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ON THE REGULARITY OF THE ROOTS OF HYPERBOLIC POLYNOMIALS

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ABSTRACT

We prove that a hyperbolic monic polynomial whose coefficients are functions of class C^r of a parameter t admits roots of class C^1 in t, if r is the maximal multiplicity of the roots as t varies. Moreover, if the coefficients are functions of t of class C^{2r}, then the roots may be chosen two times differentiable at every point in t. This improves, among others, previous results of Bronšteĭn, Mandai, Wakabayashi and Kriegl, Losik and Michor.

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1. Introduction

The problem of finding regular roots of polynomials whose coefficients depend on parameters has been long studied. Apart from its intrinsic interest, its solution could help shed some light on problems in several fields ranging from algebraic geometry to partial differential equations.

A real polynomial whose roots are all real is called **hyperbolic**. In this paper we consider polynomials whose coefficients depend on one real parameter t and that are hyperbolic for all the values of the parameter.

Maybe the first classical result in this field can be considered Glaeser's theorem in [8]: the square root of a nonnegative function of class C^∞ is locally Lipschitz continuous, and in general it is not possible to have higher regularity. Some better results can be found if we are allowed to choose carefully the (square) root as t varies: for some results, see [10, 3, 4, 5].

Another interesting result is contained in the Fefferman and Phong proof of their famous inequality on positive operators in [7]: a nonnegative function of any number of variables of class $C^{3,1}$ can be written as a finite sum of squares of functions of class $C^{1,1}$. It is in general not possible to do better than that, but for an improvement in dimensions 1 and 2, see [2].

Generalizing the problem to an arbitrary hyperbolic polynomial depending on one parameter, we consider a system of roots, i.e., a set of real functions of the parameter, maybe only defined near some value t_0, that enumerate the roots of the polynomial (with multiplicity) as the parameter varies.

The most important and well-known result in this case is probably that of Bronšteĭn (in [6]): it is possible to find a system of locally Lipschitz continuous roots of such a polynomial provided its coefficients are at least of class C^r where r is the maximal multiplicity of the roots over all values of t. For each t_0 a system of roots differentiable at t_0 is also found.

Later, a different proof of this statement was given by Wakabayashi [14]; for another result in the same line, see also Tarama [13].

A stronger result, with heavier hypotheses, was proved by Mandai in [10]: if the coefficients are of class C^{2r} it is possible to find a global system of roots of class C^1 (here global means that the roots are defined on the same domain of parameters as the coefficients of P). Kriegl, Losik and Michor in [9] proved also that if the coefficients are of class C^{3n} (where n is the degree of the polynomial) it is possible to find a global system of roots twice differentiable at every point.

We refine here the results of Bronšteĭn, Mandai and Kriegl, Losik and Michor. Indeed, we prove

THEOREM 1.1: *Let us consider a monic hyperbolic polynomial*

$$P(t,\tau) = \tau^m + a_1(t)\tau^{m-1} + a_2(t)\tau^{m-2} + \cdots + a_m(t),$$

with coefficients $a_1, \ldots, a_m \in C^r(\mathbb{R})$, where r is the maximal multiplicity of the roots of P. Then it is possible to find a system of roots $\tau_1, \ldots, \tau_m : \mathbb{R} \to \mathbb{R}$ of class C^1.

and

THEOREM 1.2: *Let us consider a monic hyperbolic polynomial*

$$P(t,\tau) = \tau^m + a_1(t)\tau^{m-1} + a_2(t)\tau^{m-2} + \cdots + a_m(t),$$

with coefficients $a_1, \ldots, a_m$ of class C^{2r} on $\mathbb{R}$, where r is the maximal multiplicity of the roots of P. Then it is possible to choose a system of roots $\tau_1, \ldots, \tau_m : \mathbb{R} \to \mathbb{R}$ so that they are twice differentiable at every point.

Section 2 and Section 3 are devoted respectively to these two results. These are somehow the best possible results with these hypotheses; in Section 4 we give some examples to show that they cannot be improved.

2. Roots of class C^1

In this paper $f'^{-}(t_0)$ (resp. $f'^{+}(t_0)$) will denote the left (resp. right) derivative of function f at point t_0 (if it exists).

First, we recall the following well-known results (see, e.g., Mandai [10] and Bronšteĭn [6]) that we will need below.

THEOREM 2.1: *Let*

$$P(t,X) = X^m + B_1(t)X^{m-1} + \cdots + B_m(t)$$

be a hyperbolic polynomial for every t in the interval $I = (a,b)$. Assume that the multiplicity of its roots does not exceed r and $B_j \in C^r(I)$, $j = 1, \ldots, m$. Then the following hold:

1. *If $\lambda(t) \in C^0(I)$ and $P(t,\lambda(t)) = 0$ on I, then for any $t_0 \in I$ there exist $\lambda'^{\pm}(t_0)$. Further, for any compact set $K \subset I$, the set $\{\lambda'^{\pm}(t), t \in K\}$ is bounded.*

(2) *Let the multiplicity of* $X = \lambda(t_0)$ *be* q. *If*

$$\lambda_j(t) \in C^0(I), \quad \lambda_j(t_0) = \lambda(t_0), \quad j = 1, \dots, q$$

and $P(t, X)$ *is divisible by* $(X - \lambda_1(t)) \cdot \ldots \cdot (X - \lambda_q(t))$ *as a polynomial of* X, *then the sets* D^+ *and* D^- *defined as*

$$D^{\pm} = \{\lambda_j'^{\pm}(t_0),\ j = 1, \dots, q\}$$

are respectively the roots of the same equation

$$a_0 X^q + a_1 X^{q-1} + \cdots + a_q = 0$$

where $a_i = \partial_t^i \partial_X^{q-i} P(t_0, \lambda(t_0))/(q-i)!i!$, $i = 0, 1, \dots, q$.

(3) *There exist* $\lambda_1, \dots, \lambda_m \in C^0(I)$ *such that*

$$P(t, X) = (X - \lambda_1(t)) \cdots (X - \lambda_m(t))$$

and λ_j, $j = 1, \dots, m$ *are differentiable on* I.

From part two we easily deduce

Corollary 2.2: *Let* $\tau_{11}, \dots, \tau_{1m}$ *and* $\tau_{21}, \dots, \tau_{2m}$ *be two systems of differentiable roots of* $P(t, X)$ *on a neighbourhood of a point* t_0. *Then there exists a bijection* $g : \{1, \dots, m\} \to \{1, \dots, m\}$ *such that* $\tau_{1j}'(t_0) = \tau_{2g(j)}'(t_0)$ *for every* $j = 1, \dots, m$.

