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ON THE TOTAL CURVATURE OF CURVES IN A MINKOWSKI SPACE

BY

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ABSTRACT

We consider simple closed curves in a Minkowski space. We give bounds of the total Minkowski curvature of the curve in terms of the total Euclidean curvature and of normal curvatures on the indicatrix (supposed to be a central symmetric hypersurface) of the Minkowski norm. Corollaries of this result provide analogues to Fenchel and Fary–Milnor theorems. We also give an upper bound of the Minkowski length of a simple closed curve contained in a Minkowski ball of radius R, in terms of the total Minkowski curvature and of normal curvatures on the indicatrix. Whenever the Minkowski space is Euclidean our results reduce to the classical ones.

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1. Introduction

In this paper we prove some properties for closed curves in a Minkowski space by considering the definition of the Minkowski curvature of a curve given by Rund [13]. We consider a Minkowski space M^{n+1}, i.e., a pair (V, F), where V^{n+1} is an $(n+1)$-dimensional vector space and F is a Minkowski metric. We assume that the indicatrix of F is a central symmetric hypersurface.

We first relate (see Theorem 1) the Minkowski total curvature of a simple closed L in M^{n+1} with its total Euclidean curvature, considering L as a curve in V with the Euclidean metric. As an immediate consequence, we get a Fenchel's type theorem, giving a lower bound for the total Minkowski curvature (see Corollary).

The Fary–Milnor Theorem, proved independently by Fáry [9] and Milnor [11], states that the total curvature of a knot in the Euclidean space E^3 is grater than 4π. A different proof was given by Brickell–Hsiung [5], that also works in the hyperbolic space H^3. More recently, Alexander–Bishop [1] and Schmitz [14] extended the Fary–Milnor Theorem to curves in a Riemannian 3-manifold of nonpositive sectional curvature.

In this paper, as a consequence of Theorem 1, we obtain an analogue to the Fary–Milnor Theorem for a simple closed curve in a 3-dimensional Minkowski space (Corollary 2). We also get an upper bound for the Minkowski length of a simple closed curve contained in a (Minkowski) ball, in terms of its Minkowski total curvature (Theorem 2).

We observe that, whenever the Minkowski space is Euclidean, our results reduce to the classical ones.

2. On the total curvature of a curve in a Minkowski space

Let $\gamma(s)$ be a curve in a Finsler space M^{n+1}, where s is the arc length parameter in the Finsler space. One defines the covariant derivative ∇ [15] of vector fields along a curve and $\gamma(s)$ is said to be a geodesic when $\nabla_{\dot\gamma}\dot\gamma = 0$. We define the **Minkowski curvature** of a curve L in M^{n+1} as

$$k_M = \|\nabla_{\dot X}\dot X\|, \tag{2.1}$$

where $X = X(s)$ is a parametrization of the curve and $\|\cdot\|$ is a Finsler norm.

For a Finsler space and a Minkowski space there are other definitions of curvature given by Finsler [10] and Busemann [6]. The definition of the curvature for curves given by Finsler coincides with Cartan's definition [7]. In this article, our definition of curvature coincides with the one given by Rund [12] (see also [13]),

$$k_M^2 = g_{ij}\left(x, \frac{d^2X}{ds^2}\right)\frac{d^2x^i}{ds^2}\frac{d^2x^j}{ds^2}.$$

Busemann's definition of curvature for curves in Minkowski space [6] is different from the definitions above. In [8], there is still another definition for curves in a Minkowski plane.

A Minkowski space M^{n+1} is a pair (V, F), where V^{n+1} is an $(n+1)$-dimensional vector space and F is a nonnegative function $F : V \to [0, \infty)$ which has the following properties:

(1) F is C^∞ on $V^{n+1} \setminus \{0\}$.
(2) $F(\lambda y) = \lambda F(y)$, for all $\lambda > 0$ and $y \in V^{n+1}$.
(3) The symmetric bilinear form $g_{\alpha\beta} = \frac{1}{2}\partial^2 F^2/\partial y^\alpha \partial y^\beta$ is an inner product on V, where on the vector space V we are considering coordinates $y^1, \ldots, y^{n+1}$ which are orthogonal in the standard Euclidean metric.

