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AN INFINITARY PROBABILITY LOGIC FOR TYPE SPACES

BY

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ABSTRACT

Type spaces in the sense of Harsanyi (1967/68) play an important role in the theory of games of incomplete information. They can be considered as the probabilistic analog of Kripke structures. By an infinitary propositional language with additional operators "individual i assigns probability at least α to" and infinitary inference rules, we axiomatize the class of (Harsanyi) type spaces. We prove that our axiom system is strongly sound and strongly complete. To the best of our knowledge, this is the very first strong completeness theorem for a probability logic with σ-additive probabilities. We show this by constructing a canonical type space whose states consist of all maximal consistent sets of formulas. Furthermore, we show that this canonical space is universal (i.e., a terminal object in the category of type spaces) and beliefs complete.

1. Introduction

1.1. GAMES OF INCOMPLETE INFORMATION AND HARSANYI TYPE SPACES. Consider players that are uncertain about a set S, called the **space of states of nature**, each element of which can be thought of as a complete list of the players' strategy sets and payoff functions, that is, a state of nature consists of

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a complete specification of the "rules" of the game. (Other interpretations are also possible. For example, if a game of complete information is given, a state $s \in S$ could be the strategy profile that the players are actually going to choose (see the analysis of epistemic conditions for Nash equilibrium by Aumann and Brandenburger (1995)).) In such a situation, following a Bayesian approach, each player will base his choice of a strategy on his subjective beliefs (i.e., a probability measure) on S. Since a player's payoff depends also on the choices of the other players, and these are based on their beliefs as well, each player must also have beliefs on the other players' beliefs on S. For the same reason, he must also have beliefs on the other players' beliefs on his beliefs on S, beliefs on the other players' beliefs on his beliefs on their beliefs on S, and so on. So, in analyzing such a situation, it seems to be unavoidable to work with infinite hierarchies of beliefs. Thus, the resulting model is complicated and cumbersome to handle. In fact, this was the reason that for a long time prevented the analysis of games of incomplete information.

A major breakthrough took place with three articles of Harsanyi (1967/68), where he succeeded in finding another, more workable model to describe interactive uncertainty. He invented the notions of **type** and **type space**: With each point in a type space, called a **state of the world**, are associated a state of nature and, for each player, a probability measure on the type space itself (that is, that player's **type** in this state of the world). Usually it is assumed that the players "know their own type", that is, a type of a player in a state assigns probability one to the set of those states where this player is of this type. This is the formalization of the idea that the players should be introspective. Since each state of the world is associated with a state of nature, each player's type in a state of the world induces a probability measure on S. But also, since with each state of the world there is associated a type for each player (and hence indirectly a probability measure on S for this player), the type of a player in a state of the world induces a probability measure on the other players' probability measures on S. Proceeding like this, one obtains in each state of the world a hierarchy of beliefs for each player, in the sense described above.

The advantages of Harsanyi's model are obvious: Since we have in each state of the world just one probability measure for each player, contrary to the hierarchical description of beliefs, this model fits in the classical Bayesian framework of describing beliefs by one probability measure, and provides therefore all its advantages (for example, it allows for integration with respect to beliefs).

Type spaces have become the predominant structures to describe incomplete information in an interactive context in game theory. See Aumann and Heifetz (2001) for a nice and well-accessible introduction to the subject (see Siniscalchi (2008) for a more recent overview article).

1.2. Different approaches to define a space of all possible beliefs, beliefs about beliefs … However, if the analyst uses some particular type space to analyze such a situation, he—informally—assumes that the type space itself is "commonly known" or "mutually agreed on" by the players. Otherwise, the type space itself would be a new source of mutual uncertainty for the players. This problem can be avoided if there is a "largest" type space that "contains all types". The definition of a type depends on its context, that is, the type space it belongs to. For this reason it is not clear what "all types" (respectively "all possible states of the world") are. In the literature, there are three ways to formalize what a space of all types should be:

1.2.1. *The explicit approach—the canonical model* [1]*.* The first manner to define a space of all types is to describe in minute detail, first, the space of underlying uncertainty, that is, the space of states of nature, then, for each player, the set of all possible beliefs about the space of states of nature, then, the set of all possible beliefs about the product of the space of states of nature and profiles of beliefs of the other players about the space of states of nature, and so on. After that, one carries out the reverse of Harsanyi's project and shows that the space of all profiles of such hierarchies can naturally be endowed with the structure of a Harsanyi type space. This was done by Mertens and Zamir (1985) under the assumption that the underlying space of states of nature is a compact Hausdorff space and all involved functions are continuous. A construction in this vein had already been proposed before by Armbruster and Böge (1979). The topological assumption of Mertens and Zamir (1985) was relaxed by Brandenburger and Dekel (1993), Heifetz (1993), Mertens, Sorin and Zamir (1994), Battigalli and Siniscalchi (1999) and Pinter (2005)[2] to more general topological assumptions. However, since probability measures are already quite complicated objects, it

[1] The distinction of the explicit versus the implicit description of mutual uncertainty follows Aumann and Heifetz (2001)

[2] Pinter starts with a measurable space of states of nature, but then endows the spaces of higher-order beliefs with suitably chosen topologies. This allows him to apply a projective limit construction (or an appropriate version of the Kolmogorov extension theorem) as in

would be interesting to know whether it is possible to carry out a more natural construction than the ones mentioned here, where a language would be devised in which the players convey whether or not the plausibility they attach to already defined expressions falls short of a given level.

In this article we want—in the spirit of the aforementioned approach—to describe the states in a type space explicitly, however, in a different, more basic way. Here, we want to describe an epistemic situation—a state of the world—with a simpler vocabulary: Using a modal language, where the beliefs of the individuals are described by modal operators that simply tell whether the individual attaches probability $\geq \alpha$ to the event described by some already defined formula of the language.

It is well-known that Kripke structures (and in particular Knowledge spaces, see Hintikka (1962) or Aumann (1976)) can be axiomatized in terms of modal logic (see, for example, Kripke (1963), Aumann (1995), Fagin et al. (1995), Heifetz (1997), and Aumann (1999a)). In this paper we aim to do the same for type spaces in the sense of Harsanyi (1967/68), which can be considered as the probabilistic analog of Kripke structures.

We define an infinitary modal language with operators p_i^α, "individual i assigns probability at least α" for rational $\alpha \in [0,1]$, and then a system of infinitary axioms and inference rules, which we prove to be strongly sound and strongly complete with respect to the class of (Harsanyi) type spaces (Theorem 1). Strongly complete means that, if a formula φ holds whenever a (possibly infinite) set of formulas Γ holds, then there is a proof of φ from Γ. We construct (Proposition 3) a **canonical model** whose states consist of the maximal consistent sets of formulas. In a very natural way, the maximal consistent sets of formulas determine already the structure of this space.

1.2.2. *The implicit approach I—the universal type space.* The second possibility to define a space of "all possible types" is to construct a **universal type space**.[3] That is, a type space to which every type space (on the same space of states of nature and for the same set of players, of course) can be mapped, preferably always in a unique way, by a map that preserves the structure of the type space,

the aforementioned papers to endow the space of coherent hierarchies with the structure of a type space.

[3] Siniscalchi (2008) calls these type spaces "terminal" and reserves the term "universal" for the space of all hierarchies mentioned above.

i.e., the manner in which types and states of nature are associated with states of the world, a so-called **type morphism**. The type spaces—on a fixed set of states of nature and for a fixed player set—as objects and the type morphisms as morphisms form a category. If we always require the map from a type space to the universal type space to be unique, then, if it exists, such a universal type space is a **terminal object** of this category. A terminal object of a category is known to be unique up to isomorphism. Hence, we are justified to talk about *the* universal type space.

The existence of a universal type space was proved by Mertens and Zamir (1985), who were preceded by Böge and Eisele (1979). They were followed by Heifetz (1993), and Mertens, Sorin and Zamir (1994) and Battigalli and Siniscalchi (1999)[4] who all showed that their spaces of hierarchies of beliefs mentioned under 1. constitute universal type spaces with respect to the classes of type spaces satisfying the corresponding topological assumptions. Finally, the general measure theoretic case was solved by Heifetz and Samet (1998b), who showed that there also exists a universal type space in this case.[5] However, they could not use a hierarchical construction as done in the above-mentioned topological cases: As Heifetz and Samet have shown in another paper (1999), there are hierarchies of σ-additive beliefs that do not give rise to a σ-additive probability measure on the space of all profiles of such hierarchies, but only to finitely additive probability measures. Hence such hierarchies cannot be induced by types in Harsanyi type spaces where beliefs are σ-additive probability measures. Heifetz and Samet proved the existence of a universal type space in the general measure theoretic case in a very elegant manner collecting all the profiles of hierarchies of beliefs that *are* induced by some state of some type space. However, since their construction uses all type spaces (on the same space of states on nature, and for the same set of players) to construct the universal type space, this construction does not give much independent information about the inner structure of the universal type space (as, for example, in Mertens and

[4] The space constructed in Brandenburger and Dekel (1993) is also universal. This follows from the results in Battigalli and Siniscalchi (1999).

[5] Meier (2006) proved the existence of a universal type space if beliefs are described by finitely additive probability measures, and Meier (2008) shows the existence of a universal knowledge-belief space, where a knowledge-belief space is a type space with an additional knowledge operator for each player.

Zamir (1985), where we know that the universal type space is the space of all coherent hierarchies).

We show here in our Theorem 2 that the canonical model and the universal space are one and the same. Hence, we provide here a (up to now missing) characterization of the universal type space in the general measure theoretic case, in the sense that we show that the universal type space "is" the space of all maximal consistent sets of formulas of our logic, endowed in a natural way with the structure of a type space.

The results of this paper show that the pathological coherent hierarchies in Heifetz and Samet (1999) that are not induced by types in a type space are excluded by the infinitary axioms and inference rules put forward here (but not by the finitary ones in Heifetz and Mongin 2001), since these rules imply σ-additivity.

Furthermore, everywhere in the literature, except in Fagin and Halpern (1994) and Heifetz and Mongin (2001), only type spaces are considered where the players know their own beliefs (we call these spaces, like Heifetz and Mongin (2001), "Harsanyi type spaces"). We construct our canonical model with and without this property and establish the first proof of the existence of a universal type space for the class of type spaces without introspection.

1.2.3. *The implicit approach II—(beliefs) complete type spaces.* In the literature "type spaces" usually are what we call here "product type spaces". Other authors who considered the more general version (as we do here) are Heifetz and Mongin (2001), who called it also "type spaces", and Mertens and Zamir (1985), who called these spaces "beliefs spaces". As it turned out in their topological setting, the universal type space of Mertens and Zamir is a product type space.

Also, our canonical model is—up to isomorphism of type spaces—a product type space (Theorem 3).

A product type space is a product of the space of states of nature and, for each player, a parameter space that "encodes" the types of this player. That is, to each element of the parameter space corresponds a probability measure on the product of the space of states of nature and the parameter spaces of the players. In the introspective case of a Harsanyi type space, one imposes the additional condition that the marginal of a type of this player on his parameter space is the delta measure of the element to which this type of the player corresponds to. (The alternative possibility in the case of a Harsanyi type space is to define a type of a player to be only a probability measure on the product of the space of states of nature and the parameter spaces of the *other* players.)

If one does not have (or does not want to define) a notion of beliefs-preserving maps between different type spaces, one can still define a notion of "all possible types within a product type space". In this case one would say that all types that are possible *within this space* are present, if for every player and every probability measure on this product space (respectively, for every probability measure on the product of the space of states of nature and the parameter spaces of the other players, in the introspective case), there is a type of this player that coincides with this probability measure (resp. whose marginal on the product of the space of states of nature and the parameter spaces of the other players coincides with this probability measure). Such a type space would be called a (**beliefs) complete type space** (Brandenburger 2003).

The existence of a beliefs complete type space was proved by Mertens and Zamir (1985), Brandenburger and Dekel (1993), Heifetz (1993), and Mertens, Sorin and Zamir (1994) and Battigalli and Siniscalchi (1999) and Pinter (2005) who all showed that their spaces of hierarchies of beliefs mentioned under 1. (resp. their universal type spaces mentioned under 2.) constitute beliefs complete type spaces.

Heifetz and Samet (1998b) did not explore whether their general measure theoretic universal type space is beliefs complete, hence this remained an open issue up to now. We show here in Theorem 4 that this is also still true in the general measure theoretic setting, in the introspective as well as in the non-introspective case. Moreover, the parameter space of each player is—as a measurable space—isomorphic to the space of probability measures on the whole type space in the non-introspective case, and in the introspective case, the parameter space of each player is isomorphic to the space of probability measures on the product of the space of states of nature and the parameter spaces of the other players.

1.3. Technical difficulties of a strongly complete axiomatization.

Heifetz and Mongin (2001)—and before Fagin, Halpern and Megiddo (1990) for a much richer syntax also expressing valuations for linear combinations of formulas—axiomatized the class of type spaces in terms of a purely finitary logic.

They showed that their axiomatization is sound and complete with respect to the class of (Harsanyi) type spaces.

A purely finitary axiomatization cannot be used to get strong soundness and strong completeness for this class of models. This was noted by Aumann (1999b) and Heifetz and Mongin (2001). Consider the following set of formulas:

$$\left\{p_i^{\frac{1}{2}-\frac{1}{n}}(\varphi):\ n \geq 2,\ n \in \mathbb{N}\right\} \cup \left\{\neg p_i^{\frac{1}{2}}(\varphi)\right\},$$

where $p_i^{\alpha}(\varphi)$ means "individual i assigns probability at least α to (the event defined by) φ". They argue that for this set of formulas each finite subset has a model (a type space and a state in it, such that each formula in this finite subset is true in this state), while the whole set itself has no model. Another reason why one seems to need to allow for infinitary formulas is σ-additivity.

However, type spaces cannot be axiomatized by an infinitary logic in the sense of Heifetz (1997): An example by Karp (1964) in a purely propositional setting shows already that, in the presence of $\aleph_\gamma$ many formulas whose truth values can be chosen independently of one another, if one allows for infinite conjunctions of $\aleph_\gamma$ many formulas, then one must also allow for conjunctions of $2^{\aleph_\gamma}$ many formulas and proofs of length of cardinality $\leq 2^{\aleph_\gamma}$ to get strong completeness. This fact is reflected by the infinitary version of the distributive de Morgan law (A6),[6] which requires to allow for uncountable conjunctions of formulas.

This conflicts with measurability conditions that must be met: When we want to define the validity relation "$\models$" for a type space τ and some state ω in τ, then, for a formula φ in our language, $(\tau, \omega) \models p_i^{\alpha}(\varphi)$ can be defined if $[\varphi]^{\tau}$, the set of states in τ where φ is true, is a measurable set. Since conjunctions of formulas correspond to intersections of subsets of the structure, uncountable conjunctions cannot be guaranteed to interpret measurable sets, unless we do assume that the σ-fields of the type spaces are closed under uncountable intersections (i.e., they would be κ-fields for some $\kappa > \aleph_1$, see Meier (2006)), which, of course, would strongly restrict the class of type spaces we could consider.

We resolve this problem by defining a language which takes the advantages and avoids the disadvantages of both the finitary and the infinitary languages. We start with a finitary language $\mathcal{L}_0$ à la Aumann (1995) and Heifetz and Mongin (2001) with the operators p_i^{α}, "individual i assigns probability at least α to". Then, we define an infinitary propositional language $\mathcal{L}$, the primitive

[6] (A6) is defined in the list of Axioms at the beginning of Section 3.

propositions of which are the formulas in $\mathcal{L}_0$. So, $\mathcal{L}_0$ is a sublanguage of $\mathcal{L}$ and infinite conjunctions and disjunctions never appear under the scope of a belief operator. Archimedianity is expressed by a continuity axiom (P3) and σ-additivity is expressed by an inference rule (Continuity at $\emptyset$). In that way, the measurability problem is avoided.

2. Preliminaries

We fix a nonempty set X of primitive propositions (to be interpreted as statements about nature, i.e., the primary source of uncertainty for the players)[7] and a nonempty set I of players. We assume without loss of generality that $0 \notin I$ and define $I_0 := I \cup \{0\}$. For a set M, denote by $|M|$ the cardinality of M.

2.1. Syntax. Let α and β denote rational numbers $\in [0,1]$, φ, χ, ψ formulas, and ω formulas that are conjunctions of maximal consistent sets of finitary formulas.

Definition 1: We define

$$\aleph_\gamma := \max\{|I|, |X|, \aleph_0\}.$$

Definition 2: The set $\mathcal{L}_0$ of **finitary formulas** is the least set such that:

1. each $x \in X \cup \{\top\}$ is a finitary formula,
2. if φ is a finitary formula, then $(\neg\varphi)$ is a finitary formula,
3. if φ and ψ are finitary formulas, then $(\varphi \wedge \psi)$ is a finitary formula,
4. if φ is a finitary formula, then for every $i \in I$ and rational $\alpha \in [0,1]$: $(p_i^\alpha(\varphi))$ is a finitary formula.

$\mathcal{L}_0$ coincides with the language in Heifetz and Mongin (2001) and is a sublanguage of the language in Aumann (1999b).

Remark 1:

$$|\mathcal{L}_0| = \max\{|I|, |X|, \aleph_0\} = \aleph_\gamma.$$

Definition 3: The set $\mathcal{L}$ of **formulas** is the least set such that:

1. each $\varphi \in \mathcal{L}_0$ is a formula,

[7] Defined in this way, the space of states of nature in the literature corresponds to $\mathrm{Pow}(X)$ and the σ-field on the space of states of nature corresponds to the σ-field on $\mathrm{Pow}(X)$ generated by the sets $\{s \subseteq X \mid x \in s\}$, where $x \in X$.

2. if φ is a formula, then $(\neg\varphi)$ is a formula,
3. if Φ is a set of formulas of cardinality $\leq 2^{\aleph_\gamma}$, then $(\bigwedge_{\varphi\in\Phi}\varphi)$ is a formula.[8]

Intuitively, $\mathcal{L}$ is "like" an infinitary propositional language whose primitive propositions are the formulas in $\mathcal{L}_0$. Note, however, that $\mathcal{L}$ is not really a propositional language, since it contains $\mathcal{L}_0$, which is not propositional. The propositional part $\mathcal{L}^0$ of $\mathcal{L}$ can be seen as the set of those formulas which are statements about nature:

Definition 4: The set $\mathcal{L}^0$ of 0**-formulas** is the set of (infinitary) propositional formulas in $\mathcal{L}$. More formally, it is the least set of formulas (and obviously, subset of $\mathcal{L}$) such that:

1. each $x \in X \cup \{\top\}$ is a 0-formula,
2. if φ is a 0-formula, then $\neg\varphi$ is a 0-formula,
3. if Φ is a set of 0-formulas of cardinality $\leq 2^{\aleph_\gamma}$, then $\bigwedge_{\varphi\in\Phi}\varphi$ is a 0-formula.