We can now prove Theorem 1.1.

Proof of Theorem 1.1. It follows from Theorem 2.1 that under our hypotheses it is possible to find a system of differentiable roots $\tau_1, \dots, \tau_m$; if we prove that for every j the function τ_j' has a limit at every point t_0, the thesis will follow by well-known results on differentiability.

Hence, we can take $t_0 = 0$ and restrict ourselves to an interval $U_\delta = [-2\delta, 2\delta]$, for a sufficiently small δ (positive and smaller than $1/2$) which will be determined below.

It is also well-known that if δ is small enough, in U_δ we may "separate" the roots according to their values at 0; that is, if $x_1, \dots, x_k$ are the distinct roots of $P(0, \tau)$, we can write

$$P(t, \tau) = P_1(t, \tau) \cdots P_k(t, \tau),$$

choosing factors so that each P_i is monic, has x_i as its unique root at $t = 0$ and its coefficients are still functions of class C^r of t. This implies that those τ_j's such that $\tau_j(0) = x_i$ can be chosen also as differentiable roots of P_i (note also that the degree of each of these factors is $\leq r$ by our hypothesis).

The proof is then mostly contained in the following two lemmas, where we suppose without loss of generality that $x_1 = 0$.

Lemma 2.3: *Let $P(t,\tau)$ be a polynomial of degree m in τ with coefficients of class C^m in t for t near 0, such that P is hyperbolic for every t and has 0 as unique root for $t = 0$; let $\tau_1, \ldots, \tau_m$ be a differentiable system of roots of P. Suppose that $\tau_j'(0) = \lambda$ for $j = 1, \ldots, q$ and $\tau_j'(0) \neq \lambda$ for $j > q$; then, for every sufficiently small δ the polynomial*

$$Q(t,\tau) = \prod_{j=1}^{q} (\tau - \tau_j(t) + \lambda t) = \tau^q + b_1(t)\tau^{q-1} + \cdots + b_q(t)$$

satisfies:

(1) *the coefficients $b_j \in C^j(U_\delta) \cap C^m(U_\delta \setminus \{0\})$;*
(2) *the ratio $b_j^{(k)}(t)/t^{j-k}$ has a finite limit as $t \to 0$ for $j = 1, \ldots, q$, $k = 0, \ldots, m$, and for $k \leq j$ this limit is 0.*

Proof of Lemma. Up to a linear change of variables, we may reduce ourselves to the case where $\lambda = 0$. We take δ sufficiently small so that in $U_\delta \setminus \{0\}$ the roots τ_j with $j \leq q$ never take the same value as those with $j > q$ (recall that the two groups have different derivatives at 0).

Writing $P(t,\tau) = \tau^m + a_1(t)\tau^{m-1} + \cdots + a_m(t)$, since all the roots vanish at 0, $a_i^{(k)}(0) = 0$ for $i = 1, \ldots, m$, $k < i$; moreover, under our hypotheses (see [6]), $a_i^{(i)}(0) = 0$ for $i = m - q + 1, \ldots, m$. Indeed, looking at the Taylor expansion of a_i, it is easy to show that, more generally,

$$\lim_{t \to 0} \frac{a_i^{(k)}(t)}{t^{i-k}}$$

exists and is finite for $i = 1, \ldots, m$, $k = 0, \ldots, m$.

Define for $j = 1, \ldots, q$

$$c_j(t) = \frac{1}{2\pi i} \int_{C_t} \frac{\partial_\tau P(t,\zeta)}{P(t,\zeta)} \zeta^j \, d\zeta$$

where C_t is the circle $\zeta = \varepsilon |t| e^{\mathrm{i}s}, 0 \leq s \leq 2\pi$ for some small $\varepsilon > 0$ and ∂_τ denotes as usual the partial derivative with respect to τ; we have that

$$c_j(t) = \tau_1^j(t) + \cdots + \tau_q^j(t),$$

since for small ε these are the only roots of P inside C_t.

Claim: *There exists some $\delta > 0$ such that $c_j \in C^j(U_\delta) \cap C^m(U_\delta \setminus \{0\})$ and $c_j^{(h)}(t)/t^{j-h}$ has a finite limit, say γ_{jh}, as $t \to 0$ for $j = 1, \ldots, q$, $h = 0, \ldots, m$.*

Indeed, take any value of ε smaller than half of the minimum value of $|\tau_i'(0)|$ for $i > q$ and δ so small that for every $t \neq 0$ in U_δ all the roots with $i < q$ are contained in the interior of C_t; every c_j is then obviously of class C^m on $U_\delta \setminus \{0\}$, since $P(t, \zeta)$ does not vanish on C_t for $t \neq 0$, and is of class C^j on U_δ thanks to the presence of the ζ^j, as one can easily see making the change of variable $\zeta = \varepsilon t e^{\mathrm{i}s}$.

To make computations easier, we observe that if we fix any point t different from 0 and take t' in a neighbourhood of t, we can assume that $C_{t'} = \varepsilon t e^{\mathrm{i}s}$; indeed, the value of the integral is the same as long as these circumferences contain the same roots of $P(t', \zeta)$. Thus, to compute the derivatives of c_j we need only take care of the derivatives of the quotient $\frac{\partial_\tau P}{P}$. If we differentiate it h times with respect to t we get (for some choice of the constants C_H where $H = (h_1, \ldots, h_l)$ is an l-tuple)

$$\partial_t^h \left[\frac{\partial_\tau P(t,\zeta)}{P(t,\zeta)} \right] = \sum_{l=1}^{h} \sum_{h_1 + \cdots + h_l = h} C_H \frac{\partial_\tau \partial_t^{h_1} P}{P} \cdot \frac{\partial_t^{h_2} P}{P} \cdots \frac{\partial_t^{h_l} P}{P}.$$

We thus obtain a sum of integrals of the form

$$\int_{C_t} C_H \frac{\partial_\tau \partial_t^{h_1} P}{P} \cdot \frac{\partial_t^{h_2} P}{P} \cdots \frac{\partial_t^{h_l} P}{P} \zeta^j \, d\zeta. \tag{2.1}$$

