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HYPOELLIPTICITY IN SPACES OF ULTRADISTRIBUTIONS—STUDY OF A MODEL CASE*

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ABSTRACT

In this work we study C^∞-hypoellipticity in spaces of ultradistributions for analytic linear partial differential operators. Our main tool is a new a-priori inequality, which is stated in terms of the behaviour of holomorphic functions on appropriate wedges. In particular, for sum of squares operators satisfying Hörmander's condition, we thus obtain a new method for studying analytic hypoellipticity for such a class. We also show how this method can be explicitly applied by studying a model operator, which is constructed as a perturbation of the so-called Baouendi–Goulaouic operator.

1. Introduction

Given a sum of squares operator $P(x, D)$, defined in an open set of $\mathbb{R}^N$ and satisfying Hörmander's condition, it is well known that $P(x, D)$ is hypoelliptic

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for distribution solutions: this is a classical result due to Hörmander [H, 1967]. However, when $P(x, D)$ is real-analytic, hypoellipticity for more general solutions may fail. For instance, in [CH, 2009], the authors have exhibited explicit hyperfunction solutions for the so-called Baouendi–Goulaouic operator which are not distributions.

In order to make our presentation more apparent it is convenient at this very beginning to recall the basic definitions of the standard ultradistribution spaces in $\mathbb{R}^N$. A function $f \in C^\infty(\Omega)$ ($\Omega \subset \mathbb{R}^N$ open set) belongs to $G^s(\Omega)$ ($s \in \mathbb{R}$, $s > 1$) if for each $K \subset \Omega$ compact there is $C_K > 0$ such that $\sup_K |D^\alpha f| \le C_K^{|\alpha|+1} \alpha!^s$ for all multi-indices α; f belongs to $G^{(s)}(\Omega)$ if for each compact set $K \subset \Omega$ and each $\varepsilon > 0$ there is $C_{K,\varepsilon} > 0$ such that $\sup_K |D^\alpha f| \le C_{K,\varepsilon} \varepsilon^{|\alpha|} \alpha!^s$ for all α. Let $G_c^s(\Omega)$, respectively $G_c^{(s)}(\Omega)$, denote the subspace of $G^s(\Omega)$, respectively $G^{(s)}(\Omega)$, formed by all functions with compact support. All these function spaces are provided with their natural locally convex space topologies (see [K, 1973]) and

$$G^{(s)}(\Omega) \subset G^s(\Omega),\ G_c^{(s)}(\Omega) \subset G_c^s(\Omega) \subset C_c^\infty(\Omega),\ G_c^\sigma(\Omega) \subset G_c^{(s)}(\Omega) \quad \text{if } 1 < \sigma < s,$$

with continuous imbeddings and dense images. The space $\mathcal{D}^{\{s\}'}(\Omega)$, respectively $\mathcal{D}^{(s)'}(\Omega)$, which is the dual space of $G_c^s(\Omega)$, respectively of $G_c^{(s)}(\Omega)$, are the *spaces of ultradistributions of order* s *in* Ω. Notice that $\mathcal{D}'(\Omega) \subset \mathcal{D}^{\{s\}'}(\Omega) \subset \mathcal{D}^{(s)'}(\Omega) \subset \mathcal{D}^{\{\sigma\}'}(\Omega)$ if $\sigma < s$.

Even if we work in the framework of ultradistribution solutions, hypoellipticity may still fail (cf. [Ma, 1987]). Since, however, every such $P(x, D)$ is Gevrey hypoelliptic of order $s \in [s_0, \infty[$, where s_0 depends on its type (see [ABC, 2009], [DZ, 1973]), and since also Gevrey hypoellipticity of order s for $P(x, D)$ implies C^∞-hypoellipticity for ${}^tP(x, D)$ in the $\mathcal{D}^{\{s\}\prime}$-sense (cf. Lemma 2.1 below), it seems reasonable to try to determine the optimal Gevrey regularity of $P(x, D)$ by examining for which values of s there is $u \in \mathcal{D}^{\{s\}'} \setminus \mathcal{D}'$ such that ${}^tP(x, D)u$ is a smooth function.

In this work we address this question by introducing a new necessary condition for C^∞-hypoellipticity in the ultradistribution sense. This is based on an a-priori inequality involving the complexification of the operator acting on holomorphic functions defined in appropriate wedges (cf. Proposition 3.1 below). Such a condition is presented in Section 3 and its derivation requires, as usual, some standard functional analytic methods. Furthermore, by recalling the results obtained in [CH, 2009], we also show that such an a-priori inequality is necessary for the analytic hypoellipticity (in the distribution framework) for operators belonging to a quite general class which includes the sum of squares operators alluded to above.

In the remaining part of the article we show how such a method can be applied by studying a model operator, constructed as a perturbation of the Baouendi–Goulaouic operator in three dimensions.

Let us consider the operator in $\mathbb{R}^3$ given by

$$\mathrm{P} = {}^t\mathrm{P} = \partial_{x_1}^2 + \partial_{x_2}^2 + x_1^2 g(x_2)^2 \partial_{x_3}^2.$$

Here g is a real-analytic function which extends as a holomorphic function to the complex disc $|z_2| < r_0$. We assume g real on the real axis and $g(0) = 1$. Thus P, which is defined in $\Omega = \mathbb{R}^2 \times]-r_0, r_0[$, satisfies Hörmander's condition. Hence P is hypoelliptic on Ω for distribution solutions. Neverthless, we show that when we allow ultradistribution solutions the situation changes drastically:

Theorem 1.1: *The following properties hold:*

(a) *for any $U \subset\subset \Omega$ open containing the origin, and each $1 < s < 2$, there is $u \in \mathcal{D}^{(s)\prime}(U) \setminus \mathcal{D}'(U)$ such that $\mathrm{P}u \in C^\infty(U)$;*

(b) *P is not analytic hypoelliptic (for distributions).*

Since $\mathcal{D}^{(s)'}(U) \subset \mathcal{D}^{\{\sigma\}'}(U)$ if $\sigma < s$ we obtain immediately from Theorem 1.1.(a) the following result:

Corollary 1.1: *For any $U \subset\subset \Omega$ open containing the origin, and each $1 < s < 2$, there is $u \in \mathcal{D}^{\{s\}\prime}(U) \setminus \mathcal{D}'(U)$ such that $\mathrm{P}u \in C^\infty(U)$.*

Since it is known that P is G^s-hypoelliptic for distributions if $s \geq 2$ ([DZ, 1973, Theorem 2.7]; see also [ABC, 2009]), the result stated in Corollary 1.1 is sharp, according to Lemma 2.1.