Definition 5: Let $i \in I$. The set $\mathcal{L}^i$ of i**-formulas** is the least set of formulas (and obviously, subset of $\mathcal{L}$) such that:

1. if $\varphi \in \mathcal{L}_0$, then for every rational $\alpha \in [0,1] : p_i^\alpha(\varphi)$ is an i-formula,
2. if φ is an i-formula, then $\neg\varphi$ is an i-formula,
3. if Φ is a set of i-formulas of cardinality $\leq 2^{\aleph_\gamma}$, then $\bigwedge_{\varphi\in\Phi}\varphi$ is an i-formula.

A formula in $\mathcal{L}^i$ is a statement about the beliefs of player i.

Convention 1:
- As usual, "$\bigvee$", "$\vee$", "$\rightarrow$" and "$\leftrightarrow$" are abbreviations, defined in the usual way:

$$\Big(\bigvee_{\varphi\in\Phi}\varphi\Big) := \Big(\neg\Big(\bigwedge_{\varphi\in\Phi}(\neg\varphi)\Big)\Big),$$
$$(\varphi\rightarrow\psi) := ((\neg\varphi)\vee\psi),$$
$$(\varphi\leftrightarrow\psi) := ((\varphi\rightarrow\psi)\wedge(\psi\rightarrow\varphi)).$$

[8] By convention, we set: $\bigwedge_{\varphi\in\emptyset}\varphi := \top$, and accordingly: $\bigvee_{\varphi\in\emptyset}\varphi := \neg\top$. Furthermore, if we write "$\varphi\wedge\psi$", where φ or $\psi \in \mathcal{L}\setminus\mathcal{L}_0$, we mean implicitly the formula $\bigwedge_{\chi\in\{\varphi,\psi\}}\chi$. Accordingly, for $\varphi,\psi\in\mathcal{L}_0$, we indentify $\varphi\wedge\psi$ with $\bigwedge_{\chi\in\{\varphi,\psi\}}\chi$.

- To avoid the use of too many brackets, we apply the usual convention of decreasing priority: $\neg$, $\bigwedge$, $\wedge$, $\bigvee$, $\vee$, $\rightarrow$, $\leftrightarrow$. This means, for example, that "$\neg\varphi \wedge \psi$" is an abbreviation for "$((\neg\varphi) \wedge \psi)$".

2.2. Semantics.

Definition 6: Let M be a nonempty set and let Σ be a σ-field on M. We denote by $\Delta(M, \Sigma)$—or in short: $\Delta(M)$—the set of all σ-additive probability measures on (M, Σ). Unless stated differently, we consider $\Delta(M, \Sigma)$ as a measurable space with the σ-field Σ_Δ generated by all the sets of the form $b^\alpha(E) := \{\mu \in \Delta(M, \Sigma) \mid \mu(E) \geq \alpha\}$, where $E \in \Sigma$ and $\alpha \in [0,1] \cap \mathbb{Q}$.

Note that if $r \in [0,1]$ and $E \in \Sigma$, then $b^r(E) = \bigcap_{\alpha \in [0,r] \cap \mathbb{Q}} b^\alpha(E) \in \Sigma_\Delta$. Therefore Σ_Δ is also generated by all the sets $b^r(E)$, where $E \in \Sigma$ and $r \in [0,1]$.

We define now type spaces, that is, the semantic objects which we will study in this paper.

Definition 7: A **type space on X for player set I** is a 4-tuple

$$\underline{M} := \langle M, \Sigma, (T_i)_{i \in I,} v \rangle,$$

where

- M is a nonempty set,
- Σ is a σ-field on M,
- for $i \in I : T_i$ is a $\Sigma - \Sigma_\Delta$-measurable function from M to $\Delta(M, \Sigma)$, the space of probability measures on (M, Σ),
- v is a function from $M \times (X \cup \{\top\})$ to $\{0,1\}$, such that $v(\cdot, x)$ is $\Sigma - \mathrm{Pow}(\{0,1\})$-measurable, for every $x \in X$, and such that $v(m, \top) = 1$, for all $m \in M$.

This structure is interpreted as follows: M is the set of states of the world. Such a state determines completely the objective parameters of the player's interaction, that is, the set of primitive propositions x such that $v(m, x) = 1$, as well as the player's beliefs about the true state of the world. In general, in a state of the world $m \in M$, player i will not know the true state of the world m; he will just have a probability measure $T_i(m)$ over the set of states of the world. $T_i(m)$ describes his beliefs in state m, that is, the **type of player i in state m**. (Knowing m would mean that $T_i(m) = \delta_m$, where δ denotes the Kronecker-delta.)

Definition 8: For a type space $\langle M,\, \Sigma,\, (T_i)_{i\in I,}\ v\rangle$ on X for player set I define

$$[T_i\,(m)] := \{m' \in M \mid T_i\,(m') = T_i\,(m)\}\,,$$

for $m \in M$ and $i \in I$. The type space $\langle M, \Sigma, (T_i)_{i\in I,}\ v\rangle$ is called a **Harsanyi type space on X for player set I** iff for all $A \in \Sigma$, $m \in M$ and $i \in I$: $A \supseteq [T_i\,(m)]$ implies $T_i\,(m)\,(A) = 1$.[9]

We will refer to the property that for all $i \in I$, $m \in M$ and $A \in \Sigma : [T_i\,(m)] \subseteq A$ implies $T_i\,(m)\,(A) = 1$ as the **introspection property** of Harsanyi type spaces. This expresses the self-consciousness of the players: In a state of the world m a player does not attribute a positive probability to states where he has a different belief from the belief he has in the present state m.

The following lemma, which will be needed in the proof of the Completeness Theorem, is a slightly changed version of Lemma 2.1 of Heifetz and Samet (1999):

LEMMA 1: *Let M be a nonempty set, let $\mathcal{F}$ be a field on M that generates the σ-field Σ on M and let $\mathcal{F}_\Delta$ be the σ-field on $\Delta(M,\Sigma)$ generated by the sets of the form*

$$b^p(E) := \{\mu \in \Delta\,(M,\Sigma) \mid \mu\,(E) \geq p\}\,,$$

where $E \in \mathcal{F}$ and $p \in [0,1] \cap \mathbb{Q}$. Then

$$\mathcal{F}_\Delta = \Sigma_\Delta.$$

Proof. The proof is the same as the proof of Lemma 2.1 of Heifetz and Samet (1999), if we replace there "such that $b^p(F) \in \mathcal{F}_\Delta$ for all $0 \leq p \leq 1$" by "such that $b^p(F) \in \mathcal{F}_\Delta$ for all $p \in [0,1] \cap \mathbb{Q}$". ∎

We define now the model relation, that is, how formulas are linked to subsets of a semantic structure. Intuitively, to each formula we associate the set of states where the formula is satisfied.

Definition 9: Let $\underline{M} = \langle M,\, \Sigma,\, (T_i)_{i\in I,}\ v\rangle$ be a type space on X for player set I. We define:

- $(\underline{M}, m) \models \top$ in any case,
- for every $x \in X : (\underline{M}, m) \models x$ iff $v\,(m,x) = 1$,
- for all $\varphi, \psi \in \mathcal{L}$: $(\underline{M}, m) \models \varphi \wedge \psi$ iff $(\underline{M}, m) \models \varphi$ and $(\underline{M}, m) \models \psi$,

[9] Note that if $[T_i\,(m)]$ is measurable, then this condition reduces to: $T_i\,(m)\,([T_i\,(m)]) = 1$.

- for every $\varphi \in \mathcal{L}$: $(\underline{M}, m) \models \neg\varphi$ iff $(\underline{M}, m) \not\models \varphi$,
- for $\varphi \in \mathcal{L}_{0,}$ such that $[\varphi]^{\underline{M}} := \{m \in M \mid (\underline{M}, m) \models \varphi\} \in \Sigma$, and for $i \in I$ and rational $\alpha \in [0,1]$: $(\underline{M}, m) \models p_i^{\alpha}(\varphi)$ iff $T_i(m)([\varphi]^{\underline{M}}) \geq \alpha$.

It is easy to show by induction on the formation of the formulas in $\mathcal{L}_0$ that $[\varphi]^{\underline{M}} \in \Sigma$, for every $\varphi \in \mathcal{L}_0$ (in particular, since $T_i : M \to \Delta(M)$ is $\Sigma - \Sigma_{\Delta}$-measurable, it follows that $[\varphi]^{\underline{M}} \in \Sigma$ implies $[p_i^{\alpha}(\varphi)]^{\underline{M}} \in \Sigma$). So, the relation "$(\underline{M}, m) \models \varphi$" is well-defined for every type space $\underline{M}$ on X for player set I, every $m \in M$, and every $\varphi \in \mathcal{L}_0$.

- If $\Phi \subseteq \mathcal{L}$ and $|\Phi| \leq 2^{\aleph_\gamma}$, then:
  $(\underline{M}, m) \models \bigwedge_{\varphi \in \Phi} \varphi$ iff for every $\varphi \in \Phi : (\underline{M}, m) \models \varphi$.

It is now easy to show that the relation "$(\underline{M}, m) \models \varphi$" is well-defined, for every type space $\underline{M}$ on X for player set I, every $m \in M$, and every $\varphi \in \mathcal{L}$.

Definition 10: A formula $\varphi \in \mathcal{L}$ is **valid** in the class of type spaces (resp. Harsanyi type spaces) on X for player set I iff for every type space (resp. Harsanyi type space) $\underline{M} = \langle M, \Sigma, (T_i)_{i \in I,} v \rangle$ on X for player set I and every $m \in M$:

$$(\underline{M}, m) \models \varphi.$$

Notation 1:

1. Let $\Gamma \subseteq \mathcal{L}$, let $\underline{M} = \langle M, \Sigma, (T_i)_{i \in I,} v \rangle$ be a type space (resp. Harsanyi type space) on X for player set I, and let $m \in M$. We write
   $$(\underline{M}, m) \models \Gamma$$
   iff for every $\psi \in \Gamma$:
   $$(\underline{M}, m) \models \psi.$$
2. Let $\Gamma \subseteq \mathcal{L}$. We say Γ **has a model** in the class of type spaces (resp. Harsanyi type spaces) on X for player set I iff there is a type space (resp. Harsanyi type space) $\underline{M}$ on X for player set I and a $m \in M$ such that $(\underline{M}, m) \models \Gamma$. If $(\underline{M}, m) \models \Gamma$ (resp. $(\underline{M}, m) \models \varphi$) holds, we say that $(\underline{M}, m)$ **is a model of** Γ (resp. φ).
3. Let $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$. We write
   $$\Gamma \models \varphi$$
   iff for every type space (resp. Harsanyi type space) $\underline{M}$ on X for player set I and every $m \in M$:
   $$(\underline{M}, m) \models \Gamma \text{ implies } (\underline{M}, m) \models \varphi.$$

If $\Gamma \models \varphi$ holds, we say that Γ **implies** φ **semantically**.

3. Strong completeness and construction of the canonical (Harsanyi) type space

In this section we define our axioms and inference rules, our notion of "proof" (in the sense of our logic) and prove (in the corresponding appendix) strong soundness and, by constructing the canonical model, strong completeness. As already said, α and β denote rational numbers in $[0, 1]$.

The List of Axioms

(A0) $\top$,

(A1) $\varphi \to (\psi \to \varphi)$, for $\varphi, \psi \in \mathcal{L}$,

(A2) $(\varphi \to (\psi \to \varrho)) \to ((\varphi \to \psi) \to (\varphi \to \varrho))$, for $\varphi, \psi, \varrho \in \mathcal{L}$,

(A3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$, for $\varphi, \psi \in \mathcal{L}$,

(A4) $\bigwedge_{\varphi\in\Phi}(\psi \to \varphi) \to (\psi \to \bigwedge_{\varphi\in\Phi}\varphi)$, for $\psi \in \mathcal{L}$ and $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \le 2^{\aleph_\gamma}$,

(A5) $\bigwedge_{\varphi\in\Phi}\varphi \to \psi$, for $\psi \in \Phi$, where $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \le 2^{\aleph_\gamma}$,

(A6) $\bigwedge_{a\in A}(\bigvee_{b\in A}\varphi_{a,b}) \to \bigvee_{g\in A^A}(\bigwedge_{a\in A}\varphi_{a,g(a)})$, whenever $|A| \le \aleph_\gamma$,[10]

(P1) $p_i^0(\varphi)$, for $\varphi \in \mathcal{L}_0$,

(P2) $p_i^1(\top)$,

(P3) $\bigwedge_{\alpha<\beta} p_i^\alpha(\varphi) \to p_i^\beta(\varphi)$, for $\varphi \in \mathcal{L}_0$,

(P4) $\left(p_i^\alpha(\varphi \wedge \psi) \wedge p_i^\beta(\varphi \wedge \neg\psi)\right) \to p_i^{\alpha+\beta}(\varphi)$, for α, β with $\alpha + \beta \le 1$ and $\varphi, \psi \in \mathcal{L}_0$,

(P5) $\left(\neg p_i^\alpha(\varphi \wedge \psi) \wedge \neg p_i^\beta(\varphi \wedge \neg\psi)\right) \to \neg p_i^{\alpha+\beta}(\varphi)$, for α, β with $\alpha + \beta \le 1$ and $\varphi, \psi \in \mathcal{L}_0$,

(P6) $p_i^\alpha(\varphi) \to \neg p_i^\beta(\neg\varphi)$, for α, β with $\alpha + \beta > 1$ and $\varphi \in \mathcal{L}_0$,

(P7) $p_i^\alpha(\varphi) \to p_i^\beta(\varphi)$, for α, β with $\beta < \alpha$ and $\varphi \in \mathcal{L}_0$,

(P8) $p_i^1(\varphi \to \psi) \to (p_i^\alpha(\varphi) \to p_i^\alpha(\psi))$, for $\varphi, \psi \in \mathcal{L}_0$,

(I1) $p_i^\alpha(\varphi) \to p_i^1(p_i^\alpha(\varphi))$, for $\varphi \in \mathcal{L}_0$,

(I2) $\neg\, p_i^\alpha(\varphi) \to p_i^1(\neg\, p_i^\alpha(\varphi))$, for $\varphi \in \mathcal{L}_0$.

Except (A0) and (P2), all the above axioms are in fact axiom schemes, i.e., lists of axioms.

[10] A^A denotes here the set of all functions from A to A.

We adopt the following **inference rules**:

- *Modus Ponens*: From φ and $\varphi \to \psi$ infer ψ.
- *Conjunction*: From Φ infer $\bigwedge_{\varphi\in\Phi} \varphi$, if $\Phi \subseteq \mathcal{L}$ such that $\mid \Phi \mid \leq 2^{\aleph_\gamma}$.
- *Necessitation*: From φ infer $p_i^1(\varphi)$, if $\varphi \in \mathcal{L}_0$.
- *Continuity at $\emptyset$* : From $\bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top$, where $\varphi_n \in \mathcal{L}_0$, for all $n \in \mathbb{N}$, infer $\bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}(\bigwedge_{n\leq l} \varphi_n)$.
- *Uncountable Introspection*: From $\varphi \to \bigvee_{n\in\mathbb{N}} \varphi_n$, where $\varphi \in \mathcal{L}^i$ and $\varphi_n \in \mathcal{L}_0$ for all $n \in \mathbb{N}$, infer $\varphi \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}(\bigvee_{n\leq l} \varphi_n)$.

(A0)–(A6) are the axioms and "Modus Ponens" and "Conjunction" are the inference rules for infinitary propositional logic, where the language is the propositional part, $\mathcal{L}^0$, of our infinitary language $\mathcal{L}$. Karp, has proved strong soundness and strong completeness (Karp (1964, Theorem 5.5.4)) for this logic. We will use this result, sometimes without referring to it explicitly.

Most of the axioms (P1)–(P8), (I1), (I2) above can be found in Aumann (1995) and Heifetz and Mongin (2001).

Definition 11: 1. The **system** P consists of the axioms (A0)–(A6), (P3)–(P6), (P8), and the inference rules "Modus Ponens", "Conjunction", "Necessitation", and "Continuity at $\emptyset$".

2. The **system** H is the system P together with the additional axiom (I1), if $\aleph_\gamma = \aleph_0$. If $\aleph_\gamma > \aleph_0$, the system H is the system P together with the inference rule "Uncountable Introspection".

Definition 12: (1) The **set of theorems of the system** P is the minimal set of formulas that contains the axioms (A0)–(A6), (P3)–(P6), (P8), and that is closed under "Modus Ponens", "Conjunction", "Necessitation" and "Continuity at $\emptyset$".

(2) The **set of theorems of the system** H is the minimal set of formulas that contains the axioms (A0)–(A6), (P3)–(P6), (P8), (I1), and that is closed under "Modus Ponens", "Conjunction", "Necessitation" and "Continuity at $\emptyset$", in the case $\aleph_\gamma = \aleph_0$. If $\aleph_\gamma > \aleph_0$, the set of theorems of the system H is the minimal set of formulas that contains the axioms (A0)–(A6), (P3)–(P6), (P8), and that is closed under "Modus Ponens", "Conjunction", "Necessitation", "Continuity at $\emptyset$" and "Uncountable Introspection".

Note that the above set of axioms is not minimal:

- (P1) follows from (P3), (A0) and Modus Ponens, if we adopt the usual convention that $\bigwedge_{\varphi\in\emptyset}\varphi := \top$.
- (P2) follows from (A0) and Necessitation.
- Heifetz and Mongin (2001) proved that (P7) follows from (A0)–(A6), (P3)–(P6) and (P8).
- The proof of the Completeness Theorem will also show that (A0)–(A6), (P3)–(P6), (P8) and (I1) imply (I2). The basic reason behind this is that $\neg p_i^\alpha(\varphi)$ holds in a state, iff there is a $\varepsilon > 0$ such that $p_i^{1+\varepsilon-\alpha}(\neg\varphi)$ holds in this state.
- "Uncountable Introspection" implies (together with (A0)–(A6), (P3)–(P6) and (P8) and the other inference rules) the axioms (I1) and (I2).

  We only show the most crucial steps: $p_i^\alpha(\varphi)$ is an i-formula. And $p_i^\alpha(\varphi) \to \bigvee_{n\in\mathbb{N}} p_i^\alpha(\varphi)$ is an instance of a tautology of propositional calculus, to which we apply the inference rule "Uncountable Introspection" to get $p_i^\alpha(\varphi) \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}}\bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}(\bigvee_{n\leq l} p_i^\alpha(\varphi))$. Applying necessitation to the tautology $\bigvee_{n\leq l} p_i^\alpha(\varphi) \to p_i^\alpha(\varphi)$, then (P8) followed by 3 of Lemma 2, one gets $p_i^\alpha(\varphi) \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}}\bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}(p_i^\alpha(\varphi))$. Using 3 of Lemma 2 and first (P7) and then (P3) one gets $p_i^\alpha(\varphi) \to p_i^1((p_i^\alpha(\varphi))$, that is (I1). (I2) is obtained by replacing $p_i^\alpha(\varphi)$ with $\neg p_i^\alpha(\varphi)$.
- The Completeness Theorem implies that, in the case of $\aleph_\gamma = \aleph_0$, (I1) (together with (A0)–(A6), (P3)–(P6), and (P8) and the other inference rules) implies the inference rule "Uncountable Introspection". "Uncountable Introspection" is a valid inference rule in the class of Harsanyi type spaces on X for player set I, also when $|I|, |X| \leq \aleph_0$. Therefore, the completeness theorem implies that "Uncountable Introspection" is implied by (I1) together with (A0)–(A6), (P3)–(P6), and (P8), when $|I|, |X| \leq \aleph_0$.