Now

$$P(t,\zeta) = (\zeta - \tau_1(t)) \cdots (\zeta - \tau_m(t)) = t^m \left(\varepsilon e^{\mathrm{i}s} - \frac{\tau_1(t)}{t} \right) \cdots \left(\varepsilon e^{\mathrm{i}s} - \frac{\tau_m(t)}{t} \right),$$

and so

$$\begin{aligned}
\partial_\tau P(t,\zeta) =& m\zeta^{m-1} + (m-1)a_1(t)\zeta^{m-2} + \cdots + a_{m-1}(t) \\
=& t^{m-1} \left(m(\varepsilon e^{\mathrm{i}s})^{m-1} + (m-1)\frac{a_1(t)}{t}(\varepsilon e^{\mathrm{i}s})^{m-2} + \cdots + \frac{a_{m-1}(t)}{t^{m-1}} \right), \\
\partial_\tau \partial_t^{h_1} P(t,\zeta) =& (m-1)a_1^{(h_1)}(t)\zeta^{m-2} + (m-2)a_2^{(h_1)}(t)\zeta^{m-3} + \cdots + a_{m-1}^{(h_1)}(t) \\
=& t^{m-1-h_1} \left((m-1)\frac{a_1^{(h_1)}(t)}{t^{1-h_1}}(\varepsilon e^{\mathrm{i}s})^{m-2} + \cdots + \frac{a_{m-1}^{(h_1)}(t)}{t^{m-1-h_1}} \right),
\end{aligned}$$

and

$$\begin{aligned}\partial_t^{h_j} P(t,\zeta) =& a_1^{(h_j)}(t)\zeta^{m-1} + a_2^{(h_j)}(t)\zeta^{m-2} + \cdots + a_m^{(h_j)}(t)\\ =& t^{m-h_j}\left(\frac{a_1^{(h_j)}(t)}{t^{1-h_j}}(\varepsilon e^{\mathrm{i}s})^{m-1} + \cdots + \frac{a_m^{(h_j)}(t)}{t^{m-h_j}}\right).\end{aligned}$$

Note that by hypothesis each ratio $\tau_i(t)/t$ admits a finite limit at 0 (whose value is $\tau_i'(0)$) as also does each ratio $a_i^{(h_j)}/t^{i-h_j}$, as noted above; moreover, the factor $\zeta^j d\zeta$ provides an additional term t^{j+1}, hence we can write each integrand in (2.1) as

$$t^{-h+j} g_{jH}(t,s)$$

for suitable continuous functions g_{jH}; if we call $\gamma_{jH} = \int_0^{2\pi} g_{jH}(0,s)ds$ and set $\gamma_{jh} = \sum_{l,H} C_H \gamma_{jH}$, we have that

$$\lim_{t\to 0} \frac{c_j^{(h)}(t)}{t^{j-h}} = \gamma_{jh},$$

which proves our claim.

We stress the fact that this holds for $0 \le h \le m$, even in the cases where $h > j$, as we will indeed use this below.

Now, $b_1, \ldots, b_q$ are the elementary symmetric functions in $\tau_1, \ldots, \tau_q$. The functions c_j and the functions b_j are homogeneous of degree j in the τ_i's and, as is well-known, the b_j's may be written as polynomials in the c_l's, with constant coefficients, involving only those c_l's with $l \le j$: this proves in particular that $b_j \in C^m(U_\delta \setminus \{0\})$, a part of property (1). Each monomial of these polynomials is a coefficient times $c_1^{d_1} c_2^{d_2} \cdots c_j^{d_j}$ (where $d_1 + 2d_2 + \cdots + jd_j = j$): let us call $D = d_1 + \cdots + d_j$ its degree.

In order to compute the k-th derivative of b_j we have to differentiate $c_1^{d_1} c_2^{d_2} \cdots c_j^{d_j}$ k times which, for suitable constants C_K', produces

$$\sum_K C_K' c_1^{(k_1)} c_1^{(k_2)} \cdots c_1^{(k_{d_1})} c_2^{(k_{d_1+1})} \cdots c_2^{(k_{d_1+d_2})} \cdots c_j^{(k_{d_1+\cdots+d_j})}$$

where we sum over the D-tuples K of integers such that $k_i \ge 0$ and $\sum_{i=1}^D k_i = k$. But since we can factor t^{i-h} from each term $c_i^{(h)}$, we can divide each summand by

$$(t^{1-k_1})(t^{1-k_2})\cdots(t^{1-k_{d_1}})(t^{2-k_{d_1+1}})\cdots(t^{2-k_{d_1+d_2}})\cdots(t^{j-k_D}) = t^{\sum l d_l - \sum k_i} = t^{j-k},$$

obtaining a finite limit at $t = 0$; therefore, summing up we have that

$$\lim_{t\to 0} \frac{b_j^{(k)}(t)}{t^{j-k}}$$

exists for any $j = 1, \dots, q$, $k = 0, \dots m$, which is the first part of property (2). As for the second part, now we know that these limits exist, we obtain them for $k \le j$ by l'Hôpital's rule from the case $k = 0$, since

$$\lim_{t\to 0} \frac{b_j(t)}{t^j} = 0$$

because the functions b_j are symmetric functions of degree j of the roots τ_i and $\tau_i(0) = \tau_i'(0) = 0$ for all i.

Finally, from what we just proved we deduce that for $k \le j$

$$\lim_{t\to 0} b_j^{(k)} = 0$$

and so, since $b_j(0) = 0$ for every j, $b_j \in C^j(U_\delta)$, which was the part of property (1) that remained to be proved. ∎

Lemma 2.4: *In the hypotheses of Lemma 2.3, let us call $\bar\tau_1, \dots, \bar\tau_q$ the system of (continuous) roots of $Q(t,\tau)$ chosen in increasing order at every point; then there exist monic polynomials $Q_\varepsilon(t,\tau)$ of degree q (whose coefficients as $\varepsilon \to 0$ converge uniformly on U_δ to the coefficients of $Q(t,\tau)$) with differentiable roots $\tau_{j,\varepsilon}$ converging uniformly to the $\bar\tau_j$'s and such that*

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \sup_{[-\delta,\delta]} |\tau_{j,\varepsilon}'(t)| = 0.$$

Proof of Lemma. Let

$$Q_\varepsilon(t,\tau) = Q(t,\tau) + C_1\varepsilon t\partial_\tau Q(t,\tau) + \dots + C_{q-1}\varepsilon^{q-1}t^{q-1}\partial_\tau^{q-1}Q(t,\tau);$$

by [11], [14] and [12] there exist suitable constants $C, C_1, \dots, C_{q-1}$ such that for every sufficiently small $\varepsilon, \delta > 0$, $Q_\varepsilon(t,\tau)$ is a strictly hyperbolic polynomial when $t \neq 0$; there is therefore a unique system of continuous roots $\tau_{j,\varepsilon}$ that turn out to be differentiable and, if appropriately reordered, satisfy also

$$|\tau_{j,\varepsilon}(t) - \bar\tau_j(t)| \le C\varepsilon|t|. \tag{2.2}$$

It is therefore clear that the coefficients $b_{i,\varepsilon}(t)$ of Q_ε and its roots converge uniformly on U_δ respectively to the corresponding coefficients of Q and to the roots of Q as $\varepsilon \to 0$, since

$$b_{i,\varepsilon}(t) = \sum_{k=0}^{i} \frac{(q-i+k)!}{(q-i)!} C_k \varepsilon^k t^k b_{i-k}(t) \tag{2.3}$$

(here we have set $C_0 = 1$ and $b_0 \equiv 1$).