Finally, we briefly describe the proof of Theorem 1.1. We shall proceed by contradiction: we will violate the a-priori inequality for P mentioned before, by constructing a family of asymptotic, holomorphic solutions to the equation $\mathrm{P}u = 0$. Such solutions will be obtained after applying a version of the Ovcyannikov theorem presented in Section 4, and for this we will build, in Section 5, appropriate scales of Banach spaces of entire functions based on the harmonic oscillator operator (see also [M, 1981] for the use of similar scales). Finally, in Section 6, we will show that the solutions so obtained satisfy the required growth conditions.

2. An abstract result

In this section we consider an arbitrary linear partial differential operator with real-analytic coefficients $P = P(x, D_x)$ defined on an open set Ω of $\mathbb{R}^N$. The following result can be regarded as an ultradistribution version of the main abstract result in [CH, 2009].

LEMMA 2.1: *Assume that $P(x,D)$ is L^2-solvable on any relatively compact open subset of Ω and also that, for some $s > 1$, $P(x,D)$ is Gevrey hypoelliptic of order s in Ω. Then given $u \in \mathcal{D}^{\{s\}\prime}(\Omega)$, if ${}^tP(x,D)u \in L^2_{\rm loc}(\Omega)$ it follows that $u \in L^2_{\rm loc}(\Omega)$. In particular, if in addition ${}^tP(x,D)$ is C^∞-hypoelliptic, then ${}^tP(x,D)$ is C^∞-hypoelliptic in $\mathcal{D}^{\{s\}\prime}(\Omega)$.*

Proof. Replacing Ω by one of its relatively compact open subsets allows us to assume that $P(x,D)$ is L^2-solvable in Ω. Thus there is $K : L^2(\Omega) \to L^2(\Omega)$ bounded such that $P(x,D)K =$ identity in $L^2(\Omega)$. Observe that, since $P(x,D)$ is Gevrey hypoelliptic of order s, the inclusion $K(G^s_c(\Omega)) \subset G^s(\Omega) \cap L^2(\Omega)$ holds.

Let then $u \in \mathcal{D}^{\{s\}\prime}(\Omega)$ be as in the statement and let $U \subset\subset \Omega$ open. By the Riesz Representation Theorem we must show that $\lambda : G^s_c(U) \to \mathbb{C}$, $\lambda(\phi) = \langle u, \phi \rangle$, is continuous when we consider in $G^s_c(U)$ the topology induced by $L^2(U)$.

We take $\chi \in G^s_c(\Omega)$, $\chi \equiv 1$ in an open neighborhood of the closure of U. It follows that $\operatorname{supp} \mathrm{d}\chi \subset W$, where W is open and $\bar{U} \cap W = \emptyset$. Then

$$\lambda(\phi) = \langle \chi u, \phi \rangle = \langle \chi u, P(x,D)K\phi \rangle = \langle\, {}^tP(x,D)(\chi u), K(\phi) \rangle.$$

Consequently, we can write $\lambda(\phi) = \lambda_1(\phi) + \lambda_2(\phi)$, where

$$\lambda_1(\phi) = \langle \chi\, {}^tP(x,D)u, K(\phi) \rangle,\ \lambda_2(\phi) = \langle v, K(\phi) \rangle.$$

Here $v \in \mathcal{E}^{\{s\}\prime}(W)$.

Now, since ${}^tP(x,D)u \in L^2_{\rm loc}(\Omega)$, the Cauchy-Schwarz inequality and the $L^2(\Omega)$-continuity of K shows that $\phi \mapsto \lambda_1(\phi)$ is continuous with respect to the $L^2(U)$-norm. On the other hand, by using again that $P(x,D)$ is Gevrey hypoelliptic of order s, we have $K(g)|_W \in G^s(W)$ if $g \in L^2_c(\bar{U})$. We then obtain a linear map $\mu : L^2_c(\bar{U}) \to G^s(W)$, $\mu(g) = K(g)|_W$, whose graph is easily seen to be sequentially closed. Applying the version of the closed graph theorem presented in [Kö, 1979, p. 56],[1] we conclude that μ is continuous. Since

[1] $G^s(W)$ is a webbed space, a property that follows from [Kö, 1979, p. 55(4) and p. 63(7,8)].

$\lambda_2(\phi) = \langle v, \mu(\phi)\rangle$, it then follows that $\phi \mapsto \lambda_2(\phi)$ is also continuous with respect to the $L^2(U)$-norm. ∎

3. A new necessary condition for hypoellipticity

In this section we continue to consider an arbitrary linear partial differential operator with real-analytic coefficients $P = P(x, D_x)$ defined on an open set Ω of $\mathbb{R}^N$. We fix a complex neighborhood $\Omega_\bullet$ of Ω in $\mathbb{C}^n$ to which the coefficients of P extend as holomorphic functions. We write $P(z, D_z)$ for the extended operator.

If $U \subset\subset \Omega$ is open, Γ is an open convex cone in $\mathbb{R}^N \setminus \{0\}$ and $\delta > 0$ we set

$$\mathcal{W}_\delta(U;\Gamma) = \{z = x + iy : x \in U,\, y \in \Gamma,\, |y| < \delta\},$$

$$\overline{\mathcal{W}}_\delta(U;\Gamma) = \mathcal{W}_\delta(U;\Gamma) \cup (U + i\{0\}).$$

We take $\delta > 0$ appropriately small in order that $\overline{\mathcal{W}}_\delta(U;\Gamma) \subset\subset \Omega_\bullet$.

Let $\eta > 0$. We shall consider the Fréchet space $\mathcal{O}_\eta(\mathcal{W}_\delta(U;\Gamma))$ of all holomorphic functions F on $\mathcal{W}_\delta(U;\Gamma)$ such that, for any compact K of $\overline{\mathcal{W}}_\delta(U;\Gamma)$,

$$|f|_{\eta,K} \doteq \sup_{\mathcal{W}_\delta(U;\Gamma)\cap K} |F(x+iy)|\, e^{-1/|y|^\eta} < \infty\,.$$

By [K, 1973, Theorem 11.5] it follows that

$$b_\Gamma\left(\mathcal{O}_\eta(\mathcal{W}_\delta(U;\Gamma))\right) \subset \mathcal{D}^{(1+1/\eta)\prime}(U),$$

where b_Γ is the hyperfunction boundary value map.

