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AUTOMORPHIC PLANCHEREL DENSITY THEOREM

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ABSTRACT

Let F be a totally real field, G a connected reductive group over F, and S a finite set of finite places of F. Assume that $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ has a discrete series representation. Building upon work of Sauvageot, Serre, Conrey–Duke–Farmer and others, we prove that the S-components of cuspidal automorphic representations of $G(\mathbb{A}_F)$ are equidistributed with respect to the Plancherel measure on the unitary dual of $G(F_S)$ in an appropriate sense. A few applications are given, such as the limit multiplicity formula for local representations in the global cuspidal spectrum and a quite flexible existence theorem for cuspidal automorphic representations with prescribed local properties. When F is not a totally real field or $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ has no discrete series, we present a weaker version of the above results.

1. Introduction

Serre ([Ser97]), and independently Conrey–Duke–Farmer ([CDF97]), proved that for a fixed prime p, the T_p-eigenvalues in the space of cuspforms $S_k(\Gamma_0(N))$, where $(p, N) = 1$, are equidistributed with respect to the Plancherel measure (on the set of unitary unramified representations of $PGL_2(\mathbb{Q}_p)$) as $k + N$ grows to infinity. Around the same time, Sauvageot ([Sau97]) obtained a similar result in the representation-theoretic setting, which says roughly that if G is an anisotropic group over $\mathbb{Q}$ (equivalently if the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$ is compact) then the p-components of automorphic representations are equidistributed with respect to the Plancherel measure as the level gets deeper and deeper (to be

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made more precise in the next paragraph). Sauvageot's work is built upon that of Corwin, De George, Wallach and Delorme. Sauvageot and Serre also proved variants of what we just mentioned; in particular, the same conclusion is shown to hold if one considers finitely many places simultaneously instead of a single prime p. Unfortunately the generalization has not been worked out to our knowledge,[1] by either Sauvageot or others, even though he apparently had a plan to carry it out in the case of noncompact quotients more than ten years ago ([Sau97, p. 153]). The modest goal of our paper is to extend the intrinsically beautiful results of Sauvageot and Serre to a reasonably general setting and to exhibit a few useful applications. Let us mention at the outset that the representation-theoretic formulation of the problem in our paper is strongly influenced by Sauvageot's work.

In order to describe our results more precisely, we introduce some notation. Let G be a connected reductive group over $\mathbb{Q}$ and S a finite nonempty set of finite primes. (Our discussion will remain valid when $\mathbb{Q}$ is replaced with any totally real field; see below.) Let $A_{G,\infty}$ denote the connected real group coming from a maximal $\mathbb{Q}$-split torus in the center of G (§2.1). For a technical reason (cf. Remark 4.1), we assume that the $\mathbb{Q}$-rank of the latter torus is the same as the $\mathbb{R}$-rank of the maximal $\mathbb{R}$-split torus in the center of $G \times_{\mathbb{Q}} \mathbb{R}$. Denote by $\widehat{G(\mathbb{Q}_S)}$ the unitary dual of $G(\mathbb{Q}_S) = \prod_{v\in S} G(\mathbb{Q}_v)$ equipped with Fell topology. The Plancherel measure $\widehat{\mu}^{\mathrm{pl}}_S$ is defined on $\widehat{G(\mathbb{Q}_S)}$, depending (up to a scalar) on the choice of a Haar measure on $G(\mathbb{Q}_S)$. Let $d(G_\infty)$ denote the cardinality of (any) real discrete L-packet of $G(F_\infty)$ (§2.4). Fix an irreducible (finite dimensional) algebraic representation ξ of G over $\mathbb{C}$. Given an open compact subgroups $U \subset G(\mathbb{A}^{S,\infty})$, define

$$\widehat{\mu}_{U,\xi} := \frac{1}{\mathrm{vol}(G(\mathbb{Q})A_{G,\infty}\backslash G(\mathbb{A}))\cdot \dim\xi}\sum_{\pi^0_S} m_{\mathrm{cusp}}(\pi^0_S;U,\xi)\cdot\delta_{\pi^0_S},$$

where $\delta_{\pi^0_S}$ is the dirac delta measure supported on π^0_S, and $m_{\mathrm{cusp}}(\pi^0_S;U,\xi)$ counts (up to scaling by the volume of U) the multiplicity of π^0_S in the space of ξ-cohomological cuspidal automorphic representations of $G(\mathbb{A})$ with level U

[1] While this paper was being written, we learned from Blasius that Margaret Upton and he were writing up a result in a similar direction for some particular groups or with a simplifying hypothesis at a finite place. On the other hand, a few people informed us that Sarnak (e.g., [Sar87]) had considered the problem well before the results of Serre, Conrey–Duke–Farmer and Sauvageot.

(outside $S \cup \{\infty\}$). The reader should not worry about the appearance of $\dim \xi$, which represents the total Plancherel mass of ξ-cohomological representations at ∞. (See Remark 3.9.)

We are going to evaluate measures at functions in $\mathscr{F}(\widehat{G(\mathbb{Q}_S)})$ (Definition 2.1), which is a reasonable space of $\widehat{\mu}^{\mathrm{pl}}_S$-measurable functions on $\widehat{G(\mathbb{Q}_S)}$. Thanks to Sauvageot's density theorem ([Sau97, Thm. 7.3]), it is known that the trace map from the Hecke algebra $C_c^\infty(G(\mathbb{Q}_S))$ to $\mathscr{F}(\widehat{G(\mathbb{Q}_S)})$ has dense image (in the sense of Proposition 2.5). Our main result is

Theorem 1.1 (Theorem 4.4, Theorem 4.11): *Suppose that $G(\mathbb{R})$ has a discrete series representation. Let $\widehat{f}_S \in \mathscr{F}(\widehat{G(\mathbb{Q}_S)})$.*

(i) *Fix ξ. If $U_n \to 1$ as $n \to \infty$ in the sense of Definition 3.1, then*

$$\lim_{n\to\infty} \widehat{\mu}_{U_n,\xi}(\widehat{f}_S) = \widehat{\mu}^{\mathrm{pl}}_S(\widehat{f}_S).$$

(ii) *Fix U. Assume that the center of G is trivial. If $\xi_n \to \infty$ as $n \to \infty$ in the sense of Definition 3.5, then*

$$\lim_{n\to\infty} \widehat{\mu}_{U,\xi_n}(\widehat{f}_S) = \widehat{\mu}^{\mathrm{pl}}_S(\widehat{f}_S).$$

(Both sides in the above equalities assume finite values.)

The above theorem is the precise version of the equidistribution property we alluded to earlier, namely that the S-components of ξ-cohomological discrete automorphic reprsentations are equidistributed with respect to $\widehat{\mu}^{\mathrm{pl}}_S$ on $\widehat{G(\mathbb{Q}_S)}$. The condition on $G(\mathbb{R})$ in the theorem is satisfied, for instance, when G is a unitary, symplectic or orthogonal group, but not when $G = GL_n$ for $n > 2$. (See a comment below for what we can do without any assumption on $G(\mathbb{R})$.)

There are three main ingredients in the proof of (i): the trace formula, the Plancherel formula and Sauvageot's density theorem. By the density theorem, the proof is reduced to the case when $\widehat{f}_S$ comes from the Hecke algebra, or more precisely when $\widehat{f}_S$ is equal to $\widehat{\phi}_S : \pi_S \mapsto \operatorname{tr} \pi_S(\phi_S)$ for some $\phi_S \in C_c^\infty(G(\mathbb{Q}_S))$. The main part of the proof is to show that the geometric side of the trace formula is dominated by the orbital integral on the identity element. Namely, if $\phi = \phi^S \cdot \phi_S$ is a test function on $G(\mathbb{A})$, then the geometric side is asymptotically equal to $\phi(1)$ multiplied by the volume of $G(\mathbb{Q})A_{G,\infty}\backslash G(\mathbb{A})$. By the Plancherel formula, $\phi(1)$ is nothing other than $\phi^S(1) \cdot \widehat{\mu}^{\mathrm{pl}}_S(\widehat{\phi}_S(1))$. From this it is not hard to deduce that

$$\lim_{n\to\infty} \widehat{\mu}_{U_n,\xi}(\widehat{\phi}_S) = \widehat{\mu}^{\mathrm{pl}}_S(\widehat{\phi}_S).$$

The strategy outlined above seems to have been well-known. Our minor contribution is to make the asymptotic argument work in the case of noncompact quotients, when we restrict ourselves to cohomological representations at infinite places. Once a suitable formulation is set up, the proof often proceeds on its own momentum.

The proof of (ii) is similar to that of (i) but requires additional input, most notably the character formula for finite dimensional representations of reductive Lie groups and some facts on stable discrete series characters. To our knowledge, (ii) was treated in the literature only in the case of elliptic modular forms ([Ser97]), although we believe that experts have known such a result for some time. Recently, Chenevier and Clozel ([CC09]) studied the effect of sending the parameter ξ to infinity in the trace formula but did not consider the distribution problem for local components.

We assume in the theorem that $G(\mathbb{R})$ has a discrete series representation, since it allows us to use the so-called Euler–Poincaré function associated with ξ in the trace formula. The Euler–Poincaré function not only singles out ξ-cohomological representations, but also simplifies the trace formula so that we need not resort to the simple trace formula of Deligne and Kazhdan. In particular, for the purpose of Theorem 1.1, we do not restrict ourselves to automorphic representations which are square-integrable or supercuspidal at some finite place. Finally, it is worth noting that the exact analogues of the previous two theorems are true if the base field $\mathbb{Q}$ is replaced with a totally real field.

If the base field is an arbitrary number field or if the existence of discrete series at ∞ is no longer assumed, we have an analogous result in a weaker form. Refer to §4.3 for details.

Let us mention a few implications of our theorem for automorphic representations. The following corollary is deduced from the proof of Theorem 1.1.(ii).

Corollary 1.2 (Theorem 5.8): *Suppose that $G(\mathbb{R})$ has a discrete series and that G has trivial center. Let $\widehat{U}$ be a $\widehat{\mu}^{\mathrm{pl}}_S$-regular relatively quasi-compact subset of $\widehat{G(\mathbb{Q}_S)}$ such that $\widehat{\mu}^{\mathrm{pl}}_S(\widehat{U}) > 0$. Let ξ be as before. Then there exist (infinitely many isomorphism classes of) cuspidal automorphic representations π of $G(\mathbb{A})$ such that*

- $\pi^{S\cup\{\infty\}}$ *is unramified,*
- $\pi_S \in \widehat{U}$,
- π_∞ *is a discrete series representation.*

The set $\widehat{U}$ prescribes a local condition on π_S. For instance, π_S may be required to belong to a particular Bernstein component (see [BD84] for this notion). A qualitative interpretation of the corollary is that a cuspidal automorphic representation with prescribed local properties exists whenever the local property is satisfied by a set of positive Plancherel measure at each place. As a special case of Corollary 1.2, when G is a split group over $\mathbb{Q}$ (without assuming the center of G to be trivial), there exist infinitely many cuspidal automorphic representations of $G(\mathbb{A})$ which are unramified everywhere and discrete series at ∞. See §§5.2–5.3 for variants of Corollary 1.2. When G is a unitary group, variants of Corollary 1.2 can be combined with the quadratic base change (e.g., [Lab]) and the construction of Galois representations (e.g., [Shia], [CH]) to produce (global) Galois representations with prescribed local conditions (such as local inertia types).

Another application of Theorem 1.1 is a limit multiplicity formula. See §5 for detailed discussion and §6 for an example in the case of Hilbert modular forms. Note that the limit multiplicity formula for discrete series representations at finite places (cf. [Clo86, §4]) is a special case of our result when the local condition (prescribed by $\widehat{U}$) consists of only one element which is a discrete series. For a discrete series at an infinite place (in which case the limit multiplicity formula is due to De George, Wallach, Clozel, Rohlfs–Speh, Savin and others), our formula is weaker in that we count the limit multiplicity not for each discrete series but for a discrete L-packet as a whole.

In fact our initial motivation for this work was to prove Corollary 1.2 or its variants, stronger than similar results in the literature. It is a fruitful problem to find a cuspidal (and non-endoscopic, if needed) automorphic representation whose local component is close to any given local representation (e.g., in the same Bernstein component) while maintaining enough control at the other places. The solution of this type of problem seems almost unavoidable when one wishes to use a global method to prove a local result.[2] One can appeal

[2] In fact the author encountered this problem in the computation of the cohomology of local moduli spaces ([Shib]), and gave a solution by a somewhat different use of the trace formula. See Section 3 of that article.

to the limit multiplicity formulas cited above when the local representation is a discrete series, but few results have been available in the general case (for a large class of reductive groups). To our knowledge [CC09, Thm. 1.3] has been among the best results so far in the non-discrete series case.

Finally, we ought to apologize to the reader that we have not strived for maximum generality. Besides our inability, the reason is that the results of this paper can be kept at a reasonable technical difficulty and still seem to suffice for various arithmetic applications. One could compute the contribution from an individual ξ-cohomological discrete series, rather than from all ξ-cohomological representations as we do here, if one works more diligently with the trace formula using pseudocoefficients for discrete series in place of Euler–Poincaré functions.[3] On the other hand, one could try to prove the full version of Theorem 1.1 without any assumption on $G(\mathbb{R})$.

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2. Preliminaries

We advise that the reader skip to §4 and use §2 and §3 as references. Most basic notations are introduced in §2.1 and §2.4.

2.1. Notation and convention.

- F is a number field.
- $\mathcal{V}_F$ (resp. $\mathcal{V}_F^\infty$) is the set of all (resp. finite) places of F.
- S is a nonempty finite subset of $\mathcal{V}_F$ (often $S \subset \mathcal{V}_F^\infty$).