It only remains to prove the statement on the derivatives $\tau'_{j,\varepsilon}$. Let us fix from now on an index j, a point $t_0 \neq 0$ in $[-\delta, \delta]$, and let

$$Q_{\varepsilon,t_0}(t,\tau) = Q_\varepsilon\big(t, \tau + \frac{\tau_{j,\varepsilon}(t_0)}{t_0} t\big);$$

if we call its coefficients b_{i,ε,t_0}, we have

$$b_{i,\varepsilon,t_0}(t) = \sum_{h=0}^{i} \binom{q-i+h}{q-i} \left(\frac{\tau_{j,\varepsilon}(t_0)}{t_0} t\right)^h b_{i-h,\varepsilon}(t) \tag{2.4}$$

and, if τ_{l,ε,t_0} are the roots of $Q_{\varepsilon,t_0}(t,\tau)$,

$$\tau_{l,\varepsilon,t_0}(t) = \tau_{l,\varepsilon}(t) - \frac{\tau_{j,\varepsilon}(t_0)}{t_0} t$$

and so there is now a unique $\bar{l}$ such that $\tau_{\bar{l},\varepsilon,t_0}(t_0) = 0$.

Let us denote by $\|\cdot\|$ the uniform norm on U_δ and set

$$\omega_{\varepsilon,t_0}(\delta) = \|b'_{1,\varepsilon,t_0}\| + \|b^{(2)}_{2,\varepsilon,t_0}\|^{1/2} + \cdots + \|b^{(q)}_{q,\varepsilon,t_0}\|^{1/q}.$$

It is easy to see that this quantity does not vanish for $\delta \neq 0$: indeed, otherwise $Q_{\varepsilon,t_0}(t,\tau)$ would not be strictly hyperbolic outside 0.

Setting $\sigma = \tau/\omega_{\varepsilon,t_0}(\delta)$ (and as a consequence defining the function $\sigma_{\bar{l}}$ by $\sigma_{\bar{l}}(t) = \tau_{\bar{l},\varepsilon,t_0}(t)/\omega_{\varepsilon,t_0}(\delta)$) the polynomial equation $Q_{\varepsilon,t_0}(t,\tau) = 0$ becomes

$$\omega_{\varepsilon,t_0}(\delta)^q \sigma^q + \omega_{\varepsilon,t_0}(\delta)^{q-1} b_{1,\varepsilon,t_0}(t) \sigma^{q-1} + \cdots + b_{q,\varepsilon,t_0}(t) = 0$$

or, equivalently,

$$\tilde{Q}_{\varepsilon,t_0}(t,\sigma) = \sigma^q + \frac{b_{1,\varepsilon,t_0}(t)}{\omega_{\varepsilon,t_0}(\delta)} \sigma^{q-1} + \cdots + \frac{b_{q,\varepsilon,t_0}(t)}{\omega_{\varepsilon,t_0}(\delta)^q} = 0.$$

Using (2.3), (2.4) and property (2) in Lemma 2.3, we have that

$$\left| \frac{b^{(k)}_{i,\varepsilon,t_0}(t)}{\omega_{\varepsilon,t_0}(\delta)^i} \right| \leq 1$$

for $t \in U_\delta$ and $k \leq i$. We can then apply Lemma 4 in [6] that we will call

BRONŠTEĬN'S LEMMA: *We consider the polynomial*

$$P(t,X)=\sum_{j=0}^{m-r-1}B_j(t)X^{m-j}+\sum_{j=0}^{r}A_j(t)X^{r-j},$$

which is hyperbolic for all $t\in[-1,1]$ *where* $B_i(t)$ *are bounded functions of* $t\in[-1,1]$ *and* $A_i(t)$ *are* i*-times differentiable functions. Let* $A_0(t)\neq 0\ \forall\, t\in[-1,1]$, $A_{r-1}(0)\neq 0, A_r(0)=0$. *Then for some constant* $C>0$ *depending only on the degree of the polynomial* P,

$$\left|\frac{A_r'(0)}{A_{r-1}(0)}\right|\le\Big(\sup_{i,t}|B_i(t)|+\max_{\substack{i,j,t\\ j\le i}}|A_i^{(j)}(t)|+\max_t|A_0(t)|^{-1}+2\Big)^C.$$

We apply this lemma to the polynomial $\tilde Q_{\varepsilon,t_0}(t_0+s,\sigma)$ (with $B_i\equiv 0$, $A_i(s)=b_{i,\varepsilon,t_0}(t_0+s)/\omega_{\varepsilon,t_0}(\delta)^i$, $s\in[-1,1]$).

A close inspection of the proof of Bronšteĭn's Lemma shows that the estimate holds actually using the maximum attained by the coefficients A_i on an interval of radius

$$\left|\frac{A_1(0)}{M_1A_0(0)}\right|=\frac{|b_{1,\varepsilon,t_0}(t_0)|}{M_1\omega_{\varepsilon,t_0}(\delta)}\quad\text{if }\frac{|b_{2,\varepsilon,t_0}(t_0)|}{\omega_{\varepsilon,t_0}(\delta)^2}\le\left(\frac{b_{1,\varepsilon,t_0}(t_0)}{\omega_{\varepsilon,t_0}(\delta)}\right)^2,$$

or of radius

$$\left|\frac{A_2(0)}{M_2^2A_0(0)}\right|^{1/2}=\frac{|b_{2,\varepsilon,t_0}(t_0)|^{1/2}}{M_2\omega_{\varepsilon,t_0}(\delta)}\quad\text{if }\frac{|b_{2,\varepsilon,t_0}(t_0)|}{\omega_{\varepsilon,t_0}(\delta)^2}\ge\left(\frac{b_{1,\varepsilon,t_0}(t_0)}{\omega_{\varepsilon,t_0}(\delta)}\right)^2,$$

where $M_1,M_2\ge 1$ are suitable constants.

Since here $|t_0|\le\delta$, a quick verification shows that

$$\frac{|b_{1,\varepsilon,t_0}(t_0)|}{\omega_{\varepsilon,t_0}(\delta)}\le\delta\quad\text{and}\quad\frac{|b_{2,\varepsilon,t_0}(t_0)|}{\omega_{\varepsilon,t_0}(\delta)^2}\le\delta^2;$$

therefore the estimate takes into consideration only points in the interval $|t-t_0|\le\delta$, which is contained in U_δ.