In fact we have here two articles in one: Given a nonempty set of players I and a nonempty set of primitive propositions X, if nothing else is said, we do all what follows for the system P on the syntactic side and for the class of type spaces on X for player set I on the semantic side. And we also do all what follows for the system H on the syntactic side and for the class of Harsanyi type spaces on X for player set I on the semantic side. We only specify the system, if there is a difference between the two cases in the proofs or in the statements of the Definitions, Lemmas, Propositions or Theorems.

Definition 13: Let Γ be a set of formulas in $\mathcal{L}$. A **proof of** φ **from** Γ in the system P (resp. in the system H) is a sequence whose length is strictly smaller than $(2^{\aleph_\gamma})^+$ and whose last formula is φ, such that each formula in the proof is in Γ, a theorem of the system P (resp. of the system H), or inferred from the previous formulas by "Modus Ponens" or "Conjunction".[11]

If there is a proof of φ from Γ, we write $\Gamma \vdash \varphi$ and say that Γ **implies** φ **syntactically**. In particular, "$\vdash \varphi$" (which stands for $\emptyset \vdash \varphi$) means that φ is a theorem.

Definition 14:

- The system P (resp. H) is **strongly sound** iff for every $\Gamma \subseteq \mathcal{L}$ and every $\varphi \in \mathcal{L}$:

$$\Gamma \vdash \varphi \text{ implies } \Gamma \models \varphi.$$

- The system P (resp. H) is **strongly complete** iff for every $\Gamma \subseteq \mathcal{L}$ and every $\varphi \in \mathcal{L}$:

$$\Gamma \models \varphi \text{ implies } \Gamma \vdash \varphi.$$

Together, strong soundness and strong completeness mean that the notions of syntactic and semantic implication coincide.

Definition 15: Γ is **consistent** in the system P (resp. in the system H) iff there is no formula $\varphi \in \mathcal{L}$ such that there are proofs of φ and $\neg\varphi$ from Γ in the system P (resp. in the system H).

Lemma 2: *Let $\varphi, \psi, \widetilde{\psi} \in \mathcal{L}$. Then:*

1. *If $\Phi \subseteq \mathcal{L}$ and $|\Phi| \leq 2^{\aleph_\gamma}$, then $\Phi \vdash \varphi$ iff $\{\bigwedge_{\chi\in\Phi} \chi\} \vdash \varphi$.*
2. *$\{\psi\} \vdash \varphi$ iff $\vdash \psi \to \varphi$.*
3. *If $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \psi \to \widetilde{\psi}$, then $\Gamma \vdash \varphi \to \widetilde{\psi}$.*
4. *$\vdash \varphi \to \neg(\neg\varphi)$.*
5. *If $\psi \in \Phi$, $\Phi \subseteq \mathcal{L}$, and $|\Phi| \leq 2^{\aleph_\gamma}$, then $\vdash \neg\psi \to \neg \bigwedge_{\chi\in\Phi} \chi$.*

[11] Of course, whether φ is a theorem of the system, resp., whether there is a proof of φ from Γ, depends on the system under consideration. That is, for some φ there might be a proof of φ from Γ in the system H, but not in the system P. It follows also that the notion of consistency depends on the system.

Proposition 1:

- *The system P is strongly sound with respect to the class of type spaces on X for player set I.*
- *The system H is strongly sound with respect to the class of Harsanyi type spaces on X for player set I.*

3.1. Strong completeness. The idea of the proof of strong completeness is as follows: We build a (Harsanyi) type space $\underline{\Omega}$ whose underlying set of states of the world Ω "is" the set of all maximal consistent sets of formulas, such that $\underline{\Omega}$ has the following additional property: For a maximal consistent set of formulas $\omega \in \Omega$ and a formula $\varphi \in \mathcal{L}$ we have $(\underline{\Omega}, \omega) \models \varphi$ iff $\varphi \in \omega$. This implies then that any consistent set Φ of formulas has a model: First one shows using Zorn's Lemma that Φ can be extended to a maximal consistent set of formulas $\omega \supseteq \Phi$. The above-mentioned property implies then that $(\underline{\Omega}, \omega) \models \Phi$. That any consistent set of formulas has a model implies (in fact, is equivalent to) strong completeness.

The first step of the construction of our canonical model is to define a set of states of the world, whose states "are" maximal consistent sets of formulas (though, formally, they are themselves formulas (of a very special form)).

Definition 16:

- $\Omega :=$

$$\left\{ \bigwedge_{\varphi \in \Phi_0} \varphi \wedge \bigwedge_{\psi \in \mathcal{L}_0 \setminus \Phi_0} \neg\psi \mid \Phi_0 \subseteq \mathcal{L}_0, \text{ s.t. } \Phi_0 \cup \{\neg\psi \mid \psi \in \mathcal{L}_0 \setminus \Phi_0\} \text{ is consistent} \right\},$$

- for $\psi \in \mathcal{L}$, define

$$[\psi] := \{\omega \in \Omega \mid \vdash \omega \to \psi\},$$

- for $\Gamma \subseteq \mathcal{L}$, define

$$[\Gamma] := \bigcap_{\psi \in \Gamma} [\psi],$$

- for $\omega \in \Omega$, such that $\omega = \bigwedge_{\varphi \in \Phi_0} \varphi \wedge \bigwedge_{\psi \in \mathcal{L}_0 \setminus \Phi_0} \neg\psi$, define

$$\Psi_\omega := \Phi_0 \cup \{\neg\psi \mid \psi \in \mathcal{L}_0 \setminus \Phi_0\}.$$

Note that although we write "Ω", we define in fact two Ω's, one corresponding to the system P, and one corresponding to the system H. By the definitions of the system P and of the system H, it follows that a set of $\mathcal{L}$-formulas that is consistent in the system H is also consistent in the system P. Hence the Ω

corresponding to the system H is a subset of the Ω corresponding to the system P.

Now, we show that Ω is nonempty.

Remark 2: 1. The class of Harsanyi type spaces on X for player set I is nonempty. And hence, the class of type spaces on X for player set I is nonempty.

2. The set Ω is nonempty.

The next proposition constitutes, together with Lemmata 3 and 4, the technical heart of the construction of the canonical model. 1. shows that $\bigvee_{\omega\in\Omega}\omega$ is a theorem and 2. that each ω implies a maximal consistent set of formulas.

Proposition 2: 1. $\vdash \bigvee_{\omega\in\Omega}\omega$.

2. *For every formula $\psi\in\mathcal{L}$ and for every $\omega\in\Omega$:*

$$\textit{Either } \vdash \omega\to\psi \textit{ or } \vdash\omega\to\neg\psi,$$

but not both.

3. *For every formula $\psi\in\mathcal{L}$:*

$$\vdash \psi \leftrightarrow \bigvee_{\omega\in[\psi]}\omega.$$

4. *If $\Phi\subseteq\mathcal{L}$ such that $|\Phi|\leq 2^{\aleph_\gamma}$, then*

$$\vdash \bigwedge_{\varphi\in\Phi}\varphi \leftrightarrow \bigvee_{\omega\in[\Phi]}\omega.$$

5. *For every formula $\psi\in\mathcal{L}$:*

$$\vdash \neg\psi \leftrightarrow \bigvee_{\omega\in\Omega\setminus[\psi]}\omega.$$

6. *For every formula $\psi\in\mathcal{L}$:*

$$[\neg\psi]=\Omega\setminus[\psi]\,.$$

7. *If $\Phi\subseteq\mathcal{L}$ such that $|\Phi|\leq 2^{\aleph_\gamma}$, then*

$$[\Phi]=\left[\bigwedge_{\varphi\in\Phi}\varphi\right].$$

The next step of the construction of the canonical type space is to define a measurable space:

Definition 17: Let Σ be the σ-field on Ω generated by the set

$$\{[\psi] \mid \psi \in \mathcal{L}_0\}.$$

By (A0) and 2 of Lemma 2, it follows that $\Omega = [\top]$, and by 2 of Proposition 2, it follows that $\Omega \backslash [\psi] = [\neg\psi]$, for $\psi \in \mathcal{L}_0$. By Conjunction, (A4) and Modus Ponens, and by (A5) and 3 of Lemma 2, it follows that $[\varphi] \cap [\psi] = [\varphi \wedge \psi]$, for $\varphi, \psi \in \mathcal{L}_0$. Hence:

Remark 3: The set

$$\mathcal{F} := \{[\psi] \mid \psi \in \mathcal{L}_0\}$$

is a field on Ω.

For each state $\omega \in \Omega$ and each player $i \in I$, we need to define eventually a probability measure. To do this, we define now a real-valued function $T_i'(\omega)$ on $\mathcal{F}$, for each state ω and each player i.

Also, we need to define a valuation function that tells us in which states which primitive propositions are satisfied.

Definition 18:

- For $\omega \in \Omega$ and $\psi \in \mathcal{L}_0$, define

$$T_i'(\omega)([\psi]) := \sup\{\alpha \in [0,1] \cap \mathbb{Q} \mid\vdash \omega \to p_i^\alpha(\psi)\}.$$

- For $\omega \in \Omega$ and $x \in X \cup \{\top\}$, define

$$v(\omega, x) := \begin{cases} 1, & \text{if } \omega \in [x], \\ 0, & \text{if } \omega \notin [x]. \end{cases}$$

Obviously, we have:

Remark 4: $v(\cdot, x)$ is $\mathcal{F} - \mathrm{Pow}(\{0,1\})$-measurable, for every $x \in X$.

The following two lemmas are necessary steps to show that $T_i'(\omega)$ is a countably additive measure on $\mathcal{F}$. By Carathéodory's theorem, this ensures the existence of a unique countably additive extension $T_i(\omega)$ of $T_i'(\omega)$ to Σ, the σ-field on Ω generated by $\mathcal{F}$.

Lemma 3: *Let $\psi \in \mathcal{L}_0$, $\omega \in \Omega$, and $\alpha \in [0,1] \cap \mathbb{Q}$ such that $\vdash \omega \to \neg p_i^\alpha(\psi)$. Then,*

$$T_i'(\omega)([\psi]) < \alpha.$$

LEMMA 4: *For every $i \in I$ and $\omega \in \Omega$:*

$$T_i'(\omega)(\cdot)$$

is well-defined, non-negative, and a countably additive measure on $\mathcal{F}$. Furthermore, for every $i \in I$ and $\omega \in \Omega$:

$$T_i'(\omega)(\Omega) = 1.$$

The following proposition says that the set of maximal consistent sets of formulas induces the structure of a type space in the P-system case, resp. the structure of a Harsanyi type space in the H-system case. Furthermore, this type space together with a state ω is a model of a formula φ iff "$\varphi \in \omega$" (that is $\vdash \omega \to \varphi$).

PROPOSITION 3:

1. *For every $i \in I$ and $\omega \in \Omega$, there is a unique extension of $T_i'(\omega)$ to a σ-additive probability measure $T_i(\omega)$ on (Ω, Σ).*
2. *For every $i \in I$, T_i is a $\Sigma - \Sigma_\Delta$-measurable function from Ω to $\Delta(\Omega, \Sigma)$, the space of probability measures on (Ω, Σ), which is endowed with the σ-field Σ_Δ generated by the sets $\{\mu \in \Delta(\Omega, \Sigma) \mid \mu(E) \geq \alpha\}$, where $E \in \Sigma$ and $\alpha \in [0,1] \cap \mathbb{Q}$.*
3. $$\underline{\Omega} := \left\langle \Omega, \Sigma, (T_i)_{i \in I}, v \right\rangle$$
   is a type space on X for player set I.
4. *For every $\psi \in \mathcal{L}$ and $\omega \in \Omega$:*
   $$\left(\left\langle \Omega, \Sigma, (T_i)_{i \in I}, v \right\rangle, \omega \right) \models \psi \text{ iff } \omega \in [\psi].$$
5. *If the Axiom (I1) is added in the case of $\aleph_\gamma = \aleph_0$, and if the inference rule Uncountable Introspection is added in the case of $\aleph_\gamma > \aleph_0$ (i.e., in the H-system case), then*
   $$\left\langle \Omega, \Sigma, (T_i)_{i \in I}, v \right\rangle$$
   is a Harsanyi type space on X for player set I.

Theorem 1 is a corollary of Proposition 3.

THEOREM 1:

1. *The system P is strongly sound and strongly complete with respect to the class of type spaces on X for player set I.*
2. *The system H is strongly sound and strongly complete with respect to the class of Harsanyi type spaces on X for player set I.*

COROLLARY 1: *Let $\Gamma \subseteq \mathcal{L}$.*

- *The set of formulas Γ is consistent in the system P (resp. in the system H) iff Γ has a model in the class of type spaces (resp. Harsanyi type spaces) on X for player set I.*
- *Furthermore, if Γ is consistent in the system P (resp. in the system H), then there is a $\omega \in \Omega$, where Ω is the Ω corresponding to the system P (resp. to the system H), such that $(\underline{\Omega}, \omega) \models \Gamma$.*

4. Universality of the canonical (Harsanyi) type space

In this section we prove that the canonical (Harsanyi) type space on X for player set I is (up to type isomorphism) the universal (Harsanyi) type space on X for player set I. This gives a characterization of the universal (Harsanyi) type space on X for player set I and shows that our language is rich enough to describe the states in the universal (Harsanyi) type space on X for player set I. (So, in some sense, the language is rich enough to capture "all relevant information".)

We define now the beliefs preserving maps between type spaces.

Definition 19: Let $\underline{M} = \langle M, \Sigma, (T_i)_{i\in I,}\ v\rangle$ and $\underline{N} = \langle N, \Sigma^N, \left(T_i^N\right)_{i\in I,}\ v^N\rangle$ be type spaces on X for player set I. A function $f : M \to N$ is a **type morphism** iff it satisfies the following conditions:

1. f is $\Sigma - \Sigma^N$-measurable,
2. for all $m \in M$ and $x \in X$:
$$v\left(m, x\right) = v\left(f\left(m\right), x\right),$$
3. for all $m \in M$, $E \in \Sigma^N$, and $i \in I$:
$$T_i^N\left(f\left(m\right)\right)\left(E\right) = T_i\left(m\right)\left(f^{-1}\left(E\right)\right).$$

Definition 20: A type morphism f is a **type isomorphism** iff it is one-to-one, onto, and the inverse of f is also a type morphism.

LEMMA 5: *Type morphisms preserve the validity of formulas, i.e., if f is a type morphism from $\underline{M}$ to $\underline{N}$, $m \in M$, and $\varphi \in \mathcal{L}$, then*

$$\left(\underline{M}, m\right) \models \varphi \text{ iff } \left(\underline{N}, f\left(m\right)\right) \models \varphi.$$

An easy check shows:

Remark 5:
- The type spaces on X for player set I—as objects—together with the type morphisms—as morphisms—form a category.
- The Harsanyi type spaces on X for player set I—as objects—together with the type morphisms—as morphisms—form a category.

Definition 21: A type space (resp. Harsanyi type space) $\underline{M}$ on X for player set I is **universal** iff for every type space (resp. Harsanyi type space) $\underline{N}$ on X for player set I there is exactly one type morphism from $\underline{N}$ to $\underline{M}$.

It is obvious that a type morphism $f : \underline{M} \to \underline{N}$ is a type isomorphism iff there is a type morphism $g : \underline{N} \to \underline{M}$ such that $g \circ f = \mathrm{id}_{\underline{M}}$ and $f \circ g = \mathrm{id}_{\underline{N}}$. Hence, type isomorphisms coincide with the isomorphisms of the category of type spaces on X for player set I. In category theoretic terms, a universal (Harsanyi) type space on X for player set I is a terminal object of the category of (Harsanyi) type spaces on X for player set I. Terminal objects, if they exist, are known to be unique up to isomorphism, hence (but it is also easily seen directly):

Remark 6: If there exists a universal type space (resp. Harsanyi type space) on X for player set I, then it is unique up to type isomorphism.

The following theorem says that the canonical model is the universal (Harsanyi) type space. That is, the explicit (or syntactic) and implicit (or semantic) notion of all states of the world coincide.

THEOREM 2: *The (Harsanyi) type space*

$$\underline{\Omega} = \left\langle \Omega, \Sigma, (T_i)_{i \in I}, v \right\rangle$$

on X for player set I is universal.

For a (Harsanyi) type space $\underline{M}$, the unique type morphism $f : M \to \Omega$ is defined by mapping $m \in M$ to the set of formulas (resp. the conjunction of the finitary formulas) for which $(\underline{M}, m)$ is a model.

For other models of interactive uncertainty, it is not always the case that there exists a universal space. For example, for knowledge spaces Heifetz and Samet (1998a) have shown that, without further restrictions on the structure, there is no universal knowledge space (for at least two players and at least two states of nature). This result was extended by Meier (2005) to the more general context of Kripke structures. The results of Meier (2008) imply that if one endows the knowledge spaces with a measurable structure, then a universal knowledge space does exist.

5. Product type spaces

The aim of this section is to show that the canonical (Harsanyi) type space on X for player set I is (up to isomorphism) a product (Harsanyi) type space. Since this space is then a universal (Harsanyi) type space in the category of product (Harsanyi) type spaces on X for player set I, this implies that—in the case of Harsanyi type spaces (i.e., the H-system case)—our canonical model is, up to isomorphism, the universal Harsanyi type space on X for player set I constructed by Heifetz and Samet (1998b).

In the literature often only type spaces are considered that have the form of a product space:

Definition 22: A **product type space on X for player set I** is a 4-tuple

$$\underline{M} := \langle M, \Sigma, (T_i)_{i\in I}, v\rangle$$

such that there are measurable spaces (M_j, Σ_j), for $j \in I_0$, so that (up to type isomorphism):

- $M_0 = \mathrm{Pow}\,(X)$,
- Σ_0 is the σ-field on $\mathrm{Pow}\,(X)$ generated by the sets

$$\{m_0 \subseteq X \mid x \in m_0\}, \text{ where } x \in X,$$

- $M = \mathrm{Pow}\,(X) \times \Pi_{i\in I} M_i$, where all the M_i are nonempty,
- Σ is the product σ-field on M which is generated by the σ-fields Σ_j, $j \in I_0$,
- for $i \in I$: T_i is a $\Sigma_i - \Sigma_\Delta$-measurable function from M_i to $\Delta(M, \Sigma)$, the space of probability measures on (M, Σ),
- for $x \in X$: $v(m_0, x) = \begin{cases} 1, & \text{if } x \in m_0, \\ 0, & \text{if } x \notin m_0, \end{cases}$
  and $v(m_0, \top) = 1$ in any case.

Obviously, T_i, for $i \in I$, can be viewed as a $\Sigma - \Sigma_\Delta$-measurable function from M to $\Delta(M, \Sigma)$, and $v(\cdot, x)$ can be viewed as a $\Sigma - \mathrm{Pow}\,(\{0,1\})$-measurable function from M to $\{0,1\}$, for every $x \in X \cup \{\top\}$. So every product type space on X for player set I is a type space on X for player set I.

We define now the finitary i-formulas.