The **indicatrix** (unit sphere) of a Minkowski space is the compact convex hypersurface defined by $F_0 = \{y^1, \ldots, y^{n+1}, F(y^1, \ldots, y^{n+1}) = 1\}$. The **Minkowski ball** of radius R is the set of points such that $F(y^1, \ldots, y^{n+1}) \leq R$.

Since the indicatrix F_0 is a compact hypersurface of the Euclidean space E^{n+1}, there exists k_1 and k_2 such that the normal curvatures k of the indicatrix F_0, in the Euclidean space, satisfy

$$0 < k_1 \leq k \leq k_2. \tag{2.2}$$

In this paper we will assume that the indicatrix is a central symmetric hypersurface.

Let $X(s)$ be a smooth parametrization of a curve L in M^{n+1} such that s is the Minkowski arc length of the curve. Then the Minkowski curvature of the curve, defined by (2.1), can be rewritten as

$$k_M = \left\| \frac{d^2X}{ds^2} \right\|, \tag{2.3}$$

where $\|\cdot\|$ is the Minkowski norm. Let σ be the Euclidean arc length of the curve L as a curve in the Euclidean space E^{n+1}. We denote by k_E the curvature of the curve L, as a curve in E^{n+1}.

We first want to consider the following question: how does the Minkowski curvature k_M of a curve L relate to the curvature k_E? For a simple closed curve L, from now on, we will denote by $\omega_M(L)$ and $\omega_E(L)$ the total Minkowski curvature and the total Euclidean curvature of L, respectively, i.e.,

$$\omega_M(L) = \int_L k_M ds, \quad \omega_E(L) = \int_L k_E d\sigma.$$

We will prove the following result.

THEOREM 1: *Let M^{n+1} be a Minkowski space whose indicatrix is central symmetric and has Euclidean normal curvature between k_1 and k_2, $0 < k_1 \leq k_2$. Then the total Minkowski curvature $\omega_M(L)$ of a simple closed curve L in M and its total Euclidean curvature $\omega_E(L)$ satisfy the following inequality:*

$$\left(\frac{k_1}{k_2}\right) \omega_E(L) \leq \omega_M(L) \leq \left(\frac{k_2}{k_1}\right)^2 \omega_E(L). \tag{2.4}$$

One recalls that in the Euclidean space Fenchel's theorem says that the total curvature ω_E of a simple closed curve L in Euclidean space E^{n+1} satisfies the inequality

$$\omega_E(L) \geq 2\pi. \tag{2.5}$$

Therefore, as an immediate consequence of Theorem 1 we obtain

COROLLARY 1: *The total curvature ω_M of a simple closed curve L in a Minkowski space satisfies the inequality*

$$\omega_M(L) \geq 2\pi \frac{k_1}{k_2}.$$

Moreover, the Fary–Milnor Theorem [9], [11] states that if the total curvature ω_E of a closed curve L in E^3 satisfies

$$\omega_E(L) \leq 4\pi, \tag{2.6}$$

then the curve is unknotted (i.e., l is the boundary of an embedded disk). An analogue to the Fary–Milnor Theorem holds for closed curves in a Minkowski space. More precisely,

COROLLARY 2: *If the total curvature of a simple closed curve L in a 3-dimensional Minkowski space satisfies the inequality*

$$\omega_M(L) \leq 4\pi \left(\frac{k_1}{k_2}\right),$$

then the curve is unknotted.

In fact, as a consequence of (2.4), we have $\omega_E(L) \leq 4\pi$ and then we apply the Fary–Milnor Theorem.

The Euclidean length $\ell_E(L)$ of a simple closed curve L, contained in a Euclidean ball of radius R of E^{n+1}, and its total curvature satisfy the inequality

$$\ell_E(L) \leq R\omega_E(L).$$

The corresponding result for the Minkowski length $\ell_M(L)$ of a closed curve in Minkowski space M^{n+1} and its total Minkowski curvtaure is given in the following theorem.

Theorem 2: *If a simple closed curve L in a Minkowski space is contained in a (Minkowski) ball of radius R, then its Minkowski length satisfies the inequality*

$$\ell_M(L) \leq \left(\frac{k_2}{k_1}\right)^4 R\omega_M(L). \tag{2.7}$$

We observe that if the Minkowski space is Euclidean, then $k_1 = k_2$ and hence Theorem 1 is an identity, while Corollaries 1 and 2 and Theorem 2 reduce to the classical results.