Now, since $\tilde Q_{\varepsilon,t_0}(t_0+s,\sigma_{\bar l}(t_0+s))$ vanishes identically, we get that

$$\begin{aligned}&\frac{d}{ds}\tilde Q_{\varepsilon,t_0}(t_0+s,\sigma_{\bar l}(t_0+s))\\&\qquad=\partial_t\tilde Q_{\varepsilon,t_0}(t_0+s,\sigma_{\bar l}(t_0+s))+\partial_\tau\tilde Q_{\varepsilon,t_0}(t_0+s,\sigma_{\bar l}(t_0+s))\sigma_{\bar l}'(t_0+s)=0,\end{aligned}$$

hence

$$|\sigma_{\bar l}'(t_0+s)|=\left|\frac{\partial_t\tilde Q_{\varepsilon,t_0}(t_0+s,\sigma_{\bar l}(t_0+s))}{\partial_\tau\tilde Q_{\varepsilon,t_0}(t_0+s,\sigma_{\bar l}(t_0+s))}\right|$$

which for $s=0$ corresponds exactly to the quotient $\left|\frac{A_q'(0)}{A_{q-1}(0)}\right|$; we can then write, independently of ε and δ,

$$|\sigma_{\bar{l}}'(t_0)| \le 4^C = M.$$

Using the definition of σ we see that

$$\frac{|\tau_{\bar{l},\varepsilon,t_0}'(t_0)|}{\omega_{\varepsilon,t_0}(\delta)} \le M,$$

that is,

$$|\tau_{\bar{l},\varepsilon,t_0}'(t_0)| \le M\omega_{\varepsilon,t_0}(\delta).$$

But $\tau_{j,\varepsilon}'(t_0) = \tau_{\bar{l},\varepsilon,t_0}'(t_0) + \frac{\tau_{j,\varepsilon}(t_0)}{t_0}$, so that

$$|\tau_{j,\varepsilon}'(t_0)| \le M\omega_{\varepsilon,t_0}(\delta) + \left|\frac{\tau_{j,\varepsilon}(t_0)}{t_0}\right|.$$

Setting

$$\omega_\varepsilon(\delta) = \|b_{1,\varepsilon}'\| + \|b_{2,\varepsilon}^{(2)}\|^{1/2} + \cdots + \|b_{q,\varepsilon}^{(q)}\|^{1/q},$$

we have

$$\omega_{\varepsilon,t_0}(\delta) \le \omega_\varepsilon(\delta) + \sup_{U_\delta\setminus\{0\}} \left\{ \left|\frac{\tau_{j,\varepsilon}(t_0)}{t_0}\right|, \left|\frac{\tau_{j,\varepsilon}(t_0)}{t_0}\right|^{1/q} \right\} \phi_\varepsilon(\delta),$$

where the supremum is finite for given ε and δ, since $\tau_{j,\varepsilon}$ is differentiable, and $\phi_\varepsilon(\delta)$ is a positive constant taking into account combinatorial factors, the C_i's and all the uniform norms $\left\|\frac{b_i^{(k)}(t)}{t^{i-k}}\right\|$ (that are indeed bounded thanks to Lemma 2.3); note that this constant is globally bounded for ε and δ small and is independent of t_0.

We conclude that for ε and δ small and $t \in [-\delta,\delta]$

$$|\tau_{j,\varepsilon}'(t)| \le M' \left(\omega_\varepsilon(\delta) + \sup_{U_\delta\setminus\{0\}} \left\{ \left|\frac{\tau_{j,\varepsilon}(t_0)}{t_0}\right|^{1/q} \right\} \right) = \Omega_j(\delta,\varepsilon).$$

Note that this expression holds also in the case $t=0$ and that, since the right-hand side does not depend on t, it gives indeed an estimate for $\sup_{[-\delta,\delta]} |\tau_{j,\varepsilon}'(t)|$.

Now, looking at the expression of the $b_{i,\varepsilon}$'s, we see that

$$\lim_{\varepsilon\to 0} \omega_\varepsilon(\delta) = \omega_0(\delta) = \|b_1'\| + \|b_2^{(2)}\|^{1/2} + \cdots + \|b_q^{(q)}\|^{1/q},$$

while by (2.2)

$$\lim_{\varepsilon\to 0} \sup_{U_\delta\setminus\{0\}} \left|\frac{\tau_{j,\varepsilon}(t_0)}{t_0}\right| = \sup_{U_\delta\setminus\{0\}} \left|\frac{\bar{\tau}_j(t_0)}{t_0}\right|.$$

We have then proved that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{[-\delta,\delta]}|\tau'_{j,\varepsilon}(t)| \le M' \lim_{\delta\to 0}\left(\omega_0(\delta)+\sup_{U_\delta\setminus\{0\}}\left|\frac{\bar\tau_j(t_0)}{t_0}\right|^{1/q}\right).$$

But on one side, by Lemma 2.3, $\lim_{\delta\to 0}\omega_0(\delta)=0$; on the other side, by Theorem 2.1, the ordered roots $\bar\tau_j$ have right and left derivatives at 0 that coincide (as a set) with the set of derivatives at 0 of the differentiable roots τ_j of $Q(t,\tau)$ and all these derivatives are 0, therefore

$$\lim_{t_0\to 0}\frac{\bar\tau_j(t_0)}{t_0}=0$$

and so also

$$\lim_{\delta\to 0}\sup_{[-\delta,\delta]}\left|\frac{\bar\tau_j(t_0)}{t_0}\right|=0,$$

which ends our proof of Lemma 2.4. ∎

We now come back to the proof of Theorem 1.1: thanks to our reductions, it is sufficient to study the roots of the polynomial $Q(t,\tau)$ relative to one value of λ (namely, we shall take $\lambda=0$), where we can use Lemma 2.3 and Lemma 2.4.