[3] One of the referees made an insightful suggestion that one should be able to do this by using the technique of Axel Ferrari ([Fer07]) and showing that the endoscopic contribution is negligible.

- $F_S := \prod_{v\in S} F_v$, $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R}$.
- $\mathbb{A}_F^S := \prod'_{v\in\mathcal{V}_F\setminus S} F_v$, $\mathbb{A}_F^{S,\infty} := \prod'_{v\in\mathcal{V}_F^\infty\setminus S} F_v$.
- G is a connected reductive group over F.
- $C_c^\infty(G(\mathbb{A}_F^{S,\infty}))$ is the space of locally constant compactly supported $\mathbb{C}$-valued functions on $G(\mathbb{A}_F^{S,\infty})$. Similarly $C_c^\infty(G(F_S))$ is defined.
- $\widehat{G(F_S)}$ is the unitary dual of $G(F_S)$ equipped with Fell topology ([Fel60]).
- $\widehat{\mu}_S^{\mathrm{pl}}$ is the Plancherel measure on $\widehat{G(F_S)}$.
- ϕ_S usually denotes a function in $C_c^\infty(G(F_S))$.
- $\widehat{\phi}_S$ is the function on $\widehat{G(F_S)}$ associated to ϕ_S which is given by $\pi_S \mapsto \operatorname{tr}\pi_S(\phi_S)$.
- $\widehat{f}_S$ denotes a $\widehat{\mu}_S^{\mathrm{pl}}$-measurable function on $\widehat{G(F_S)}$ (there need not be a function f_S).
- $Z(G)$ is the center of G.
- A_G is a maximal $\mathbb{Q}$-split torus in the center of $\mathrm{Res}_{F/\mathbb{Q}}G$, and set $A_{G,\infty} := A_G(\mathbb{R})^0$.

If M is a Levi subgroup of G over F_S, and π is an irreducible smooth representation of $M(F_S)$, we write $\text{n-ind}_M^G(\pi)$ for the normalized parabolic induction (when the choice of the parabolic subgroup does not matter). For a continuous (quasi-)character $\chi : A_{G,\infty} \to \mathbb{C}^\times$, define $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$ as the space of $\mathbb{C}$-valued functions on $G(\mathbb{A}_F)$ which are square-integrable modulo $A_{G,\infty}$ and transform under $A_{G,\infty}$ by χ. The space $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$ is equipped with an action of $G(\mathbb{A}_F)$ via right translation. There are discrete and cuspidal subspaces in $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$ which are stable under the action of $G(\mathbb{A}_F)$.

$$L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}_F), \chi) \subset L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F), \chi) \subset L^2(G(F)\backslash G(\mathbb{A}_F), \chi).$$

For an irreducible admissible representation π of $G(\mathbb{A}_F)$, we often write π_∞ for the infinite component $\otimes_{v|\infty}\pi_v$. The multiplicity of π in $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F), \chi)$ (resp. $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}_F), \chi)$) is denoted by $m_{\mathrm{disc}}(\pi)$ (resp. $m_{\mathrm{cusp}}(\pi)$).

2.2. Choice of Haar measures. By choosing Haar measures μ_v on $G(F_v)$ for each $v \in \mathcal{V}_F$ such that μ_v assigns volume 1 to a hyperspecial subgroup for almost all v, we form a measure $\mu := \prod_{v\in\mathcal{V}_F} \mu_v$ on $G(\mathbb{A}_F)$. By giving the point-counting measure on $G(F)$ and the Lebesgue measure on $A_{G,\infty}$, we get the quotient measure $\overline{\mu}$ on $G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)$.

In this article we will often suppose that

(i) F is a totally real field and
(ii) $G(F_\infty)$ has a discrete series representation.

In that case, we make a particular choice of μ_∞. Note that there is the so-called Euler–Poincaré measure $\overline{\mu}^{\mathrm{EP}}_\infty$ on $G(F_\infty)/A_{G,\infty}$, which is a particular Haar measure. On the other hand, $A_{G,\infty}$ is equipped with the usual multiplicative Lebesgue measure (as there is a natural identification $A_{G,\infty} \simeq (\mathbb{R}^\times_{>0})^r$ for some $r \in \mathbb{Z}_{>0}$). Define a unique Haar measure μ^{EP}_∞ on $G(F_\infty)$ to be compatible with the above measures via the exact sequence

$$1 \to A_{G,\infty} \to G(F_\infty) \to G(F_\infty)/A_{G,\infty} \to 1.$$

Then we will always assume (when the above conditions on F and $G(F_\infty)$ are in effect) that

$$\mu_\infty = \mu^{\mathrm{EP}}_\infty.$$

Sometimes we assume in addition to (i) and (ii) above that G is unramified at all finite places. (This will be considered in §6.2 for $G = GL_2$.) In that case, choose μ_v for each $v \in \mathcal{V}^\infty_F$ such that a hyperspecial subgroup of $G(F_v)$ has volume 1. Then the product measure $(\prod_{v\nmid\infty} \mu_v) \times \mu^{\mathrm{EP}}_\infty$ will be written as μ^{EP}. Let $\overline{\mu}^{\mathrm{EP}}$ denote the quotient measure on $G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)$ (with respect to the measures on $G(F)$ and $A_{G,\infty}$ as above).

2.3. Density theorem for functions on unitary dual. We set up some notation first. As before G is a connected reductive group over F. When M is a Levi subgroup of G over F_S, let $\Psi_u(M)$ denote the set of "unramified" unitary characters of $M(F_S)$, which naturally has the structure of a real torus. (For the precise definition, see [Wal03, p. 239] where $\Psi_u(M)$ is denoted by $\mathrm{Im}\, X(M)$.) When σ is an admissible representation of $M(F_S)$, write $\text{n-ind}^G_M(\sigma)$ for the normalized parabolic induction. Although the semisimplification of $\text{n-ind}^G_M(\sigma)$ is well-defined, its isomorphism class depends on the choice of a parabolic subgroup. We will use this notation only when $\text{n-ind}^G_M(\sigma)$ is irreducible or when we are only interested in computing traces.

The Plancherel measure $\widehat{\mu}^{\mathrm{pl}}_S$ is a positive Borel measure on $\widehat{G(F_S)}$. Note that $\widehat{\mu}^{\mathrm{pl}}_S$ depends on a choice of a Haar measure on $G(F_S)$. Precisely, if the latter Haar measure is multiplied by a scalar $c \in \mathbb{C}^\times$ then $\widehat{\mu}^{\mathrm{pl}}_S$ is multiplied by c^{-1}. Denote by $\mathscr{B}_c(\widehat{G(F_S)})$ the space of bounded $\widehat{\mu}^{\mathrm{pl}}_S$-measurable functions $\widehat{f}_S$ on $\widehat{G(F_S)}$ such that the support of $\widehat{f}_S$ has compact image in the space of infinitesimal characters via the map denoted by inf. ch. in [BDK86, 2.1]. In our paper a measure on $\widehat{G(F_S)}$ will be viewed as a linear functional on the subspace $\mathscr{F}(\widehat{G(F_S)})$ of $\mathscr{B}_c(\widehat{G(F_S)})$ defined below.

Definition 2.1: Define $\mathscr{F}(\widehat{G(F_S)})$ to be the space of functions $\widehat{f}_S \in \mathscr{B}_c(\widehat{G(F_S)})$ such that for every F_S-rational Levi subgroup M of G and every discrete series σ of $M(F_S)$, the function

$$\Psi_u(M) \to \mathbb{C} \quad \text{given by} \quad \chi \mapsto \widehat{f}_S(\text{n-ind}_M^G(\sigma \otimes \chi))$$

has the property that its discontinuous points are contained in a measure zero set. By definition, $\widehat{f}_S(\text{n-ind}_M^G(\sigma \otimes \chi))$ is the sum of $\widehat{f}_S(\sigma')$ as σ' runs over the irreducible subquotients of $\text{n-ind}_M^G(\sigma \otimes \chi)$ with multiplicity. (Note that any subquotient σ' is unitary.)

Lemma 2.2: *For any $\phi_S \in C_c^\infty(G(F_S))$, the function $\widehat{\phi}_S$ (defined in §2.1) belongs to $\mathscr{F}(\widehat{G(F_S)})$.*

Proof. This is an easy fact and essentially the inclusion $F_{\text{tr}} \subset F_{\text{good}}$ in [BDK86, 1.2]. ∎

Example 2.3: There are many functions in $\mathscr{F}(\widehat{G(F_S)})$ which are not of the form $\widehat{\phi}_S$ for any $\phi_S \in C_c^\infty(G(F_S))$. Any characteristic function on a $\widehat{\mu}_S^{\text{pl}}$-regular relatively quasi-compact subset $\widehat{U}$ of $\widehat{G(F_S)}$ belongs to $\mathscr{F}(\widehat{G(F_S)})$. (Lemma 7.2 of [Sau97] proves this when $\widehat{U}$ is open, but the same proof works for non-open sets as well.) An example of such a $\widehat{U}$ is the set of $\pi \in \widehat{G(F_S)}$ in the same Bernstein component.

The well-known Plancherel formula by Harish-Chandra says:

Proposition 2.4 (Harish-Chandra): *For $\phi_S \in C_c^\infty(G(F_S))$, we have $\widehat{\mu}_S^{\text{pl}}(\widehat{\phi}_S) = \phi_S(1)$.*

The following fundamental theorem due to Sauvageot tells us roughly that the image of $C_c^\infty(G(F_S))$ in $\mathscr{F}(G(F_S))$ via $\phi_S \mapsto \widehat{\phi}_S$ is dense.

Proposition 2.5 ([Sau97, Thm. 7.3]): *Let $\widehat{f}_S \in \mathscr{F}(\widehat{G(F_S)})$. For any $\epsilon > 0$, there exist $\phi_S, \psi_S \in C_c^\infty(G(F_S))$ such that*

$$|\widehat{f}_S(\pi) - \widehat{\phi}_S(\pi)| \leq \widehat{\psi}_S(\pi),\ \forall \pi \in \widehat{G(F_S)} \quad \textit{and} \quad \widehat{\mu}_S^{\text{pl}}(\widehat{\psi}_S) \leq \epsilon.$$

Remark 2.6: In fact, Sauvageot also proved the converse that any $\widehat{f}_S \in \mathscr{B}_c(\widehat{G(F_S)})$ with the above property belongs to $\mathscr{F}(\widehat{G(F_S)})$. This will not be needed as our theorems will be stated only for $\widehat{f}_S \in \mathscr{F}(\widehat{G(F_S)})$.

The analogous density theorem at archimedean places is also proved by Sauvageot and will be referred to in §4.3. As the result is basically the same, we chose not to copy it here.

2.4. Euler–Poincaré function at infinity. The purpose of this subsection is to recall necessary facts concerning Euler–Poincaré functions in the theory of real Lie groups. We assume (i) and (ii) of §2.2 throughout §2.4. Note that (ii) holds precisely when the ranks of G_∞ and K_∞ (given below) are the same. The following notation will be used.

- $G_\infty := (\mathrm{Res}_{F/\mathbb{Q}} G) \times_{\mathbb{Q}} \mathbb{R}$. So $G_\infty(\mathbb{R}) = G(F_\infty)$.
- ξ is an irreducible algebraic representation of G_∞ over $\mathbb{C}$.
- A_G is the maximal $\mathbb{Q}$-split torus in the center of $\mathrm{Res}_{F/\mathbb{Q}} G$, and similarly $A_{G,\mathbb{R}}$ is the maximal $\mathbb{R}$-split torus in the center of G_∞. We assume $A_G \times_{\mathbb{Q}} \mathbb{R} = A_{G,\mathbb{R}}$ and set $A_{G,\infty} := A_G(\mathbb{R})^0$.
- $\chi : A_{G,\infty} \to \mathbb{C}^\times$ is any continuous character.
- $\chi_\xi : A_{G,\infty} \to \mathbb{C}^\times$ is the character obtained by restricting ξ to $A_{G,\infty}$.
- $\mathfrak{g}_\infty$ is the Lie algebra of $G(F_\infty)$.
- $K_\infty = K^1_\infty A_{G,\infty}$ where K^1_∞ is a maximal compact subgroup of G_∞.
- $C^\infty_c(G(F_\infty), \chi)$ is the space of K_∞-bi-finite smooth $\mathbb{C}$-valued functions which are compactly supported modulo $A_{G,\infty}$ and transform under $A_{G,\infty}$ by χ.
- $d(G_\infty) := |W(T,G)|/|W_{\mathbb{R}}(T,G)|$ where T is a maximal $\mathbb{R}$-torus contained in K_∞ and W (resp. $W_{\mathbb{R}}$) denotes the Weyl group over $\mathbb{C}$ (resp. over $\mathbb{R}$).
- $q(G_\infty) := \dim(G(F_\infty)/K_\infty)$.
- $\Pi(G_\infty, \chi)$ is the set of isomorphism classes of irreducible admissible representations π_∞ of $G(F_\infty)$ such that the central character of π_∞ on $A_{G,\infty}$ is χ.
- $\Pi_u(G_\infty, \chi)$ is the subset of $\Pi(G_\infty, \chi)$ consisting of unitary representations.

- $\Pi_{\mathrm{disc}}(G_\infty, \xi^\vee)$ is the set of $\pi_\infty \in \Pi(G_\infty, \chi_\xi^{-1})$ which has the same central character and infinitesimal character as $\xi^\vee$. It is well known that $|\Pi_{\mathrm{disc}}(G_\infty, \xi^\vee)| = d(G_\infty)$.