Before we prove Theorems 1 and 2 in Section 4, we will mention some properties of the Minkowski space and a lemma for compact convex hypersurfaces in Euclidean space that will be useful in the proofs.

3. Preliminaries

We will be using greek letters α, β for indices from 1 to $n+1$ and latin letters i, j for indices from 1 to n. Moreover, we will use the Einstein summation convention for repeated indices.

We start by observing that it follows from item (2) of the definition of a Minkowski norm F that

$$F(y) = F_\alpha(y)y^\alpha, \tag{3.1}$$

$$F_{\alpha\beta}(y)y^\alpha y^\beta = 0, \tag{3.2}$$

where $y = (y^1, \ldots, y^{n+1})$ and F_α denotes the derivative of F with respect to y^α. In fact, these relations follow from taking the first and second derivatives of $F(\lambda y) = \lambda F(y)$ with respect to λ, and then considering $\lambda = 1$. Moreover, taking the derivative of (3.1) with respect to y^β one gets

$$F_{\alpha\beta}(y)y^\alpha = 0, \quad \forall\beta. \tag{3.3}$$

Now assume that a neighborhood U of the indicatrix F_0 is a graph of a function, i.e., U is given by

$$F(y^1, \ldots, y^n, f(y^1, \ldots, y^n)) = 1. \tag{3.4}$$

Then it follows from (3.1) that

$$1 = F_i y^i + F_{n+1} f. \tag{3.5}$$

Taking the derivative of (3.4) with respect to y^i and then with respect to y^j we have

$$F_i + F_{n+1} f_i = 0, \quad \forall i, \tag{3.6}$$

$$F_{ij} + F_{in+1} f_j + F_{n+1j} F_i + F_{n+1n+1} f_j f_i + F_{n+1} F_{ij} = 0, \quad \forall i, j. \tag{3.7}$$

We now consider a curve $y(s) = (y^1(s), \ldots, y^{n+1}(s))$ on the indicatrix F_0, i.e., we have $F(y(s)) = 1$. Then (3.1) reduces to

$$1 = F_\alpha(y(s))y^\alpha(s), \tag{3.8}$$

and its derivative with respect to s gives

$$F_{\alpha\beta}(y(s))\frac{dy^\beta}{ds}y^\alpha + F_\alpha \frac{dy^\alpha}{ds} = 0. \tag{3.9}$$

Finally, it follows from (3.3) restricted to the curve and (3.9) that

$$F_\alpha(y(s))\frac{dy^\alpha}{ds} = 0. \tag{3.10}$$

In the proofs of Theorems 1 and 2, we will consider a parametrization of the indicatrix F_0 of the form $r(\nu)\nu$, where ν is a unit vector of the sphere S^n in the Euclidean space, and $r(\nu) > 0$. It follows from (2.2) that $\forall \nu \in S^n$, we have

$$\frac{1}{k_2} \leq r(\nu) \leq \frac{1}{k_1}. \tag{3.11}$$

We will also need the following lemma, which is analogous to the results in the Riemannian manifolds of negative sectional curvature obtained in [2], [3], [4].

LEMMA 1: *Let M^n be a compact, convex hypersurface in the Euclidean space E^{n+1}. Let O be a point in the interior of the region bounded by M and let h be the distance from O to M. Suppose the normal curvature k of M satisfies the inequalities $0 < k_1 \leq k \leq k_2$. If α is the angle between the position vector of M and the exterior normal direction, then*

$$\cos\alpha \geq hk_1. \tag{3.12}$$

If M^n is also symmetric and O is the center of symmetry, then

$$\cos\alpha \geq \frac{k_1}{k_2}. \tag{3.13}$$

Proof. We will first prove (3.12) when $n = 1$, i.e., when M^1 is a closed convex curve in the plane E^2, and h is the distance from O to the curve. Then we will prove the result in any dimension n.