Now we have that the functions $\tau_{j,\varepsilon}$ (roots of $Q_\varepsilon(t,\tau)$) are Lipschitz continuous on $[-\delta,\delta]$ with a Lipschitz constant $\Omega(\delta,\varepsilon)$ that can be taken independent of j; and so also the ordered roots $\bar\tau_j$ are Lipschitz continuous with Lipschitz constant $\lim_{\varepsilon\to 0}\Omega(\delta,\varepsilon)=\Omega(\delta)$, which implies that for every $t\in[-\delta,\delta]$

$$|\bar\tau_j'^{-}(t)|\le\Omega(\delta)\quad\text{ and }\quad|\bar\tau_j'^{+}(t)|\le\Omega(\delta).$$

But since the set of derivatives is the same, then also

$$|\tau_j'(t)|\le\Omega(\delta)$$

in $[-\delta,\delta]$, and since $\lim_{\delta\to 0}\Omega(\delta)=0$,

$$\lim_{t\to 0}\tau_j'(t)$$

exists (and is 0) for every j, which ends our proof. ∎

Remark 2.5: It may be worth noting that we have indeed proved that in these hypotheses any system of differentiable roots is already of class C^1: in this sense, this result improves the corresponding one in [10], where coefficients needed be of class C^{2r}.

3. Twice differentiable roots

In this section we will consider a monic hyperbolic polynomial

$$P(t,\tau) = \tau^m + a_1(t)\tau^{m-1} + a_2(t)\tau^{m-2} + \cdots + a_m(t),$$

with coefficients $a_1, \ldots, a_m$ of class C^{2r} on $\mathbb{R}$, where r is the maximal multiplicity of the roots of P as t varies.

We start with a few lemmas; first, a very simple one about continuous functions.

Lemma 3.1: *Let $f \in C^0((0,1))$ and $h_i \in C^0([0,1))$ for $i = 1, \ldots, k$; set $x_i = h_i(0)$ and suppose that, for every $t \in (0,1)$, $f(t)$ equals some $h_{i(t)}(t)$. Then $\lim_{t\to 0^+} f(t)$ exists and equals one of the x_i's.*

If all the x_i's coincide this holds even without the assumption of continuity on f.

Moreover, suppose that $f_i \in C^0((0,1))$ for $i = 1, \ldots, k$ and that for every $t \in (0,1)$ there exists a bijection $g_t : \{1, \ldots, k\} \to \{1, \ldots, k\}$ such that $f_i(t) = h_{g_t(i)}(t)$; if there are exactly n functions h_i with the same value $\bar{x}$ at 0, there are exactly n functions f_i with that value as a limit at 0.

Proof. The first two statements follow directly applying the definition of continuity.

The third then also follows since for small t the bijection g_t would not exist if the number of functions f_i having limit $\bar{x}$ at 0 were more than n; and since by the first part all the f_i's have a limit whose value is among the x_i's, their number cannot be less than n either. ∎

If the hypotheses of Lemma 3.1 are fulfilled, we will say that f is **pinned** by the h_i's (the same applies also to the functions f_i of the third part of the Lemma).

Now we establish a few properties of roots and of systems of roots of $P(t,\tau)$.

Lemma 3.2: *For every point t_0 there exists a system of roots $\tau_{1,t_0}, \ldots, \tau_{m,t_0}$ defined in a neighbourhood of t_0 and twice differentiable at that point.*

Proof. We take as always $t_0 = 0$ and suppose that 0 is the only root of $P(t,\tau)$ for $t = 0$ (and so $\deg P \leq r$), which is not restrictive up to separating the roots as in the proof of Theorem 1.1.

Now we notice that each ratio $a_i(t)/t^i$ has a finite limit at $t = 0$ (see Lemma 2.3) and so it is possible to extend it as a function $\bar{a}_i$ of class $C^{2r-i} \subset C^r$ on $\mathbb{R}$; therefore we can find by Theorem 1.1 a system of roots σ_j of class C^1 of the polynomial

$$\sigma^m + \bar{a}_1(t)\sigma^{m-1} + \bar{a}_2(t)\sigma^{m-2} + \cdots + \bar{a}_m(t)$$

and so the functions $\tau_{j,0}$ defined as $\tau_{j,0}(t) = t\sigma_j(t)$ are obviously a system of roots of P twice differentiable at $t = 0$. ∎

Remark 3.3: In fact the system of roots of Lemma 3.2 could be found of class C^1 on all $\mathbb{R}$; anyway, we cannot ensure that these roots will be twice differentiable at any point except that at t_0.

LEMMA 3.4: *Let $\tilde{\tau}$ be a root of $P(t,\tau)$ of class C^1 on an interval (t_0, t_1). Then it is possible to extend it to a root of class C^1 on the interval $[t_0, t_1)$.*

Proof. The proof is similar to that of Corollary 2.2: since by Theorem 1.1 there is a system of roots τ_j of class C^1, again by Theorem 2.1, at every point near t_0 their derivatives satisfy also a polynomial (with coefficients depending on the point); but the same polynomial must be satisfied also by the derivative of $\tilde{\tau}$ at all those points, that is, $\tilde{\tau}'$ is pinned by the τ_j''s and we conclude by Lemma 3.1. ∎

LEMMA 3.5: *Let $\tilde{\tau}$ be a root of $P(t,\tau)$ of class C^1 on an interval (t_0, t_1). Then the left and right second derivatives of $\tilde{\tau}$ exist at every point $t \in (t_0, t_1)$; moreover, if $\tilde{\tau}$ is suitably extended to $[t_0, t_1]$, the right second derivative exists at t_0 and the left second derivative exists at t_1.*

Proof. Since we proved that systems of roots τ_{j,t_0} with second right (left) derivative at a certain point t_0 exist, the first part of the statement follows by Lemma 3.1 after we note that the function

$$\frac{\tilde{\tau}'(t) - \tilde{\tau}'(t_0)}{t - t_0},$$

that is continuous outside t_0, is pinned by the functions

$$\frac{\tau_{j,t_0}(t) - \tau_{j,t_0}(t_0)}{t - t_0},$$

that are continuous also at t_0, if we define them there as $\tau''_{j,t_0}(t_0)$.

As for the second part, Lemma 3.4 shows that we can extend $\tilde{\tau}$ to a function of class C^1 on $[t_0, t_1]$ and the same proof applies to right (or left) second derivatives. ∎

Corollary 3.6: *Let $\tau_{11}, \dots, \tau_{1m}$ and $\tau_{21}, \dots, \tau_{2m}$ be two systems of roots of class C^1 of $P(t,\tau)$ on a neighbourhood of a point t_0. Then there exist bijections $g^-, g^+ : \{1, \dots, m\} \to \{1, \dots, m\}$ such that for every $j = 1, \dots, m$*

$$\tau''^{-}_{1j}(t_0) = \tau''^{-}_{2g^-(j)}(t_0) \quad \textit{and} \quad \tau''^{+}_{1j}(t_0) = \tau''^{+}_{2g^+(j)}(t_0).$$

Proof. The argument used in the proof of Lemma 3.5 proves this statement if we now use the third part of Lemma 3.1. ∎

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let us prove our statement arguing by induction on the maximal multiplicity k of the roots of P on any given open interval I (obviously, $k \leq r$).