Lemma 3.1: *The space*

$$\mathrm{E} = \{F \in \mathcal{O}_\eta(\mathcal{W}_\delta(U;\Gamma)) : b_\Gamma({}^tP(z, D_z)F) \in C^\infty(U)\},$$

with the locally convex topology defined by the seminorms

$$F \mapsto |F|_{\eta,K} + \|b_\Gamma({}^tP(z,D_z)F)\|_{C^M(K')}, \quad K \subset\subset \overline{\mathcal{W}}_\delta(U;\Gamma),\, K' \subset\subset U,\, M \in \mathbb{Z}_+,$$

is a Fréchet space.

Proof. Let $\{F_j\}$ be a Cauchy sequence in E. Then $\{F_j\}$ is a Cauchy sequence in $\mathcal{O}_\eta(\mathcal{W}_\delta(U;\Gamma))$ and $\{b_\Gamma({}^tP(z,D_z)F_j\}$ is a Cauchy sequence in $C^\infty(U)$. Since these spaces are Fréchet, we conclude the existence of $F \in \mathcal{O}_\eta(\mathcal{W}_\delta(U;\Gamma))$ and $v \in C^\infty(U)$ such that $F_j \to F$ in $\mathcal{O}_\eta(\mathcal{W}_\delta(U;\Gamma))$ and $b_\Gamma({}^tP(z,D_z)F_j) \to v$ in $C^\infty(U)$. Now, [K, 1973, Theorem 11.5] implies that the map

$b_\Gamma : \mathcal{O}_\eta(\mathcal{W}_\delta(U;\Gamma)) \to \mathcal{D}^{(1+1/\eta)\prime}(U)$ is continuous.[2] Hence we have $b_\Gamma(F_j) \to b_\Gamma(F)$ in $\mathcal{D}^{(1+1/\eta)\prime}(U)$ and hence

$$b_\Gamma({}^tP(z,D_z)F_j) = {}^tP(z,D_z)\,(b_\Gamma(F_j)) \to {}^tP(z,D_z)(b_\Gamma(F)) = b_\Gamma({}^tP(z,D_z)F)$$

in $\mathcal{D}^{(1+1/\eta)\prime}(U)$. Since convergence in $C^\infty(U)$ implies convergence in $\mathcal{D}^{(1+1/\eta)\prime}(U)$ it follows that $v = b_\Gamma({}^tP(z,D_z)F)$, which concludes the proof. ∎

Proposition 3.1: *Suppose that, for some $\eta > 0$, the following property holds for every open set $U \subset\subset \Omega$:*

(∗) Given $u \in \mathcal{D}^{(1+1/\eta)\prime}(U)$, then ${}^tPu \in C^\infty(U)$ implies $u \in C^\infty(U)$.

Let U, Γ, $\delta > 0$ be as before. Then given $K_0 \subset\subset U$ there are compact sets $K \subset \overline{\mathcal{W}}_\delta(U;\Gamma)$, $K' \subset U$, $M \in \mathbb{Z}_+$ and $C > 0$ such that

(1)
$$\sup_{K_0}|F| \leq C\Big(\sup_{\mathcal{W}_\delta(U;\Gamma)\cap K}|F(x+iy)|e^{-1/|y|^\eta} + \|\,{}^tP(x,D_x)F\|_{C^M(K')}\Big),\ F \in \mathcal{O}(\Omega_\bullet).$$

Proof. We consider the Fréchet space $\mathcal{O}(\overline{\mathcal{W}}_\delta(U;\Gamma))$ of all functions G which are holomorphic on $\mathcal{W}_\delta(U;\Gamma)$ and smooth up to $U + i\{0\}$, where now the topology is defined by the seminorms

$$G \mapsto \sup_K |D_z^\alpha G|, \quad K \subset\subset \overline{\mathcal{W}}_\delta(U;\Gamma),\ \alpha \in \mathbb{Z}_+^N.$$

Property (∗) implies that $\mathrm{E} \subset \mathcal{O}(\overline{\mathcal{W}}_\delta(U;\Gamma))$ and the closed graph theorem implies that this inclusion is continuous, from which (1) follows. ∎

Corollary 3.1: *Suppose that $P(x,D_x)$ is L^2-solvable on every open set $U \subset\subset \Omega$ and that ${}^tP(x,D)$ is hypoelliptic in Ω (such properties hold, for instance, if $P(x,D)$ is a sum of squares operator satisfying Hörmander's condition). Assume that P is analytic hypoelliptic (for distributions) in Ω and let U, Γ, $\delta > 0$ be as before. Then given $\eta > 0$ and $K_0 \subset\subset U$ there are compact sets $K \subset \overline{\mathcal{W}}_\delta(U;\Gamma)$, $K' \subset U$, $M \in \mathbb{Z}_+$ and $C > 0$ such that (1) holds.*

Indeed, according to the main result in [CH, 2009], *every such operator satisfies property (∗) for every $\eta > 0$.* ∎

[2] Here we must recall that $\mathcal{O}_\eta(\mathcal{W}_\delta(U;\Gamma))$ is bornological since it is metrizable ([Kö, 1979, p. 380]) and hence $b_\Gamma : \mathcal{O}_\eta(\mathcal{W}_\delta(U;\Gamma)) \to \mathcal{D}^{(1+1/\eta)\prime}(U)$, being locally bounded, is continuous ([Kö, 1979, p. 381]). See also [Tr, 1967, Proposition 14.8, p. 141].