Definition 23: For $i \in I_0$ define $\mathcal{L}_0^i := \mathcal{L}_0 \cap \mathcal{L}^i$.

We build now, step by step, a product type space. In the rest of this section we show then that this type space is isomorphic to the canonical type space.

Definition 24:

- For $j \in I_0$, define Ω_j to be the following set of formulas:

$$\left\{ \bigwedge_{\varphi\in\Phi_0^j} \varphi \wedge \bigwedge_{\psi\in\mathcal{L}_0^j\setminus\Phi_0^j} \neg\psi \middle| \Phi_0^j \subseteq \mathcal{L}_0^j, \text{ s.t. } \Phi_0^j \cup \left\{\neg\psi \mid \psi \in \mathcal{L}_0^j\setminus\Phi_0^j\right\} \text{ is consistent} \right\}.$$

- For $j \in I_0$ and $\psi_j \in \mathcal{L}_0^j$, define
$$[\psi_j]^j := \{\omega_j \in \Omega_j \mid\vdash \omega_j \to \psi_j\}.$$
- For $j \in I_0$, denote by Σ_j the σ-field on Ω_j generated by all the sets $[\psi_j]^j$, where $\psi_j \in \mathcal{L}_0^j$.
- Define
$$\Omega^* := \Pi_{j\in I_0}\Omega_j.$$
- Denote by Σ^* the product σ-field of the σ-fields Σ_j, $j \in I_0$, on Ω^*.
- For $i \in I$, define
$$\Omega_{-i} := \Pi_{j\in I_0\setminus\{i\}}\Omega_j.$$
- For $i \in I$, denote by Σ_{-i} the product σ-field of the σ-fields Σ_j, $j \in I_0 \setminus \{i\}$, on Ω_{-i}.

Remark 7: Let $j \in I_0$. By 4 of Proposition 3 and Corollary 1, for every $\omega_j \in \Omega_j$, there is a $\omega \in \Omega$ such that $\vdash \omega \to \omega_j$.

For $\omega \in \Omega$, we have
$$\vdash \omega \to \bigwedge_{\varphi_j\in\mathcal{L}_0^j\cap\Psi_\omega} \varphi_j.$$
By the definitions and the consistency of ω, it follows that $(\bigwedge_{\varphi_j\in\mathcal{L}_0^j\cap\Psi_\omega} \varphi_j) \in \Omega_j$. By definition of the $\omega_j \in \Omega_j$, two such formulas contradict each other, i.e., for $\omega_j \neq \omega_j' \in \Omega_j$ there is a $\varphi \in \mathcal{L}_0^j$ such that $\vdash \omega_j \to \varphi$ and $\vdash \omega_j' \to \neg\varphi$. Hence, since ω is consistent, for every $\omega \in \Omega$, there is exactly one $\omega_j \in \Omega_j$ such that $\vdash \omega \to \omega_j$. We denote this ω_j by $\omega(j)$.

Since Ω is nonempty and since $\omega(j) \in \Omega_j$, for $\omega \in \Omega$ and $j \in I_0$, we have that each Ω_j, for $j \in I_0$, is nonempty.

LEMMA 6: *Let $j \in I_0$, $\omega_j \in \Omega_j$ and $\psi_j \in \mathcal{L}^j$. Then,*

$$\text{either } \vdash \omega_j \to \psi_j \text{ or } \vdash \omega_j \to \neg\psi_j,$$

but not both.

Definition 25: For $i \in I_0$ and $E_i \in \Sigma_i$ define

$$E_i^* := \Pi_{j \in I_0} U_j,$$

where $U_j = \Omega_j$, for $j \neq i$ and $U_i = E_i$.

We have $E_i^* \in \Sigma^*$. Observe that $\mathcal{L}_0^i \cap \mathcal{L}_0^j = \emptyset$, for $i \neq j \in I_0$. Hence, for $\varphi_i \in \mathcal{L}_0^i$, the following is well-defined:

$$[\varphi_i]^* := ([\varphi_i]^i)^*.$$

By the definition and the consistency of the $\omega_i \in \Omega_i$, we have

$$\Omega^* \setminus [\varphi_i]^* = [\neg\varphi_i]^* \quad \text{and} \quad [\varphi_i]^* \cap [\psi_i]^* = [\varphi_i \wedge \psi_i]^*,$$

for $\varphi_i, \psi_i \in \mathcal{L}_0^i$.

Starting with the i-formulas, for $i \in I_0$, we define now recursively:

$$[\neg\varphi]^* := \Omega^* \setminus [\varphi]^* \quad \text{and} \quad [\varphi \wedge \psi]^* := [\varphi]^* \cap [\psi]^*,$$

for $\varphi, \psi \in \mathcal{L}_0$.

This is still well-defined for the finitary i-formulas, for $i \in I_0$, and it is well-defined for the other finitary formulas by the unique readability of finitary formulas as finite Boolean combinations of finitary formulas $\varphi \in \bigcup_{i \in I_0} \mathcal{L}_0^i$, which can be proved in the usual way (what we don't know at this moment is that logically equivalent finitary formulas define the same sets in Ω^*).

It is obvious that these sets form a field $\mathcal{F}^*$ on Ω^* which generates Σ^*.

Remark 8:
- Σ_0 is generated by the sets $[x]^0$, where $x \in X$.
- For every $m_0 \subseteq X$, there is exactly one $\omega_0 \in \Omega_0$ such that for every $x \in X$:

$$\vdash \omega_0 \to x \quad \text{iff} \quad x \in m_0.$$

LEMMA 7: *Let $(\omega_j)_{j \in I_0} \in \Omega^*$ and let $\{\omega_j \mid j \in I_0\}$ be consistent. Then there is exactly one $\omega \in \Omega$ such that*

$$\vdash \omega \leftrightarrow \bigwedge_{j \in I_0} \omega_j,$$

and furthermore $\omega_j = \omega(j)$, for all $j \in I_0$. Conversely, for every $\omega \in \Omega$:

$$\vdash \omega \leftrightarrow \bigwedge_{j \in I_0} \omega(j).$$

The above lemma shows that $h : \Omega \to \Omega^*$, defined by $h(\omega) := (\omega(j))_{j \in I_0}$, is one-to-one. We are now justified to identify, with some abuse of notation, $h(\Omega)$ with Ω.

Lemma 8: *Let $\varphi \in \mathcal{L}_0$. Then:*

- $$[\varphi]^* \cap h(\Omega) = h([\varphi]),$$
- $$h^{-1}\left([\varphi]^*\right) = [\varphi].$$

Definition 26:

- For $i \in I$, $\omega_i \in \Omega_i$, and $\psi \in \mathcal{L}_0$, define

$$T_i^*(\omega_i)([\psi]^*) := \sup\left\{\alpha \in [0,1] \cap \mathbb{Q} \mid \vdash \omega_i \to p_i^\alpha(\psi)\right\}.$$

- For $\omega_0 \in \Omega_0$ and $x \in X$, define

$$v^*\left(\omega_0, x\right) := \begin{cases} 1, & \text{if } \omega_0 \in [x]^0, \\ 0, & \text{if } \omega_0 \notin [x]^0, \end{cases}$$

  and

$$v^*\left(\omega_0, \top\right) := 1 \quad \text{in any case.}$$

Obviously, for every $x \in X$, $v^*(\cdot, x)$ is $\Sigma_0 - \mathrm{Pow}(\{0,1\})$-measurable, hence viewed as a function from Ω^* to $\{0,1\}$, it is $\Sigma^* - \mathrm{Pow}(\{0,1\})$-measurable.

Lemma 9: *For every $i \in I$ and $\omega_i \in \Omega_i$:*

$$T_i^*\left(\omega_i\right)(\cdot)$$

is well-defined and a countably additive measure on $\mathcal{F}^*$.

Furthermore, for every $i \in I$ and $\omega_i \in \Omega_i$:

$$T_i^*\left(\omega_i\right)\left(\Omega^*\right) = 1.$$

Proposition 4:

1. *For every $i \in I$ and $\omega_i \in \Omega_i$, there is a unique extension of $T_i^*(\omega_i)$ to a σ-additive probability measure on (Ω^*, Σ^*), which we denote also by $T_i^*(\omega_i)$.*

2. *For every* $i \in I$, *this extension* T_i^* *is a* $\Sigma_i - \Sigma_\Delta^*$*-measurable function from* Ω_i *to* $\Delta(\Omega^*, \Sigma^*)$, *the space of probability measures on* (Ω^*, Σ^*), *which is endowed with the* σ*-field* Σ_Δ^* *generated by the sets* $\{\mu \in \Delta(\Omega^*, \Sigma^*) \mid \mu(E) \geq \alpha\}$, *where* $E \in \Sigma^*$ *and* $\alpha \in [0,1] \cap \mathbb{Q}$.
3. $$\underline{\Omega}^* := \langle \Omega^*, \Sigma^*, (T_i^*)_{i \in I}, v^* \rangle$$

   is a product type space on X *for player set* I.
4. *For every* $\psi \in \mathcal{L}_0$ *and* $(\omega_j)_{j \in I_0} \in \Omega^*$:

   $$(\langle \Omega^*, \Sigma^*, (T_i^*)_{i \in I}, v^* \rangle, (\omega_j)_{j \in I_0}) \models \psi \quad \textit{iff} \quad (\omega_j)_{j \in I_0} \in [\psi]^*.$$
5. *In the* H*-system case, i.e., if the axiom* $(I1)$ *is added in the case of* $\aleph_\gamma = \aleph_0$, *and if the inference rule Uncountable Introspection is added in the case of* $\aleph_\gamma > \aleph_0$, *then*

   $$\langle \Omega^*, \Sigma^*, (T_i^*)_{i \in I}, v^* \rangle$$

   is a Harsanyi product type space on X *for player set* I.

Theorem 3: *The function*

$$h : \Omega \to \Omega^*$$

defined by

$$h(\omega) := (\omega(j))_{j \in I_0}, \quad \textit{for } \omega \in \Omega,$$

is a type isomorphism from $\underline{\Omega}$ *to* $\underline{\Omega}^*$.

That the canonical space is a product space implies the following: Given states $u_i \in \Omega$, for $i \in I_0$, there is one state $u \in \Omega$ such that: $v(u_0, x) = v(u, x)$, for all $x \in X$, and $T_i(u_i) = T_i(u)$, for $i \in I$. This fact is reflected by the axioms in the following way: There is no axiom and also no inference rule that relates the beliefs of one player to the beliefs of other players or to nature. So, whatever a player in a state of the world believes about other players or nature might be wrong (as long as this is nothing tautological, of course). This is not the case for the canonical knowledge space and the corresponding $S5$ axiom system, where there is an axiom "$k_i\varphi \to \varphi$". So, if, for example, $\varphi = k_j x$ and if $k_i\varphi$ is true in a state, then the fact that i knows that j knows x implies that j knows x in this state, and this implies that x is true in this state.

6. Beliefs completeness of the canonical (Harsanyi) type space

The aim of this section is to prove the following—somewhat surprising—theorem of appealing measure-theoretic taste, which, in some topological cases, was proved by Mertens and Zamir (1985), Brandenburger and Dekel (1993), Heifetz (1993) and Mertens, Sorin and Zamir (1994). The general measure-theoretic case proved here is original. The theorem says that, in the P-system case, the component space of each player is—up to isomorphism of measurable spaces—the space of probability measures on the space of states of the world, and in the H-system case, for each player $i \in I$, the component space of i is—up to isomorphism of measurable spaces—the space of probability measures on Ω_{-i}.

THEOREM 4:

- *In the P-system case, let $\mu \in \Delta(\Omega^*, \Sigma^*)$. For every $i \in I$, there is exactly one $\omega_i \in \Omega_i$ such that $T_i^*(\omega_i) = \mu$. Furthermore, for every $i \in I$,*

$$T_i^* : \Omega_i \to \Delta(\Omega^*, \Sigma^*)$$

is an isomorphism of the measurable spaces (Ω_i, Σ_i) and $(\Delta(\Omega^*, \Sigma^*), \Sigma^*_\Delta)$.

- *In the H-system case, let $i \in I$ and $\mu_i \in \Delta(\Omega_{-i}, \Sigma_{-i})$. Then there is exactly one $\omega_i \in \Omega_i$ such that the marginal of $T_i^*(\omega_i)$ on Ω_{-i} is μ_i. Furthermore, for every $i \in I$,*

$$\mathrm{marg}_{\Omega_{-i}} \circ T_i^* : \Omega_i \to \Delta(\Omega_{-i}, \Sigma_{-i})$$

is an isomorphism of the measurable spaces

$$(\Omega_i, \Sigma_i) \quad \textit{and} \quad (\Delta(\Omega_{-i}, \Sigma_{-i}), (\Sigma_{-i})_\Delta) .$$

Brandenburger (2003) defines possibility structures, which can be viewed as the product-space version of Kripke structures. He shows by a straightforward cardinality argument that, without any further (e.g., topological) restrictions, there is no beliefs complete possibility structure (for at least two players and at least two states of nature). Brandenburger and Keisler (2006) obtain a similar, but much more involved impossibility theorem, where a possible belief of a player is a first-order definable subset of the product of the space of states of nature and the parameter space of the other player. Mariotti, Meier and Piccione (2005) prove the existence of a beliefs complete possibility structure if suitable topological restrictions are imposed. Salonen (2006) shows that a beliefs complete structure in his framework exists, where a type of a player corresponds to a measurable knowledge operator on the product of the space of states of nature and the space of (names of) types of the other players.

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A. Proofs of Section 3

Proof of Lemma 2.

1. "If" follows by applying the inference rule "Conjunction" to Φ. "Only if" follows by replacing in the proof of φ from Φ every occurrence of a $\chi \in \Phi$ by the sequence $\bigwedge_{\chi' \in \Phi} \chi'$, $\bigwedge_{\chi' \in \Phi} \chi' \rightarrow \chi$, χ. This yields then a proof of φ from $\{\bigwedge_{\chi \in \Phi} \chi\}$.
2. "If" follows immediately by Modus Ponens. "Only if" follows by induction on the length of the proof of φ from $\{\psi\}$. There are four cases:
   (a) $\varphi = \psi$: By (A5) applied to $\{\psi\}$, it follows that $\vdash \psi \rightarrow \psi$.
   (b) φ is a theorem: By (A1), $\vdash \varphi \rightarrow (\psi \rightarrow \varphi)$, and by Modus Ponens it follows that $\vdash \psi \rightarrow \varphi$.

(c) φ follows by Modus Ponens: Then there is a χ such that χ and $\chi \to \varphi$ occur in the proof of φ. The sequences up to (and including) χ and $\chi \to \varphi$ are proofs of χ and $\chi \to \varphi$ from $\{\psi\}$ of shorter length. Hence, by the induction hypothesis, $\vdash \psi \to \chi$ and $\vdash \psi \to (\chi \to \varphi)$.

$$(\psi \to (\chi \to \varphi)) \to ((\psi \to \chi) \to (\psi \to \varphi))$$

is a theorem (A2), so by applying Modus Ponens two times we get $\vdash \psi \to \varphi$.

(d) φ follows by Conjunction: Then $\varphi = \bigwedge_{\chi\in\Phi} \chi$ with $|\Phi| \le 2^{\aleph_\gamma}$. By the induction hypothesis (since each χ must occur before φ in the proof), we have $\vdash \psi \to \chi$, for every $\chi \in \Phi$. By conjunction, we get $\vdash \bigwedge_{\chi\in\Phi} (\psi \to \chi)$ and by applying Modus Ponens to (A4), $\vdash \psi \to \bigwedge_{\chi\in\Phi} \chi$.

3. We have $\Gamma \vdash \psi \to \widetilde{\psi}$. By (A1), $(\psi \to \widetilde{\psi}) \to (\varphi \to (\psi \to \widetilde{\psi}))$ is an axiom. Modus Ponens yields $\Gamma \vdash \varphi \to (\psi \to \widetilde{\psi})$. By (A2),

$$(\varphi \to (\psi \to \widetilde{\psi})) \to ((\varphi \to \psi) \to (\varphi \to \widetilde{\psi}))$$

is an axiom. Modus Ponens yields $\Gamma \vdash (\varphi \to \psi) \to (\varphi \to \widetilde{\psi})$. Together with $\Gamma \vdash \varphi \to \psi$, Modus Ponens yields now $\Gamma \vdash \varphi \to \widetilde{\psi}$.

4. and 5. are well-known tautologies of Propositional Calculus, so, according to the Completeness Theorem of Karp (1964, Theorem 5.5.4), theorems of our system. ∎

Proof of Proposition 1.

1. For $\varphi \in \mathcal{L}$, we have to show that if $\vdash \varphi$ (i.e., φ is a theorem) in the system P (resp. in the system H) and if $\underline{M}$ is a type space (resp. Harsanyi type space) on X for player set I and $m \in M$, then $(\underline{M}, m) \models \varphi$.
2. And for $\varphi \in \mathcal{L}$ and a nonempty set $\Gamma \subseteq \mathcal{L}$, we have to show that if $\underline{M}$ is a type space (resp. Harsanyi type space) on X for player set I, $m \in M$, $(\underline{M}, m) \models \Gamma$ and if $\Gamma \vdash \varphi$ in the system P (resp. in the system H), then $(\underline{M}, m) \models \varphi$.

To:

1. It suffices to show for $\varphi, \psi \in \mathcal{L}$ and $\Phi \subseteq \mathcal{L}$:
   (a) If φ is an axiom of the system P (resp. of the system H) and if $\underline{M}$ is a type space (resp. Harsanyi type space) on X for player set I

and $m \in M$, then $(\underline{M}, m) \models \varphi$. That is, we have to show that the axioms are valid.

(b) If φ is valid and $\varphi \to \psi$ is valid, then ψ is valid.

(c) If $\varphi \in \mathcal{L}_0$ and φ is valid, then $p_i^1(\varphi)$ is valid.

(d) If $|\Phi| \leq 2^{\aleph_\gamma}$ and each $\varphi \in \Phi$ is valid, then $\bigwedge_{\varphi \in \Phi} \varphi$ is valid.

(e) If $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$, and $\bigwedge \varphi_n \to \neg\top$ is valid, then

$$\bigwedge_{k \in \mathbb{N}\setminus\{0\}} \bigvee_{l \in \mathbb{N}} \neg p_i^{\frac{1}{k}}\left(\bigwedge_{n \leq l} \varphi_n\right)$$

is valid.

(f) If $\varphi \in \mathcal{L}^i$ and $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$, then the validity of $\varphi \to \bigvee_{n \in \mathbb{N}} \varphi_n$ in the class of Harsanyi type spaces on X for player set I implies that

$$\varphi \to \bigwedge_{k \in \mathbb{N}\setminus\{0\}} \bigvee_{l \in \mathbb{N}} p_i^{1-\frac{1}{k}}\left(\bigvee_{n \leq l} \varphi_n\right)$$

is valid in the class of Harsanyi type spaces on X for player set I.

To:

(a) That the axioms are valid is an easy check, (A6) is valid, provided we include the axiom of choice (like always) in our underlying set theory. (P1) – (P8) correspond to well-known properties of probability measures.