Let $\phi_\xi \in C_c^\infty(G(F_\infty), \chi_\xi)$ denote the Euler–Poincaré function à la Clozel-Delorme such that for every $\pi_\infty \in \Pi(G_\infty, \chi_\xi^{-1})$,

$$\mathrm{tr}\, \pi_\infty(\phi_\xi) = \sum_{i\geq 0} (-1)^i \dim H^i(\mathfrak{g}_\infty, K_\infty; \pi_\infty \otimes \xi). \tag{2.1}$$

We say that π_∞ is ξ-cohomological if the summand in (2.1) is nonzero for some $i \geq 0$. Some standard properties of ϕ_ξ are recorded below.

Lemma 2.7: (i) *For $\pi_\infty \in \Pi(G_\infty, \chi_\xi^{-1})$, $\mathrm{tr}\, \pi_\infty(\phi_\xi) = 0$ unless π_∞ has the same infinitesimal character as $\xi^\vee$.*

(ii) *Suppose that the highest weight of ξ is regular. Then for $\pi_\infty \in \Pi(G_\infty, \chi_\xi^{-1})$,*

$$\mathrm{tr}\, \pi_\infty(\phi_\xi) = \begin{cases} (-1)^{q(G_\infty)}, & \text{if } \pi_\infty \in \Pi_{\mathrm{disc}}(G_\infty, \xi^\vee), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The first two properties follow from Clozel–Delorme's results on the trace Paley–Wiener theorems on real groups and standard properties of relative Lie algebra cohomology, as explained on pp. 264–266 of [Art89]. ∎

2.5. On stable discrete series characters. In this subsection, we will be concerned with real groups only and use slightly different notation from the rest of the article.

Let G be a connected reductive group over $\mathbb{R}$. Let M be an $\mathbb{R}$-rational Levi subgroup of G, and T an elliptic maximal torus of M. Let ξ be an irreducible algebraic representation of G. Write $T_{\mathrm{reg}}(\mathbb{R})$ for the set of $t \in T(\mathbb{R})$ which are regular in $G(\mathbb{R})$. Denote by W_M and W_G the Weyl group of T in M and G over $\mathbb{C}$, respectively.

Let $Q = LU$ be an $\mathbb{R}$-rational parabolic subgroup of G with Levi decomposition such that the Levi part L contains M. For $\gamma \in L(\mathbb{R})$, define

$$\delta_Q(\gamma) := \left|\det(\mathrm{Ad}(\gamma)|_{\mathrm{Lie}\,(U)})\right|, \quad D_M^L(\gamma) := \left|\det(1 - \mathrm{Ad}(\gamma)|_{\mathrm{Lie}\,(L)/\mathrm{Lie}\,(M)})\right|.$$

Let B be a Borel subgroup of $G_\mathbb{C}$ such that $T_\mathbb{C} \subset B \subset Q_\mathbb{C}$. Denote by $\lambda_B \in X^*(T)$ the B-dominant highest weight of ξ, and by $\rho_B \in X^*(T) \otimes_\mathbb{Z} \mathbb{R}$ the half sum of all B-positive roots of $T_\mathbb{C}$ in $G_\mathbb{C}$.

For each $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$, let us define

$$\Phi_M^G(\gamma,\xi) := (-1)^{q(G_\infty)}|D_M^G(\gamma)|^{1/2} \sum_{\pi\in\Pi_{\mathrm{disc}}(\xi)} \Theta_\pi(\gamma)$$

where Θ_π is the character function of π. It is known that the function $\Phi_M^G(\gamma,\xi)$ continuously extends to a W_M-invariant function on $T(\mathbb{R})$ ([Art89, Lem. 4.2], cf. [GKM97, Lem. 4.1]). We will need the following facts later.

LEMMA 2.8: (i) *If $M = G$, $\Phi_G^G(\gamma,\xi) = \mathrm{tr}\,\xi(\gamma)$ for all $\gamma \in T(\mathbb{R})$.*

(ii) *Suppose that $M \neq G$ and fix $\gamma \in T(\mathbb{R})$. Then for any ξ,*

$$(2.2)\quad |\Phi_M^G(\gamma,\xi)| \le \sum_Q \sum_w c\cdot |D_M^L(\gamma)|^{1/2}\delta_Q(\gamma)^{-1/2}\left|\mathrm{tr}\,(\gamma^{-1}|V^L_{w(\lambda_B+\rho_B)-\rho_B})\right|$$

where Q runs over the set of parabolic subgroups of G containing M (allowing $Q = G$), L denotes the Levi subgroup of Q containing M, and c is a constant which is independent of γ and ξ (and depends only on G).

Further explanation is necessary to make sense of (2.2). For each Q, choose a Borel subgroup B of $G_\mathbb{C}$ such that $T_\mathbb{C} \subset B \subset Q_\mathbb{C}$. The second sum is taken over the set of $w \in W_G$ such that the set of Q-positive roots in L is carried into the set of B-positive roots in G. We have defined λ_B and ρ_B before the lemma. The summand is independent of the choice of B.

Proof. Part (i) is an easy consequence of the character formula for discrete series (cf. [Art89, p. 271]).

Part (ii) follows from Theorem 5.1 of [GKM97]. It is immediate from the last display of page 504 of that paper that

$$|\Phi_M^G(\gamma,\xi)| \le \sum_Q |D_M^L(\gamma)|^{1/2}\delta_Q(\gamma)^{-1/2}\cdot L_Q^\nu(\gamma)$$

where ν is taken to be ν_m of page 509.

For each Q, there is an integer-valued function φ_Q defined in the appendix of [GKM97]. It is clearly a bounded function, for instance in view of Lemma A.6 in the appendix. Let c be the maximum of the supremum norm of $|\varphi_Q|$. The formula right above Theorem 5.1 of the same paper implies that

$$|L_Q^\nu(\gamma)| \le c\cdot \sum_w \left|\mathrm{tr}\,(\gamma^{-1}|V^L_{w(\lambda_B+\rho_B)-\rho_B})\right|$$

where w runs over the same set as in the statement of the lemma. The proof of (ii) is complete. ∎

3. Key definitions

In this section we provide key definitions that are needed in stating our main results in §4.

3.1. Tower of functions.

Definition 3.1: Let $\{U_n\}_{n\geq 1}$ be a sequence of open compact subgroups of $G(\mathbb{A}_F^{S,\infty})$. We write that

$$U_n \to 1 \quad \text{as } n \to \infty$$

if there exists a decreasing sequence of open compact subgroups $\{V_m\}_{m\geq 1}$ such that

- $V_1 \supset V_2 \supset \cdots$ and $\bigcap_{m=1}^{\infty} V_m = \{1\}$.
- For each m, there exists $i_m \in \mathbb{Z}_{>0}$ such that $U_n \subset V_m$ for all $n \geq i_m$.

Example 3.2: Let $v \in \mathcal{V}^\infty \backslash S$ and fix an open compact subgroup $U^{S,v,\infty} \subset G(\mathbb{A}_F^{S,v,\infty})$. Let $\{U_{v,n}\}_{n\geq 1}$ be a decreasing sequence which forms an open basis of 1 in $G(F_v)$. Then $U_n := U_{v,n}U^{S,v,\infty}$ has the property that $U_n \to 1$ as $n \to \infty$.

Remark 3.3: The condition that $U_n \to 1$ as $n \to \infty$ does not imply convergence of the sequence in the usual topological sense. Namely, given an open compact subgroup $U \subset G(\mathbb{A}_F^{S,\infty})$, there may not exist n_0 such that $U_n \subset U$ for all $n \geq n_0$, as we see in Example 3.2.

We slightly modify Sauvageot's notion of tower of functions ([Sau97, p. 155]) to suit our purpose.

Definition 3.4: Let $\{\phi_n^{S,\infty}\}_{n\geq 1}$ be a sequence of functions in $C_c^\infty(G(\mathbb{A}_F^{S,\infty}))$. We say $\{\phi_n^{S,\infty}\}_{n\geq 1}$ is a **tower of functions tending to 1** if

- For every $n \geq 1$, $\phi_n^{S,\infty}(1) = 1$.
- $\{\mathrm{supp}\,(\phi_n^{S,\infty})\} \to 1$ as $n \to \infty$ in the sense of Definition 3.1.
- For some $\phi \in C_c^\infty(G(\mathbb{A}_F^{S,\infty}))$, we have $|\phi_n(x)| \leq \phi(x)$ for every $x \in G(\mathbb{A}_F^{S,\infty})$ and every $n \geq 1$.
- For every $\pi \in \widehat{G(\mathbb{A}_F^{S,\infty})}$ and every $n \geq 1$, $\widehat{\phi_n^{S,\infty}}(\pi) \geq 0$.

3.2. Sequence of parameters at ∞. Let T_∞ be an elliptic maximal torus in G_∞. Let B be a Borel subgroup of G_∞ over $\mathbb{C}$ containing T_∞. Let $\{\xi_n\}_{n\geq 1}$ be a sequence of irreducible algebraic representations of G_∞ over $\mathbb{C}$ with highest weight vectors $\lambda_{\xi_n} \in X^*(T_\infty)$ which are B-dominant. We define the notion of the parameters ξ_n tending to infinity.

Definition 3.5: We write

$$\xi_n \to \infty \quad \text{as } n \to \infty$$

if for every $n \geq 1$ and for every B-positive root α of T_∞ in G_∞,

$$\lim_{n\to\infty} \langle \lambda_{\xi_n}, \alpha\rangle = +\infty. \tag{3.1}$$

This definition is independent of the choice of B.

Example 3.6: Let $G = GL_2$ and F be totally real so that $G_\infty = \prod_{\tau: F\hookrightarrow\mathbb{R}} GL_2$ over $\mathbb{R}$. For each $n \geq 1$, let

$$\xi_n = \bigotimes_{\tau:F\hookrightarrow\mathbb{R}} (\mathrm{Sym}^{a_{\tau,n}}(\mathbb{C}^2) \otimes \det{}^{b_{\tau,n}}(\mathbb{C}^2))$$

for $a_{\tau,n} \in \mathbb{Z}_{\geq 0}$ and $b_{\tau,n} \in \mathbb{Z}$. Then $\xi_n \to \infty$ as $n \to \infty$ if and only if $\sum_\tau a_{\tau,n} \to +\infty$. For instance, $\xi_n \to \infty$ if $a_{\tau_0,n} \to +\infty$ for some $\tau_0 : F \hookrightarrow \mathbb{R}$ and $a_{\tau,n}$ remain fixed for all the other $\tau \neq \tau_0$ as n varies.

3.3. Multiplicities and measures. Let $\phi^{S,\infty} \in C_c^\infty(G(\mathbb{A}_F^{S,\infty}))$.

Definition 3.7: The **cuspidal** $(\phi^{S,\infty},\xi)$**-multiplicity** of π_S^0 is defined as the following complex number:

$$m_{\mathrm{cusp}}(\pi_S^0;\phi^{S,\infty},\xi) := \sum_\pi m_{\mathrm{cusp}}(\pi)\cdot \mathrm{tr}\,\pi^{S,\infty}(\phi^{S,\infty})\cdot \mathrm{tr}\,\pi_\infty(\phi_\xi), \tag{3.2}$$

where π runs over isomorphism classes of irreducible admissible representations of $G(\mathbb{A}_F)$ such that $\pi_S \simeq \pi_S^0$ and π_∞ is ξ-cohomological (§2.4). Note that for each $\phi^{S,\infty}$, there are only finitely many nonzero terms in the sum. Define $m_{\mathrm{disc}}(\pi_S^0;\phi^{S,\infty},\xi)$ similarly by replacing m_{cusp} with m_{disc} in (3.2).

Example 3.8: Suppose that U is an open compact subgroup of $G(\mathbb{A}_F^{S,\infty})$. Then

$$m_{\mathrm{cusp}}(\pi_S^0;\mathrm{char}_U,\xi) := \sum_\pi m_{\mathrm{cusp}}(\pi)\cdot\mu^{S,\infty}(U)\cdot\dim(\pi^{S,\infty})^U\cdot\mathrm{tr}\,\pi_\infty(\phi_\xi), \tag{3.3}$$

where the sum runs over the same set of π as in Definition 3.7. If moreover ξ has regular highest weight, then in light of Lemma 2.7,

$$\begin{aligned} & m_{\mathrm{cusp}}(\pi_S^0; \mathrm{char}_U, \xi) \\ (3.4) \quad & = (-1)^{q(G_\infty)} \cdot \mu^{S,\infty}(U) \cdot \sum_{\pi} \left(m_{\mathrm{cusp}}(\pi) \cdot \dim (\pi^{S,\infty})^U \right) \\ & = (-1)^{q(G_\infty)} \cdot \mu^{S,\infty}(U) \cdot \sum_{\pi_\infty} m(\pi_S^0 \otimes \pi_\infty | L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}_F)/U, \chi_\xi^{-1}). \end{aligned}$$

The first sum is taken over π such that $\pi_S \simeq \pi_S^0$, $(\pi^{S,\infty})^U \neq (0)$ and $\pi_\infty \in \Pi_{\mathrm{disc}}(G_\infty, \xi^\vee)$. The second sum is taken over $\Pi_{\mathrm{disc}}(G_\infty, \xi^\vee)$.

Define a positive Borel measure $\widehat{\mu}^{\mathrm{cusp}}_{\phi^{S,\infty},\xi}$ on $\widehat{G(F_S)}$ by

$$(3.5) \quad \widehat{\mu}^{\mathrm{cusp}}_{\phi^{S,\infty},\xi} := \frac{1}{\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)) \cdot \dim \xi} \sum_{\pi_S^0} m_{\mathrm{cusp}}(\pi_S^0; \phi^{S,\infty}, \xi) \cdot \delta_{\pi_S^0}.$$

Note that this is a countable sum as there are countably many cuspidal automorphic representations in $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}), \chi_\xi^{-1})$. The analogue $\widehat{\mu}^{\mathrm{disc}}_{\phi^{S,\infty},\xi}$ is defined in an obvious manner.

