3)/2} dt \\ =& \frac{A_{n-1}}{A_{n+1}} 2^k \, \frac{r^k}{(x_0^2+1+r^2)^{k+(n+1)/2}} \\ & \times (-1)^k \frac{\Gamma(\frac{n+1}{2}+k)}{k!\Gamma(\frac{n+1}{2})} \times \frac{1}{2^k} \, k! \, \sqrt{\pi} \, \frac{\Gamma((n-1)/2)}{\Gamma(k+n/2)}\end{aligned}$$

and with some simplifications we obtain

$$\mathcal{Q}_{k,n}(x_0,r) = \mathcal{C}_{k,n} \, \frac{r^k}{(x_0^2+1+r^2)^{k+(n+1)/2}}$$

where

$$\mathcal{C}_{k,n} := \frac{(-1)^k}{\sqrt{\pi}} \frac{\Gamma(k+\frac{n+1}{2})}{\Gamma(k+\frac{n}{2})}.$$

Finally, we compute

$$\lim_{r\to 0} \frac{x_0}{r^k} \mathcal{Q}_{k,n}(x_0,r) = \mathcal{C}_{k,n} \frac{x_0}{(x_0^2+1)^{k+(n+1)/2}},$$

from which we deduce one of the limits in (13), that is $\lim_{\underline{x}\to\underline{0}} \mathcal{S}^+_{k,n}(x)$. The above computations allow us to determine also the factorization for $\mathcal{F}^-_{k,n}(x)$ and the limit of $\mathcal{S}^-_{k,n}(x)$. In fact

$$\begin{aligned}\mathcal{F}^-_{k,n}(x) &= \frac{1}{A_{n+1}} \int_{\mathbb{S}^{n-1}} \frac{x_0-\underline{x}+\underline{\omega}}{(x_0^2+\langle \underline{x}-\underline{\omega}, \underline{x}-\underline{\omega}\rangle)^{(n+1)/2}} \, \underline{\omega} \, \mathcal{P}_k(\underline{\omega}) \, dS(\underline{\omega}) \\ &= (x_0-\underline{x})\mathcal{L}_{k,n}(x_0,r) - \mathcal{J}_{k,n}(x_0,r).\end{aligned}$$

With calculations similar to those above, we deduce the factorization $\mathcal{F}^-_{k,n}(x) = \mathcal{S}^-_{k,n}(x)\mathcal{P}_k(\underline{x})$ and the limit

$$\lim_{\underline{x}\to\underline{0}} \mathcal{S}^+_{k,n}(x) = -\mathcal{C}_{k,n} \frac{1}{(x_0^2+1)^{k+(n+1)/2}}.$$

This concludes the proof. ∎

4. The Fueter primitives of the kernels $\mathcal{F}^+_{k,n}(x)$ and $\mathcal{F}^-_{k,n}(x)$

The factorization property of the kernels $\mathcal{F}^+_{k,n}(x)$ and $\mathcal{F}^-_{k,n}(x)$ (see Theorem (3.6)) require the determination only of the functions $\mathcal{S}^+_{k,n}(x)$ and $\mathcal{S}^-_{k,n}(x)$. Now we define the Fueter primitives of $\mathcal{F}^\pm_{k,n}(x)$.

Definition 4.1: Let n be an odd number. Let $\mathcal{P}_k(\underline{x})$ be an inner left spherical monogenic polynomial of degree $k \in \mathbb{N}_0$. We will denote by $\mathcal{W}^+_{k,n}(x)\mathcal{P}_k(\underline{x})$ and $\mathcal{W}^-_{k,n}(x)\mathcal{P}_k(\underline{x})$ the Fueter primitives of $\mathcal{F}^+_{k,n}(x)$ and $\mathcal{F}^-_{k,n}(x)$, that is $\mathcal{W}^+_{k,n}(x)\mathcal{P}_k(\underline{x})$ and $\mathcal{W}^-_{k,n}(x)\mathcal{P}_k(\underline{x})$ satisfy

$$\Delta^{k+\frac{n-1}{2}}(\mathcal{W}^+_{k,n}(x)\mathcal{P}_k(\underline{x})) = \mathcal{F}^+_{k,n}(x), \quad \Delta^{k+\frac{n-1}{2}}(\mathcal{W}^-_{k,n}(x)\mathcal{P}_k(\underline{x})) = \mathcal{F}^-_{k,n}(x).$$