(I1): Let $\underline{M}$ be a Harsanyi type space on X for player set I, $\varphi \in \mathcal{L}_0$ and $m \in M$. Then, $(\underline{M}, m) \models \neg p_i^\alpha(\varphi) \vee p_i^1(p_i^\alpha(\varphi))$ iff $(\underline{M}, m) \models \neg p_i^\alpha(\varphi)$ or $(\underline{M}, m) \models p_i^1(p_i^\alpha(\varphi))$. Let $(\underline{M}, m) \models p_i^\alpha(\varphi)$. This means that $T_i(m)([\varphi]^{\underline{M}}) \geq \alpha$. But then $[T_i(m)]^{\underline{M}} \subseteq [p_i^\alpha(\varphi)]^{\underline{M}}$ and hence $T_i(m)([p_i^\alpha(\varphi)]^{\underline{M}}) = 1$ and (I1) is valid. (I2) follows in the same manner.

(b)–(d) above are clear,

(e) corresponds to the continuity at $\emptyset$, a well-known property of σ-additive probability measures: Let $\underline{M}$ be a type space (resp. Harsanyi type space) on X for player set I, $m \in M$ and $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$. By the definition of "$\models$", we have

$$\left[\bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top\right]^{\underline{M}} = \Big(M \setminus \bigcap_{n\in\mathbb{N}} [\varphi_n]^{\underline{M}}\Big) \cup (M \setminus [\top]^{\underline{M}}) = \Big(M \setminus \bigcap_{n\in\mathbb{N}} [\varphi_n]^{\underline{M}}\Big).$$

If

$$\bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top$$

is valid, then $\bigcap_{n\in\mathbb{N}} [\varphi_n]^{\underline{M}} = \emptyset$. In this case, we have for

$$E_l := \bigcap_{n\leq l} [\varphi_n]^{\underline{M}} = \left[\bigwedge_{n\leq l} \varphi_n\right]^{\underline{M}},$$

that $E_l \downarrow \emptyset$. So, for every $m \in M$ and $k \in \mathbb{N} \setminus \{0\}$ there is a $l\,(k,m) \in \mathbb{N}$ such that $T_i\,(m)\,\big(E_{l(k,m)}\big) < \frac{1}{k}$. By definition of "$\models$", we have

$$(\underline{M}, m) \models \neg p_i^{\frac{1}{k}}\Big(\bigwedge_{n\leq l(k,m)} \varphi_n\Big).$$

Again by definition of "$\models$", it follows that

$$(\underline{M}, m) \models \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\Big(\bigwedge_{n\leq l} \varphi_n\Big).$$

Hence

$$\bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\Big(\bigwedge_{n\leq l} \varphi_n\Big)$$

is valid.

(f) Let $\underline{M}$ be a type space (resp. Harsanyi type space) on X for player set I and $m \in M$. Then, it is easy to see by induction on the formation of formulas $\varphi \in \mathcal{L}^i$: Either $[T_i(m)]^{\underline{M}} \subseteq [\varphi]^{\underline{M}}$ or $[T_i(m)]^{\underline{M}} \cap [\varphi]^{\underline{M}} = \emptyset$, for $\varphi \in \mathcal{L}^i$. Let $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$. Then,

$$(\underline{M}, m) \models \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}\Big(\bigvee_{n\leq l} \varphi_n\Big) \quad \text{iff} \quad \lim_{l\to\infty} T_i(m)\Big(\Big[\bigvee_{n\leq l} \varphi_n\Big]\Big) = 1,$$

which is by σ-additivity the case iff

$$T_i(m)\Big(\Big[\bigvee_{n\in\mathbb{N}} \varphi_n\Big]\Big) = 1.$$

Let $\varphi \in \mathcal{L}^i$ and assume that

$$\varphi \to \bigvee_{n \in \mathbb{N}} \varphi_n$$

is valid in the class of Harsanyi type spaces on X for player set I. Assume that $(\underline{M}, m) \models \varphi$. By the above, $[T_i(m)]^{\underline{M}} \subseteq [\varphi]^{\underline{M}}$, since φ is an i-formula. This implies that $[T_i(m)]^{\underline{M}} \subseteq [\bigvee_{n\in\mathbb{N}} \varphi_n]^{\underline{M}}$, and by the introspection property of the Harsanyi type spaces $T_i(m)([\bigvee_{n\in\mathbb{N}} \varphi_n]) = 1$. The above observation implies now that

$$(\underline{M}, m) \models \bigwedge_{k \in \mathbb{N}\setminus\{0\}} \bigvee_{l \in \mathbb{N}} p_i^{1-\frac{1}{k}} \Bigl(\bigvee_{n \leq l} \varphi_n \Bigr),$$

hence

$$\varphi \to \bigwedge_{k \in \mathbb{N}\setminus\{0\}} \bigvee_{l \in \mathbb{N}} p_i^{1-\frac{1}{k}} \Bigl(\bigvee_{n \leq l} \varphi_n \Bigr)$$

is valid in the class of Harsanyi type spaces on X for player set I.

2. Given 1., we have to show:
   (a) If $\underline{M}$ is a type space (resp. Harsanyi type space) on X for player set I, $m \in M$, $\varphi, \psi \in \mathcal{L}$, $(\underline{M}, m) \models \varphi$ and $(\underline{M}, m) \models \varphi \to \psi$, then $(\underline{M}, m) \models \psi$. But

$$(\underline{M}, m) \models \varphi \to \psi \text{ iff } m \in [\neg\varphi \vee \psi]^{\underline{M}} = (M \setminus [\varphi]^{\underline{M}}) \cup [\psi]^{\underline{M}},$$

   so $m \in [\varphi]^{\underline{M}}$ and $(\underline{M}, m) \models \varphi \to \psi$ imply $m \in [\psi]^{\underline{M}}$.
   (b) If $\underline{M}$ is a type space (resp. Harsanyi type space) on X for player set I, $m \in M$ and $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$ and such that for all $\varphi \in \Phi : (\underline{M}, m) \models \varphi$, then $(\underline{M}, m) \models \bigwedge_{\varphi\in\Phi} \varphi$, but this is clear by the definition of "$\models$". ∎

Proof of Remark 2.

1. Set

$$\begin{array}{lcll} M & := & \{m\}, & \\ \Sigma & := & \operatorname{Pow}(M), & \text{(i.e., the power set of } M\text{)}, \\ T_i(m) & := & \delta_m, & \text{for every } i \in I, \quad \text{(i.e., the delta-measure at } m\text{)}, \\ v(m,x) & := & 1, & \text{for every } x \in X \cup \{\top\}. \end{array}$$

   Then,

$$\underline{M} := \langle M, \Sigma, (T_i)_{i\in I}, v \rangle$$

   forms a Harsanyi type space on X for player set I.

2. Let $\underline{M}$ be the Harsanyi type space on X for player set I constructed above. Consider the set

$$\Phi_0 := \{\varphi \in \mathcal{L}_0 \mid (\underline{M}, m) \models \varphi\}.$$

By the definition of "$\models$", we have $(\underline{M}, m) \models \neg\psi$, for $\psi \in \mathcal{L}_0 \backslash \Phi_0$, and hence

$$\Phi_0 = \Phi_0 \cup \{\neg\psi \in \mathcal{L}_0 \mid \psi \in \mathcal{L}_0 \backslash \Phi_0\}.$$

We claim that Φ_0 is consistent in the system H (and hence also in the system P). Otherwise, $\Phi_0 \vdash \chi$ and $\Phi_0 \vdash \neg\chi$, for some $\chi \in \mathcal{L}$. But then, by Proposition 1, we have $(\underline{M}, m) \models \chi$ and $(\underline{M}, m) \models \neg\chi$. By the definition of the relation "$\models$", this is impossible. Hence,

$$\bigwedge_{\varphi \in \Phi_0} \varphi \wedge \bigwedge_{\psi \in \mathcal{L}_0 \backslash \Phi_0} \neg\psi \in \Omega \neq \emptyset. \qquad \blacksquare$$

Proof of Proposition 2.

1. By (A5), $\vdash \varphi \vee \neg\varphi$, for $\varphi \in \mathcal{L}_0$. Since $|\mathcal{L}_0| \leq \aleph_\gamma$, it follows by Conjunction that $\vdash \bigwedge_{\varphi \in \mathcal{L}_0} (\varphi \vee \neg\varphi)$. By (A6) and Modus Ponens, it follows that

$$\vdash \bigvee_{\Phi_0 \subseteq \mathcal{L}_0} \left(\bigwedge_{\varphi \in \Phi_0} \varphi \wedge \bigwedge_{\varphi \in \mathcal{L}_0 \backslash \Phi_0} \neg\varphi \right).$$

If $\Phi_0 \cup \{\neg\varphi | \varphi \in \mathcal{L}_0 \backslash \Phi_0\}$ is inconsistent (i.e., not consistent), then it follows by 1 and 2 of Lemma 2 that

$$\vdash \left(\bigwedge_{\varphi \in \Phi_0} \varphi \wedge \bigwedge_{\varphi \in \mathcal{L}_0 \backslash \Phi_0} \neg\varphi \right) \to \psi \text{ and } \vdash \left(\bigwedge_{\varphi \in \Phi_0} \varphi \wedge \bigwedge_{\varphi \in \mathcal{L}_0 \backslash \Phi_0} \neg\varphi \right) \to \neg\psi,$$

for a $\psi \in \mathcal{L}$. By Conjunction, (A4) and Modus Ponens, we get

$$\vdash \left(\bigwedge_{\varphi \in \Phi_0} \varphi \wedge \bigwedge_{\varphi \in \mathcal{L}_0 \backslash \Phi_0} \neg\varphi \right) \to (\psi \wedge \neg\psi).$$

Since $(\chi \to \rho) \to (\neg\rho \to \neg\chi)$ is a tautology of the Propositional Calculus, we get, by Modus Ponens,

$$\vdash \neg(\psi \wedge \neg\psi) \to \neg\left(\bigwedge_{\varphi \in \Phi_0} \varphi \wedge \bigwedge_{\varphi \in \mathcal{L}_0 \backslash \Phi_0} \neg\varphi \right).$$

$\neg(\psi \wedge \neg\psi)$ is a tautology of the Propositional Calculus, hence Modus Ponens yields

$$\vdash \neg\Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi \Big).$$

Let $\mathbf{C}_0$ be the set of all $\Phi_0 \subseteq \mathcal{L}_0$ such that $\Phi_0 \cup \{\neg\varphi | \varphi \in \mathcal{L}_0 \setminus \Phi_0\}$ is inconsistent. By Conjunction,

$$\bigwedge_{\Phi_0\in\mathbf{C}_0} \neg\Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi \Big)$$

is a theorem. By the definition of "$\bigvee$", "$\vee$" and "$\rightarrow$",

$$\Big(\bigvee_{\Phi_0\subseteq\mathcal{L}_0} \Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi \Big)\Big) \rightarrow \Big(\Big(\bigwedge_{\Phi_0\in\mathbf{C}_0} \neg\Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi \Big)\Big) \rightarrow \bigvee_{\omega\in\Omega} \omega \Big),$$

is a tautology of Propositional Calculus, hence a theorem. Applying Modus Ponens two times yields now

$$\vdash \bigvee_{\omega\in\Omega} \omega.$$

2. Follows by induction on the formation of the formulas in $\mathcal{L}$. Let $\omega \in \Omega$. If $\vdash \omega \rightarrow \varphi$ and $\vdash \omega \rightarrow \neg\varphi$ for some $\varphi \in \mathcal{L}$, then by 1 and 2 of Lemma 2, Ψ_ω is not consistent, a contradiction.

   For every $\psi \in \mathcal{L}_0$ we have $\psi \in \Psi_\omega$ or $\neg\psi \in \Psi_\omega$. Again by 1 and 2 of Lemma 2 it follows that $\vdash \omega \rightarrow \psi$ or $\vdash \omega \rightarrow \neg\psi$.

   If $\varphi \in \mathcal{L}$ and $\psi = \neg\varphi$, then, by the induction hypothesis, either $\vdash \omega \rightarrow \neg\varphi$, or $\vdash \omega \rightarrow \varphi$. In the second case, since $\varphi \rightarrow \neg(\neg\varphi)$ is a tautology of the Propositional Calculus, we have $\vdash \varphi \rightarrow \neg\psi$ and by 3 of Lemma 2 it follows that $\vdash \omega \rightarrow \neg\psi$.

   If $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$ and $\psi = \bigwedge_{\varphi\in\Phi} \varphi$, then, by the induction hypothesis, either $\vdash \omega \rightarrow \varphi$ for all $\varphi \in \Phi$, or there is a $\chi \in \Phi$ such that $\vdash \omega \rightarrow \neg\chi$. In the first case, by Conjunction, it follows that $\vdash \bigwedge_{\varphi\in\Phi}(\omega \rightarrow \varphi)$ and by (A4) and Modus Ponens,

$$\vdash \omega \rightarrow \bigwedge_{\varphi\in\Phi} \varphi.$$

In the second case, since $\neg\chi \to \neg \bigwedge_{\varphi\in\Phi} \varphi$ is a tautology of the Propositional Calculus, hence a theorem, we conclude by 3 of Lemma 2 that

$$\vdash \omega \to \neg \bigwedge_{\varphi\in\Phi} \varphi.$$

3. Let $\omega \in \Omega$ and $\psi \in \mathcal{L}$. If $\omega \notin [\psi]$, then $\vdash \omega \to \neg\psi$. But then, since $(\omega \to \neg\psi) \to (\psi \to \neg\omega)$ is a tautology of the Propositional Calculus, we have $\vdash \psi \to \neg\omega$. By Conjunction, (A4) and Modus Ponens, we conclude $\vdash \psi \to \bigwedge_{\omega\in\Omega\setminus[\psi]} \neg\omega$.

$$\bigwedge_{\omega\in\Omega\setminus[\psi]} \neg\omega \to \neg \bigvee_{\omega\in\Omega\setminus[\psi]} \omega$$

is a tautology of the Propositional Calculus, so we conclude that $\vdash \psi \to \neg \bigvee_{\omega\in\Omega\setminus[\psi]} \omega$.

$$\bigvee_{\omega\in\Omega} \omega \to \left(\neg \bigvee_{\omega\in\Omega\setminus[\psi]} \omega \to \bigvee_{\omega\in[\psi]} \omega\right)$$

is a tautology of the Propositional Calculus, hence a theorem, so we infer by 1 and Modus Ponens that $\vdash \neg \bigvee_{\omega\in\Omega\setminus[\psi]} \omega \to \bigvee_{\omega\in[\psi]} \omega$. By 3 of Lemma 2, it follows that

$$\vdash \psi \to \bigvee_{\omega\in[\psi]} \omega.$$

If $\omega \in [\psi]$, then $\vdash \omega \to \psi$. Since $(\omega \to \psi) \to (\neg\psi \to \neg\omega)$ is a tautology of the Propositional Calculus, hence a theorem, it follows by Modus Ponens that $\vdash \neg\psi \to \neg\omega$. By Conjunction, (A4) and Modus Ponens, we get $\vdash \neg\psi \to \bigwedge_{\omega\in[\psi]} \neg\omega$. Then, since

$$\left(\neg\psi \to \bigwedge_{\omega\in[\psi]} \neg\omega\right) \to \left(\bigvee_{\omega\in[\psi]} \omega \to \psi\right)$$

is a tautology of the Propositional Calculus, hence a theorem, it follows by Modus Ponens that

$$\vdash \bigvee_{\omega\in[\psi]} \omega \to \psi,$$

so, by Conjunction, we conclude

$$\vdash \psi \leftrightarrow \bigvee_{\omega\in[\psi]} \omega.$$

4. Let $\omega \in \Omega$ and $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$. By 3 it suffices to show that $\omega \in [\bigwedge_{\varphi\in\Phi} \varphi]$ iff $\omega \in [\Phi]$. If $\omega \in [\Phi]$, then, for every $\varphi \in \Phi$, we have $\vdash \omega \to \varphi$. By Conjunction, (A4) and Modus Ponens it follows that $\vdash \omega \to \bigwedge_{\varphi\in\Phi} \varphi$, so $\omega \in [\bigwedge_{\varphi\in\Phi} \varphi]$.

   If $\omega \in [\bigwedge_{\varphi\in\Phi} \varphi]$, then $\vdash \omega \to \bigwedge_{\varphi\in\Phi} \varphi$. For every $\psi \in \Phi$ we have, by (A5), $\vdash \bigwedge_{\varphi\in\Phi} \varphi \to \psi$, so, by 3 of Lemma 2, it follows that $\vdash \omega \to \psi$, and hence $\omega \in [\Phi]$.
5. By 2. we have $\Omega\setminus[\psi] = [\neg\psi]$; 5. follows now from 3.
6. See the Proof of 5.
7. See the Proof of 4. ∎

Proof of Lemma 3. Assume that $T_i'(\omega)([\psi]) \geq \alpha$. Then, for every $\beta' < \alpha$, there is a $\beta > \beta'$ with $\vdash \omega \to p_i^\beta(\psi)$. By (P7), $\vdash p_i^\beta(\psi) \to p_i^{\beta'}(\psi)$. Hence, by 3 of Lemma 2, $\vdash \omega \to p_i^{\beta'}(\psi)$. By Conjunction, (A4) and Modus Ponens, we have $\vdash \omega \to \bigwedge_{\beta<\alpha} p_i^\beta(\psi)$. By 3 of Lemma 2 and (P3), it follows that $\vdash \omega \to p_i^\alpha(\psi)$, a contradiction to 2 of Proposition 2. ∎

Proof of Lemma 4. Let $\varphi \in \mathcal{L}_0$. By (P1), $p_i^0(\varphi)$ is an axiom and, by (A1),

$$p_i^0(\varphi) \to \big(\omega \to p_i^0(\varphi)\big)$$

is an axiom. By Modus Ponens, it follows that $\vdash \omega \to p_i^0(\varphi)$. Hence $T_i'(\omega)([\varphi]) \geq 0$.

Let $\varphi, \psi \in \mathcal{L}_0$ with $[\varphi] = [\psi]$. (Of course, we have by (A5) and Modus Ponens that $\vdash \varphi \leftrightarrow \varphi'$ implies $\vdash \varphi \to \varphi'$ and $\vdash \varphi' \to \varphi$, and by Conjunction the opposite direction follows.) Then, $\vdash \psi \leftrightarrow \bigvee_{\omega\in[\psi]} \omega$ and $\vdash (\bigvee_{\omega\in[\psi]} \omega) \leftrightarrow \varphi$, so by Lemma 2, $\vdash \varphi \leftrightarrow \psi$. By Necessitation, (P8) and Modus Ponens, it follows that $\vdash p_i^\alpha(\varphi) \to p_i^\alpha(\psi)$ and $\vdash p_i^\alpha(\psi) \to p_i^\alpha(\varphi)$, so

$$\sup\{\alpha \mid \vdash \omega \to p_i^\alpha(\varphi)\} = \sup\{\alpha \mid \vdash \omega \to p_i^\alpha(\psi)\},$$

and $T_i'(\omega)$ is well-defined.