Remark 3.9: The value $\dim \xi$ should be viewed as the total Plancherel mass of the real L-packet $\Pi_{\mathrm{disc}}(G_\infty, \xi^\vee)$. Note that the Plancherel measure $\widehat{\mu}^{\mathrm{pl}}_\infty$ is chosen compatibly with μ^{EP}_∞. Then each discrete series in $\Pi_{\mathrm{disc}}(G_\infty, \xi^\vee)$ has Plancherel mass $\dim \xi / d(G_\infty)$ in the unitary dual $\Pi_u(G_\infty, \chi_\xi^{-1})$. Since $|\Pi_{\mathrm{disc}}(G_\infty, \xi^\vee)| = d(G_\infty)$, the total mass is $\dim \xi$.

Lemma 3.10: *If the highest weight of ξ is regular, then $m_{\mathrm{cusp}}(\pi_S^0; \phi^{S,\infty}, \xi) = m_{\mathrm{disc}}(\pi_S^0; \phi^{S,\infty}, \xi)$ and $\widehat{\mu}^{\mathrm{cusp}}_{\phi^{S,\infty},\xi} = \widehat{\mu}^{\mathrm{disc}}_{\phi^{S,\infty},\xi}$.*

Proof. Let π be an automorphic representation of $G(\mathbb{A}_F)$. Under the assumption, $\mathrm{tr}\, \pi_\infty(\phi_\xi) \neq 0$ implies that π_∞ is a discrete series. Then $m_{\mathrm{cusp}}(\pi) = m_{\mathrm{disc}}(\pi)$ by [Wal84, Thm. 4.3]. The lemma follows immediately. ∎

4. Main results

We aim to prove that the averaged "counting measure" of (3.5) tends to the Plancherel measure in two different settings: (1) along a tower of functions while weight is fixed; (2) as the weight parameter tends to infinity while the level is fixed. With the exception of §4.3, we assume the following.

- F is totally real,
- $G(F_\infty)$ has a discrete series, and
- $A_G \times_{\mathbb{Q}} \mathbb{R} = A_{G,\mathbb{R}}$. (See the notation of §2.4.)

Remark 4.1: The last condition is trivially satisfied if G is a semisimple group, and imposed in order to make use of Euler–Poincaré functions at infinity. If G satisfies only the first two conditions, one should still be able to obtain results by using the trace formula with fixed central character.

4.1. Use of the trace formula.

Proposition 4.2: *For any $\phi^{S,\infty} \in C_c^\infty(G(\mathbb{A}_F^{S,\infty}))$ and $\phi_S \in C_c^\infty(G(F_S))$,*

$$\widehat{\mu}^{\mathrm{disc}}_{\phi^{S,\infty},\xi}(\widehat{\phi}_S) = \frac{I_{\mathrm{spec}}(\phi^{S,\infty}\phi_S\phi_\xi)}{\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)) \cdot \dim \xi}.$$

Proof. We will use results of [Art89] which simplifies the trace formula when the test function at infinity is ϕ_ξ. Note that our function ϕ_ξ coincides with his f_ξ. The argument of pp. 267–268 of that paper shows that

$$I_{\mathrm{spec}}(\phi^{S,\infty}\phi_S\phi_\xi) = \sum_\pi m_{\mathrm{disc}}(\pi) \cdot \mathrm{tr}\, \pi(\phi^{S,\infty}\phi_S\phi_\xi),$$

where π runs over automorphic representations of $G(\mathbb{A}_F)$ (up to isomorphism) such that π_∞ is ξ-cohomological. The right-hand side is none other than $\widehat{\mu}^{\mathrm{disc}}_{\phi^{S,\infty},\xi}(\widehat{\phi}_S)$ in view of (3.2), (3.5) and the identity $\delta_{\pi_S^0}(\widehat{\phi}_S) = \widehat{\phi}_S(\pi_S^0) = \mathrm{tr}\, \pi_S^0(\phi_S)$. ∎

4.2. When weight is fixed and level varies. Let $\{\phi_n^{S,\infty}\}_{n\ge 1}$ be a tower of functions (Definition 3.4). The following lemma will be used in the proof of Theorem 4.4 below.

Lemma 4.3: *Let $\{U_n^{S,\infty}\}_{n\ge1}$ be a sequence in $G(\mathbb{A}_F^{S,\infty})$ such that $U_n^{S,\infty} \to 1$ as $n \to \infty$. Let U_S be an open compact subgroup of $G(F_S)$. Then there exists n_0 such that for every $n \ge n_0$, the following holds: if $x^{-1}\gamma x \in U_n^{S,\infty}U_S K_\infty$ for $x \in G(\mathbb{A}_F)$ and $\gamma \in G(F)$ then γ is unipotent.*

Proof. This is proved by the same argument as in [Clo86, Lem. 5]. ∎

Theorem 4.4: *For every $\widehat{f}_S \in \mathscr{F}(\widehat{G(F_S)})$ (cf. Definition 2.1),*

$$\lim_{n\to\infty} \widehat{\mu}^{\mathrm{cusp}}_{\phi_n^{S,\infty},\xi}(\widehat{f}_S) = \lim_{n\to\infty} \widehat{\mu}^{\mathrm{disc}}_{\phi_n^{S,\infty},\xi}(\widehat{f}_S) = \widehat{\mu}^{\mathrm{pl}}_S(\widehat{f}_S). \tag{4.1}$$

Proof. It suffices to show (4.1) for $\widehat{f}_S = \widehat{\phi}_S$ for $\phi_S \in C_c^\infty(G(F_S))$. Then the general case follows easily from this case and Proposition 2.5, by exactly the same argument as in the proof of [Sau97, Prop. 1.3].

To justify the first equality, we must show that $\widehat{\mu}^{\mathrm{res}}_{\phi_n^{S,\infty},\xi}(\widehat{\phi}_S)$ tends to zero as $n \to \infty$, where $\widehat{\mu}^{\mathrm{res}}_{\phi_n^{S,\infty},\xi}$ is defined as in (3.5) with the residual multiplicity $m_{\mathrm{disc}} - m_{\mathrm{cusp}}$ in place of m_{disc}. By the definition of a tower of functions, we can find a sequence $\{U_n\}$ and a constant $\alpha > 0$ such that $|\phi_n^{S,\infty}| \le \alpha \cdot \mathrm{char}_{U_n}$ for every $n \ge 1$. Thus the proof is easily reduced to the case where $\phi_n^{S,\infty} = \mathrm{char}_{U_n}$ for all $n \ge 1$ and $U_n \to 1$ as $n \to \infty$. Now the desired equality

$$\lim_{n\to\infty} \widehat{\mu}^{\mathrm{res}}_{\mathrm{char}_{U_n},\xi}(\widehat{\phi}_S) = 0$$

follows from Lemma 2.3 and Corollary 3.6 of [RS87] (interpreted in the adelic setting).

The rest of the proof is devoted to justifying the second equality. By [Art89, (3.5), Thm. 6.1], we know that $I_{\mathrm{spec}}(\phi_n^{S,\infty}\phi_S\phi_\xi)$ equals the following geometric expansion, where we borrow his notation.

$$(4.2)\qquad I_{\mathrm{geom}}(\phi_n^{S,\infty}\phi_S\phi_\xi) = \sum_{M\in\mathscr{L}} (-1)^{q(G_\infty)+\dim(A_M/A_G)} \frac{|W_M|}{|W_G|} \times \sum_{\gamma\in M(F)/\sim} \chi(M_\gamma)|\iota^M(\gamma)|^{-1}\Phi_M(\gamma,\xi)\cdot\phi^\infty_{n,M}(\gamma).$$

The formula should be explained. The set $\mathscr{L}$ is the set of F-rational cuspidal Levi subgroups of G containing a fixed minimal Levi subgroup, allowing $G \in \mathscr{L}$. The second sum runs over a set of representatives for semisimple conjugacy classes in $M(F)$. (The fact that only cuspidal Levi subgroups contribute in Arthur's formula is noted in [GKM97, p. 539]. For our purpose, we need not recall the definition of cuspidal Levi subgroups.) Recall the formula for $\phi^\infty_{n,M}$ (we apologize for the abuse of subscript here), which denotes the orbital integral of the constant term of ϕ_n^∞, from (6.2) of Arthur's paper:

$$(4.3)\quad \phi^\infty_{n,M}(\gamma) = \delta_P(\gamma)^{1/2}\int_{K^\infty}\int_{N_P(\mathbb{A}^\infty)}\int_{M_\gamma(\mathbb{A}_F^\infty)\backslash M(\mathbb{A}_F^\infty)} \phi_n^\infty(k^{-1}m^{-1}\gamma mnk)\cdot dm\,dn\,dk.$$

With the analogous definition of $\phi_{S,M}(\gamma)$ and $\phi^{S,\infty}_{n,M}(\gamma)$ (by integrating at S and outside S), we have $\phi^\infty_M(\gamma) = \phi^{S,\infty}_{n,M}(\gamma)\cdot\phi_{S,M}(\gamma)$. We denote by M_γ the neutral component of the centralizer of γ in M, and by $|\iota^M(\gamma)|$ the number of connected components in M_γ containing an F-point. As in (6.3) of [Art89], $\chi(M_\gamma)$ is given by

(4.4)

$$\chi(M_\gamma) = (-1)^{q(M_\gamma)}\mathrm{vol}(M_\gamma(F)A_{M_\gamma,\infty}\backslash M_\gamma(\mathbb{A}_F))\mathrm{vol}(A_{M_\gamma,\infty}\backslash \overline{M}_\gamma(F_\infty))^{-1}\cdot d(M_\gamma)$$

where $\overline{M}_\gamma$ is an inner form of M_γ over F_∞ which is compact modulo center. To compute the volume, $M_\gamma(F_\infty)$ and $\overline{M}_\gamma(F_\infty)$ are given compatible measures in the sense of [Kot88, p. 631], and the measure on $M_\gamma(\mathbb{A}_F^\infty)$ is chosen compatibly with the measure on $M(\mathbb{A}_F^\infty)$ and dm in (4.3) (in the usual sense). The quotient measures are taken with respect to the measures on $M_\gamma(F)A_{M_\gamma,\infty}$ and $A_{M_\gamma,\infty}$ as in §2.2.

We claim that there exists $n_0 \in \mathbb{Z}_{>0}$ such that for every $n \geq n_0$ and every $M \in \mathscr{L}$, the summand in the second summation of (4.2) vanishes unless $\gamma = 1$. Let us prove the claim. Note that the summand vanishes if γ is not elliptic in $M(F_\infty)$ (in particular, this is the case if $M(F)$ has no elements which are elliptic in $M(F_\infty)$). Now choose n_0 as in Lemma 4.3 for $U_n^{S,\infty} = \mathrm{supp}\,(\phi^{S,\infty})$ and $U_S = \mathrm{supp}\,(\phi_S)$. Suppose $n \geq n_0$. If the second summand of (4.2) is nonzero, then $k^{-1}m^{-1}\gamma mnk$ belongs to $\mathrm{supp}\,(\phi^{S,\infty}) \times \mathrm{supp}\,(\phi_S)$ and γ is elliptic in $M(F_\infty)$ (so it is also elliptic in $G(F_\infty)$, thus $\gamma \in K_\infty$). By the very choice of n_0, we must have that $m^{-1}\gamma mn$ is unipotent. Hence $\gamma = 1$ and the claim is justified.

Only $\phi_{n,M}^\infty(\gamma)$ varies in the summand of (4.2) as n tends to ∞. When $\gamma = 1$,

$$\phi_{n,M}^{S,\infty}(1) = \int_{K^{S,\infty}} \int_{N_P(\mathbb{A}_F^{S,\infty})} \phi_n^{S,\infty}(k^{-1}nk)\cdot dndk.$$

Let V_m and i_m ($m \geq 1$) be as in (3.1) by taking $U_n := \mathrm{supp}\,(\phi_n^{S,\infty})$. Of course we can choose V_m to be normal subgroups of $K^{S,\infty}$. For each $m \geq 1$, whenever $n \geq i_m$ the function $\phi_n^{S,\infty}$ is supported on V_m in variable $n \in N_P(\mathbb{A}_F^{S,\infty})$. Thus

$$|\phi_{n,M}^{S,\infty}(1)| \leq C \cdot \mu^{S,\infty}(K^{S,\infty}) \cdot \mu_{N_P(\mathbb{A}_F^{S,\infty})}(V_m \cap N_P(\mathbb{A}_F^{S,\infty}))$$

for every $n \geq i_m$. Since $\{V_m\}_{m\geq 1}$ is decreasing to $\{1\}$, it follows that

$$\lim_{n\to\infty} \phi_{n,M}^{S,\infty}(1) = 0$$

unless $M = G$, in which case N_P is trivial (and $\mu_{N_P(\mathbb{A}_F^{S,\infty})}(V_m \cap N_P(\mathbb{A}_F^{S,\infty})) = 1$ as the measure of a point).

Finally, consider the term in (4.2) for $M = G$ and $\gamma = 1$. It is easy to see that $|\iota(G)| = 1$ and $\Phi_G(1,\xi) = \dim \xi$, the latter following from Lemma 2.7(i).