Remark 4.2: Let us observe that, thanks to Theorem 3.6, we also have

$$\Delta^{k+\frac{n-1}{2}}(\mathcal{W}^+_{k,n}(x)\mathcal{P}_k(\underline{x})) = \mathcal{S}^+_{k,n}(x)\mathcal{P}_k(\underline{x}),$$
$$\Delta^{k+\frac{n-1}{2}}(\mathcal{W}^-_{k,n}(x)\mathcal{P}_k(\underline{x})) = \mathcal{S}^-_{k,n}(x)\mathcal{P}_k(\underline{x}).$$

Theorem 4.3 (The explicit structure of the functions $\mathcal{W}^+_{k,n}(x)$ and $\mathcal{W}^-_{k,n}(x)$): *Let n be an odd number and let $k \in \mathbb{N}_0$. Then we have*

$$\mathcal{W}^+_{k,n}(x_0) = \frac{\mathcal{C}_{k,n}}{\mathcal{H}_{k,n}}\ D^{-(2k+n-1)}\frac{x_0}{(x_0^2+1)^{k+(n+1)/2}},$$

$$\mathcal{W}^-_{k,n}(x_0) = -\frac{\mathcal{C}_{k,n}}{\mathcal{H}_{k,n}}\ D^{-(2k+n-1)}\frac{1}{(x_0^2+1)^{k+(n+1)/2}},$$

where the symbol $D^{-(n-1+2k)}$ stands for the $(2k+n-1)$ integrations with respect to x_0. Replacing x_0 by x in both $\mathcal{W}^+_{k,n}(x_0)$ and $\mathcal{W}^-_{k,n}(x_0)$ we get $\mathcal{W}^+_{k,n}(x)$ and $\mathcal{W}^-_{k,n}(x)$, respectively. Moreover, the functions $\mathcal{W}^+_{k,n}(x)$ and $\mathcal{W}^-_{k,n}(x)$ belong to $\mathcal{N}(U)$, where U is any open set in $\mathbb{R}^{n+1}\setminus\mathbb{S}^{n-1}$.

Proof. Let us observe that the kernels $\mathcal{F}^+_{k,n}(x)$ and $\mathcal{F}^-_{k,n}(x)$ defined in (10) are axially symmetric functions of degree k, so $\mathcal{S}^+_{k,n}(x)$ and $\mathcal{S}^-_{k,n}(x)$, see (12), can be written in the form $A+\underline{\omega}B$, where A, B satisfy a Vekua-type system, which is elliptic, and so $\mathcal{S}^+_{k,n}(x)$ and $\mathcal{S}^-_{k,n}(x)$ are determined by their restrictions to $\underline{x}=0$. From Theorem 3.6 we have that

$$\lim_{\underline{x}\to\underline{0}}\mathcal{S}^+_{k,n}(x) = \mathcal{C}_{k,n}\frac{x_0}{(x_0^2+1)^{k+(n+1)/2}}, \quad \lim_{\underline{x}\to\underline{0}}\mathcal{S}^-_{k,n}(x) = -\mathcal{C}_{k,n}\frac{1}{(x_0^2+1)^{k+(n+1)/2}},$$

where the constants $\mathcal{C}_{k,n}$ are explicitly determined in (14). Now recall that if $W\in\mathcal{N}(U)$, where U is an axially symmetric s-domain in $\mathbb{R}^{n+1}$, and if $\mathcal{P}_k(\underline{x})$

are inner left spherical monogenic polynomials of degree $k \in \mathbb{N}_0$, then by Proposition 2.4 we have, for $\underline{x} \to \underline{0}$,