Let $\omega \in \Omega$. To show that $T_i'(\omega)$ is countably additive it is enough to show that it is finitely additive and continuous at $\emptyset$ (see Dudley (1989), Theorem 3.1.1). Let $\varphi, \psi \in \mathcal{L}_0$ with $[\varphi] \cap [\psi] = \emptyset$. Then, $[\varphi] \subseteq \Omega \setminus [\psi]$. It follows that

$$[\varphi] = ([\varphi] \cup [\psi]) \cap (\Omega \setminus [\psi]) \text{ and } [\psi] = ([\varphi] \cup [\psi]) \cap [\psi].$$

By 6 and 7 of Proposition 2, it follows that

$$([\varphi] \cup [\psi]) \cap (\Omega \setminus [\psi]) = [(\varphi \vee \psi) \wedge \neg\psi] \text{ and } ([\varphi] \cup [\psi]) \cap [\psi] = [(\varphi \vee \psi) \wedge \psi].$$

Now, let

$$T_i'(\omega)([\varphi]) = r \text{ and } T_i'(\omega)([\psi]) = r'.$$

Assume that $r+r' > 1$. Then there are rationals $\alpha, \beta \in [0,1]$ such that $\alpha \leq r$, $\beta \leq r'$ and $\alpha + \beta > 1$. But then,

$$\vdash \omega \to p_i^{\alpha}((\varphi \vee \psi) \wedge \neg\psi) \text{ and } \vdash \omega \to p_i^{\beta}((\varphi \vee \psi) \wedge \psi).$$

We have

$$\vdash (\varphi \vee \psi) \wedge \psi \to \neg((\varphi \vee \psi) \wedge \neg\psi),$$

because this is a tautology of the Propositional Calculus. Necessitation, (P8) and Modus Ponens yield now

$$\vdash p_i^{\beta}((\varphi \vee \psi) \wedge \psi) \to p_i^{\beta}(\neg((\varphi \vee \psi) \wedge \neg\psi)).$$

By Lemma 2, we conclude that

$$\vdash \omega \to p_i^{\beta}(\neg((\varphi \vee \psi) \wedge \neg\psi)).$$

But since $\alpha + \beta > 1$, we have that

$$p_i^{\alpha}((\varphi \vee \psi) \wedge \neg\psi) \to \neg p_i^{\beta}(\neg((\varphi \vee \psi) \wedge \neg\psi))$$

is an axiom (P6), hence

$$\vdash \omega \to \neg p_i^{\beta}(\neg((\varphi \vee \psi) \wedge \neg\psi)),$$

which is by 2 of Proposition 2 a contradiction. So, it follows that $r + r' \leq 1$.

For every $\varepsilon > 0$, there are rational $\alpha, \beta \in [0,1]$ with $\alpha \leq r$ and $\beta \leq r'$ such that $\alpha \geq r - \frac{\varepsilon}{2}$ and $\beta \geq r' - \frac{\varepsilon}{2}$. For such α and β we have

$$\vdash \omega \to p_i^{\alpha}((\varphi \vee \psi) \wedge \neg\psi) \text{ and } \vdash \omega \to p_i^{\beta}((\varphi \vee \psi) \wedge \psi),$$

so, by Conjunction, (A4) and Modus Ponens,

$$\vdash \omega \to p_i^{\alpha}((\varphi \vee \psi) \wedge \neg\psi) \wedge p_i^{\beta}((\varphi \vee \psi) \wedge \psi).$$

Together with (P4) and Lemma 2, we conclude that $\vdash \omega \to p_i^{\alpha+\beta}(\varphi \vee \psi)$. This implies that

$$T_i'(\omega)([\varphi] \cup [\psi]) = T_i'(\omega)([\varphi \vee \psi]) \geq r + r'.$$

If $r + r' = 1$, then we have $T_i'(\omega)([\varphi \vee \psi]) = 1$, since by definition $T_i'(\omega)([\varphi \vee \psi]) \leq 1$. If $r + r' < 1$, then for all $\varepsilon > 0$ such that $\varepsilon + r + r' \leq 1$, there are rationals $\alpha, \beta \in [0,1]$ such that $\alpha > r$, $\beta > r'$ and $\alpha + \beta \leq \varepsilon + r + r'$. For such α, β we have

$$\vdash \omega \to \neg p_i^{\alpha}((\varphi \vee \psi) \wedge \neg\psi) \text{ and } \vdash \omega \to \neg p_i^{\beta}((\varphi \vee \psi) \wedge \psi).$$

This implies (like above, but with the use of (P5)) that $\vdash \omega \rightarrow \neg p_i^{\alpha+\beta}(\varphi \vee \psi)$. So, by Lemma 3, we have

$$T_i'(\omega)([\varphi] \cup [\psi]) = T_i'(\omega)([\varphi \vee \psi]) \leq r + r'.$$

Altogether, this shows that $T_i'(\omega)$ is finitely additive.

Since $\top$ is an axiom, we have for every $\omega \in \Omega$ that $\{\omega\} \vdash \top$. Therefore, by Lemma 2, $\vdash \omega \rightarrow \top$, and hence $[\top] = \Omega$. Since $\top$ is a theorem, Necessitation yields $\vdash p_i^1(\top)$, so, as above, we have for every $\omega \in \Omega$ that $\vdash \omega \rightarrow p_i^1(\top)$. This implies that $T_i'(\omega)(\Omega) = 1$, for every $\omega \in \Omega$.

Note that we have $[\neg\top] = \emptyset$, and since, for $\omega \in \Omega$ and $i \in I$, $T_i'(\omega)$ is finitely additive, we have

$$T_i'(\omega)(\emptyset) = T_i'(\omega)(\emptyset \cup \emptyset) = T_i'(\omega)(\emptyset) + T_i'(\omega)(\emptyset),$$

and hence $T_i'(\omega)(\emptyset) = 0$.

For $\omega \in \Omega$, it remains to show that $T_i'(\omega)$ is continuous at $\emptyset$: For $n \in \mathbb{N}$, let $E_n = [\varphi_n]$ with $\varphi_n \in \mathcal{L}_0$ and let $E_n \downarrow \emptyset$, that is, for all $n \in \mathbb{N} : E_{n+1} \subseteq E_n$ and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. Then, by 7 of Proposition 2, we have

$$[\varphi_n] = \left[\bigwedge_{m\leq n} \varphi_m\right] \text{ and } \left[\bigwedge_{n\in\mathbb{N}} \varphi_n\right] = \bigcap_{n\in\mathbb{N}} [\varphi_n] = \emptyset.$$

It follows that

$$\Omega = \left(\left(\Omega \setminus \left[\bigwedge_{n\in\mathbb{N}} \varphi_n\right] \cup [\neg\top]\right)\right) = \left[\bigwedge_{n\in\mathbb{N}} \varphi_n \rightarrow \neg\top\right],$$

by 6 and 7 of Proposition 2. By 3 and 1 of Proposition 2 and by Modus Ponens, it follows that

$$\vdash \bigwedge_{n\in\mathbb{N}} \varphi_n \rightarrow \neg\top.$$

So, by the inference rule "Continuity at $\emptyset$", we have

$$\vdash \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\left(\bigwedge_{n\leq l} \varphi_n\right).$$

Hence,

$$\{\omega\} \vdash \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\left(\bigwedge_{n\leq l} \varphi_n\right)$$

and, by 2 of Lemma 2,

$$\vdash \omega \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\left(\bigwedge_{n\le l} \varphi_n\right).$$

For $\varepsilon > 0$ fix $k \in \mathbb{N}\setminus\{0\}$ with $\frac{1}{k} \le \varepsilon$. By (A5) and 3 of Lemma 2, it follows that

$$\vdash \omega \to \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\left(\bigwedge_{n\le l} \varphi_n\right).$$

But then there is a $l \in \mathbb{N}$ such that

$$\vdash \omega \to \neg p_i^{\frac{1}{k}}\left(\bigwedge_{n\le l} \varphi_n\right),$$

for if not, it follows by 2 of Proposition 2, Conjunction, (A4) and Modus Ponens that

$$\vdash \omega \to \bigwedge_{l\in\mathbb{N}} p_i^{\frac{1}{k}}\left(\bigwedge_{n\le l} \varphi_n\right),$$

a contradiction. (Note that $\bigvee_{l\in\mathbb{N}} \neg\psi_l \to \neg \bigwedge_{l\in\mathbb{N}} \psi_l$ is a tautology of the Propositional Calculus.) By Lemma 3, it follows that

$$T_i'(\omega)\left(\left[\bigwedge_{n\le l} \varphi_n\right]\right) < \frac{1}{k} \le \varepsilon.$$

The additivity (and the fact that $T_i'(\omega)$ takes only non-negative values) implies then that

$$T_i'(\omega)\left(\left[\bigwedge_{n\le m} \varphi_n\right]\right) < \frac{1}{k} \le \varepsilon,$$

for $m \ge l$. So, we have

$$\lim_{n\to\infty} T_i'(\omega)(E_n) = 0. \qquad \blacksquare$$

Proof of Proposition 3.

1. Follows directly from Lemma 4 and Carathéodory's extension Theorem.
2. Follows from Lemma 1. Since $\mathcal{F}$ is a field that generates Σ, by that Lemma, the σ-field on $\Delta(\Omega,\Sigma)$ generated by the sets

   $$\{\mu \in \Delta(\Omega,\Sigma) \mid \mu(F) \ge \alpha\},$$

   with $F \in \mathcal{F}$ and rational $\alpha \in [0,1]$, is equal to the σ-field on $\Delta(\Omega,\Sigma)$ generated by the sets

   $$\{\mu \in \Delta(\Omega,\Sigma) \mid \mu(E) \ge \alpha\},$$

with $E \in \Sigma$ and rational $\alpha \in [0,1]$. Inverse images commute with arbitrary intersections and unions, and with complements. So, it suffices to show that $\{\omega \mid T_i(\omega)([\psi]) \geq \alpha\} \in \Sigma$, for all $\psi \in \mathcal{L}_0$, $i \in I$ and rational $\alpha \in [0,1]$. By Lemma 3, 2 of Proposition 2 and the definition of $T_i(\omega)$, it follows that

$$T_i(\omega)([\psi]) \geq \alpha \text{ iff } \vdash \omega \to p_i^\alpha(\psi).$$

But we have that

$$\vdash \omega \to p_i^\alpha(\psi) \text{ iff } \omega \in [p_i^\alpha(\psi)],$$

and $[p_i^\alpha(\psi)] \in \mathcal{F} \subseteq \Sigma$.

3. Follows from Remark 4, 2 of Remark 2, and 1. and 2. of this proposition.
4. We proceed by induction on the formation of the formulas in $\mathcal{L}$: Let $\omega \in \Omega$. Then:
   (a) For $x \in X \cup \{\top\}$:

$$\begin{aligned} (\underline{\Omega},\omega) \models x \quad &\text{iff} \quad v(\omega,x)=1 \\ &\text{iff} \quad \omega \in [x]. \end{aligned}$$

   (b) For $\varphi \in \mathcal{L}$:

$$\begin{aligned} (\underline{\Omega},\omega) \models \neg\varphi \quad &\text{iff} \quad (\underline{\Omega},\omega) \not\models \varphi \\ &\text{iff} \quad \omega \notin [\varphi] \\ &\text{iff} \quad \omega \in [\neg\varphi], \end{aligned}$$

   where the last equivalence follows from 2 of Proposition 2.
   (c) For $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$:

$$\begin{aligned} (\underline{\Omega},\omega) \models \textstyle\bigwedge_{\varphi\in\Phi}\varphi \quad &\text{iff} \quad (\underline{\Omega},\omega) \models \varphi, \quad \text{for all } \varphi \in \Phi, \\ &\text{iff} \quad \omega \in \textstyle\bigcap_{\varphi\in\Phi}[\varphi] \\ &\text{iff} \quad \omega \in \left[\textstyle\bigwedge_{\varphi\in\Phi}\varphi\right], \end{aligned}$$

   where the last equivalence follows from 7 of Proposition 2.
   (d) For $i \in I$, $\alpha \in [0,1] \cap \mathbb{Q}$, and $\varphi \in \mathcal{L}_0$:

$$\begin{aligned} (\underline{\Omega},\omega) \models p_i^\alpha(\varphi) \quad \text{iff} \quad &\{\omega' \in \Omega \mid (\underline{\Omega},\omega') \models \varphi\} \in \Sigma \text{ and} \\ &T_i(\omega)(\{\omega' \in \Omega \mid (\underline{\Omega},\omega') \models \varphi\}) \geq \alpha. \end{aligned}$$

   But, by the induction hypothesis,

$$\{\omega' \in \Omega \mid (\underline{\Omega},\omega') \models \varphi\} = [\varphi] \in \Sigma.$$

So,

$$(\underline{\Omega}, \omega) \models p_i^{\alpha}(\varphi) \quad \text{iff} \quad \sup\left\{\beta \in [0,1] \cap \mathbb{Q} \mid \vdash \omega \to p_i^{\beta}(\varphi)\right\} \geq \alpha$$
$$\text{iff} \quad \vdash \omega \to p_i^{\alpha}(\varphi),$$

where the last equivalence follows by Lemma 3 and Proposition 2.

5.

(a) Case $\aleph_\gamma = \aleph_0$: We show that, for all $\omega \in \Omega$ and $i \in I$: $[T_i(\omega)]$ is measurable and $T_i(\omega)([T_i(\omega)]) = 1$. Since $\mathcal{L}_0$ is countable and since

$$\{\omega' \in \Omega \mid T_i(\omega')([\varphi]) \geq \alpha\} = [p_i^{\alpha}(\varphi)],$$

for $\varphi \in \mathcal{L}_0$ (by 4. of this proposition), the set

$$[T_i(\omega)]_0 := \bigcap_{\alpha \in [0,1]\cap\mathbb{Q},\ \varphi \in \mathcal{L}_0,\ \text{s.t. } T_i(\omega)([\varphi]) \geq \alpha} \{\omega' \in \Omega \mid T_i(\omega')([\varphi]) \geq \alpha\}$$

is measurable. Since $\mathcal{F}$ is closed under complements, every $\omega' \in [T_i(\omega)]_0$ satisfies, for all $A \in \mathcal{F}$:

$$T_i(\omega)(A) = T_i(\omega')(A).$$

Since $\mathcal{F}$ is a field which generates Σ, by Carathéodory's extension Theorem, it follows that $T_i(\omega) = T_i(\omega')$. Hence,

$$[T_i(\omega)] = [T_i(\omega)]_0 \in \Sigma.$$

By (I1), $p_i^{\alpha}(\varphi) \to p_i^1(p_i^{\alpha}(\varphi))$ is an axiom. So, by 3. of Lemma 2, $\vdash \omega \to p_i^{\alpha}(\varphi)$ implies $\vdash \omega \to p_i^1(p_i^{\alpha}(\varphi))$. Hence, by the definition of $T_i(\omega)$, it follows that $T_i(\omega)([\varphi]) \geq \alpha$ implies $T_i(\omega)([p_i^{\alpha}(\varphi)]) = 1$. Since $T_i(\omega)$ is a σ-additive probability measure, it follows that $T_i(\omega)([T_i(\omega)]) = 1$.

(b) Case $\aleph_\gamma > \aleph_0$: We have to show that, for all $\omega \in \Omega$, $i \in I$ and $A \in \Sigma$: $[T_i(\omega)] \subseteq A$ implies $T_i(\omega)(A) = 1$.
By the definition of $T_i(\omega)$ in the proof of Carathéodory's Theorem, it is enough to show that for $(\varphi_n)_{n\in\mathbb{N}}$, where $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$:

$$\bigcup_{n\in\mathbb{N}} [\varphi_n] \supseteq A \quad \text{implies} \quad \sum_{n\in\mathbb{N}} T_i'(\omega)([\varphi_n]) \geq 1.$$

We can assume without loss of generality that the $[\varphi_n]$ are pairwise disjoint. (That

$$\inf\left\{\sum_{n\in\mathbb{N}} T_i'(\omega)([\varphi_n])\,\middle|\; \varphi_n \in \mathcal{L}_0, n\in\mathbb{N} \text{ such that } \bigcup_{n\in\ \mathbb{N}}[\varphi_n]\supseteq A\right\}\leq 1$$

is clear, because $A\subseteq\Omega$ and $T_i'(\omega)(\Omega)=1$.)

For $i\in I$ and $\omega\in\Omega$ define

$$\varphi_i(\omega) := \bigwedge_{\alpha\in[0,1]\cap\mathbb{Q},\ \chi\in\mathcal{L}_0,\text{ s.t. } \vdash\omega\to p_i^{\alpha}(\chi)} p_i^{\alpha}(\chi) \wedge \bigwedge_{\beta\in[0,1]\cap\mathbb{Q},\ \psi\in\mathcal{L}_0,\text{ s.t. } \vdash\omega\to\neg p_i^{\beta}(\psi)} \neg p_i^{\beta}(\psi)\,.$$

By the definition of $T_i(\omega)$, we have $[T_i(\omega)] = [\varphi_i(\omega)]$, where $\varphi_i(\omega)$ is an i-formula. From

$$[T_i(\omega)]\subseteq\bigcup_{n\in\mathbb{N}}[\varphi_n] = \left[\bigvee_{n\in\mathbb{N}}\varphi_n\right]$$

it follows (by 6 and 7 of Proposition 2) that

$$\Omega = \left[\varphi_i(\omega)\to\bigvee_{n\in\mathbb{N}}\varphi_n\right].$$

By 1 and 3 of Proposition 2 and Modus Ponens, it follows that $\varphi_i(\omega)\to\bigvee_{n\in\mathbb{N}}\varphi_n$ is a theorem. By the inference rule "Uncountable Introspection", we can conclude that

$$\varphi_i(\omega)\to\bigwedge_{k\in\mathbb{N}\setminus\{0\}}\bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}\left(\bigvee_{n\leq l}\varphi_n\right)$$

is a theorem. By Conjunction, (A4) and Modus Ponens, it follows that $\vdash\omega\to\varphi_i(\omega)$. This implies that

$$\vdash\omega\to\bigwedge_{k\in\mathbb{N}\setminus\{0\}}\bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}\left(\bigvee_{n\leq l}\varphi_n\right),$$

which implies that

$$1 = \lim_{l\to\infty}T_i'(\omega)\left(\left[\bigvee_{n\leq l}\varphi_n\right]\right)\leq\sum_{n\in\mathbb{N}}T_i'(\omega)([\varphi_n]). \qquad \blacksquare$$

Proof of Theorem 1. "Strongly sound" follows from Proposition 1.

According to Proposition 3, $\underline{\Omega} = \langle \Omega, \Sigma, (T_i)_{i \in I}, v \rangle$ is a type space (resp. Harsanyi type space) on X for player set I.