For $M_\gamma = G$, (4.4) can be rewritten as

$$\chi(G) = \overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))$$

by noting the following two facts: (i) $(-1)^{q(G)}d(G_\infty)$ times the Euler–Poincaré measure on $\overline{G}(F_\infty)/A_{G,\infty}$ is compatible with $\overline{\mu}^{\mathrm{EP}}_\infty$ ([Kot88, Thm. 1]) on $G(F_\infty)/A_{G,\infty}$ and (ii) the compact group $\overline{G}(F_\infty)/A_{G,\infty}$ has volume 1 under the Euler–Poincaré measure. Since $\phi^\infty(1) = \phi^{S,\infty}(1)\phi_S(1) = \widehat{\mu}^{\mathrm{pl}}_S(\widehat{\phi}_S)$ by the Plancherel formula (Proposition 2.4), the term for $M = G$ and $\gamma = 1$ in (4.2) is computed as

$$\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)) \cdot \dim \xi \cdot \widehat{\mu}^{\mathrm{pl}}_S(\widehat{\phi}_S).$$

Hence the right-hand side of (4.2) tends to the above value as $n \to \infty$. We complete the proof of Theorem 4.4 by invoking Proposition 4.2. ∎

Let $\widehat{U}$ be a $\widehat{\mu}^{\mathrm{pl}}_S$-regular relatively quasi-compact subset of $\widehat{G(F_S)}$. As remarked in Example 2.3, the function $\mathrm{char}_{\widehat{U}}$ belongs to $\mathscr{F}(\widehat{G(F_S)})$. By taking $\widehat{f}_S = \mathrm{char}_{\widehat{U}}$ in Theorem 4.4 we deduce

Corollary 4.5: *For any $\widehat{U}$ as above,*

$$\lim_{n\to\infty} \widehat{\mu}^{\mathrm{cusp}}_{\phi^{S,\infty}_n,\xi}(\mathrm{char}_{\widehat{U}}) = \lim_{n\to\infty} \widehat{\mu}^{\mathrm{disc}}_{\phi^{S,\infty}_n,\xi}(\mathrm{char}_{\widehat{U}}) = \widehat{\mu}^{\mathrm{pl}}_S(\widehat{U}).$$

Remark 4.6: It is indeed necessary to assume that $\widehat{U}$ is $\widehat{\mu}^{\mathrm{pl}}_S$-regular. For instance, let $\widehat{V}$ be an open relatively quasi-compact subset of $\widehat{G(F_S)}$ such that $\widehat{\mu}^{\mathrm{pl}}_S(\widehat{V}) > 0$. Let $\widehat{U}$ be the subset of $\widehat{V}$ consisting of those π_S which do not arise as the S-components of ξ-cohomological cuspidal automorphic representations of $G(\mathbb{A}_F)$. Then $\widehat{U}$ is a complement of a countable subset in $\widehat{V}$, and it might seem that Corollary 4.5 does not hold for this $\widehat{U}$ as the left-hand side should be zero whereas the right-hand side should be positive. In fact, such a $\widehat{U}$ is not $\widehat{\mu}^{\mathrm{pl}}_S$-regular and excluded in the corollary.

4.3. When $G(F_\infty)$ has no discrete series or F is not totally real.

Suppose that either F is not totally real or $G(F_\infty)$ has no discrete series. We can still obtain analogous results by appealing to the simple trace formula if we impose an extra condition at two auxiliary finite places. Although it would be possible to remove this unnatural restriction, we have not done so.

Choose auxiliary finite places $v_1, v_2 \in \mathcal{V}^\infty_F$ and a finite subset $S \subset \mathcal{V}_F$ such that $v_1, v_2 \notin S$ and S contains all infinite places of F. (This is a change from the previous subsection, where S contained no infinite places.) By abuse of notation, $\widehat{G(F_S)}$ will mean the unitary dual of $G(F_S)/A_{G,\infty}$ (viewed as the subset of the usual unitary dual of $G(F_S)$). The space of functions $\mathscr{F}(\widehat{G(F_S)})$, which was defined in §2.3 when S consists of finite places, makes sense in this context and the analogue of Sauvageot's density theorem (Proposition 2.5) still holds. See page 180 and Theorem 7.3 of [Sau97].

Let ϕ_{v_1} be a truncated Kottwitz function in the sense of [Lab99, 3.9] (cf. [Kot88, §2]). It is known ([Lab99, Prop. 3.9.1]) that the orbital integral of ϕ_{v_1} vanishes on non-elliptic or non-semisimple elements of $G(F_{v_1})$. Let ϕ_{v_2} be any function such that $\widehat{\phi}_{v_2}$ is positive on one supercuspidal Bernstein component and zero outside it. In particular, $\operatorname{tr}\pi_{v_2}(\phi_{v_2}) = 0$ whenever π_{v_2} is a subquotient of a parabolically induced representation of $G(F_{v_2})$. (Such a ϕ_{v_2} is easily constructed by the trace Paley–Wiener theorem of [BDK86].) By scaling, we require that

$$\phi_{v_1}(1) = \phi_{v_2}(1) = 1. \tag{4.5}$$

Set $\phi_{v_1,v_2} := \phi_{v_1}\phi_{v_2}$. As an analogue of Definition 3.7, we have

Definition 4.7: The cuspidal $(\phi_n^{S,v_1,v_2}, \phi_{v_1,v_2})$-multiplicity of π_S^0 is defined as the following complex number:

(4.6)

$$m_{\mathrm{cusp}}(\pi_S^0; \phi_n^{S,v_1,v_2}, \phi_{v_1,v_2}) := \sum_{\pi} m_{\mathrm{cusp}}(\pi) \cdot \operatorname{tr}\pi^{S,v_1,v_2}(\phi_n^{S,v_1,v_2}) \cdot \operatorname{tr}\pi_{v_1,v_2}(\phi_{v_1,v_2}),$$

where π runs over isomorphism classes of irreducible representations of $G(\mathbb{A}_F)$ in $L^2_{\mathrm{cusp}}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))$ such that $\pi_S \simeq \pi_S^0$. The sum has finitely many nonzero terms.

Define a positive Borel measure $\widehat{\mu}_n = \widehat{\mu}_{\phi_n^{S,v_1,v_2},\phi_{v_1,v_2}}$ on $\widehat{G(F_S)}$ by

$$\widehat{\mu}_n := \frac{1}{\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))} \sum_{\pi_S^0} m_{\mathrm{cusp}}(\pi_S^0; \phi_n^{S,v_1,v_2}, \phi_{v_1,v_2}) \cdot \delta_{\pi_S^0}. \tag{4.7}$$

Theorem 4.8: *For any $\widehat{f}_S \in \mathscr{F}(\widehat{G(F_S)})$,*

$$\lim_{n\to\infty} \widehat{\mu}_n(\widehat{f}_S) = \widehat{\mu}_S^{\mathrm{pl}}(\widehat{f}_S).$$

Proof. Since the outline of the proof is identical to that of Theorem 4.4, we only sketch the argument.

As before, Sauvageot's density theorem reduces the proof to the case $\widehat{f}_S = \widehat{\phi}_S$ for some $\phi_S \in C_c^\infty(G(F_S))$. In view of the properties of ϕ_{v_1} and ϕ_{v_2} stated above, Delgine–Kazhdan's simple trace formula (cf. [Art88, Cor. 7.3, Cor. 7.4]) is applicable to G and $\phi_n := \phi_n^{S,v_1,v_2}\phi_{v_1,v_2}\phi_S$. Now that the simple trace formula replaces a version of Arthur's trace formula in the proof of Theorem 4.4, the argument in the current case is only easier. It is straightforward to check that

$$\widehat{\mu}_n(\widehat{\phi}_S) = \frac{I_{\mathrm{spec}}(\phi_n)}{\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))}. \tag{4.8}$$

(By our choice of ϕ_{v_2}, only cuspidal automorphic representations contribute to $I_{\mathrm{spec}}(\phi_n)$.) The geometric side of the simple trace formula is given by

$$I_{\mathrm{geom}}(\phi_n) = \sum_{\gamma \in G(F)/\sim} \mathrm{vol}(G_\gamma(F)A_{G,\infty}\backslash G_\gamma(\mathbb{A}_F)) \cdot O_\gamma(\phi_n),$$

where the sum runs over a set of representatives for elliptic conjugacy classes in $G(F)$, and G_γ denotes the centralizer of γ in G. As in the proof of Theorem 4.4, we can prove that only the summand for $\gamma = 1$ survives as $n \to \infty$. Using (4.5) and the Plancherel formula (Proposition 2.4), we obtain

$$\begin{aligned} \lim_{n\to\infty} I_{\mathrm{geom}}(\phi_n) &= \overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)) \cdot \phi_n(1) \\ &= \overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)) \cdot \widehat{\mu}_S^{\mathrm{pl}}(\widehat{\phi}_S). \end{aligned} \tag{4.9}$$

Formulas (4.8) and (4.9) complete the proof. ∎

4.4. When weight varies and level is fixed. Only in §4.4 we assume that the center of G is trivial. (See Remark 4.13 about this assumption.) In particular, we need not worry about $A_{G,\infty}$, which is also trivial. We retain the notation from §3.2.

Lemma 4.9: *Let M be an $\mathbb{R}$-rational proper Levi subgroup of G_∞. Suppose that $\xi_n \to \infty$ as $n \to \infty$. Then:*

(i) *For any elliptic $\gamma \in G_\infty(\mathbb{R})$ such that $\gamma \notin Z(G_\infty)(\mathbb{R})$,*

$$\lim_{n\to\infty} \frac{|\mathrm{tr}\,\xi_n(\gamma)|}{\dim \xi_n} = 0.$$

(ii) *For any elliptic $\gamma \in M(\mathbb{R})$,*

$$\lim_{n\to\infty} \frac{|\Phi_M^G(\gamma, \xi_n)|}{\dim \xi_n} = 0.$$

Remark 4.10: It will be evident from the proof that the lemma is true without assuming $Z(G) = \{1\}$, which is only needed in the proof of Theorem 4.11.

Proof. We may assume $F_\infty = \mathbb{R}$ as the general case can be handled in the same way. The basic idea of the proof is to study the asymptotic behavior of the Weyl character formula. Part (i) is proved as Corollary 1.12 in [CC09]. (Even though that corollary is proved under the assumption that $G_\infty(\mathbb{R})$ is compact, the general case is easily deduced via an inner form of $G_\infty(\mathbb{R})$ which is compact mod center. This is possible because an elliptic conjugacy class always transfers to a compact mod center inner form.)

Let us deduce part (ii) essentially from Lemma 2.8. We will freely use the notation of that lemma in the rest of the proof. Let λ_B^n denote the B-dominant highest weight for ξ_n. We divide into two cases depending on γ.

The first case is when $\gamma \notin Z(G_\infty)(\mathbb{R})$. In view of (2.2), it is enough to prove that

$$\left|\mathrm{tr}\,(\gamma^{-1}|V^L_{w(\lambda_B^n+\rho_B)-\rho_B})\right| \cdot (\dim \xi_n)^{-1} \to 0 \qquad \text{as } n \to \infty.$$

The left-hand side can be decomposed as

$$\frac{\left|\mathrm{tr}\,(\gamma^{-1}|V^L_{w(\lambda_B^n+\rho_B)-\rho_B})\right|}{\dim V^L_{w(\lambda_B^n+\rho_B)-\rho_B}} \times \frac{\dim V^L_{w(\lambda_B^n+\rho_B)-\rho_B}}{\dim \xi_n}. \tag{4.10}$$

Recall the Weyl dimension formula

$$\dim \xi_n = \prod_\alpha \frac{\langle \alpha, \lambda_B^n + \rho_B\rangle}{\langle \alpha, \rho_B\rangle},$$

where α runs over the B-positive roots in G. Since a similar formula holds for $\dim V^L_{w(\lambda_B^n+\rho_B)-\rho_B}$, we see from (3.1) that the second term in (4.10) tends to zero as $n \to \infty$ unless $L = G$. In case $L = G$, the same term is equal to 1 for any n since $\xi_n = V^L_{w(\lambda_B^n+\rho_B)-\rho_B}$. Now consider the first term in (4.10). If γ is in the center of $L(\mathbb{R})$, it is clearly a (nonzero) constant. If else, the first term tends to zero as $n \to \infty$ by part (i). Finally, we deduce that the limit of (3.1) is zero.

Next consider the case $\gamma \in Z(G_\infty)(\mathbb{R})$. If $L \neq G$, we argue exactly as in the other case. When $L = G$, we simply note that the summand in (2.2) vanishes since $D^G_M(\gamma) = 0$ if $M \subsetneq G$ and γ is central in G. The proof is complete. ∎

Theorem 4.11: *Let $\phi^{S,\infty} \in C_c^\infty(G(\mathbb{A}_F^{S,\infty}))$ be such that $\phi^{S,\infty}(1) = 1$. For any $\widehat{f}_S \in \mathscr{F}(\widehat{G(F_S)})$,*

$$\lim_{n\to\infty} \widehat{\mu}^{\mathrm{cusp}}_{\phi^{S,\infty},\xi_n}(\widehat{f}_S) = \lim_{n\to\infty} \widehat{\mu}^{\mathrm{disc}}_{\phi^{S,\infty},\xi_n}(\widehat{f}_S) = \widehat{\mu}^{\mathrm{pl}}_S(\widehat{f}_S).$$

Proof. As in the proof of Theorem 4.4, it is enough to prove the theorem for $\widehat{f}_S$ which has the form $\widehat{\phi}_S$ for some $\phi_S \in C_c^\infty(G(F_S))$. Since the first equality of the theorem is an obvious consequence of Lemma 3.10, it suffices to establish the second equality. It follows from Proposition 4.2 and (4.2) that

(4.11)

$$\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)) \cdot \widehat{\mu}^{\mathrm{disc}}_{\phi^{S,\infty},\xi_n}(\widehat{\phi}_S) = \sum_{M\in\mathscr{L}} (-1)^{\dim(A_M/A_G)} \frac{|W_M|}{|W_G|}\left(\sum_{\gamma\in M(F)/\sim} \chi(M_\gamma)|\iota^M(\gamma)|^{-1}\phi_M^\infty(\gamma)\frac{\Phi_M(\gamma,\xi_n)}{\dim\xi_n}\right)$$

with the same notation as previously in (4.2). We claim that the second sum over conjugacy classes in $M(F)$ can be taken over a finite set Y_M which is independent of n. To see this, note that $\Phi_M^G(\gamma,\xi_n)$ is nonzero only if γ is elliptic in $M(F_\infty)$, or equivalently $M(F_\infty)$-conjugate to an element in $M(F_\infty)\cap K_\infty$. In order that $\phi_M^\infty(\gamma)\neq 0$, γ must be $M(\mathbb{A}_F^\infty)$-conjugate to an element in $\operatorname{supp}\phi_M^\infty$. So the summand for γ is zero unless γ is $M(\mathbb{A}_F)$-conjugate to some element in the compact set $(\operatorname{supp}\phi_M^\infty)\times(M(F_\infty)\cap K_\infty)$. The last condition on γ is clearly independent of n and satisfied by only finitely many semisimple conjugacy classes of $M(F)$ by [Kot86, Prop. 8.2].