$$\Delta_x^{k+(n-1)/2}(W(x)\mathcal{P}_k(\underline{x})) = \mathcal{H}_{n,k} V^{(2k+n-1)}(x_0)\mathcal{P}_k(\underline{x}) + \mathcal{R}(x_0, \underline{x})\mathcal{P}_k(\underline{x}),$$

where $W(x)|_{\underline{x}=0} := V(x_0)$ and $\mathcal{H}_{n,k}$ are explicitly determined by (8). If we set

$$D^{(2k+n-1)}\mathcal{W}_{k,n}^{+}(x_0) = \frac{\mathcal{C}_{k,n}}{\mathcal{H}_{k,n}} \; \frac{x_0}{(x_0^2+1)^{k+(n+1)/2}}$$

and we integrate the function $\dfrac{x_0}{(x_0^2+1)^{k+(n+1)/2}}$ $(2k+n-1)$ times, we get

$$\mathcal{W}_{k,n}^{+}(x_0) = \frac{\mathcal{C}_{k,n}}{\mathcal{H}_{k,n}} \; D^{-(2k+n-1)} \frac{x_0}{(x_0^2+1)^{k+(n+1)/2}},$$

where the symbol $D^{-(2k+n-1)}$ stands for the $(2k+n-1)$ integrations with respect to x_0. By replacing now x_0 by x in $\mathcal{S}_{k,n}^{+}(x_0)$ we get $\mathcal{S}_{k,n}^{+}(x)$. We observe that it is the required function since

$$\begin{aligned}\lim_{\underline{x}\to\underline{0}} \Delta_x^{k+(n-1)/2}(\mathcal{W}_{k,n}^{+}(x)\mathcal{P}_k(\underline{x})) &= \frac{\mathcal{C}_{k,n}}{\mathcal{H}_{k,n}}\mathcal{H}_{k,n} V^{(2k+n-1)}(x_0) \lim_{\underline{x}\to\underline{0}} \mathcal{P}_k(\underline{x}) \\ &= \mathcal{C}_{k,n} \; \frac{x_0}{(x_0^2+1)^{k+(n+1)/2}} \lim_{\underline{x}\to\underline{0}} \mathcal{P}_k(\underline{x}).\end{aligned}$$

Analogously we set

$$D^{(2k+n-1)}\mathcal{S}_{k,n}^{-}(x_0) := -\frac{\mathcal{C}_n}{\mathcal{K}_n} \; \frac{1}{(x_0^2+1)^{k+(n+1)/2}}$$

and we integrate the function $\frac{1}{(x_0^2+1)^{k+(n+1)/2}}$, $(2k+n-1)$ times. We get

$$\mathcal{W}_{k,n}^{-}(x_0) := -\frac{\mathcal{C}_{k,n}}{\mathcal{H}_{k,n}} \; D^{-(2k+n-1)} \frac{1}{(x_0^2+1)^{k+(n+1)/2}}.$$

Replacing now x_0 by x in $\mathcal{W}_{k,n}^{-}(x_0)$ we obtain $\mathcal{W}_{k,n}^{-}(x)$, which is the required function. Finally, we observe that the functions

$$x_0 \mapsto \frac{x_0}{(x_0^2+1)^{k+(n+1)/2}}, \quad x_0 \mapsto \frac{1}{(x_0^2+1)^{k+(n+1)/2}}$$

can be integrated by parts in closed form an arbitrary number of times. Such primitives contain rational functions of x_0 and $\arctan x_0$, where x_0 is a real variable.

When we replace in such functions the real variable x_0 by the paravector variable x, we clearly obtain that the functions $\mathcal{W}_{k,n}^{-}$ and $\mathcal{W}_{k,n}^{+}$ belong to $\mathcal{N}(U)$. ∎

5. The inverse Fueter mapping theorem in integral form

We now recall Cauchy's integral formula for monogenic functions.