Let $\Gamma \models \varphi$ in the class of type spaces (resp. Harsanyi type spaces) on X for player set I. Then, for all $\omega \in \Omega$:

$$(\underline{\Omega}, \omega) \models \Gamma \text{ implies } (\underline{\Omega}, \omega) \models \varphi.$$

So, by 4 of Proposition 3, $[\Gamma] \subseteq [\varphi]$, which implies, by 3 of Proposition 2, $\vdash \bigvee_{\omega \in [\Gamma]} \omega \to \varphi$, because for sets of formulas $A \subseteq B$ with $|B| \leq 2^{\aleph_\gamma}$: $\bigvee_{\chi \in A} \chi \to \bigvee_{\chi \in B} \chi$ is a tautology of the Propositional Calculus. Let $\omega \in \Omega$ and $\omega \notin [\Gamma]$. Then there is $\psi \in \Gamma$ with $\omega \notin [\psi]$, so $\vdash \omega \to \neg\psi$. But then, $\vdash \psi \to \neg\omega$ (since $(\chi \to \neg\widetilde{\chi}) \to (\widetilde{\chi} \to \neg\chi)$ is a tautology of the Propositional Calculus). By Modus Ponens, it follows that $\Gamma \vdash \neg\omega$. By Conjunction and the fact that $|\Omega| \leq 2^{\aleph_\gamma}$, we have $\Gamma \vdash \bigwedge_{\omega \notin [\Gamma]} \neg\omega$. Then, since (as in the proof of 3. of Proposition 2) $\vdash \bigwedge_{\omega \notin [\Gamma]} \neg\omega \to \bigvee_{\omega \in [\Gamma]} \omega$ is a theorem, it follows that $\Gamma \vdash \bigvee_{\omega \in [\Gamma]} \omega$, hence, by Modus Ponens, we have $\Gamma \vdash \varphi$. ∎

Proof of Corollary 1. Assume that $(\underline{\Omega}, \omega) \not\models \Gamma$, for every $\omega \in \Omega$, where Ω is the Ω corresponding to the system P (resp. to the system H). Hence, for every ω there is a $\varphi_\omega \in \Gamma$ such that $(\underline{\Omega}, \omega) \models \neg\varphi_\omega$. By 4 of Proposition 3, it follows that $\omega \in [\neg\varphi_\omega]$, that is $\vdash \omega \to \neg\varphi_\omega$, hence $\vdash \varphi_\omega \to \neg\omega$. Since $|\Omega| \leq 2^{\aleph_\gamma}$, we have $\vdash \bigwedge_{\omega' \in \Omega} \varphi_{\omega'} \to \neg\omega$ (because $\vdash \bigwedge_{\omega' \in \Omega} \varphi_{\omega'} \to \varphi_\omega$ is an axiom). It follows (by Conjunction, (A4) and Modus Ponens) that $\vdash \bigwedge_{\omega' \in \Omega} \varphi_{\omega'} \to \bigwedge_{\omega \in \Omega} \neg\omega$. By 4 of Proposition 3 and the definition of "$\models$", we have $[\neg(x \wedge \neg x)] = \Omega$, for $x \in X$. So, by Proposition 2, it follows that $\vdash \neg(x \wedge \neg x) \to \bigvee_{\omega \in \Omega} \omega$ and hence $\vdash \bigwedge_{\omega \in \Omega} \neg\omega \to x \wedge \neg x$, which implies $\vdash \bigwedge_{\omega' \in \Omega} \varphi_{\omega'} \to x \wedge \neg x$, and we conclude $\Gamma \vdash x \wedge \neg x$. By (A5) and Modus Ponens, it follows that $\Gamma \vdash x$ and $\Gamma \vdash \neg x$, so Γ is inconsistent in the system P (resp. in the system H).

If Γ is not consistent in the system P (resp. in the system H), then there is a φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ in the system P (resp. in the system H). So, by the strong soundness, for every type space (resp. Harsanyi type space) $\underline{M}$ on X for player set I and every $m \in M$: If $(\underline{M}, m) \models \Gamma$, then $(\underline{M}, m) \models \varphi$ and $(\underline{M}, m) \models \neg\varphi$. By the definition of the relation "$\models$", there is no $(\underline{M}, m)$ such that $(\underline{M}, m) \models \varphi$ and $(\underline{M}, m) \models \neg\varphi$. So Γ has no model in the class of type spaces (resp. Harsanyi type spaces) on X for player set I. ∎

B. Proofs of Section 4

Proof of Lemma 5. By induction on the formation of the formulas in $\mathcal{L}$:

1. Let $x \in X \cup \{\top\}$. Then,

$$(\underline{M}, m) \models x \text{ iff } v(m, x) = 1 \text{ iff } v(f(m), x) = 1 \text{ iff } (\underline{N}, f(m)) \models x.$$

2. Let $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$. Then, by the induction hypothesis,

$$\begin{aligned} (\underline{M}, m) \models \textstyle\bigwedge_{\varphi \in \Phi} \varphi \quad & \text{iff} \quad (\underline{M}, m) \models \varphi, && \text{for all } \varphi \in \Phi, \\ & \text{iff} \quad (\underline{N}, f(m)) \models \varphi, && \text{for all } \varphi \in \Phi, \\ & \text{iff} \quad (\underline{N}, f(m)) \models \textstyle\bigwedge_{\varphi \in \Phi} \varphi. \end{aligned}$$

3. Let $\varphi \in \mathcal{L}$. Then, by the induction hypothesis,

$$(\underline{M}, m) \models \neg\varphi \text{ iff } (\underline{M}, m) \not\models \varphi \text{ iff } (\underline{N}, f(m)) \not\models \varphi \text{ iff } (\underline{N}, f(m)) \models \neg\varphi.$$

4. Let $\psi \in \mathcal{L}_0$. As remarked in the definition of the relation "$\models$", $[\psi]^{\underline{M}}$ is measurable in $\underline{M}$ and $[\psi]^{\underline{N}}$ is measurable in $\underline{N}$. By the induction hypothesis, we have $f^{-1}([\psi]^{\underline{N}}) = [\psi]^{\underline{M}}$, so we have

$$\begin{aligned} (\underline{M}, m) \models p_i^\alpha(\psi) \quad & \text{iff} \quad \alpha \leq T_i(m)([\psi]^{\underline{M}}) \\ & \text{iff} \quad \alpha \leq T_i^N(f(m))([\psi]^{\underline{N}}) \\ & \text{iff} \quad (\underline{N}, f(m)) \models p_i^\alpha(\psi). \qquad \blacksquare \end{aligned}$$

Proof of Theorem 2. Let $\underline{M} = \langle M, \Sigma^M, (T_i^M), v^M \rangle$ be a type space (resp. Harsanyi type space) on X for player set I. By Lemma 5, 2 of Proposition 2, and 4 of Proposition 3, it follows that there is at most one type morphism from $\underline{M}$ to $\underline{\Omega}$.

Let

$$\Phi_m := \{\varphi \in \mathcal{L}_0 \mid (\underline{M}, m) \models \varphi\},$$

for $m \in M$, and define

$$f(m) := \bigwedge_{\varphi \in \Phi_m} \varphi.$$

By Corollary 1, Φ_m is consistent and, by the definition of the relation "$\models$", it follows for $\psi \in \mathcal{L}_0$ that

$$\psi \in \Phi_m \text{ iff } \neg\psi \notin \Phi_m.$$

This implies that $f(m) \in \Omega$. It remains to show that $f : M \to \Omega$ is a type morphism:

1. It is enough to show that for every $\psi \in \mathcal{L}_0 : f^{-1}([\psi]) \in \Sigma^M$, since the set $\{[\varphi] \mid \varphi \in \mathcal{L}_0\}$ is a field that generates Σ. We have
$$f(m) \in [\psi] \text{ iff } \vdash f(m) \to \psi \text{ iff } \psi \in \Phi_m \text{ iff } m \in [\psi]^{\underline{M}}.$$
But $[\psi]^{\underline{M}} \in \Sigma^M$ (see the definition of "$\models$").
2. Let $x \in X \cup \{\top\}$. Then,
$$v^M(m,x) = 1 \text{ iff } x \in \Phi_m \text{ iff } \vdash f(m) \to x \text{ iff } f(m) \in [x] \text{ iff } v(f(m),x) = 1.$$
3. Let $i \in I$ and $m \in M$. Since $f : M \to \Omega$ is $\Sigma^M - \Sigma$-measurable, $T_i^M(m)\left(f^{-1}(\cdot)\right)$ is a σ-additive probability measure on (Ω, Σ). Since $\mathcal{F}$ is a field that generates Σ, by Carathéodory's Extension Theorem,
$$T_i^M(m)\left(f^{-1}([\varphi])\right) = T_i(f(m))([\varphi]),$$
for all $\varphi \in \mathcal{L}_0$, implies
$$T_i^M(m)\left(f^{-1}(E)\right) = T_i(f(m))(E),$$
for all $E \in \Sigma$. As shown in 1, we have $f^{-1}([\varphi]) = [\varphi]^{\underline{M}}$, for $\varphi \in \mathcal{L}_0$, and hence
$$\begin{aligned} T_i^M(m)\left(f^{-1}([\varphi])\right) &= T_i^M(m)([\varphi]^{\underline{M}}) \\ &= \sup\{\alpha \mid (\underline{M}, m) \models p_i^\alpha(\varphi)\} \\ &= \sup\{\alpha \mid p_i^\alpha(\varphi) \in \Phi_m\} \\ &= \sup\{\alpha \mid \vdash f(m) \to p_i^\alpha(\varphi)\} \\ &= T_i(f(m))([\varphi]). \quad \blacksquare \end{aligned}$$

C. Proofs of Section 5

Proof of Lemma 6. The "either" follows by the consistency of ω_j, while the "or" follows by an easy induction on the formation of the formulas in $\mathcal{L}^j$, which is done in the same way as in the proof of 2 of Proposition 2. ∎

Proof of Remark 8. For $\omega_0 \in \Omega_0$ and $\varphi_0, \psi_0 \in \mathcal{L}_0^0$, we have by Conjunction, (A4) and Modus Ponens that
$$\omega_0 \in [\varphi_0]^0 \cap [\psi_0]^0 \quad \text{implies} \quad \omega_0 \in [\varphi_0 \wedge \psi_0]^0,$$
and by (A5) and 3 of Lemma 2
$$\omega_0 \in [\varphi_0 \wedge \psi_0]^0 \quad \text{implies} \quad \omega_0 \in [\varphi_0]^0 \cap [\psi_0]^0,$$

and hence

$$[\varphi_0 \wedge \psi_0]^0 = [\varphi_0]^0 \cap [\psi_0]^0 .$$

By Lemma 6, we have

$$\Omega_0 \setminus [\varphi_0]^0 = [\neg\varphi_0]^0 ,$$

for $\varphi_0 \in \mathcal{L}_0^0$. It follows that the generators of Σ_0 are finite Boolean combinations of the sets $[x]^0$, where $x \in X$. Hence Σ_0 is generated by the sets $[x]^0$, where $x \in X$.

For the second point: Existence: Take a type space $\underline{M}$ on X for player set I consisting of one point (i.e., $M = \{m\}$, note that $\underline{M}$ is necessarily a Harsanyi type space) such that $v(m, x) = 1$ iff $x \in m_0$. By Corollary 1,

$$\omega_0 := \bigwedge_{\varphi^0 \in \mathcal{L}_0^0 : (\underline{M}, m) \models \varphi^0} \varphi^0$$

is consistent in the system H (and hence in the system P) and, by the definition of the relation "$\models$", $\omega_0 \in \Omega_0$. By (A5) and Lemma 6, we have

$$\vdash \omega_0 \to x \ \text{ iff } \ x \in m_0.$$

Hence by Lemma 6 we have

$$\vdash \omega_0 \to \neg y \ \text{ iff } \ y \notin m_0.$$

So, we have by Conjunction, (A4) and Modus Ponens,

$$\vdash \omega_0 \to \bigwedge_{x \in X : x \in m_0} x \wedge \bigwedge_{y \in X : y \notin m_0} \neg y.$$

Uniqueness: An easy induction on the formation of the formulas in $\mathcal{L}_0^0$ shows that for all $\varphi^0 \in \mathcal{L}_0^0$:

$$\vdash \bigwedge_{x \in m_0} x \wedge \bigwedge_{y \in X \setminus m_0} \neg y \to \varphi^0 \ \text{ or } \ \vdash \bigwedge_{x \in m_0} x \wedge \bigwedge_{y \in X \setminus m_0} \neg y \to \neg\varphi^0.$$

Hence, by Conjunction, (A4) and Modus Ponens, we have either

$$\bigwedge_{x \in X : x \in m_0} x \wedge \bigwedge_{y \in X : y \notin m_0} \neg y \to \omega_0$$

or

$$\bigwedge_{x \in X : x \in m_0} x \wedge \bigwedge_{y \in X : y \notin m_0} \neg y \to \neg\omega_0.$$

By 3 of Lemma 2 and the consistency of ω_0, this implies

$$\vdash \omega_0 \leftrightarrow \bigwedge_{x \in X : x \in m_0} x \wedge \bigwedge_{y \in X : y \notin m_0} \neg y.$$

Notice that since ω_0 is consistent, we cannot have $\vdash \omega_0 \to \omega_0'$ for some $\omega_0' \in \Omega_0$ with $\omega_0 \neq \omega_0'$, because by the definitions $\{\omega_0, \omega_0'\}$ is an inconsistent set of formulas. ∎

Proof of Lemma 7. An easy induction on the formation of the formulas shows that for all $\varphi \in \mathcal{L}$:

$$\text{Either } \vdash \left(\bigwedge_{j \in I_0} \omega_j \right) \to \varphi \ \text{ or } \ \vdash \left(\bigwedge_{j \in I_0} \omega_j \right) \to \neg\varphi,$$

but not both. By the consistency of $\bigwedge_{j \in I_0} \omega_j$, Corollary 1 and 4 of Proposition 3, the rest is now obvious. ∎

Proof of Lemma 8. The first assertion is true for i-formulas, according to Lemmas 6 and 7. The rest of the first assertion follows from the definition of $[\cdot]^*$ and 6 and 7 of Proposition 2.

To the second assertion: Since

$$[\varphi] \subseteq h^{-1}\left([\varphi]^*\right) \quad \text{and} \quad \Omega \setminus [\varphi] = [\neg\varphi] \subseteq h^{-1}\left([\neg\varphi]^*\right),$$

and since

$$h^{-1}\left([\varphi]^*\right) \cap h^{-1}\left([\neg\varphi]^*\right) = h^{-1}\left([\varphi]^*\right) \cap h^{-1}\left(\Omega^* \setminus [\varphi]^*\right) = \emptyset,$$

we have

$$[\varphi] = h^{-1}\left([\varphi]^*\right). \qquad ∎$$

Proof of Lemma 9. Let $i \in I$. For $\omega_i \in \Omega_i$, choose $\omega \in \Omega$ such that $\omega(i) = \omega_i$. By the definitions and Lemma 6,

$$T_i^*(\omega_i)([\varphi]^*) = T_i'(\omega)([\varphi]),$$

for all $\varphi \in \mathcal{L}_0$. This implies in particular that $T_i^*(\omega_i)(\cdot)$ is non-negative.

If for $\varphi, \psi \in \mathcal{L}_0$: $[\varphi]^* = [\psi]^*$, then $[\varphi] = [\psi]$, by Lemma 8. It follows that

$$T_i^*(\omega_i)([\varphi]^*) = T_i'(\omega)([\varphi]) = T_i'(\omega)([\psi]) = T_i^*(\omega_i)([\psi]^*),$$

hence $T_i^*(\omega_i)(\cdot)$ is well-defined.

If for $\varphi, \psi \in \mathcal{L}_0$: $[\varphi]^* \cap [\psi]^* = \emptyset$, then $[\varphi] \cap [\psi] = \emptyset$, by Lemma 8. By the definition of $[\cdot]^*$, it follows that $[\varphi]^* \cup [\psi]^* = [\varphi \vee \psi]^*$. Hence,

$$\begin{aligned} T_i^*(\omega_i)([\varphi]^*) + T_i^*(\omega_i)([\psi]^*) =& T_i'(\omega)([\varphi]) + T_i'(\omega)([\psi]) \\ =& T_i'(\omega)([\varphi \vee \psi]) \\ =& T_i^*(\omega_i)([\varphi \vee \psi]^*) \\ =& T_i^*(\omega_i)([\varphi]^* \cup [\psi]^*). \end{aligned}$$

Therefore, $T_i^*(\omega_i)(\cdot)$ is additive on $\mathcal{F}^*$.

Since $\top$ and $p_i^1(\top)$ are theorems, we have $\Omega_0 = [\top]^0$ and, by the definition of the $\omega_i \in \Omega_i$, $\vdash \omega_i \to p_i^1(\top)$. It follows that $T_i^*(\omega_i)(\Omega^*) = 1$.

Let $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$, and let $[\varphi_n]^* \downarrow \emptyset$. It follows by Lemma 8 that $[\varphi_n] \downarrow \emptyset$ and therefore

$$\lim_{n\to\infty} T_i^*(\omega_i)([\varphi_n]^*) = \lim_{n\to\infty} T_i'(\omega)([\varphi_n]) = 0. \qquad \blacksquare$$

Proof of Proposition 4.

1. Follows from Carathéodory's Extension Theorem and Lemma 9.
2. Follows from Lemma 1: It suffices to show that for every $\psi \in \mathcal{L}_0$, rational $\alpha \in [0,1]$ and $i \in I$:

   $$\{\omega_i \mid T_i^*(\omega_i)([\psi]^*) \geq \alpha\} \in \Sigma_i.$$

   Choose a $\omega \in \Omega$ such that $\omega(i) = \omega_i$. By Lemma 6, we have

   $$\vdash \omega \to p_i^\alpha(\psi) \quad \text{iff} \quad \vdash \omega_i \to p_i^\alpha(\psi).$$

   By Lemma 3, it follows that

   $$T_i^*(\omega_i)([\psi]^*) \geq \alpha \quad \text{iff} \quad \vdash \omega_i \to p_i^\alpha(\psi).$$

   Hence,

   $$\{\omega_i \mid T_i^*(\omega_i)([\psi]^*) \geq \alpha\} = [p_i^\alpha(\psi)]^i \in \Sigma_i.$$
3. Follows from Remark 8, and 1 and 2 of this proposition.
4. We proceed by induction on the formation of the formulas in $\mathcal{L}_0$:
   - For $x \in X \cup \{\top\}$ and $(\omega_j)_{j\in I_0} \in \Omega^*$:

     $$\begin{aligned} (\underline{\Omega}^*, (\omega_j)_{j\in I_0}) \models x \quad & \text{iff} \quad v^*(\omega_0, x) = 1 \\ & \text{iff} \quad \omega_0 \in [x]^0 \\ & \text{iff} \quad (\omega_j)_{j\in I_0} \in [x]^*. \end{aligned}$$

- The induction steps "$\neg\varphi$", for $\varphi \in \mathcal{L}_0$, and "$\varphi \wedge \psi$", for φ and $\psi \in \mathcal{L}_0$, are clear by the definition of "$\models$" and "$[\cdot]^*$".
- So there remains the step "$p_i^\alpha(\varphi)$", for $i \in I$, $\alpha \in [0,1] \cap \mathbb{Q}$, and $\varphi \in \mathcal{L}_0$. By the induction hypothesis, we have

$$[\varphi]^* = \left\{ (\omega'_j)_{j \in I_0} \in \Omega^* \middle| \ \left(\underline{\Omega}^*, (\omega'_j)_{j\in I_0}\right) \models \varphi \right\}.$$

It follows that, for $(\omega_j)_{j\in I_0} \in \Omega^*$:

$$\begin{aligned} (\underline{\Omega}^*, (\omega_j)_{j\in I_0}) \models p_i^\alpha(\varphi) \quad &\text{iff} \quad T_i^*(\omega_j)\left([\varphi]^*\right) \geq \alpha \\ &\text{iff} \quad \sup\left\{ \beta \in [0,1]\cap\mathbb{Q} \middle| \ \vdash \omega_i \to p_i^\beta(\varphi) \right\} \geq \alpha \\ &\text{iff} \quad \vdash \omega_i \to p_i^\alpha(\varphi) \\ &\text{iff} \quad \omega_i \in [p_i^\alpha(\varphi)]^i \\ &\text{iff} \quad (\omega_j)_{j\in I_0} \in [p_i^\alpha(\varphi)]^*, \end{aligned}$$

where the third equivalence follows by the axioms (P7) and (P3).