4. Proof of Theorem 1.1

We now return to the operator P defined in the Introduction. We let

$$F_\lambda(z) = e^{i\lambda z_3} f_\lambda(\lambda^{1/2} z_1, z_2), \quad \lambda \geq 1,$$

with $f_\lambda = f_\lambda(\zeta, z_2) \in \mathcal{O}(\mathbb{C}^2)$. Then

$$\mathrm{P} F_\lambda(z) = e^{i\lambda z_3} (Q_\lambda f_\lambda)(\lambda^{1/2} z_1, z_2),$$

where we have written

(2) $$Q_\lambda = \partial_{z_2}^2 - \lambda \left\{ \zeta^2 g(z_2)^2 - \partial_\zeta^2 \right\}.$$

In what follows we shall use the following notation: if $r > 0$, we denote by $D(r)$ the open disc centered at the origin in $\mathbb{C}$ and with radius r. The main goal of the present work will be to prove the following result, from which our main result (Theorem 1.1) follows:

PROPOSITION 4.1: *There is $0 < \rho < r_0$ such that for each $1/2 < \kappa < 1$ there is $f_\lambda \in \mathcal{O}(\mathbb{C} \times D(\rho))$ satisfying the following properties:*

(i) $f_\lambda(0,0) = 1$;

(ii) *There are $a > 0$, $C > 0$ such that*

(3) $$|f_\lambda(\zeta, z_2)| \leq C\, e^{a(|\Im \zeta|^2 + \lambda^\kappa)}, \; (\zeta, z_2) \in \mathbb{C} \times D(\rho).$$

(iii) $\forall M \in \mathbb{Z}_+,\ \lambda^M \sum_{p+q \leq M} \|(\partial_\xi^p \partial_x^q Q_\lambda f_\lambda)(\xi, x)\|_{L^\infty(\mathbb{R} \times \{x \in \mathbb{R} : |x| \leq \rho\})} \longrightarrow 0$ *when $\lambda \to \infty$.*

Proof of Theorem 1.1. We assume that (1) holds for P with $U \subset\subset \mathbb{R}^2 \times D(\rho)$ an open set containing the origin, $K_0 = \{0\}$, $\Gamma \subset \{y_3 - a|y_1| > \varepsilon |y|\} (\varepsilon > 0)$, $0 < \delta \leq 1$ and $\eta > 1$. We obtain

$$c \leq \sup_{y \in \Gamma,\, |y_1| \leq \delta\, , |z_2| < \rho} |e^{i\lambda z_3} f_\lambda(\sqrt{\lambda} z_1, z_2)| e^{-1/|y|^\eta} + R(\lambda),$$

where $c > 0$ and $R(\lambda) \to 0$ when $\lambda \to \infty$. Hence

(4) $$c \leq e^{-\lambda(y_3 - a|y_1|) + a\lambda^\kappa - |y|^{-\eta}} + R(\lambda).$$

Choose $1/2 < \kappa < 1$ with $1 + 1/\eta < 1/\kappa$. If $\kappa/\eta < \alpha < 1 - \kappa$ we estimate the exponent in (4) as follows:

- If $y \in \Gamma$ and $|y| \leq \lambda^{-\alpha}$, then $|y|^{-\eta} \geq \lambda^{\alpha\eta}$ and the exponent is $\leq -\lambda^{\alpha\eta} + a\lambda^\kappa$.

- If $y \in \Gamma$ and $|y| > \lambda^{-\alpha}$, then $(y_3 - a|y_1|) \geq \epsilon\lambda^{-\alpha}$ and the exponent is now $\leq -\epsilon\lambda^{1-\alpha} + a\lambda^{\kappa}$.

Thus, for any $\eta > 1$, we have contradicted the validity of estimate (1). Consequently, for each $1 < s < 2$ there exists $u \in \mathcal{D}^{(s)\prime}(U) \setminus \mathcal{D}'(U)$ such that $\mathrm{P}u \in C^{\infty}(U)$. Finally, by Corollary 3.1, it follows that P is not analytic hypoelliptic. ∎

A PROPERTY OF THE BAOUENDI–GOUALOUIC OPERATOR. When $g = 1$, then P equals P_0, the well-known Baouendi–Goulaouic operator [BG, 1972]. In this case we can even derive the existence of a solution to the homogeneous equation $\mathrm{P}_0 u = 0$ which belongs to $\mathcal{D}^{(2)\prime} \setminus \mathcal{D}'$ (this statement is analogous to a result of Matsuzawa [Ma, 1987] concerning the heat operator). Indeed, in complex variables as before we have

$$\mathrm{P}_0 = \partial_{z_1}^2 + \partial_{z_2}^2 + z_2^2\partial_{z_3}^2 = (\partial_{z_2} - iz_2\partial_{z_3})(\partial_{z_2} + iz_2\partial_{z_3}) + \partial_{z_1}^2 - i\partial_{z_3}.$$

If $F(\zeta, z_1)$ is holomorphic, then $g(z) = F(z_2^2/2 - iz_3, z_1)$ satisfies $\mathrm{P}_0 g = 0$ if

$$\partial_{z_1}^2 F - \partial_{\zeta} F = 0.$$

We obtain a solution of this equation by setting

$$F(\zeta, z_1) = \zeta^{-1/2} e^{z_1^2/4\zeta}.$$

Hence

$$u(z) = \frac{1}{(z_2^2/2 - iz_3)^{1/2}} \exp\left\{\frac{z_1^2}{2z_2^2 - 4iz_3}\right\}$$

satisfies $\mathrm{P}_0 u = 0$.

In the truncated cone $\Gamma = \{y_3 > |y_2|,\ |y_2| < 1\}$, we have the estimate

$$|2z_2^2 - 4iz_3| \geq \Re(2z_2^2 - 4iz_3) = 2(x_2^2 - y_2^2) + 4y_3 \geq 2y_3$$

and hence, still in Γ,

$$|u(z)| \leq \frac{1}{(2y_3)^{1/2}} \exp\left\{\frac{|z_1|^2}{2y_3}\right\}.$$

Since in any cone $\Gamma' \subset\subset \Gamma$ we can dominate $y_3 \geq c|y|$, we conclude $b_\Gamma(u)$ belongs to ${\mathcal{D}^{(2)}}' \setminus \mathcal{D}'$ and satisfies $\mathrm{P}_0 b_\Gamma(u) = 0$. ∎

5. A version of the Ovcyannikov Theorem

We pause to discuss an abstract Cauchy problem which will produce the sought family f_λ. The result is known (see, e.g., [Tr, 1968]) but it is worth recalling its proof, mainly in order to derive an estimate for the solution.

Let us consider a scale of Banach spaces $\{E_s\}$, where $0 \le a \le s \le b < \infty$. As usual we have $E_{s'} \subset E_s$ if $s \le s'$ and this inclusion is continuous, with norm ≤ 1.