If M^1 is a circle centered at O, then (3.12) holds trivially. So we may assume that M^1 is not such a circle. We consider u, θ polar coordinates in E^2 with pole O. Let $u(s)$, $\theta(s)$, $0 \leq s \leq \ell$ be a parametrization by arc length of the curve. For later arguments in this proof, we will need to consider intervals contained in $[0, \ell]$, where the distance function from O to the points of the curve, $u(s)$, is monotone. Since $u(s)$ is not constant, there exists a subdivision $0 \leq s_0 < s_1 < \cdots < s_r \leq \ell$ such that in each interval (s_i, s_{i+1}), we have $u'(s) \neq 0$. Since $u(s)$ is stricly monotone in such an interval, there exists the inverse function $s(u)$ for $u \in J$. Hence we can locally reparametrize the curve by $(x(u), y(u)) = (u\cos\theta(u), u\sin\theta(u))$, $u \in J$. The curvature and the external unit normal are respectively given by

$$k(u) = \frac{-x''y' + x'y''}{((x')^2 + (y')^2)^{3/2}}, \quad n(u) = \frac{(y', -x')}{((x')^2 + (y')^2)^{1/2}}.$$

Let $\alpha(u)$ be the angle between the position vector and the normal $n(u)$. Then

$$\cos\alpha(u) = \frac{u\theta'(u)}{\sqrt{1 + u^2(\theta'(u))^2}}, \quad u \in J, \tag{3.14}$$

where without loss of generality we may assume $\theta' > 0$. Moreover,

$$k(u) = \frac{2\theta' + u^2(\theta')^3 + u\theta''}{(1 + u^2(\theta')^2)^{3/2}}. \tag{3.15}$$

By computing θ'' from (3.14), we get

$$k(u) = \frac{\cos\alpha}{u} - \sin\alpha\frac{d\alpha}{du}, \quad u \in J. \tag{3.16}$$

Now consider a circle of radius $1/k_1$, centered at the point $(1/k_1-h, 0)$, locally parametrized by $(x(u), y(u)) = (u\cos\theta_2(u), u\sin\theta_2(u))$, $u \in J$, where without loss of generality we may assume $\theta_2' > 0$. Then $\theta_2(u)$ satisfies the equation

$$h^2 + u^2 - \frac{2h}{k_1} + 2\Big(h - \frac{1}{k_1}\Big)\cos\theta_2 = 0. \tag{3.17}$$

Let $\beta(u)$ be the angle between the position vector of the circle and the exterior normal. Then, the same arguments as before give

$$\cos\beta(u) = \frac{u\theta_2'(u)}{\sqrt{1+u^2(\theta_2'(u))^2}}, \quad k_1 = \frac{\cos\beta}{u} - \sin\beta\frac{d\beta}{du}. \tag{3.18}$$

It follows from (3.17) that

$$\theta_2' = \frac{1}{\sin\theta_2(hk_1-1)}\left(\frac{h}{u^2}(2-hk_1) + \frac{k_1}{2}\right).$$

Therefore,

$$\cos\beta = \frac{h(2-hk_1)}{2u} + \frac{uk_1}{2}.$$

Since $u \in J$ and $h \le u \le 2/k_1 - h$, we conclude that

$$\cos\beta \ge hk_1. \tag{3.19}$$

Subtracting (3.18) from (3.16) we get, $\forall u \in J$,

$$0 \le k(u) - k_1 = \frac{1}{u}(\cos\alpha - \cos\beta) - \sin\alpha\frac{d\alpha}{du} + \sin\beta\frac{d\beta}{du}.$$

We consider the function $f(u) = \cos\alpha(u) - \cos\beta(u)$. Then

$$\frac{1}{u}f(u) + f'(u) \ge 0, \quad \forall u \in J. \tag{3.20}$$

We now observe that the distance u from O to the points of the curve M^1 is monotone for $u \in J$. Moreover, if u_0 is a point on the boundary ∂J of J, then u_0 is a critical point of the distance function and $\cos\alpha(u_0) = 1$. Hence, $\lim_{u\to u_0} f(u) \ge 0$.

Claim: *If $u_0 \in \partial J$ then for all $\epsilon > 0$ there exists $u_1 \in J$ such that $0 < |u_1 - u_0| < \epsilon$ and $f(u_1) \ge 0$.*

In fact, otherwise, there exists $\epsilon_1 > 0$ such that $f(u) < 0$ for all $u \in J_1 = J \cap (u_0 - \epsilon_1, u_0 + \epsilon_1)$. Since the limit of f when u tends to u_0 is nonnegative, we conclude that there exists $\epsilon_2 > 0$ such that f is strictly decreasing for all $u \in J_2 = J \cap (u_0 - \epsilon_2, u_0 + \epsilon_2)$, i.e., $f'(u) < 0$ for $u \in J_2$. Let $J_0 = J_1 \cap J_2$. Then for all $u \in J_0$, we have $f(u) < 0$ and $f'(u) < 0$, and hence $f(u)/u + f'(u) < 0$, $\forall u \in J_0$, which contradicts (3.20). This proves our claim.