5. We have to show that, for $i \in I$ and $\omega_i \in \Omega_i$:

$$A \in \Sigma^* \quad \text{and} \quad [T_i^*(\omega_i)]^* := \{(\omega'_j)_{j\in I_0} \mid T_i^*(\omega'_i) = T_i^*(\omega_i)\} \subseteq A$$

imply

$$T_i^*(\omega_i)(A) = 1.$$

By the definition of $\mathcal{F}^*$, Σ^*, and $T_i^*(\omega_i)$, it suffices to show that $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{L}_0$ and $\bigcup_{n\in\mathbb{N}}[\varphi_n]^* \supseteq A$ imply

$$\lim_{l\to\infty} T_i^*(\omega_i)\left(\left[\bigvee_{n\leq l}\varphi_n\right]^*\right) = 1.$$

Choose $\omega \in \Omega$ with $\vdash \omega \to \omega_i$.

By an easy induction on the formation of the i-formulas $\varphi^i \in \mathcal{L}^i$, we have that for $\omega', \widetilde{\omega} \in \Omega$ and $i \in I$: $T_i(\omega') = T_i(\widetilde{\omega})$ implies

$$(\underline{\Omega}, \omega') \models \varphi^i \quad \text{iff} \quad (\underline{\Omega}, \widetilde{\omega}) \models \varphi^i.$$

Hence, by 4 of Proposition 3, $T_i(\omega') = T_i(\widetilde{\omega})$ implies

$$\vdash \omega' \to \omega_i \quad \text{iff} \quad \vdash \widetilde{\omega} \to \omega_i.$$

It follows that $h^{-1}([T_i^*(\omega_i)]^*) \supseteq [T_i(\omega)]$. Hence, by Lemma 8, $\bigcup_{n\in\mathbb{N}}[\varphi_n] \supseteq [T_i(\omega)]$. This implies

$$\lim_{l\to\infty} T_i^*(\omega_i)\left(\left[\bigvee_{n\leq l}\varphi_n\right]^*\right) = \lim_{l\to\infty} T_i(\omega)\left(\left[\bigvee_{n\leq l}\varphi_n\right]\right) = 1. \qquad \blacksquare$$

Proof of Theorem 3. By Lemma 7, h is one-to-one.

Let $(\omega_j)_{j\in I_0} \in \Omega^*$. By 4 of Proposition 4, Lemma 6 and the definition of $[\cdot]^*$, we have for $i \in I_0$ and $\varphi_i \in \mathcal{L}_0^i$:

$$\vdash \omega_i \to \varphi_i \ \text{ iff } \ (\omega_j)_{j\in I_0} \in [\varphi_i]^* \ \text{ iff } \ \langle \underline{\Omega}^*, (\omega_j)_{j\in I_0}\rangle \models \varphi_i.$$

By definition of "$\models$", this implies that $\langle \underline{\Omega}^*, (\omega_j)_{j\in I_0}\rangle \models \omega_i$, for $i \in I_0$. Hence, we have

$$\langle \underline{\Omega}^*, (\omega_j)_{j\in I_0}\rangle \models \bigwedge_{j\in I_0} \omega_j.$$

So $\bigwedge_{j\in I_0} \omega_j$ is consistent for every $(\omega_j)_{j\in I_0} \in \Omega^*$, hence, by Lemma 7, h is onto.

For $(\omega_j)_{j\in I_0} \in \Omega^*$ let $\omega \in \Omega$ such that $\vdash \omega \leftrightarrow \bigwedge_{j\in I_0} \omega_j$. Then it follows that

$$\langle \underline{\Omega}^*, (\omega_j)_{j\in I_0}\rangle \models \omega.$$

Hence, h^{-1} is the type morphism from the proof of Theorem 2.

h is a type morphism: For $\varphi \in \mathcal{L}_0$, we have by Lemma 8:

$$h^{-1}([\varphi]^*) = [\varphi] \in \Sigma.$$

Since $\mathcal{F}^*$ is a field that generates Σ^*, it follows that h is measurable.

Let $\omega \in \Omega$. For $x \in X \cup \{\top\}$, we have by Lemma 6, Lemma 7 and the definitions

$$v^*(h(\omega), x) = v^*((\omega(j))_{j\in I_0}, x) = v^*(\omega(0), x),$$

and

$$\begin{aligned} v^*(\omega(0), x) = 1 \quad & \text{iff} \quad \vdash \omega(0) \to x \\ & \text{iff} \quad \vdash \omega \to x \\ & \text{iff} \quad v(\omega, x) = 1. \end{aligned}$$

By Carathéodory's Extension Theorem, it is enough to show that for $\omega \in \Omega$, $i \in I$ and $\varphi \in \mathcal{L}_0$:

$$T_i^*\left(\omega(j)_{j\in I_0}\right)([\varphi]^*) = T_i\left(\omega\right)\left(h^{-1}([\varphi]^*)\right).$$

Since $h^{-1}([\varphi]^*) = [\varphi]$, this is clear by Lemma 6, Lemma 7 and the definitions. ∎

D. Proofs of Section 6

Proof of Theorem 4. We prove the P-system case and sketch the differences for the proof of the H-system case. For the P-system case:

Let $i \in I$. For $\omega_i \in \Omega_i$ define

$$\varphi_i(\omega_i) := \left(\bigwedge_{\chi \in \mathcal{L}_0,\ \alpha \in [0,1]\cap\mathbb{Q},\ \text{s.t. } \vdash \omega_i \to p_i^\alpha(\chi)} p_i^\alpha(\chi) \right) \wedge \left(\bigwedge_{\psi \in \mathcal{L}_0,\ \beta \in [0,1]\cap\mathbb{Q},\ \text{s.t. } \vdash \omega_i \to \neg p_i^\beta(\psi)} \neg p_i^\beta(\psi) \right).$$

An easy induction on the formation of the i-formulas shows that for every i-formula $\chi_i \in \mathcal{L}^i$:

$$\text{Either } \ \vdash \varphi_i(\omega_i) \to \chi_i \ \text{ or } \ \vdash \varphi_i(\omega_i) \to \neg\chi_i.$$

Since $\vdash \omega_i \to \varphi_i(\omega_i)$, by the consistency of ω_i, it follows that $\vdash \omega_i \leftrightarrow \varphi_i(\omega_i)$. Hence, $\omega_i' \neq \omega_i'' \in \Omega_i$ implies that $\varphi_i(\omega_i') \neq \varphi_i(\omega_i'')$. Therefore, by the definitions of $T_i^*(\omega_i')$ and $T_i^*(\omega_i'')$, we have $T_i^*(\omega_i')([\varphi]^*) \neq T_i^*(\omega_i'')([\varphi]^*)$, for some $\varphi \in \mathcal{L}_0$. We conclude that

$$T_i^* : \Omega_i \to \Delta(\Omega^*, \Sigma^*)$$

is one-to-one.

It follows—and in the same manner, also in the H-system case—that for $i \in I$ and $\omega_i \in \Omega_i : [T_i^*(\omega_i)]^* = \{\omega_i\} \times \Omega_{-i}$. Hence, in the H-system case, the introspection property of the canonical Harsanyi type space on X for player set I implies that

$$\mathrm{marg}_{\Omega_i} \circ T_i^*(\omega_i) = \delta_{\omega_i},$$

for $i \in I$ and $\omega_i \in \Omega_i$.

By 2 of Proposition 4, T_i^* is measurable, for $i \in I$.

Let $\mu \in \Delta(\Omega^*, \Sigma^*)$ and fix $i \in I$. Consider the following set of formulas:

$$\begin{aligned} \Phi^\mu := &\{p_i^\alpha(\varphi) \mid \varphi \in \mathcal{L}_0,\ \alpha \in [0,1]\cap\mathbb{Q} \text{ s.t. } \mu([\varphi]^*) \geq \alpha\} \\ &\cup \{\neg p_i^\beta(\psi) \mid \psi \in \mathcal{L}_0,\ \beta \in [0,1]\cap\mathbb{Q} \text{ s.t. } \mu([\psi]^*) < \beta\}. \end{aligned}$$

If this set of formulas is consistent in the system P, then by Corollary 1, there is a $\omega \in \Omega$, where Ω is the Ω belonging to the system P, such that

$$(\underline{\Omega}, \omega) \models \Phi^\mu.$$

But then, from 4 of Proposition 3, the definition of $\omega(i)$, the fact that $\vdash \omega \to \omega(i)$, and the consistency of ω, it follows that $\vdash \omega(i) \to \chi$, for all $\chi \in \Phi^\mu$. The definition of $T_i^*(\omega(i))$ implies then that

$$T_i^*(\omega(i))\left([\varphi]^*\right) = \mu\left([\varphi]^*\right),$$

for all $\varphi \in \mathcal{L}_0$. Hence, since $T_i^*\left(\omega\left(i\right)\right)$ and μ are σ-additive probability measures on Σ^* that coincide on the field $\mathcal{F}^*$, and since $\mathcal{F}^*$ generates Σ^*, Carathéodory's Extension Theorem implies then that

$$T_i^*\left(\omega\left(i\right)\right) = \mu.$$

In the following, we show that Φ^μ is consistent in the system P. Let $u \notin \Omega_i$. Define

$$\begin{aligned}
\Omega_i^\mu :=& \Omega_i \cup \{u\},\\
\Omega_j^\mu :=& \Omega_j, \quad \text{for } j \in I_0 \setminus \{i\},\\
\Sigma_i^\mu :=& \Sigma_i \cup \{E \cup \{u\} \mid E \in \Sigma_i\},\\
\Sigma_j^\mu :=& \Sigma_j, \quad \text{for } j \in I_0 \setminus \{i\},\\
\Omega^\mu :=& \Pi_{j \in I_0} \Omega_j^\mu,\\
\Sigma^\mu :=& \text{the product } \sigma\text{-field of the } \Sigma_j^\mu,\ j \in I_0.
\end{aligned}$$

Note that Σ_i^μ is a σ-field, $\Sigma^* \subseteq \Sigma^\mu$, and $E \cap \Omega^* \in \Sigma^*$, for $E \in \Sigma^\mu$. Note furthermore that, since each Ω_j, for $j \in I_0$, is nonempty, each Ω_j^μ, for $j \in I_0$, is nonempty.

For $j \in I_0 \setminus \{i\}$, choose $u_j \in \Omega_j$, set $u_i := u$ and define

$$\overline{u} := \left(u_j\right)_{j \in I_0}.$$

For $j \in I$, $\omega_j \in \Omega_j^\mu$ and $E \in \Sigma^\mu$ define

$$\begin{aligned}
T_j^\mu(\omega_j)(E) :=& T_j^*(\omega_j)(E \cap \Omega^*), \quad \text{if } j \neq i \text{ or if } i = j \text{ and } \omega_i \neq u,\\
T_i^\mu(\omega_i)(E) :=& \mu(E \cap \Omega^*), \qquad\qquad \text{if } \ \omega_i = u.
\end{aligned}$$

By this definition, $T_j^\mu(\omega_j)$ is a σ-additive probability measure on (Ω^μ, Σ^μ), for $j \in I$ and $\omega_j \in \Omega_j^\mu$.

For $x \in X$ and $\omega_0 \in \Omega_0^\mu$ define:

$$\begin{aligned}
v^\mu(\omega_0, x) :=& v^*(\omega_0, x),\\
v^\mu(\omega_0, \top) :=& 1, \qquad\qquad \text{in any case.}
\end{aligned}$$

By this definition, it is clear that $v^\mu(\cdot, x)$ is $\Sigma_0 - \mathrm{Pow}\left(\{0,1\}\right)$-measurable, for $x \in X \cup \{\top\}$.

Let $E \in \Sigma^\mu$, $j \in I \setminus \{i\}$, $\alpha \in [0,1] \cap \mathbb{Q}$, and

$$b^\alpha(E) := \{\nu \in \Delta(\Omega^\mu, \Sigma^\mu) \mid \nu(E) \geq \alpha\}.$$

Then, by the definitions:

$$(T_j^{\mu})^{-1}(b^{\alpha}(E)) = (T_j^{*})^{-1}(b^{\alpha}(E \cap \Omega^*)) \in \Sigma_j.$$

Hence,

$$T_j^{\mu} : \Omega_j^{\mu} \to \Delta(\Omega^{\mu}, \Sigma^{\mu})$$

is $\Sigma_j^{\mu} - \Sigma_{\Delta}^{\mu}$-measurable, for $j \in I \setminus \{i\}$.

Note that

$$T_i^{\mu}(u)(E) = \mu(E \cap \Omega^*) = T_i^{\mu}(u)(E \cap \Omega^*),$$

for $E \in \Sigma^{\mu}$. So, for all $j \in I$, $\omega_j \in \Omega_j^{\mu}$ and $E \in \Sigma^{\mu}$:

$$T_j^{\mu}(\omega_j)(E) = T_j^{\mu}(\omega_j)(E \cap \Omega^*).$$

By definition, we have

$$(T_i^{\mu})^{-1}(b^{\alpha}(E)) = \begin{cases} (T_i^{*})^{-1}(b^{\alpha}(E \cap \Omega^*)) \in \Sigma_i \subseteq \Sigma_i^{\mu}, & \text{if } \mu(E \cap \Omega^*) < \alpha, \\ \{u\} \cup (T_i^{*})^{-1}(b^{\alpha}(E \cap \Omega^*)) \in \Sigma_i^{\mu}, & \text{if } \mu(E \cap \Omega^*) \geq \alpha. \end{cases}$$

Hence,

$$T_i^{\mu} : \Omega_i^{\mu} \to \Delta(\Omega^{\mu}, \Sigma^{\mu})$$

is $\Sigma_i^{\mu} - \Sigma_{\Delta}^{\mu}$-measurable.

Now, we have proved that

$$\underline{\Omega}^{\mu} := \left\langle \Omega^{\mu}, \Sigma^{\mu}, (T_j^{\mu})_{j \in I}, v^{\mu} \right\rangle$$

is a product type space on X for player set I.

Next, we show by induction on the formation of the formulas $\varphi \in \mathcal{L}_0$ that, for $\omega \in \Omega^*$:

$$(\underline{\Omega}^{\mu}, \omega) \models \varphi \text{ iff } (\underline{\Omega}^{*}, \omega) \models \varphi.$$

An equivalent statement is:

$$[\varphi]^* = [\varphi]^{\mu} \cap \Omega^*, \quad \text{where } [\varphi]^{\mu} := \{\omega \in \Omega^{\mu} \mid (\underline{\Omega}^{\mu}, \omega) \models \varphi\}.$$

(Recall that, by 4 of Proposition 4, $[\varphi]^{\underline{\Omega}^*} = [\varphi]^*$, for $\varphi \in \mathcal{L}_0$.)

Since

$$T_i^{\mu}(u)([\varphi]^{\mu}) = \mu([\varphi]^{\mu} \cap \Omega^*),$$

it follows then that

$$(\underline{\Omega}^{\mu}, \overline{u}) \models \Phi^{\mu}.$$

(And this then implies that Φ^{μ} is consistent.)

Let $\omega \in \Omega^*$. By definition, we have for $x \in X \cup \{\top\}$:

$$\begin{aligned}(\underline{\Omega}^\mu, \omega) \models x &\text{ iff } v^\mu (\omega(0), x) = 1 \\ &\text{ iff } v^* (\omega(0), x) = 1 \\ &\text{ iff } (\underline{\Omega}^*, \omega) \models x.\end{aligned}$$

The steps "$\wedge$" and "$\neg$" are trivial.

Let $j \in I$. For $\varphi \in \mathcal{L}_0$, we have by the induction hypothesis $[\varphi]^* = [\varphi]^\mu \cap \Omega^*$, and hence, for $\alpha \in [0,1] \cap \mathbb{Q}$:

$$\begin{aligned}[p_j^\alpha(\varphi)]^* =& \{\omega \in \Omega^* \mid T_j^*(\omega(j))([\varphi]^*) \geq \alpha\} \\ =& \{\omega \in \Omega^* \mid T_j^*(\omega(j))([\varphi]^\mu \cap \Omega^*) \geq \alpha\} \\ =& \{\omega \in \Omega^* \mid T_j^\mu(\omega(j))([\varphi]^\mu) \geq \alpha\} \\ =& \{\omega \in \Omega^\mu \mid T_j^\mu(\omega(j))([\varphi]^\mu) \geq \alpha\} \cap \Omega^* \\ =& [p_j^\alpha(\varphi)]^\mu \cap \Omega^*.\end{aligned}$$

Now, we have shown that

$$T_i^* : \Omega_i \to \Delta(\Omega^*, \Sigma^*)$$

is onto, for $i \in I$.

For $i \in I$, it remains to prove that $(T_i^*)^{-1}$ is measurable: The sets $[p_i^\alpha(\varphi)]^i$, where $\varphi \in \mathcal{L}_0$ and $\alpha \in [0,1] \cap \mathbb{Q}$, generate the σ-field Σ_i. So it is enough to show that $T_i^*([p_i^\alpha(\varphi)]^i)$ is a measurable set in $\Delta(\Omega^*, \Sigma^*)$. But we have

$$T_i^*([p_i^\alpha(\varphi)]^i) = \{\nu \in \Delta(\Omega^*, \Sigma^*) \mid \nu([\varphi]^*) \geq \alpha\} \in \Sigma_\Delta^*.$$

The H-system case is proved similarly, but for the "onto part" one starts with $\mu \in \Delta(\Omega_{-i}, \Sigma_{-i})$ and defines $T^\mu := \delta_u \times \mu$, where "$\times$" denotes here the product of measures and δ_u is the delta measure at $u \in \Omega_i^\mu$.

And for the "one-to-one" part, one uses the following fact, which holds in general for σ-additive probability measures on product spaces (endowed with the product σ-field): If $\nu \in \Delta(\Omega^*, \Sigma^*)$ and for some $\omega_i \in \Omega_i$: $\mathrm{marg}_{\Omega_i}(\nu) = \delta_{\omega_i}$, then $\nu = \delta_{\omega_i} \times \mathrm{marg}_{\Omega_{-i}}(\nu)$. This fact together with Carathéodory's Extension Theorem is then used to show by induction on the formation of the i-formulas that $\mathrm{marg}_{\Omega_{-i}} \circ T_i^*$ is one-to-one. The measurability of $\mathrm{marg}_{\Omega_{-i}} \circ T_i^*$ is straightforward, while, using Lemma 1, the measurability of $(\mathrm{marg}_{\Omega_{-i}} \circ T_i^*)^{-1}$ follows also by induction on the formation of the i-formulas. ∎

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