We shall assume we are given a holomorphic map $A(z)$, defined for $z \in \mathbb{C}$, $|z| < \rho$ and valued in $\mathcal{L}(E_{s'}, E_s)$, the space of bounded linear operators from $E_{s'}$ into E_s, for every pair $s < s'$. We also assume that

$$\|A(z)\| \le \frac{\vartheta}{(s'-s)^\theta}, \quad |z| < \rho, \tag{5}$$

where $\vartheta > 0$ and $0 < \theta < 1$. The norm in (5) is of course the one in $\mathcal{L}(E_{s'}, E_s)$. *We shall refer to this property by saying that A is an endomorphism of the scale $\{E_s\}$ of type θ.*

THEOREM 5.1: *Let $h \in \mathcal{O}(\{|z| < \rho\}; E_b)$. Under the preceding hypotheses the Cauchy problem*

$$u'(z) = A(z)u(z) + h(z), \quad |z| < \rho, \quad u(0) = 0 \in E_b, \tag{6}$$

has a (unique) solution u which belongs to $\mathcal{O}(\{|z| < \rho\}; E_s)$ for every $a \le s < b$.

Proof. We define by induction the following sequence $u_n \in \mathcal{O}(\{|z| < r\}; E_a)$: we set $u_0(z) = 0$ and

$$u_{n+1}(z) = \int_0^z h(\sigma)\mathrm{d}\sigma + \int_0^z A(\sigma)u_n(\sigma)\,\mathrm{d}\sigma.$$

We shall prove by induction the following estimates, for $s \in [a, b[$:

$$\|u_n(z) - u_{n-1}(z)\|_s \le M\,\frac{\vartheta^n e^{\theta n}|z|^n}{(b-s)^{\theta n}(n!)^{1-\theta}}, \quad |z| < \rho. \tag{7}$$

Here we have set

$$M = \sup_{|z|<\rho} \int_0^z \|h(\sigma)\|_b\,|\mathrm{d}\sigma|$$

and $u_{-1} = 0$, and thus (7) is valid for $n = 0$. We then assume (7) valid for $n-1$. Take $a \leq s < s+\delta < b$. We have

$$\|u_n(z) - u_{n-1}(z)\|_s \leq \int_0^t \frac{\vartheta}{\delta^\theta} \|u_{n-1}(\sigma) - u_{n-2}(\sigma)\|_{s+\delta} |\mathrm{d}\sigma|$$
$$\leq \frac{M\vartheta^n e^{\theta(n-1)}|z|^n}{\delta^\theta (b-s-\delta)^{\theta(n-1)} n(n-1)!^{1-\theta}}.$$

If as usual we take $\delta = (b-s)/n$, then (7) follows immediately.

It follows, in particular, that u_n converges, in $\mathcal{O}(\{|z| < \rho\}; E_s)$, to an element $u \in \mathcal{O}(\{|z| < \rho\}; E_s)$ which clearly satisfies

$$u(z) = \int_0^z h(\sigma)\,\mathrm{d}\sigma + \int_0^z A(\sigma)u(\sigma)\mathrm{d}\sigma.$$

In particular $u \in \mathcal{O}(\{|z| < r\}; E_s)$, for every $a \leq s < b$ and $u'(z) = A(z)u(z) + h(z)$.

The proof of the uniqueness is standard. ∎

An estimate for the solution $u(z)$. Observe that, for $|z| < \rho$, we have

$$\|u(z) - u_0\|_s \leq \sum_{n=1}^{\infty} \|u_n(z) - u_{n-1}(z)\|_s \leq M \sum_{n=1}^{\infty} \frac{(\gamma\rho\vartheta)^n}{n!^{(1-\theta)}},$$

where $\gamma \doteq (e/(b-s))^\theta$. Applying Lemma A.1.2 in Appendix 1 gives

$$\|u(z)\|_s \leq MKe^{K(\rho\vartheta)^{1/(1-\theta)}/(b-s)^{\theta/(1-\theta)}}, \quad |z| < \rho, \tag{8}$$

where $K > 0$ depends only on θ. ∎

6. A scale of Banach spaces of entire functions

Write $\zeta = \xi + i\eta$ and consider the harmonic oscillator operator in $\mathbb{R}$:

$$T = \xi^2 - \partial_\xi^2.$$

As is well known and obvious by formula (25) below, T has an inverse $S \in \mathcal{L}(L^2(\mathbb{R}))$, the ring of bounded linear oparators in $L^2(\mathbb{R})$.

If $s > 0$ and $\theta \in]0,1[$, we shall denote by $G_{s,\theta}$ the vector space of all $h = h(\xi) \in \mathcal{S}(\mathbb{R})$ for which

$$\|h\|_{s,\theta} = \sup_{n \geq 0} \left\{ \frac{\|T^n h\|_0 \, s^{\theta n}}{n!^\theta} \right\} < \infty.$$

Each $G_{s,\theta}$ is a Banach space. Moreover, $G_{s',\theta} \subset G_{s,\theta}$ if $s' \geq s$ and these inclusions have norm ≤ 1. Notice furthermore that $\psi(\xi) = e^{-\xi^2/2}$ belongs to $G_{s,\theta}$ for every s and θ, for $T\psi = \psi$.

In the next result we summarize the key properties of this scale of Banach spaces.