We consider the differential equation $g/u + g' = 0$ with initial conditions $g(u_1) = f(u_1) \geq 0$, where u_1 is fixed as in the Claim. The unique solution to this equation in J is $g(u) = f(u_1)u_1/u \geq 0$. Comparing solutions of this equation with solutions of (3.20), we conclude that $f(u) \geq g(u) \geq 0$. Therefore, $\cos\alpha(u) \geq \cos\beta(u)$, $\forall u \in J$. It follows from (3.19) that $\cos\alpha(u) \geq hk_1$ in J. Since J corresponds to any interval s_i, s_{i+1} where $u'(s) \neq 0$, we conclude that (3.12) holds for M^1.

Now we prove the n-dimensional case. Let $M^n \subset E^{n+1}$ be a compact and convex hypersurface. We consider O as the origin of E^{n+1}. We observe that the critical points of the distance function from O to M are the points where the angle between the position vector and the exterior normal is zero. If M is a ball centered at O, then (3.12) holds trivially. Assuming M is not such a ball, we consider a point $P \in M$ which is not a critical point of the distance function from O. Let Π be the plane through the origin determined by the position vector and the exterior normal at P. The intersection of the plane Π with M is a closed and convex plane curve $\mathcal{C}$. Let h_0 be the distance from O to $\mathcal{C}$ and let $k_0 > 0$ be the lower bound of the curvature of $\mathcal{C}$. It follows from the case $n = 1$ that $\cos\alpha(P) \geq h_0k_0$. Since $k_1 \leq k_0$ and $h \leq h_0$, we conclude that (3.12) for M^n.

If the manifold M^n is also symmetric and O is the center of symmetry, then $h \geq 1/k_2$. Hence, as an immediate consequence of (3.12), we obtain (3.13). ∎

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let $X(s)$ be a smooth parametrization of a curve L in a Minkowski space, where s is the Minkowski arc length of the curve. Let σ be the Euclidean arc length of the curve L as a curve in the Euclidean space E^{n+1}. Then $X(s) = X(\sigma(s))$ and $ds = F(X_\sigma)d\sigma$, i.e.,

$$\frac{d\sigma}{ds} = \frac{1}{F(X_\sigma)}, \tag{4.1}$$

where $dX/d\sigma = X_\sigma$.

We consider a parametrization of the indicatrix F_0 of the form $r(\nu)\nu$, where ν is a unit vector of the sphere S^n in the Euclidean space and $r(\nu) > 0$. Moreover, we consider the unit sphere $S^n \subset E^{n+1}$ to be parametrized by the angles

$\varphi_1, \dots, \varphi_n$. Since the derivative of X with respect to s, denoted by X_s, is a curve on the indicatrix F_0, we have $X_s = r(X_\sigma)X_\sigma$ and

$$\frac{d\sigma}{ds} = r(X_\sigma) = r. \tag{4.2}$$

From (2.3), we have $k_M = F\left(\frac{d^2X}{ds^2}\right)$. We need to compute d^2X/ds^2.

$$\frac{dX}{ds} = \frac{dX}{d\sigma}\frac{d\sigma}{ds};$$

$$\frac{d^2X}{ds^2} = \frac{d^2X}{d\sigma^2}\left(\frac{d\sigma}{ds}\right)^2 + \frac{dX}{d\sigma}\frac{d^2\sigma}{ds^2}. \tag{4.3}$$

Observe that

$$\frac{d^2\sigma}{ds^2} = \frac{d}{ds}\left(r(X_\sigma)\right) = \frac{d}{ds}\left(r(\varphi_1, \dots, \varphi_n)\right) = \frac{\partial r}{\partial \varphi_i}\frac{\partial \varphi_i}{\partial \sigma}\frac{d\sigma}{ds}, \tag{4.4}$$

where $X_\sigma(s) = (\varphi_1(s), \dots, \varphi_n(s))$ is a curve on the unit sphere S^n. Moreover,

$$\frac{\partial r}{\partial \varphi_i}\frac{\partial \varphi_i}{\partial \sigma} = \langle \text{grad } r, X_{\sigma\sigma}\rangle_{S^n}, \tag{4.5}$$