We will write

$$I_{\mathrm{geom}}(\phi^{S,\infty}\phi_S\phi_{\xi_n}) = I_{1,n} + I_{2,n} + I_{3,n} \tag{4.12}$$

where $I_{1,n}$, $I_{2,n}$ and $I_{3,n}$ are partial sums in (4.11) defined as:

- $I_{1,n}$ is the term for $M=G$ and $\gamma=1$,
- $I_{2,n}$ is the sum over $M=G$ and all $\gamma\neq 1$,
- $I_{3,n}$ is the sum over all $M\subsetneq G$ and all γ.

It is easy to compute that (recall Lemma 2.8.(i) and the assumption $\phi^{S,\infty}(1)=1$)

$$I_{1,n} = \overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))\cdot\phi_S(1). \tag{4.13}$$

We have the following expressions for $I_{2,n}$ and $I_{3,n}$, where $a(\gamma), a_M(\gamma)\in\mathbb{C}$ are constants independent of n:

$$I_{2,n} = \sum_{\gamma\neq 1} a(\gamma)\cdot \operatorname{tr}\xi_n(\gamma)/\dim\xi_n,$$

$$I_{3,n} = \sum_M\sum_\gamma a_M(\gamma)\cdot\Phi_M^G(\gamma,\xi_n)/\dim\xi_n.$$

In the expression for $I_{2,n}$ (resp. $I_{3,n}$), γ runs over the fixed finite set $Y_G\backslash\{1\}$ (resp. Y_M). The assumption in the beginning of §4.4 ensures that no $\gamma \in Y_G\backslash\{1\}$ is contained in the center of G. By Lemmas 2.8 and 4.9,

$$\lim_{n\to\infty} I_{2,n} = 0, \quad \lim_{n\to\infty} I_{3,n} = 0. \tag{4.14}$$

Formulas (4.11), (4.12), (4.13) and (4.14) imply that

$$\begin{aligned}\lim_{n\to\infty} \widehat{\mu}^{\mathrm{disc}}_{\phi^{S,\infty},\xi_n}(\widehat{\phi}_S) &= \lim_{n\to\infty} \overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))^{-1}\cdot I_{1,n}\\ &= \phi_S(1) \;=\; \widehat{\mu}^{\mathrm{pl}}_S(\widehat{\phi}_S). \qquad \blacksquare\end{aligned}$$

Corollary 4.12: *Let $\widehat{U}$ be a $\widehat{\mu}^{\mathrm{pl}}_S$-regular relatively quasi-compact subset of $\widehat{G(F_S)}$. Then*

$$\lim_{n\to\infty} \widehat{\mu}^{\mathrm{cusp}}_{\phi^{S,\infty},\xi_n}(\mathrm{char}_{\widehat{U}}) = \lim_{n\to\infty} \widehat{\mu}^{\mathrm{disc}}_{\phi^{S,\infty},\xi_n}(\mathrm{char}_{\widehat{U}}) = \widehat{\mu}^{\mathrm{pl}}_S(\widehat{U}).$$

Proof. Take $\widehat{f}_S = \mathrm{char}_{\widehat{U}}$ in Theorem 4.11. $\blacksquare$

We conclude this subsection with two remarks.

Remark 4.13: We need to say a word about the condition that $Z(G) = \{1\}$. The triviality of the center was imposed to ensure that the limit of $I_{2,n}$ vanishes in the course of proving Theorem 4.11 by appealing to Lemma 4.9(i). If $Z(G) \neq \{1\}$, the summands for other central elements may not die as n tends to infinity. (If $Z(G)(\mathbb{Q})$ is finite and its projection into $G(\mathbb{A}^{S,\infty})$ meets $\mathrm{supp}\,\phi^{S,\infty}$ in $\{1\}$, then this problem does not occur.) In order to avoid this issue in the general case, it seems best to fix a central character for automorphic representations. Then we expect that the analogue of Theorem 4.11 is true but have not attempted to prove it.

Remark 4.14: When $G(F_\infty)$ has no discrete series, it would be an interesting problem to prove an analogue of Theorem 4.11. For this, one may formulate the problem in terms of a sequence $\{\widehat{U}_{\infty,n}\}_{n\geq 1}$ in $\widehat{G(F_\infty)}$, as a substitute for $\{\xi_n\}_{\geq 1}$, such that (i) $\widehat{\mu}^{\mathrm{pl}}_\infty(\widehat{U}_{\infty,n}) > 0$ for every n, (ii) each $\widehat{U}_{\infty,n}$ is bounded and relatively compact, and (iii) the infinitesimal characters for $\widehat{U}_{\infty,n}$ in $X^*(T)$ tend to infinity in a uniform manner.

5. Limit multiplicities and existence of automorphic representations

This section is devoted to a few applications of our main results in the last section. In §5.1, we provide a limit multiplicity formula for a reasonable subset $\widehat{U}$ of $\widehat{G(F_S)}$ in the ξ-cohomological discrete (or cuspidal) spectrum for $G(\mathbb{A}_F)$ in terms of the Plancherel mass of $\widehat{U}$. In §§5.2–5.3, an existence theorem for cuspidal automorphic representations with prescribed local properties will be presented. An immediate corollary, to be given in §5.4, is a result of Burger–Li–Sarnak and Clozel–Ullmo that the automorphic points are dense in the local tempered spectrum.

5.1. Limit multiplicity formula via Plancherel measure. We assume the three conditions at the start of §4. Let $\widehat{U}$ be a $\widehat{\mu}^{\mathrm{pl}}_S$-regular relatively quasi-compact subset of $\widehat{G(F_S)}$.

Definition 5.1: The $(\widehat{U},\xi)$-limit multiplicity[4] in $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}_F),\chi_\xi^{-1})$ is defined as

$$\begin{aligned} m_{\mathrm{cusp,lim}}(\widehat{U},\xi) &:= \dim\xi\cdot\lim_{n\to\infty}\widehat{\mu}^{\mathrm{cusp}}_{\mathrm{char}_{U_n},\xi}(\widehat{U})\\ &= \lim_{n\to\infty}\left(\frac{\sum_{\pi_S\in\widehat{U}} m_{\mathrm{cusp}}(\pi_S;\mathrm{char}_{U_n},\xi)}{\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))}\right) \end{aligned}\tag{5.1}$$

for any decreasing sequence $\{U_n\}_{n\geq 1}$ in $G(\mathbb{A}_F^{S,\infty})$ such that $U_n\to 1$ as $n\to\infty$ (Definition 3.1), provided that the limit is independent of the choice of $\{U_n\}_{n\geq1}$. (The independence will be shown below.) Note that the sum has only finitely many nonzero terms for each n by Harish-Chandra's finiteness theorem. Similarly define $m_{\mathrm{disc,lim}}(\widehat{U},\xi)$.

Remark 5.2: If ξ has regular highest weight, then (5.1) may be rewritten as follows in light of (3.4):

$$\begin{aligned} &m_{\mathrm{cusp,lim}}(\widehat{U},\xi)\\ &= (-1)^{q(G_\infty)}\cdot\lim_{n\to\infty}\left(\sum_{\substack{\pi_S\in\widehat{U}\\ \pi_\infty}}\frac{m(\pi_S\otimes\pi_\infty|L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}_F)/U_n,\chi_\xi^{-1}))}{\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))\cdot\mu^{S,\infty}(U_n)^{-1}}\right)\end{aligned}$$

[4] It can be seen from the definition that this number depends on the Haar measure at S, or $\widehat{\mu}^{\mathrm{pl}}_S$. In fact the dependence is obvious in Proposition 5.3.

where π_∞ runs over the L-packet $\Pi_{\mathrm{disc}}(G_\infty,\xi^\vee)$. The identity seems to be true without the regularity condition on highest weight. To justify this, we need to show that nontempered ξ-cohomological representations of $G(F_\infty)$ do not contribute to the limit multiplicity formula. This is proved in [Clo86, Lem. 8] (based on the method of DeGeorge and Wallach) but under a mild restriction (2.3 of that paper) on the sequence $\{U_n\}_{n\geq 1}$.

PROPOSITION 5.3: $m_{\mathrm{cusp,lim}}(\widehat{U},\xi) = m_{\mathrm{disc,lim}}(\widehat{U},\xi) = \widehat{\mu}^{\mathrm{pl}}_S(\widehat{U})\cdot \dim\xi$.

Proof. This is immediate from Corollary 4.5. ∎

Remark 5.4: If the Haar measure μ_S on $G(F_S)$ is multiplied by a scalar $c\in\mathbb{C}^\times$ then $m_{\mathrm{cusp,lim}}(\widehat{U},\xi)$ is multiplied by c^{-1}. In the last expression of (5.1), the numerator is unchanged and the denominator is multiplied by c. Since $\widehat{\mu}^{\mathrm{pl}}_S$ is also multiplied by c^{-1}, we see that the proposition is not affected.

Remark 5.5: Proposition 5.3 may be restated as

$$\sum_{\pi^0_S\in\widehat{U}} m_{\mathrm{cusp}}(\pi^0_S;\mathrm{char}_{U_n},\xi)$$
$$= \overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))\cdot \mu^{S,\infty}(U_n)^{-1}\cdot \widehat{\mu}^{\mathrm{pl}}_S(\widehat{U})\cdot\dim\xi \ + \ o(1),$$

where the $o(1)$-term multiplied by $\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)/U_n)$ tends to 0 as $n\to\infty$. The same is true with discrete multiplicity. Note that $\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))\cdot\mu^{S,\infty}(U_n)^{-1}$ computes the volume of the double quotient $G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)/U_n$ if U_n is sufficiently small.

The constant $\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))$ can be made explicit ([Gro97, Thm. 9.9]) if the connected center of G is anisotropic. (For an arbitrary reductive group G, some modification is necessary.) See §6.2 where the case of $G=GL_2$ is studied in detail.

5.2. EXISTENCE THEOREM (I). As in the previous subsection F is a totally real field, $G(F_\infty)$ is assumed to have a discrete series and $A_G\times_{\mathbb{Q}}\mathbb{R}=A_{G,\mathbb{R}}$ throughout §5.2. Let $\widehat{U}$ be a $\widehat{\mu}^{\mathrm{pl}}_S$-regular relatively quasi-compact subset of $\widehat{G(F_S)}$ such that $\widehat{\mu}^{\mathrm{pl}}_S(\widehat{U})>0$.

Example 5.6: Here are two useful examples of $\widehat{U}$. First, we can take $\widehat{U}$ to be the characteristic function on the set of tempered representations $\pi\in\widehat{G(F_S)}$ which belong to a particular Bernstein component (equivalently, a particular inertia equivalence class). For the next example, let M be a Levi subgroup of G over F_S. Let $\mathcal{O}$ be an orbit of discrete series representations of $M(F_S)$ under the twist by characters of $\Psi_u(M)$ (defined in §2.3). By collecting irreducible subquotients of $\text{n-ind}_M^G(\pi_M)$ for all $\pi_M \in \mathcal{O}$, we get another example of $\widehat{U}$.

Theorem 5.7: *Let v be a finite place of F not contained in S. Assume that G is unramified outside $S \cup \{v, \infty\}$. (In other words, at each $w \in \mathcal{V}_F$ not contained in $S \cup \{v, \infty\}$, G is quasi-split over F_w and split over an unramified extension of F_w.) Then there exist infinitely many cuspidal automorphic representations π of $G(\mathbb{A}_F)$ such that*

- *$\pi^{S,v,\infty}$ is unramified,*
- *$\pi_S \in \widehat{U}$, and*
- *π_∞ is ξ-cohomological.*

Proof. If the theorem is false, the S-components of π as above form a finite subset of $\widehat{U}$. By shrinking $\widehat{U}$, we can assume that no π as above exists while retaining the condition $\widehat{\mu}^{\text{pl}}_S(\widehat{U}) > 0$.