To start with, if $k = 1$ we know that continuous roots are automatically of class C^{2r}: indeed, this is more generally true whenever the multiplicity is constant on I. Namely, the set of points where the multiplicity of each root is locally minimal is open (by the semicontinuity of multiplicity) and so on any open interval contained in it any system of continuous roots will be of class C^{2r}.

Now, suppose we have already proved the theorem for all the intervals where the maximal multiplicity is at most $k-1$ and take an open interval I where the multiplicity of the roots is at most k.

For every point $\bar{t} \in I$ we can again factor P to separate the roots, with the new coefficients in the same class as the old ones, in a suitable open interval $I_{\bar{t}} \subset I$.

First step. *In each $I_{\bar{t}}$ it is possible to find a system of twice differentiable roots.*

For this, we can suppose that the polynomial has degree k and has just one root of multiplicity k at $\bar{t}$ (if the multiplicity were less than k we would conclude by the induction hypothesis).

Let us call X the set of t's in $I_{\bar{t}}$ at which the multiplicity of the roots is exactly k; X is closed in $I_{\bar{t}}$ (though in general not closed in $\mathbb{R}$, because we ignore what happens outside I). Let us write $X = X_1 \cup X_2$, where X_1 is the set of isolated points of X and X_2 is its complement in X; X_2 is closed in $I_{\bar{t}}$, too.

Up to the translation $\tau \to \tau - \frac{1}{k}a_1(t)$, we can assume that $a_1 \equiv 0$ on $I_{\bar{t}}$, which means, in particular, that at all points of X in this interval all the roots coincide and are equal to 0.

Now let us write

$$I_{\bar{t}} \setminus X = \bigcup_{\alpha} I_\alpha$$

and

$$I_{\bar{t}} \setminus X_2 = \bigcup_{\beta} J_\beta$$

for suitable disjoint open intervals I_α and J_β; note that on each I_α the maximal multiplicity is at most $k-1$.

Claim: *On every J_β there is a system of twice differentiable roots.*

Indeed, by our hypotheses each interval J_β is a union of intervals I_α and points of X_1 (that cannot accumulate). In particular, for every β we can choose a point $t_\beta \in J_\beta \setminus X_1$. Now, t_β belongs to some I_α, where by induction there is a system of k twice differentiable roots.

For every point t in J_β the interval $[t_\beta, t]$ (or $[t, t_\beta]$) is covered by a finite number of intervals I_α and points of X_1. We use induction on the number of points of X_1: clearly, if $J_\beta \cap X_1 = \emptyset$, we already have the roots on the whole J_β.

Thanks to Lemmas 3.2—3.6, if on the left and on the right of a point t_1 of X_1 we have systems of twice differentiable roots σ_j^-, σ_j^+ of P (defined, say, on two intervals I_{α^-} and I_{α^+}), there is a bijection $g_{t_1} : \{1, \dots, m\} \to \{1, \dots, m\}$ such that the functions τ_j defined as

$$\tau_j(t) = \begin{cases} \sigma_j^-(t) & \text{if } t \in I_{\alpha^-}, \\ \lim_{t \to t_1} \sigma_j^-(t) & \text{if } t = t_1, \\ \sigma_{g_{t_1}(j)}^+(t) & \text{if } t \in I_{\alpha^+}, \end{cases}$$

are a system of twice differentiable roots of P on $I_{\alpha^-} \cup \{t_1\} \cup I_{\alpha^+}$. Indeed, the σ_j^-'s and σ_j^+'s can be extended with second derivative to t_1 and their derivatives and their left and right second derivatives are pinned by the corresponding derivatives of the system of twice differentiable roots existing near t_1: therefore they must be the same on the left and on the right of t_1, up to a reordering. This proves our claim.

To complete the first step in the proof, we now have to deal with points $t_2 \in X_2$ (let us suppose $t_2 = 0$ as usual).

We cannot use the same argument as above, since there could be no interval on the left or on the right of 0 where we know that the roots are twice differentiable: indeed, the roots at the moment could even be not defined at infinitely many points accumulating at 0.

Anyway, we can extend them to functions τ_j defined also at points of X_2 in $I_{\bar{t}}$ with the value 0. Since we have chosen twice differentiable (hence continuous) roots on every interval J_β, these functions are pinned by any system of continuous roots near 0 and so are continuous on their part on the whole $I_{\bar{t}}$ by Lemma 3.1.

Since now all the τ_j's vanish simultaneously infinitely many times near 0 (and hence so does each of the twice differentiable roots $\tau_{j,0}$ given by Lemma 3.2), if we denote by $\Delta f(t)$ the difference quotient of the function f calculated between t and 0, we have that

$$\lim_{t\to 0} \Delta\tau_{j,0}(t) = \lim_{t\to 0} \frac{\tau_{j,0}(t)}{t} = 0$$

because $\tau_{j,0}$ is twice differentiable; but each $\Delta\tau_j$ is pinned by the functions $\Delta\tau_{j,0}(t)$, so also

$$\lim_{t\to 0^\pm} \Delta\tau_j(t) = \lim_{t\to 0} \frac{\tau_j(t)}{t} = 0.$$

Thus, τ_j' exists in $I_{\bar{t}}$ and vanishes on X_2.

Similarly, we see that the τ_j''s are pinned by the $\tau_{j,0}'$'s that share the same limit at 0 and therefore, by the second part of Lemma 3.1, τ_j' is continuous (also) at the points of X_2: τ_j is then indeed of class C^1 on $I_{\bar{t}}$.

But again, since $\tau_{j,0}$ (and hence $\tau_{j,0}'$) vanishes infinitely many times near 0 and $\tau_{j,0}$ is twice differentiable,

$$\lim_{t\to 0} \frac{\tau_{j,0}'(t)}{t} = 0;$$

now, since the value $\tau_j'(t)/t$ must be among the values $\tau_{j,0}'(t)/t$ at every point $t \neq 0$ (we already know it outside X_2, and on $X_2 \setminus \{0\}$ all these values are 0), again by Lemma 3.1 also

$$\lim_{t\to 0^\pm} \frac{\tau_j'(t)}{t} = 0.$$

Thus τ_j is twice differentiable at points of X_2 and our first step is completed.

Second step. *There is a global system of twice differentiable roots on I.*

For this, we first choose a subcovering $\{I_{t_\gamma}\}_{\gamma\in\Gamma} \subset \{I_{\bar{t}}\}_{\bar{t}\in I}$ of I (where Γ is a finite or infinite interval of $\mathbb{Z}$ containing 0) whose members satisfy:

(1) no one of them is contained in the union of the others, and
(2) no one of them intersects more than two of the others.