Proposition 6.1: (1) *The operator T defines endomorphisms of the scale $\{G_{s,\theta}\}$ of type θ. More precisely, if $0 < s < s'$ we have*

$$\|Th\|_{s,\theta} \leq \frac{(s'/s)^\theta}{(s'-s)^\theta} \|h\|_{s',\theta} \,. \tag{9}$$

(2) *If $h \in G_{s,\theta}$, then h extends as an entire function of $\zeta = \xi + i\eta$ and*

$$|h(\zeta)| \leq A \|h\|_{s,\theta} e^{A|\eta|^2}. \tag{10}$$

Proof. For (1) we observe that

$$\begin{aligned} \|Th\|_{s,\theta} &= \sup_{n\geq 0} \left\{ \frac{\|T^{n+1}h\|_0 \, s^{\theta n}}{n!^\theta} \right\} \\ &\leq \max_{n\geq 0} \left\{ (n+1)s^n/(s')^{n+1} \right\}^\theta \|h\|_{s',\theta}, \end{aligned}$$

and hence to conclude the proof of (1) it suffices to notice that for every $m \in \mathbb{N}$ we have

$$\frac{m}{s}\left(\frac{s}{s'}\right)^m = \frac{m}{s}\, e^{-m\log(s'/s)} \leq \frac{s'/s}{s'-s}.$$

The proof of (2) will be presented in Appendix 2. ∎

Let now M denote the operator multiplication by ξ and let $S \in \mathcal{L}(L^2(\mathbb{R}))$ be the inverse of T. Let also

$$\Theta_n \doteq T^n M^2 S^{n+1}.$$

In Appendix 2 we shall also present the proof of the following result:

Lemma 6.1: *For each $n = 0, 1, 2, \dots$ we have $\Theta_n \in \mathcal{L}(L^2(\mathbb{R}))$ and there is $\mu > 1$ such that $\|\Theta_n\| \leq \mu^{n+1}$, $n = 0, 1, 2, \dots$.*

We can then prove:

Lemma 6.2: *If $s' > s > 0$, and if μ is the constant given by Lemma 6.1, then M^2 maps $G_{\mu^{1/\theta}s',\theta}$ continuously into $G_{s,\theta}$ and*

$$\|M^2 f\|_{s,\theta} \leq \frac{(s'/s)^\theta}{(s'-s)^\theta} \|f\|_{\mu^{1/\theta}s',\theta}. \tag{11}$$

Proof. Since

$$\|T^n M^2 f\|_0 = \|\Theta_n T^{n+1} f\|_0 \le \mu^{n+1} \|T^{n+1} f\|_0,$$

we have

$$\|M^2 f\|_{s,\theta} = \sup_{n\ge 0}\left\{\frac{\|T^n M^2 f\|_0 s^{\theta n}}{n!^\theta}\right\} \le \mu \sup_{n\ge 0}\left\{\frac{\|T^{n+1} h\|_0\,(\mu^{1/\theta} s)^{\theta n}}{n!^\theta}\right\}$$

and the argument concludes as in the proof of Proposition 2. ∎

7. Proof of Proposition 4.1.

We shall consider the equation $Q_\lambda f_\lambda = 0$ in the variables $(\xi, z_2) \in \mathbb{R} \times D(r_0)$ and write it in the form of a system. If we set

$$u_\lambda = \begin{bmatrix} f_\lambda \\ \lambda^{-1/2} f'_\lambda \\ T f_\lambda \end{bmatrix},$$

then $Q_\lambda f_\lambda = 0$ is equivalent to the first order system

(12) $$\partial_{z_2} u_\lambda = \lambda^{1/2} D(z_2, \xi, \partial_\xi) u_\lambda,$$

where

$$D(z_2, \xi, \partial_\xi) \doteq \begin{bmatrix} 0 & I & 0 \\ g_1(z_2)\xi^2 & 0 & I \\ 0 & T & 0 \end{bmatrix},$$

and $g_1 \doteq g^2 - 1$. Notice that

(13) $$g_1(z_2) = z_2\, g_\star(z_2).$$

At this point we make a crucial remark: this first order system can be interpreted as an ODE valued in the scale $E_{s,\theta} \doteq G_{s,\theta} \times G_{s,\theta} \times G_{s,\theta}$. We shall view u_λ as a holomorphic function of z_2 valued in $E_{s,\theta}$ which, as we have seen, is a space of entire functions of ζ. We rewrite (12) as

(14) $$u'_\lambda(z_2) = \lambda^{1/2} A u_\lambda(z_2) + \lambda^{1/2} g_1(z_2) B u_\lambda,$$

where now

$$A \doteq \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & T & 0 \end{bmatrix}, \quad B \doteq \begin{bmatrix} 0 & 0 & 0 \\ M^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have the following estimates, which follow from Proposition 6.1 and Lemma 6.2:

$$\|A\| \le \vartheta/(s'-s)^\theta \qquad \text{in } \mathcal{L}(E_{s',\theta}; E_{s,\theta}),$$

$$\|B\| \le \vartheta/(s'-s)^\theta \qquad \text{in } \mathcal{L}(E_{\mu^{1/\theta}s',\theta}; E_{s,\theta}),$$

where $\vartheta = C(s'/s)^\theta$.

On $D(\rho)$, with $0 < \rho < r_0$ to be chosen, we shall construct a formal solution to (14) in the form

$$u_\lambda(z_2) = \sum_{j\ge 0} \lambda^{-j/2} v_{\lambda,j}(\sqrt{\lambda}\, z_2).$$

Making use of (13) we have the recursion formulae

$$v'_{\lambda,0}(w) - Av_{\lambda,0}(w) = 0, \quad w \in D(\sqrt{\lambda}\rho), \tag{15}$$

$$v'_{\lambda,j}(w) - Av_{\lambda,j}(w) = wg_{\lambda,\star}(w)Bv_{\lambda,j-1}(w), \quad w \in D(\sqrt{\lambda}\rho),\ j \ge 1. \tag{16}$$

Here we have written $g_{\lambda,\star}(w) = g_\star(w/\sqrt{\lambda})$. We take, as a solution of (15), the function $v_{\lambda,0}(w) = e^w v_0$, where

$$v_0 = \begin{bmatrix} \psi \\ \psi \\ \psi \end{bmatrix},$$

and solve (16) with initial condition $v_{\lambda,j}(0) = 0$, $j \ge 1$. We apply Theorem 5.1 and then obtain a sequence

$$\{v_{\lambda,j}\}_{j\ge 0} \subset \bigcap_{s>0} \mathcal{O}(D(\sqrt{\lambda}\rho), E_{s,\theta})$$

solving (16) and satisfying $v_{\lambda,j}(0) = 0$ when $j \ge 1$.

For $s>0$ fixed we apply (8) taking $(E_{\tau,\theta})_{s\leq\tau\leq 2s}$ as the scale of Banach spaces. Since, for this particular scale, we can bound $\vartheta\leq C2^{\theta}$, estimate (8) gives

(17) $$\|v_{\lambda,j}(w)\|_{s,\theta}\leq$$
$$K\sup_{|w|\leq\sqrt{\lambda}\rho}\left\{\int_0^w|\sigma g_{\lambda,\star}(\sigma)|\|Bv_{\lambda,j-1}(\sigma)\|_{2s,\theta}\,|\mathrm{d}\sigma|\right\}\exp\left\{\frac{K\lambda^{1/(2-2\theta)}}{s^{\theta/(1-\theta)}}\right\},\ |w|<\sqrt{\lambda}\rho,$$

where K is a constant that depends only on θ if we restrict $\rho\leq 1$.