Let $U^{S,v,\infty}$ be a hyperspecial maximal compact subgroup of $G(\mathbb{A}_F^{S,v,\infty})$. Consider a sequence $U_n = U_{v,n}U^{S,v,\infty}$ as in Example 3.2. It is easy to see from (3.3) that $m_{\text{cusp}}(\pi_S^0; \text{char}_{U_n}, \xi) = 0$ for every n and every $\pi_S^0 \in \widehat{U}$. This implies $\widehat{\mu}^{\text{cusp}}_{\text{char}_{U_n},\xi}(\widehat{U}) = 0$, which contradicts Corollary 4.5. ∎

Theorem 5.8: *Suppose that G is unramified outside S and ∞ and that $Z(G) = \{1\}$. There exist infinitely many cuspidal automorphic representations π of $G(\mathbb{A}_F)$ such that*

- *$\pi^{S,\infty}$ is unramified,*
- *$\pi_S \in \widehat{U}$, and*
- *π_∞ is a discrete series.*

Proof. Let $U^{S,\infty}$ be a hyperspecial subgroup of $G(\mathbb{A}_F^{S,\infty})$. Let ξ_n be any sequence such that $\xi_n \to \infty$ as $n \to \infty$. Suppose that the theorem is false. As in the proof of Theorem 5.7, we may even assume that there exist no π as in the theorem by shrinking $\widehat{U}$. Then $m_{\text{cusp}}(\pi_S^0; \text{char}_{U^{S,\infty}}, \xi_n) = 0$ for all $n \geq 1$, which implies that $\widehat{\mu}^{\text{cusp}}_{\text{char}_{U^{S,\infty}},\xi_n}(\widehat{U}) = 0$. This contradicts Proposition 4.12. ∎

Corollary 5.9: *Suppose that G is quasi-split over F. Then there exist infinitely many cuspidal automorphic representations of $G(\mathbb{A}_F)$ which are unramified at all finite places and discrete series at infinity.*

Proof. The proof is easily reduced to the case $Z(G) = \{1\}$. Fix a finite place v of F. Take $S = \{v\}$ and $\widehat{U}$ to be the set of unramified tempered representations of $G(F_v)$. It is well-known that $\widehat{\mu}_v^{\mathrm{pl}}(\widehat{U}) > 0$. (One can prove this by applying the Plancherel formula to the characteristic function on a hyperspecial subgroup.) The corollary follows from Theorem 5.8. ∎

Remark 5.10: Of course, Theorem 5.8 implies more than Corollary 5.9: we can even arrange that cuspidal automorphic representations in the corollary have their Satake parameters at finitely many places in a particular region (to be prescribed by a choice of $\widehat{U}$).

Remark 5.11: It would be nice to have an effective lower (resp. upper) bound for the parameter at infinity which ensures the existence (resp. non-existence) of the automorphic representations as in the corollary. Our approach does not seem to offer a clue.

Remark 5.12: It can be asked whether Theorems 5.7 and 5.8 continue to hold if π is also required to be "stable" in the sense that (loosely speaking) π is not in the image of the (conjectural) transfer from representations of any elliptic endoscopic subgroup of G which is not a quasi-split inner form. In fact, one can even ask whether the theorems in §4 hold if we replace the discrete or cuspidal automorphic multiplicity with the corresponding multiplicity in the stable part of the spectrum only.

5.3. Existence theorem (II). Let F be any number field and drop the assumptions in §5.2. We prove a weaker version of Theorem 5.7.

Theorem 5.13: *Let v_1, v_2 be a finite place of F not contained in S. There exist infinitely many cuspidal automorphic representations π of $G(\mathbb{A}_F)$ such that*

- *π^{S,v_1,v_2} is unramified and*
- *$\pi_S \in \widehat{U}$, and*
- *π_{v_1} is square integrable and π_{v_2} is supercuspidal.*

Proof. This is deduced from Theorem 4.8 (with $\widehat{f}_S = \mathrm{char}_{\widehat{U}}$) by the same argument proving Theorem 5.7 from Corollary 4.5. ∎

5.4. Density of automorphic points in a local tempered spectrum. We impose no restriction on F and G, exactly as in §5.3. Let $\widehat{G(F_S)}_{\mathrm{temp}}$ denote the subset of $\widehat{G(F_S)}$ consisting of tempered representations.

Corollary 5.14 (Burger–Li–Sarnak, Clozel–Ullmo, cf. [Sar05, (28)]): *The set of the S-components π_S of cuspidal automorphic representations π meets $\widehat{G(F_S)}_{\mathrm{temp}}$ in a dense subset.*

Proof. For any $\widehat{\mu}^{\mathrm{pl}}_S$-regular relatively quasi-compact open subset $\widehat{U}$ of $\widehat{G(F_S)}$ which intersects $\widehat{G(F_S)}_{\mathrm{temp}}$ nontrivially, we have that $\widehat{U} \cap \widehat{G(F_S)}_{\mathrm{temp}}$ is $\widehat{\mu}^{\mathrm{pl}}_S$-regular relatively quasi-compact and that $\widehat{\mu}^{\mathrm{pl}}_S(\widehat{U} \cap \widehat{G(F_S)}_{\mathrm{temp}}) > 0$. Theorem 5.13 tells us that $\widehat{U} \cap \widehat{G(F_S)}_{\mathrm{temp}}$ contains the S-component of some cuspidal automorphic representation of $G(\mathbb{A}_F)$. ∎

6. Example: Hilbert modular case

This section provides an asymptotic formula for the dimension of the spaces of Hilbert modular forms (Proposition 6.4) with prescribed local conditions.

6.1. Explicit Plancherel measure on GL_2 over a p-adic field. In this subsection we use the following notation.

- K is a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and a uniformizer ϖ_K,
- $q := |\mathcal{O}_K/\varpi_K|$,
- $G = GL_2$ over K,
- $T = GL_1 \times GL_1$ is the subgroup of diagonal matrices,
- $S^1 = \{z \in \mathbb{C} : |z| = 1\}$,
- $d(\pi)$ denotes the formal degree when π is a discrete series of $G(K)$.

We fix a Haar measure μ on $G(K)$ such that $\mu(GL_2(\mathcal{O}_K)) = 1$. The formal degrees of discrete series of $G(K)$ will be computed with respect to μ.

Let π be an irreducible tempered representation of $G(K)$. There exists a pair $(M, \mathcal{O})$ where $M \in \{G, T\}$ and $\mathcal{O}$ is an $\Psi_u(M)$-orbit of (unitary) discrete series representations of $M(K)$ such that $\pi = \text{n-ind}^G_M(\sigma)$ for some $\sigma \in \mathcal{O}$. It is well known that $\text{n-ind}^G_M(\sigma)$ is irreducible whenever σ is a unitary representation. This condition determines $(M, \mathcal{O})$ uniquely. To be more concrete, we divide into four disjoint cases, namely Case (i) through Case (iv) below.

Let $d\omega$ denote the canonical measure of Harish-Chandra on $\mathcal{O}$, whose description on page 31 of [AP05] allows us to compute the volume $\omega(\mathcal{O})$ under this measure. Following [AP05], we give below an explicit description of the Plancherel measure $\widehat{\mu}^{\mathrm{pl}}$ on $\mathcal{O}$ as well as the Plancherel mass of $\mathcal{O}$. This is a slight abuse of notation: the measure $\widehat{\mu}^{\mathrm{pl}}$ here is identified with our previous Plancherel measure via the map $\mathcal{O} \to \widehat{G(K)}$ given by $\sigma \mapsto \mathrm{Ind}_B^G(\sigma)$.

From here on, we freely adopt their notation. The measure $\widehat{\mu}^{\mathrm{pl}}$ on $\mathcal{O}$ is given by the formula

$$\widehat{\mu}^{\mathrm{pl}} = \gamma(G|M) \cdot j(\sigma)^{-1} \cdot d\omega,$$

where σ is any member of $\mathcal{O}$. By [AP05, Thm. 3.1], $\gamma(G|G) = 1$ and $\gamma(G|T) = (q+1)/q$. An explicit formula for $j(\sigma)^{-1}$ of [AP05, Thm. 5.4] leads to formulas for $\widehat{\mu}^{\mathrm{pl}}$ and the Plancherel mass $\widehat{\mu}^{\mathrm{pl}}(\mathcal{O})$, except in Case (ii) below. In Case (ii), use Theorem 4.4, the second formula on page 32 and formula (4) on page 33 from the same paper.

We summarize the computation below.

Case (i) $\pi = \text{n-ind}_M^G(\sigma)$ where $\sigma = \chi \otimes \chi$ up to an unramified twist.

$\mathcal{O} \simeq (S^1 \times S^1)/\mathfrak{S}_2$, $\omega(\mathcal{O}) = 1/2$.

$$\widehat{\mu}^{\mathrm{pl}} = \left|\frac{1 - z_1^{-1}z_2}{1 - z_1^{-1}z_2q^{-1}}\right|^2 d\omega = \left|\frac{1 - z_1^{-1}z_2}{1 - z_1^{-1}z_2q^{-1}}\right|^2 \cdot \frac{1}{2(2\pi i)^2} \cdot \frac{z_1}{dz_1}\frac{z_2}{dz_2}.$$

$\widehat{\mu}^{\mathrm{pl}}(\mathcal{O}) = 1$.

Case (ii) $\pi = \text{n-ind}_M^G(\sigma)$ where $\sigma = \chi_1 \otimes \chi_2$; χ_1 and χ_2 differ by a ramified character.

$\mathcal{O} \simeq S^1 \times S^1$, $\omega(\mathcal{O}) = 1$.

$\widehat{\mu}^{\mathrm{pl}} = q^{f(\chi_1^{-1}\chi_2)} \cdot \frac{q+1}{q} \cdot d\omega$.

$\widehat{\mu}^{\mathrm{pl}}(\mathcal{O}) = q^{f(\chi_1^{-1}\chi_2)} \cdot \frac{q+1}{q}$.

Case (iii) π is a Steinberg representation up to a (unitary) character twist of GL_2.

$\mathcal{O} \simeq S^1$, $\omega(\mathcal{O}) = 2$, $d(\pi) = \frac{q-1}{2}$.

$\widehat{\mu}^{\mathrm{pl}} = \frac{q-1}{2} \cdot d\omega$.

$\widehat{\mu}^{\mathrm{pl}}(\mathcal{O}) = 2d(\pi) = q - 1$.

Case (iv) π is (unitary) supercuspidal.

$\mathcal{O} \simeq S^1$, $\omega(\mathcal{O}) = 2/r(\pi)$.

$\widehat{\mu}^{\mathrm{pl}} = d(\pi) \cdot d\omega$.

$\widehat{\mu}^{\mathrm{pl}}(\mathcal{O}) = 2d(\pi)/r(\pi)$.

6.2. Limit multiplicity formula for Hilbert modular forms. Let $G = GL_2$ and F a totally real field. Then the first two conditions at the start of §4 are satisfied but the last one is true only when $F = \mathbb{Q}$. So, strictly speaking, our results below will hold only when $F = \mathbb{Q}$. However, we pretend that F could be a general totally real field. This is justified by the fact that all three conditions of §4 are fulfilled if GL_2 is replaced with either PGL_2 or SL_2, in which case the results below indeed make sense with very minor modifications. (Also see Remark 4.1.)

We use the notation of §5.1 and §6.1. The choice of Haar measures is as in the last paragraph of §2.2. At each $v \in S$, choose any pair $(M_v, \mathcal{O}_v)$ as in §6.1. In particular, $\mathcal{O}_v$ is an orbit of discrete series of $M_v(F_v)$. Put

$$\widehat{U} := \prod_{v\in S} \text{n-ind}_{M_v}^{GL_2} \mathcal{O}_v. \tag{6.1}$$

Suppose that $U_1 = \prod_{v\in \mathcal{V}_F^\infty \backslash S} GL_2(\mathcal{O}_{F_v})$ (for convenience) and that $\{U_n\}_{n\geq 1}$ is a decreasing sequence in $G(\mathbb{A}_F^{S,\infty})$ tending to 1 in the sense of Definition 3.1. Our choice of $\mu = \prod_{v\in\mathcal{V}_F} \mu_v$ is such that $\mu^{S,\infty}(U_1) = 1$. Thus $\mu^{S,\infty}(U_n) = [U_1 : U_n]^{-1}$. It is straightforward to check $d(G_\infty) = 1$ and $q(G_\infty) = [F : \mathbb{Q}]$ in the current case. Let π_ξ denote the unique representation in $\Pi_{\mathrm{disc}}(G_\infty, \xi^\vee)$.

Remark 5.5 tells us that

$$\begin{aligned} &\sum_{\pi_S^0 \in \widehat{U}} m_{\mathrm{cusp}}(\pi_S^0; \mathrm{char}_{U_n}, \xi) \\ &\qquad = \overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)) \cdot \mu^{S,\infty}(U_n)^{-1} \cdot \widehat{\mu}_S^{\mathrm{pl}}(\widehat{U}) \cdot \dim \xi \ + \ o(1). \end{aligned} \tag{6.2}$$

Formula (3.4) for GL_2, which is true even when the highest weight of ξ is not regular,[5] tells us that $m_{\mathrm{cusp}}(\pi_S^0; \mathrm{char}_{U_n}, \xi)$ is a rational number with sign $(-1)^{q(G_\infty)} = (-1)^{[F:\mathbb{Q}]}$ and that the left-hand side of (6.2) is $(-1)^{[F:\mathbb{Q}]}$ times the sum of $\dim(\pi^{S,\infty})^{U_n}$ over the set of cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ such that

- $\pi_v \in \text{n-ind}_{M_v}^{GL_2} \mathcal{O}_v$ for all $v \in S$,
- $(\pi^{S,\infty})^{U_n} \neq (0)$, and
- $\pi_\infty \simeq \pi_\xi$.

[5] For this, one can appeal to the fact that cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$ always have discrete series at infinity.

Equivalently, the left-hand side of (6.2) is the dimension of S-new Hilbert modular forms generated by cuspidal automorphic representations of level U_n (outside S and ∞), weight ξ and prescribed condition $\widehat{U}$ at S.