THEOREM 5.1 (Cauchy's integral representation theorem for monogenic functions): *Let $\breve{f}$ be a left monogenic function in $U \subseteq \mathbb{R}^{n+1}$. Then, for every $\Sigma \subset U$ and for $x \in \Sigma$, we have*

$$\breve{f}(x) = \int_{\partial\Sigma} \mathcal{G}(y-x)d\sigma(y)\breve{f}(y), \tag{22}$$

where $\partial\Sigma$ is an n-dimensional compact smooth manifold in U, and the differential form $d\sigma(y)$ is given by $d\sigma(y) = \eta(y)dS(y)$, where $\eta(y)$ is the outer unit normal to $\partial\Sigma$ at point y and $dS(y)$ is the scalar element of surface area on $\partial\Sigma$.

We are now in position to state and prove the main result of this paper.

THEOREM 5.2 (The inverse Fueter mapping theorem): *Let n be an odd number and let $\mathcal{P}_k(\underline{x})$ be an inner left spherical monogenic polynomial of degree $k \in \mathbb{N}_0$. Let*

$$\breve{f}(x)\mathcal{P}_k(\underline{x}) = (A(x_0,\rho) + \underline{\omega}B(x_0,\rho))\mathcal{P}_k(\underline{x})$$

be an axially monogenic function of degree k defined on an axially symmetric open set $U \subseteq \mathbb{R}^{n+1}$. Let Γ be the boundary of an open bounded subset $\mathcal{V}$ in the half plane $\mathbb{R} + \underline{\omega}\mathbb{R}^+$ and let $V \subset U$ be the open set in $\mathbb{R}^{n+1}$ induced by $\mathcal{V}$. Moreover, suppose that Γ is a regular curve whose parametric equations $y_0 = y_0(s)$, $\rho = \rho(s)$ are expressed in terms of the arc-length $s \in [0, L]$, $L > 0$ and consider the manifold

$$\Sigma := \{y_0 + \underline{\omega}\rho \,|\, (y_0,\rho) \in \Gamma, \quad \underline{\omega} \in \mathbb{S}^{n-1}\}. \tag{23}$$

Then the function

$$\begin{aligned} & f(x)\mathcal{P}_k(\underline{x}) \\ &= \int_\Gamma \mathcal{W}^-_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{\rho}\Big)\rho^{2k+n-2}[dy_0\, A(y_0,\rho) - d\rho\, B(y_0,\rho)] \\ &\quad - \int_\Gamma \mathcal{W}^+_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{\rho}\Big)\rho^{2k+n-2}[dy_0 B(y_0,\rho) + d\rho A(y_0,\rho)]. \end{aligned} \tag{24}$$

is a Fueter primitive of $\breve{f}(x)\mathcal{P}_k(\underline{x})$ on V.

Proof. We represent axially monogenic functions $\breve{f}$ by the Cauchy formula (22) using the manifold Σ in (23). We specify the notations: ds is the infinitesimal arc-length, $dS(\underline{\omega})$ is the infinitesimal element of surface area on $\mathbb{S}^{n-1}$; $\mathbf{t} = \frac{d}{ds}(y_0 + \underline{\omega}\rho)$ is the unit tangent vector at a point of Γ, while the normal unit vector is given by

$$\mathbf{n} = -\underline{\omega}\mathbf{t} = \frac{d}{ds}[\rho(s) - \underline{\omega}y_0(s)].$$

The scalar infinitesimal element of the manifold Σ, expressed in terms of ds and dS, is given by

$$d\Sigma = \rho^{n-1}\, ds\, dS(\underline{\omega});$$

finally, the oriented infinitesimal element of manifold $d\sigma(s, \underline{\omega})$ is given by

$$d\sigma(s, \underline{\omega}) = \mathbf{n} d\Sigma = \frac{d}{ds}[\rho(s) - \underline{\omega}y_0(s)]\, \rho^{n-1}\, ds\, dS(\underline{\omega}).$$

So finally we get

$$d\sigma(s, \underline{\omega}) = [d\rho(s) - \underline{\omega}dy_0(s)]\rho^{n-1} dS(\underline{\omega}).$$

Thanks to the above considerations we have

$$\breve{f}(x_0 + Ir)\mathcal{P}_k(I) = \int_\Gamma \int_{S^{n-1}} \mathcal{G}(y_0 + \underline{\omega}\rho - x_0 - rI)\, d\sigma(s, \underline{\omega})\, \breve{f}(y_0 + \underline{\omega}\rho)\mathcal{P}_k(\underline{\omega}).$$