From now on we shall write

$$|\!|\!|v_{\lambda,j}|\!|\!|_{s,\theta}=\sup_{|w|\leq\sqrt{\lambda}\rho}\|v_{\lambda,j}(w)\|_{s,\theta}.$$

If we further take $\beta\geq 3$ and notice that

$$\sup_{|w|\leq\sqrt{\lambda}\rho}\int_0^w|\sigma g_{\lambda,\star}(\sigma)|\,|\mathrm{d}\sigma|\leq\lambda\rho\|g_1\|_{L^\infty(D(\rho))}$$

we obtain, for $|w|<\sqrt{\lambda}\rho$,

$$|\!|\!|v_{\lambda,j}|\!|\!|_{s,\theta}\leq\rho K_\bullet\frac{\lambda\beta^\theta}{(\beta-2)^\theta s^\theta}\exp\left\{\frac{K\lambda^{1/(2-2\theta)}}{s^{\theta/(1-\theta)}}\right\}|\!|\!|v_{\lambda,j-1}|\!|\!|_{\mu^{1/\theta}\beta s,\theta}\,, \tag{18}$$

where $K_\bullet$ is a new constant depending only on θ and μ is given by Lemma 6.1. We emphazise that this inequality holds for every $s>0$ and for every $\beta\geq 3$. Since $\beta/(\beta-2)\leq 3$, if $\beta\geq 3$ we can further write, after redefining $K_\bullet$,

$$|\!|\!|v_{\lambda,j}|\!|\!|_{s,\theta}\leq\rho K_\bullet\frac{\lambda}{s^\theta}\exp\left\{\frac{K\lambda^{1/(2-2\theta)}}{s^{\theta/(1-\theta)}}\right\}|\!|\!|v_{\lambda,j-1}|\!|\!|_{\mu^{1/\theta}\beta s,\theta}. \tag{19}$$

Let now $\omega_\theta>0$ be such that

$$t\leq\omega_\theta\exp\left\{Kt^{1/(2-2\theta)}\right\},\quad t\geq 0.$$

Then

$$\frac{\lambda}{s^\theta}\leq\omega_\theta\exp\left\{\frac{K\lambda^{1/(2-2\theta)}}{s^{\theta/(2-2\theta)}}\right\}.$$

Since for $s\geq 1$ we have $s^{\theta/(2-2\theta)}\leq s^{\theta/(1-\theta)}$ we obtain, with a new constant K that depends only on θ and with a redefinition of β,

$$|\!|\!|v_{\lambda,j}|\!|\!|_{s,\theta}\leq\rho K\exp\left\{\frac{K\lambda^{1/(2-2\theta)}}{s^{\theta/(2-2\theta)}}\right\}|\!|\!|v_{\lambda,j-1}|\!|\!|_{\beta s,\theta}\,. \tag{20}$$

Notice that (20) holds for every $s\geq 1$, $\beta\geq 3\mu^{1/\theta}$, $\lambda\geq 1$, $j\geq 1$.

We start by estimating $v_{\lambda,0}$. For any $s > 0$ and $\theta > 0$ we have

$$\|v_{\lambda,0}\|_{s,\theta} \le \sup_{n\ge 0} \frac{s^{n\theta}}{n!^{\theta}} \le e^{s\theta}.$$

If we iterate (18) and assume $\rho K \le 1$ we obtain

$$\|v_{\lambda,j}\|_{s,\theta} \le e^{s\theta} \exp\left\{ \frac{\iota K \lambda^{1/(2-2\theta)}}{s^{\theta/(2-2\theta)}} \right\}, \tag{21}$$

where

$$\iota = \sum_{j=0}^{\infty} \beta^{-\frac{j\theta}{2-2\theta}}.$$

We remark that this inequality holds for every $s \ge 1$, $\lambda \ge 1$.

Finally we shall set

$$u_\lambda(z_2) \doteq \sum_{j<\lambda^{1/(1-\theta)}} \frac{1}{\lambda^{j/2}} v_{\lambda,j}(\sqrt{\lambda} z_2).$$

From estimate (21) we derive

$$\|u_\lambda(z_2)\|_{s,\theta} \le e^{s\theta} \lambda^{1/(1-\theta)} \exp\left\{ \frac{\iota K \lambda^{1/(2-2\theta)}}{s^{\theta/(2-2\theta)}} \right\}, \quad z_2 \in D(\rho),\ s \ge 1,$$

and thus $f_\lambda(z_2)$, the first component of $u_\lambda(z_2)$, satisfies (3) with $\kappa = 1/(2-2\theta)$ (cf. Proposition 6.1 (2)).

If we denote

$$\mathcal{L} = d/dz_2 - \lambda^{1/2} A - \lambda^{1/2} g_1(z_2) B,$$

then a computation, which makes use of (16), gives

$$\mathcal{L} u_\lambda = -\frac{g_1(z_2)}{\lambda^{(q-1)/2}} B v_{\lambda,q}(\sqrt{\lambda} z_2),$$

where q is the integer part of $\lambda^{1/(1-\theta)} - 1$. Since $Q_\lambda f_\lambda$ is the second component of the vector $\mathcal{L} u_\lambda$ we derive an estimate of the kind

$$|Q_\lambda f_\lambda(\xi, z_2)| \le C e^{C\lambda^{1/(2-2\theta)} - c\lambda^{1/(1-\theta)} \log \lambda}, \quad \xi \in \mathbb{R},\ z_2 \in D(\rho).$$

Property (iii) in Proposition 4.1 then follows easily and our argument is complete. ∎

Appendix 1

In this appendix we prove Lemma A.1.2 used earlier. For this we need

Lemma A.1.1: *If $A, \omega > 0$, then*

$$\sum_{n=0}^{\infty}\left(\frac{A^n}{n^n}\right)^{\omega} \leq (3A/e + C_\omega)e^{\omega A/e},$$

where $C_\omega = [1-(e/3)^\omega]^{-1}$.