Let us reorder the I_{t_γ} so that the indices respect the ordering of $\mathbb{R}$.

We prove the following statement by induction on N: *there is a system of twice differentiable roots on the interval* $\bigcup_{|\gamma|<N} I_{t_\gamma}$. Note that the "first step" above covers the case $N = 1$.

Choose a point (say 0) that belongs only to I_{t_0} and a point s_γ in every intersection $I_{t_\gamma} \cap I_{t_{\gamma+1}}$: at these points we use the same argument as above for points of X_1, this time to connect the roots defined on $\bigcup_{|\gamma|<N} I_{t_\gamma}$ to those defined on I_{t_N} and $I_{t_{-N}}$. Indeed, we have again two systems of twice differentiable roots on intervals on the sides of s_γ, except that in this case we have to take also care of many possible different limits x_j (which is nevertheless easily done thanks to the third part of our usual Lemma 3.1). Also the second step is then proved.

This ends also the inductive step on the multiplicity k; of course, when we reach the maximal multiplicity r of P, we can take $I = \mathbb{R}$ and our thesis follows. ∎

Remark 3.7: In the previous section we proved that differentiable roots were indeed of class C^1 on $\mathbb{R}$; in this section, instead, to have roots twice differentiable at every point we might be forced to choose them more carefully. For example, the choice of $\tau_j(t) = \pm t|t|$ as a system of roots of $\tau^2 - t^4$ is legitimate in Bronšteĭn's theorem and in Theorem 1.1, but gives only roots of class C^1; to obtain a system of twice differentiable roots the choice of $\tau_j(t) = \pm t^2$ must be made instead.

4. Counterexamples

In [1] the following examples are given, which show that the regularity assumptions in Theorem 1.1 cannot be weakened, even in the simple case $P(t,\tau) = \tau^2 - f(t)$ (whose roots will be called **admissible square roots of** f):

Example 4.1: Let

$$f(t) = \begin{cases} t^2 \sin^2(\log|t|) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Then $f \in C^{1,1}$ (and $f \notin C^2$) but no admissible square root of f is differentiable.

Example 4.2: Let

$$f(t) = \begin{cases} t^4 \sin^2(\log|t|) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Then $f \in C^{3,1}$ (and $f \notin C^4$), but its admissible square roots (that can be of class C^1) are never twice differentiable.

Analogously, for equations of degree three we can give the following examples:

Example 4.3: In the equation

$$\tau^3 - t^2\tau = \frac{t^3}{3}\sin(\log|t|)$$

the coefficients are of class $C^{2,1}$, but not of class C^3, and no root is of class C^1.

Example 4.4: In the equation

$$\tau^3 - t^4\tau = \frac{t^6}{3}\sin(\log|t|)$$

the coefficients are of class $C^{5,1}$, but not of class C^6, and no root is twice differentiable at $t = 0$.

Moreover, it is possible to find examples of equations for which the lack of regularity of the solutions is due to the low regularity of the coefficient of some monomial of positive degree. For example, if the equation is of degree two and there is a term of degree one whose coefficient is not of class C^2, it is possible that the roots cannot be differentiable.

Example 4.5: In the equation

$$\tau^2 + (t^2\sin^2(\log|t|) - 2)\tau + 1 = 0$$

the coefficients are of class $C^{1,1}$, but no root is differentiable at $t = 0$.

In a similar way it is possible to give an example of an equation of degree three where the term of degree two is "responsible" for the low regularity of the roots.

Example 4.6: In the equation

$$(\tau + 1)^3 - t^2(\tau + 1) = \left(\frac{t^3}{3}\sin(\log|t|)\right)\tau^2$$

the coefficients are of class $C^{2,1}$, but no root is differentiable at $t=0$.

Proof. Replacing $\tau+1$ with τ we have

$$\tau^3 - t^2\tau = \left(\frac{t^3}{3}\sin(\log|t|)\right)(\tau-1)^2.$$

If $t=0$ then $\tau=0$ is a triple root; putting $\sigma=\tau/t$, that is, $\tau=t\sigma$, we have the equation

$$\sigma^3-\sigma = \left(\frac{1}{3}\sin(\log|t|)\right)(t\sigma-1)^2.$$

This equation has three real roots for t near 0, but they do not have limit as $t\to 0$, since in the neighbourhood of 0 they take as values, e.g., both the roots of

$$\sigma^3-\sigma=\frac{1}{3}(e^{-\frac{4k+3}{2}\pi}\sigma-1)^2$$

and the roots of

$$\sigma^3-\sigma=-\frac{1}{3}(e^{-\frac{4k+1}{2}\pi}\sigma-1)^2,$$

that by continuity converge respectively to the roots of

$$\sigma^3-\sigma=\frac{1}{3} \quad \text{and of} \quad \sigma^3-\sigma=-\frac{1}{3}$$

as the natural number k goes to infinity. ∎

Going back to $P(t,\tau)=\tau^2-f(t)$, the following examples show that, on the other hand, even for smooth functions f the results given by Theorems 1.1 and 1.2 cannot be improved.

In [8] there is a well-known example of a function of class C^∞ whose square root is not of class C^2. A very similar function can be chosen to show that it is possible that no admissible square root be of class $C^{1,\alpha}$ for any α: namely (see [3]), we can set

$$f(t)=\begin{cases} e^{-1/|t|}(\sin^2(\frac{\pi}{|t|})+e^{-1/t^2}) & \text{if } t\neq 0,\\ 0 & \text{if } t=0.\end{cases}$$

Indeed, grossly speaking, the smaller are the positive minimum values near 0, the less regular are the admissible square roots. This observation leads to a refinement of the previous result: let us fix a positive real number λ and a continuous, increasing, concave function $\omega:[0,\lambda]\to\mathbb{R}$ such that $\omega(0)=0$ (which we assume satisfies $\omega(s)\geq s$). If $I\subset\mathbb{R}$ is an interval, a function

$f : I \to \mathbb{R}$ will be called **ω-continuous** on I if there is a positive constant C such that for any $|s| \le \lambda$ when $t, t+s \in I$

$$|f(t+s) - f(t)| \le C\omega(|s|),$$

and ω in this case will be called a **modulus of continuity** for f.

We then have

Theorem 4.7 (see [3]): *Given a function ω as above there exists a nonnegative function $f \in C^\infty(\mathbb{R})$ that vanishes only at 0 and such that $h = (\sqrt{f})'$ is not ω-continuous on $\mathbb{R}$ and therefore f has no admissible square root whose derivative is ω-continuous.*

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