Remark 6.1: By the local Langlands correspondence, the first condition on π is equivalent to prescribing the inertia action and monodromy operator for the Weil–Deligne representation corresponding to π_v.

We can calculate $\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F))$ in formula (6.2).

Lemma 6.2: *$\overline{\mu}(G(F)A_{G,\infty}\backslash G(\mathbb{A}_F)) = \zeta_F(-1)\cdot h_F \cdot 2^{[F:\mathbb{Q}]-1}$, where ζ_F is the Dedekind zeta function for F and h_F is the class number of F.*

Remark 6.3: It is known that $\zeta_F(-1)$ is a nonzero rational number (cf. [Gro, Thm. 1.1, Lem. 1.3].

Proof. Let $Z := GL_1$ and $G_{\mathrm{ad}} := PGL_2$. By Hilbert 90 it is easy to see that

(6.3) $$1 \to Z(\mathbb{A}_F)/A_{Z,\infty} \to G(\mathbb{A}_F)/A_{G,\infty} \to G_{\mathrm{ad}}(\mathbb{A}_F) \to 1$$

is a short exact sequence of topological groups. For each group $H \in \{Z, G, G_{\mathrm{ad}}\}$ over F, the measure $\mu^{EP,\infty}$ defined on $H(\mathbb{A}_F)/A_{H,\infty}$ as in §2.2 will be denoted by μ_H^{EP}, and the Tamagawa measure on $H(\mathbb{A}_F)$ will be denoted by μ_H^{Tama}. The Tamagawa number $\tau(H)$ is the volume of $H(F)A_{H,\infty}\backslash H(\mathbb{A}_F)$ under the measure induced by μ_H^{Tama}. We have the formula ([Kot88, p. 629])

(6.4) $$\tau(H) = |\pi_0(Z(\widehat{H})^{\mathrm{Gal}(\overline{F}/F)})| \cdot |\ker^1(F, Z(\widehat{H}))|^{-1}.$$

Note the following.

(i) μ_Z^{EP}, μ_G^{EP} and $\mu_{G_{\mathrm{ad}}}^{\mathrm{EP}}$ are compatible with respect to (6.3).
(ii) $\overline{\mu}_Z^{\mathrm{EP}}(Z(F)\backslash Z(\mathbb{A}_F)/A_{Z,\infty}) = h_F$.
(iii) $\overline{\mu}_{G_{\mathrm{ad}}}^{\mathrm{EP}}(G_{\mathrm{ad}}(F)\backslash G_{\mathrm{ad}}(\mathbb{A}_F)) = \zeta_K(-1)\cdot 2^{[F:\mathbb{Q}]}$.
(iv) μ_Z^{Tama}, μ_G^{Tama} and $\mu_{G_{\mathrm{ad}}}^{\mathrm{Tama}}$ are compatible with respect to (6.3).
(v) $\tau(Z) = \tau(G) = 1$, $\tau(G_{\mathrm{ad}}) = 2$.

We check (i) at finite places and infinite places separately from the definition of μ^{EP}. It is elementary to verify (ii). Part (iii) follows from [Gro97, Thm. 9.9]. For (iv), we refer to page 75 of [San81]. The last part is immediate from (6.4).

Now (i), (iv) and (v) imply that $\overline{\mu}_G^{\mathrm{EP}}(G(F)\backslash G(\mathbb{A}_F)/A_{G,\infty})$ equals

$$2^{-1}\cdot \overline{\mu}_Z^{\mathrm{EP}}(Z(F)\backslash Z(\mathbb{A}_F)/A_{Z,\infty})\cdot \overline{\mu}_{G_{\mathrm{ad}}}^{\mathrm{EP}}(G_{\mathrm{ad}}(F)\backslash G_{\mathrm{ad}}(\mathbb{A}_F)).$$

Applying (ii) and (iii), we conclude the proof. ∎

Formula (6.2) and Lemma 6.2 yield:

Proposition 6.4: *Let $\{U_n\}_{n\geq 1}$ be a decreasing sequence in $GL_2(\mathbb{A}_F^{S,\infty})$ tending to 1 in the sense of Definition 3.1. Let $\widehat{U}$ be as at the start of §6.2. Then the dimension of S-new Hilbert cuspforms of level U_n (outside S and ∞), weight ξ and prescribed condition $\widehat{U}$ at S (cf. the paragraph preceding Remark 6.1) is given by*

$$(-1)^{[F:\mathbb{Q}]} \cdot \zeta_F(-1) \cdot 2^{1-[F:\mathbb{Q}]} \cdot h_F \cdot [U_1 : U_n] \cdot \widehat{\mu}^{\mathrm{pl}}_S(\widehat{U}) \cdot \dim \xi \; + \; o(1),$$

where $\widehat{\mu}^{\mathrm{pl}}_S(\widehat{U})$ is explicitly given by (6.1) *and the list of* §6.1.

Remark 6.5: When $n = 1$, the leading term appears to be the same as the one in [Wei09, Thm. 1.1]. (It suffices to compare our $\widehat{\mu}^{\mathrm{pl}}_S(\widehat{U}) \cdot \dim \xi$ with his $d(\tau)$. This comes down to matching our local computation at $v \in S$ as in §6.1 with his local invariants defined in [Wei09, §2.1]. The task is easy in the non-supercuspidal case. In the supercuspidal case we use the formal degree formula in [Car84, §5] and [CMS90, §2.2].) As n grows, our formula is different from Weinstein's because the dimension we are counting is different from his. Namely, he is concerned with the number of cuspforms (or corresponding representations π of $GL_2(\mathbb{A}_F)$) while we estimate the sum of $\dim(\pi^{S,\infty})^{U_n}$ over the same set of π as $n \to \infty$. If U_1 is hyperspecial then the latter dimension equals 1, which explains why the two results are related.

Appendix A. On Sauvageot's paper

Our paper relies on Sauvageot's work ([Sau97]) in an essential way. Although his paper is correct and beautifully written in our opinion, it seems to contain a handful of minor errors, which we think would be helpful to list here. We are fully responsible for any possible misinterpretation of part of his work or introduction of new errors.

- On p. 158, Lemme 2.1, it should be assumed in addition that X is Hausdorff to validate the lemma, as the proof requires the Stone–Weierstrass theorem, which is applicable to locally compact Hausdorff spaces. (Remark: Lemme 2.1 is applied in section 7 where $X = \Theta(G)$. Since the latter is Hausdorff, the added assumption does not affect the results of section 7. Here $\Theta(G)$ is the set of infinitesimal characters as defined on p. 164.)

- In the first line of the proof of Lemme 2.1, it should have been said that h_0 is a ν-integrable function.
- p. 161, Lemme 2.6, in (3), "Tout point de C" should be strengthened to "Tout point de $\bigcup_{i\in I}\mathcal{W}_i$". The proof given there is already enough to imply the latter.
- On p. 167, line 8, "donc l'induite $\mathcal{I}_{P',\tau',\lambda}$ est irréductible pour λ assez petit." can be a circulation of logic unless one is careful, but it is not clear where the author gets this information from. One solution would be to consider τ'_λ as an irreducible quotient of $r_{P''G}\Pi$ (rather than $r_{P'G}\Pi$). Then $\mathcal{I}_{P'',\tau',\lambda}$ is irreducible for small enough λ by the induction hypothesis, thus one obtains $\Pi = \mathcal{I}_{P'',\tau',\lambda}$. The rest of the argument remains the same. (Concerning this remark, the referee pointed out the following: "It is possible to refer to [Wal03, Prop. IV.2.2], where Theorem 3.2 of Sauvageot is treated when σ is cuspidal.")
- On p. 173, in the paragraph preceding Lemme 5.1, the reference (Casselman, 1989, Theorem 6.6.1) should be (Casselman, preprint, Theorem 6.6.1). In fact Theorem 6.6.1 does not even exist in the former reference. The latter is available at
  `http://www.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf`
  Casselman deals only with p-adic groups, but Sauvageot needs the result for archimedean F too, in which case the desired irreducibility of parabolic induction is found in [KZ82, Cor. 9.2].
- On p. 176, line -4, the displayed inequality is deduced from the stronger version of Lemme 2.6 as we stated above. (The original Lemme 2.6 shows the weaker inequality in which $c''_G 1_C$, not $c''_G 1_{C_1}$, is placed on the right-hand side.)
- On p. 176, line -5, it is worth recalling that not only $\mathcal{W}_i \subset \mathcal{V}_{\theta_i,\epsilon_1}$ but $\overline{\mathcal{W}_i} \subset \mathcal{V}_{\theta_i,\epsilon_1}$ can be ensured (by Lemme 2.6). This allows us to strengthen p. 177, line 4 as "tel que $\overline{\mathcal{W}_i} \subset \mathcal{W}'_i \subset \mathcal{V}_{\theta_i,\epsilon_1}$ et".
- On p. 177, line -3, add the condition that "$\widehat{\phi}_0$ is ≥ 0 on $\Pi_u(G)$", which can be arranged by Lemme 5.3.
- On p. 177, Lemme 5.5, the second inequality is not proved and its validity is unclear. For the purpose of proving Théorème 5.4, it suffices to observe that the very last display of p. 177 is positive everywhere on C_1 and > 1 on $\mathcal{N} \cap C$. This is exactly asserted in the first paragraph

of p. 178 (in line 4, C_1 should be replaced with C). Although the assertion is supposed to follow from the questionable second inequality of Lemme 5.5, the assertion can be shown directly without that lemma. (In the argument we need the last two corrections above as well as the correction to p. 175, Lemma 5.2 found below.)

- On p. 178, lines 6–9, there is a missing term on the right hand side. To be more precise, it results from $\widehat{\Psi}_1 = \epsilon_3 N(\epsilon_1)M(\epsilon_1)\widehat{\phi}_0 + \sum_{i=1}^{N(\epsilon_1)} f_i\widehat{\phi}_{\theta_i,\epsilon_1}$ that

$$\mu^G(|\widehat{\Psi}_1|) \leq \epsilon_3 N(\epsilon_1)M(\epsilon_1)\mu^G(|\widehat{\phi}_0|) + \mu^G\left(\left|\sum_{i=1}^{N(\epsilon_1)} f_i\widehat{\phi}_{\theta_i,\epsilon_1}\right|\right).$$

  The second term on the right0hand side is less than or equal to the right hand side of p. 178, lines 7–9 thanks to the first inequality of Lemme 5.5. The correction here does not change the fact (p. 178, lines 10–11) that one can choose ϵ_1, ϵ_2, ϵ_3 such that $\mu^G(\widehat{\Psi}_1) < \epsilon/2$.
- About the proof of Theorem 5.4: As a supplement to the last two corrections, we orient the reader by summarizing how (1), (2) and (3) of Theorem 5.4 are checked on pp. 178–179.
  (1) $\widehat{\Psi}_1$ is shown to be positive on C_1 on p. 178, line 2. This is used to show that $\widehat{\Psi} \geq 0$ on C_1. Outside C_1, $\widehat{\Psi} \geq 0$ (p. 179, line 3) thanks to the last inequality of p. 178.
  (2) $\mu^G(\widehat{\Psi}) \leq \epsilon$ is proved on p. 179, lines 4–6, whose essential input is p. 178, lines 6–9.
  (3) This follows from the fact that $\widehat{\Psi}_1 \geq 1$ on $\mathcal{N} \cap C$ (p. 178, lines 3–5) and that $\widehat{\Psi} \geq \widehat{\Psi}_1$ on $\mathcal{N} \cap C$ by construction.
- In Section 7, the author refers to Section 2 for some results on $\mathfrak{B}_c$, but Section 2 deals with $\mathfrak{B}$ rather than its subspace $\mathfrak{B}_c$. (The latter space consists of the functions whose supports have compact images in $\Theta(G)$.) Thus it should be remarked that results in Section 2 are still valid with $\mathfrak{B}_c$ in place of $\mathfrak{B}$ (though this is not difficult to check).
- On p. 181, line 9, “Théorème 6.1” should be “Corollaire 6.1”.

One of the referees kindly sent us further remarks on Sauvageot’s paper, which we list below without change.

- Section 2, line 1, add “séparé” to localement compact. Same in line 1 of Lemma 2.1.

- In Lemma 2.1, add that "the elements of A are ν-integrable" and that "f has bounded support".
- p. 159, line 5: Instead of "Donc" put "Le début de la preuve du Théorème 62, p. 557 de Schwartz montre que".
- p. 160, line 8: add "Ce qui veut dire que A sépare les points de Θ et que pour tout $\theta \in \Theta$, il existe $f \in A$ avec $f(\theta) \neq 0$."
- p. 168, line -3: Replace "et donc..." by "Il existe $\phi_i \in \mathcal{H}$, $\lambda_i \in \mathbb{C}$, $i = 1, ..., n$ tel que pour toute représentation de longueur finie π:

$$\mathrm{tr}\,(\pi(\phi)) = \sum_i \lambda_t r(\pi(\phi_i)\pi(\phi_i)^*).$$

- First line 2 of the proof of Théorème 5.4, put $\theta \in C$ instead of $\theta \in \Theta(G)$.
- p. 181, lines 4–5: It would the best to rewrite this part as "Par hypothèse, c'est une fonction $\widetilde{\mu}^G_{L,\sigma} d\chi$-Riemann-intégrable. Notons $\mu^*_{L,\sigma}$ l'image directe sur $\Theta(G)$ de la mesure $\widetilde{\mu}^G_{L,\sigma} d\chi$. D'après le Lemme 2.1, dont les hypothèses sont satisfaites car tout $h \in \mathcal{A}(G)$ est $\mu^*_{L,\sigma}$-intégrable, étant donné...".
- p. 175: In Lemma 5.2, one has "f est à valeurs positives ou nulles".

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