Proof. Consider the function $\Lambda(t) = A^t/t^t = e^{t\log(A/t)}$, defined for $t > 0$. The maximum of Λ is attained at the point $t_0 = A/e$. Thus

$$\Lambda(t) \leq \Lambda(t_0) = e^{A/e}.$$

We split the sum as

$$\sum_{n=0}^{\infty}\left(\frac{A^n}{n^n}\right)^{\omega} = \underbrace{\sum_{n\leq 3A/e}\left(\frac{A^n}{n^n}\right)^{\omega}}_{\doteq S_1} + \underbrace{\sum_{n>3A/e}\left(\frac{A^n}{n^n}\right)^{\omega}}_{\doteq S_2}$$

and see that

$$S_1 \leq (3A/e)e^{\omega A/e},$$
$$S_2 \leq \sum_{n=0}^{\infty}(e/3)^{n\omega} \doteq C_\omega. \qquad \blacksquare$$

Lemma A.1.2: *If $R, \omega > 0$, then*

$$\sum_{n=0}^{\infty}\frac{R^n}{n!^\omega} \leq (3R^{1/\omega} + C_\omega)e^{\omega R^{1/\omega}}$$

Proof. We have

$$\sum_{n=0}^{\infty}\frac{R^n}{n!^\omega} \leq \sum_{n=0}^{\infty}\frac{R^n e^{n\omega}}{n^{n\omega}} = \sum_{n=0}^{\infty}\left(\frac{R^{n/\omega}e^n}{n^n}\right)^{\omega}. \qquad \blacksquare$$

Appendix 2

In this appendix we shall apply some well known facts concerning the sequence of Hermite functions $\{\psi_p\}_{p\geq 0}$. Each ψ_p can be written as

$$\psi_p(\tau) = c_p h_p(\tau)e^{-\tau^2/2}, \quad c_p = \pi^{-1/4}(2^p p!)^{-1/2},$$

where $\{h_p(\tau)\}_{p\geq 0}$ is the sequence of Hermite polinomials.

The following properties are well known and will be crucial for us:

$$T\psi_p = (2p+1)\psi_p, \quad p \geq 0; \tag{22}$$

$$h_{p+1}(\tau) + 2\tau h_p(\tau) + 2p h_{p-1}(\tau) = 0, \quad p \geq 1. \tag{23}$$

Proof of Proposition 6.1 part (2). Let $f \in G_{s,\theta}$ and write

$$f = \sum_{p=0}^{\infty} a_p \psi_p.$$

Then

$$T^n f = \sum_{p=0}^{\infty} (2p+1)^n a_p \psi_p$$

and hence

$$\sum_{p=0}^{\infty} (2p+1)^{2n} |a_p|^2 \leq \frac{n!^{2\theta}}{s^{2\theta n}} \|f\|_{s,\theta}^2, \quad n \geq 0.$$

In particular, taking $p = n$ in the summation gives

$$|a_n| \leq \frac{n!^{\theta}}{s^{\theta n}(2n+1)^n} \leq \frac{\|f\|_{s,\theta}}{s^{\theta n} n!^{1-\theta}}.$$

Now, according to [M, 1980, p. 842] there is a constant $L > 0$ such that

$$|\psi_p^{(j)}(\xi)| \leq j!^{1/2} L^{p+j} e^{-\xi^2/4}$$

and thus

$$|\psi_p(\xi + i\eta)| \leq \sum_{j=0}^{\infty} \frac{|\psi_p^{(j)}(\xi)|}{j!} |\eta|^j \leq L^p \sum_{j=0}^{\infty} \frac{L^j}{j!^{1/2}} |\eta|^j \leq L_1 L^p e^{L_1 |\eta|^2}.$$

Hence we can estimate (recall that $0 < \theta < 1$)

$$\begin{aligned} |f(\xi + i\eta)| \leq & \sum_{p=0}^{\infty} |a_p| |\psi_p(\xi + i\eta)| \leq L_1 \left\{ \sum_{p=0}^{\infty} |a_p| L^p \right\} e^{L_1 |\eta|^2} \\ \leq & L_1 \left\{ \sum_{p=0}^{\infty} \frac{L^p}{s^{\theta p} p!^{1-\theta}} \right\} \|f\|_{s,\theta} \, e^{L_1 |\eta|^2}. \qquad \blacksquare \end{aligned}$$

Proof of Lemma 6.1. Notice that a simple computation shows that (23) is equivalent to

$$\tau \psi_p = -\frac{1}{\sqrt{2}} \left\{ \sqrt{p}\,\psi_{p-1} + \sqrt{p+1}\,\psi_{p+1} \right), \tag{24}$$

which gives

$$M^2\psi_p = \frac{1}{2}\left\{\sqrt{p(p-1)}\psi_{p-2} + (2p+1)\psi_p + \sqrt{(p+2)(p+1)}\psi_{p+2}\right\}.$$

The inverse S of T can be defined by

$$S(\psi_p) = (\psi_p/(2p+1)). \tag{25}$$

A simple computation shows that

$$\Theta_n\psi_p = a_{n,p-2}\psi_{p-2} + \psi_p + a_{n,p+2}\psi_{p+2},$$

where

$$a_{n,p-2} = \frac{(2p-3)^n p^{1/2}(p-1)^{1/2}}{(2p+1)^n},\ a_{n,p+2} = \frac{(2p+5)^n(p+2)^{1/2}(p+1)^{1/2}}{(2p+1)^{n+1}}.$$

We then obtain the estimates $|a_{n,p-2}| \leq 1$, $|a_{n,p+2}| \leq 4^{n+1}$ and hence, by Schur's Lemma,[3] it then follows that $\Theta_n \in \mathcal{L}(L^2(\mathbb{R}))$ and that $\|\Theta_n\| \leq 3\times 4^{n+1}$. ∎

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[3] If (a_{pq}) is such that

$$\sup_p \sum_q |a_{pq}| \leq M, \quad \sup_q \sum_p |a_{pq}| \leq M,$$

then $K : \ell_2(\mathbb{Z}_+) \to \ell_2(\mathbb{Z}_+)$ defined by $K((x_p)) = (\sum_q a_{pq}x_q)$ is bounded with norm $\leq M$.

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