So we can split the integral in the following way:

$$\begin{aligned}\breve{f}(x_0 + Ir)\mathcal{P}_k(I) = &-\int_\Gamma \left[\int_{\mathbb{S}^{n-1}} \mathcal{G}(y_0 + \underline{\omega}\rho - x_0 - rI)\, \underline{\omega}\, \mathcal{P}_k(\underline{\omega})\, dS(\underline{\omega})\right] \\ &\qquad \times \rho^{n-1}[dy_0\, A(y_0, \rho) - d\rho\, B(y_0, \rho)] \\ &+ \int_\Gamma \left[\int_{S^{n-1}} \mathcal{G}(y_0 + \underline{\omega}\rho - x_0 - rI)\, \mathcal{P}_k(\underline{\omega})\, dS(\underline{\omega})\right] \\ &\qquad \times \rho^{n-1}[dy_0 B(y_0, \rho) + d\rho A(y_0, \rho)];\end{aligned}$$

keeping in mind the property $\mathcal{G}(tx) = t^{-n}\mathcal{G}(x)$ for $t > 0$, with a change of variables, we have

$$\begin{aligned}\breve{f}(x_0 + Ir)\mathcal{P}_k(I) = &\int_\Gamma \left[\int_{\mathbb{S}^{n-1}} \rho^{-n}\mathcal{G}\Big(\frac{x_0 - y_0}{\rho} + \frac{r}{\rho}I - \underline{\omega}\Big)\, \underline{\omega}\, \mathcal{P}_k(\underline{\omega})\, dS(\underline{\omega})\right] \\ &\qquad \times \rho^{n-1}[dy_0\, A(y_0, \rho) - d\rho\, B(y_0, \rho)] \\ &- \int_\Gamma \left[\int_{\mathbb{S}^{n-1}} \rho^{-n}\mathcal{G}\Big(\frac{x_0 - y_0}{\rho} + \frac{r}{\rho}I - \underline{\omega}\Big)\, \mathcal{P}_k(\underline{\omega})\, dS(\underline{\omega})\right] \\ &\qquad \times \rho^{n-1}[dy_0 B(y_0, \rho) + d\rho A(y_0, \rho)].\end{aligned}$$

Recalling the definitions of $\mathcal{F}^+_{k,n}$ and $\mathcal{F}^-_{k,n}$ we get

$$\breve{f}(x_0+Ir)\mathcal{P}_k(I)=\int_\Gamma \mathcal{F}^-_{k,n}\Big(\frac{x_0-y_0}{\rho}+\frac{r}{\rho}I\Big)\,\rho^{-1}[dy_0\,A(y_0,\rho)-d\rho\,B(y_0,\rho)]$$
$$-\int_\Gamma \mathcal{F}^+_{k,n}\Big(\frac{x_0-y_0}{\rho}+\frac{r}{\rho}I\Big)\,\rho^{-1}[dy_0 B(y_0,\rho)+d\rho A(y_0,\rho)].$$

Let us observe that, since $x=x_0+Ir$, we have

$$\breve{f}(x_0+Ir)\mathcal{P}_k(I)=\int_\Gamma \mathcal{F}^-_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\,\rho^{-1}[dy_0\,A(y_0,\rho)-d\rho\,B(y_0,\rho)]$$
$$-\int_\Gamma \mathcal{F}^+_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\,\rho^{-1}[dy_0 B(y_0,\rho)+d\rho A(y_0,\rho)].$$

By setting

$$x':=\frac{x-y_0}{\rho},$$

by Definition (4.1), we obtain

$$\breve{f}(x_0+Ir)\mathcal{P}_k(I)$$
$$=\int_\Gamma \Delta_{x'}^{k+(n-1)/2}(\mathcal{W}^-_{k,n}(x')\mathcal{P}_k(\underline{x}'))\rho^{-1}[dy_0 A(y_0,\rho)-d\rho B(y_0,\rho)]$$
$$-\int_\Gamma \Delta_{x'}^{k+(n-1)/2}(\mathcal{W}^+_{k,n}(x')\mathcal{P}_k(\underline{x}'))\rho^{-1}[dy_0 B(y_0,\rho)+d\rho A(y_0,\rho)],$$

and since $\Delta_{x'}^{k+(n-1)/2}=\rho^{2k+n-1}\Delta_x^{k+(n-1)/2}$ we get

$$\breve{f}(x_0+Ir)\mathcal{P}_k(I)=\Delta_x^{k+(n-1)/2}\Big[\int_\Gamma \mathcal{W}^-_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{\rho}\Big)\rho^{2k+n-2}$$
$$\times[dy_0\,A(y_0,\rho)-d\rho\,B(y_0,\rho)]$$
$$-\int_\Gamma \mathcal{W}^+_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{\rho}\Big)\rho^{2k+n-2}$$
$$\times[dy_0 B(y_0,\rho)+d\rho A(y_0,\rho)]\Big],$$

where

$$f(x)\mathcal{P}_k(\underline{x})=\int_\Gamma \mathcal{W}^-_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{\rho}\Big)\rho^{2k+n-2}$$
$$\times[dy_0\,A(y_0,\rho)-d\rho\,B(y_0,\rho)]$$
$$-\int_\Gamma \mathcal{W}^+_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{\rho}\Big)\rho^{2k+n-2}\tag{25}$$
$$\times[dy_0 B(y_0,\rho)+d\rho A(y_0,\rho)].\qquad\blacksquare$$

From the proof of the above theorem one can easily see that the following result holds.

Corollary 5.3: *Under the hypothesis of the above theorem, the Cauchy integral formula for axially monogenic functions $\breve{f}$ of degree k can be written on V in the form*

$$\breve{f}(x)\mathcal{P}_k(\underline{x}) = \int_\Gamma \mathcal{F}^-_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\,\rho^{-1}[dy_0\,A(y_0,\rho) - d\rho\,B(y_0,\rho)] \\ - \int_\Gamma \mathcal{F}^+_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\,\rho^{-1}[dy_0 B(y_0,\rho) + d\rho A(y_0,\rho)].$$

Let us denote by $\mathcal{AM}_k(U)$ the set of axially symmetric functions on the axially symmetric open set U and let us introduce the set

$$\mathcal{N}_k(U) \\ = \Big\{\varphi_k = \sum_{j=1}^{m_k} f_j(x)\mathcal{P}_{k,j}(\underline{x})|\, f_j \in \mathcal{N}(U), \mathcal{P}_{k,j} \text{ spherical monogenic of degree } k\Big\},$$

where $m_k = \dim \mathcal{AM}_k$. We have the following important corollary which corresponds to the inverse Fueter mapping theorem for any monogenic function defined on an axially symmetric open set:

Corollary 5.4: *Let n be an odd number and let U be an axially symmetric open set in $\mathbb{R}^{n+1}$. There is a map of $\mathbb{R}_n$-modules*

$$\mathcal{AM}_k(U) \to \mathcal{N}_k(U)$$

such that, given $(A_k + \underline{\omega}B_k)\mathcal{P}_k \in \mathcal{AM}_k(U)$, we have

$$(A_k + \underline{\omega}B_k)\mathcal{P}_k = \Delta^{k+\frac{n-1}{2}}((\alpha_k + \underline{\omega}\beta_k)\mathcal{P}_k),$$

with $\alpha_k + \underline{\omega}\beta_k \in \mathcal{N}(U)$. Moreover, there is a map

$$\mathcal{M}(U) \to \bigoplus_k \Delta^k \mathcal{N}_k(U)$$

such that, given $\breve{f} = \sum_k \breve{f}_k \in \mathcal{M}(U)$, $f_k \in \mathcal{AM}_k(U)$, there are $\varphi_k \in \mathcal{N}_k$ such that

$$\breve{f} = \Delta^{\frac{n-1}{2}} \sum_k \Delta^k \varphi_k.$$

Proof. It is a consequence of the inverse Fueter mapping theorem and of Theorem 1.4. ∎

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