where the last expression is the inner product of $X_{\sigma\sigma}$ with the gradient of r on S^n. Now we consider two Euclidean orthonormal vector fields τ and ν along the curve as follows: $\tau = X_\sigma$ and ν such that $X_{\sigma\sigma} = k_E\nu$ (assuming $k_E \neq 0$). Then it follows from (4.3), (4.4) and (4.5) that

$$\frac{d^2X}{ds^2} = rk_E(r\,\nu + |\text{grad } r|_{S^n}\,\beta\,\tau),$$

where $0 \le \beta \le 1$ is the cosine of the angle between ν and grad r. Therefore, we have

$$\frac{d^2X}{ds^2} = rk_E\sqrt{r^2 + |\text{grad } r|^2_{S^n}\,\beta^2}\;(\cos\gamma\nu + \sin\gamma\tau),$$

where

$$\cos\gamma = \frac{r}{\sqrt{r^2 + |\text{grad } r|^2_{S^n}\,\beta^2}}, \quad \sin\gamma = \frac{|\text{grad } r|_{S^n}\,\beta}{\sqrt{r^2 + |\text{grad } r|^2_{S^n}\,\beta^2}}.$$

Therefore,

$$k_M = F\left(\frac{d^2X}{ds^2}\right) = rk_E\sqrt{r^2 + |\text{grad } r|^2_{S^n}\,\beta^2}\;F(e), \tag{4.6}$$

where

$$e = \cos\gamma\nu + \sin\gamma\tau = \frac{\frac{d^2X}{ds^2}}{|\frac{d^2X}{ds^2}|_E}$$

is a unit Euclidean vector field.

Since we are assuming that the indicatrix F_0 is a central symmetric hypersurface, it follows from (2.2) that

$$k_2 \geq F(e) = \frac{1}{r(e)} \geq k_1. \tag{4.7}$$

Moreover, considering the polar coordinates in E^{n+1}, with the origin at the center of F_0, it follows, by a straightforward computation, that the Euclidean angle α between the position vector X_s and the outward normal to F_0 is given by

$$\cos\alpha = \frac{r}{\sqrt{r^2 + |\text{grad } r|^2_{S^n}}}.$$

Hence, from (3.13), we have

$$\sqrt{r^2 + |\text{grad } r|^2_{S^n}} \leq \frac{k_2}{k_1}\, r(X_\sigma).$$

Therefore, since $0 \leq \beta \leq 1$, we have

$$\sqrt{r^2 + |\text{grad } r|^2_{S^n}\, \beta^2} \leq \sqrt{r^2 + |\text{grad } r|^2_{S^n}} \leq \frac{k_2}{k_1}\, r. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8) we get

$$k_E r\left(\frac{k_1}{k_2}\right) \leq k_M \leq k_E r\left(\frac{k_2}{k_1}\right) F(e) r(X_\sigma) \leq k_E r\left(\frac{k_2}{k_1}\right)^2. \tag{4.9}$$

Using (4.2) and (4.9) we get (2.4), and this concludes the proof of Theorem 1. ∎

Proof of Theorem 2. Let L be a closed curve in a Minkowski space. We consider an orthogonal system of coordinates $x^1, \ldots, x^{n+1}$ in the auxiliary Euclidean space E^{n+1} and the curve L parametrized by Minkowski arc length $X(s) = (x^1(s), \ldots, x^{n+1}(s))$. Then $F(X_s) = 1$ and $y(s) = X_s$ is a closed curve on the indicatrix F_0. It follows from (3.8) and integration by parts that its length is given by

$$\ell_M(L) = \int_L ds = \int_L F_\alpha(X_s)\frac{dx^\alpha}{ds} ds = -\int_L F_{\alpha\beta}\frac{\partial^2 x^\beta}{d^2 s} x^\alpha ds. \tag{4.10}$$

From item (3) of the definition of the Minkowski norm we have $F_{\alpha\beta} a^\alpha a^\beta \geq 0$, for any vector $a = (a^1, \ldots, a^{n+1})$. Hence, it follows from the Cauchy inequality that

$$(4.11) \qquad -\int F_{\alpha\beta}\frac{d^2x^\beta}{ds^2}x^\alpha ds \leq \int \sqrt{F_{\alpha\beta}x^\alpha x^\beta}\sqrt{F_{\alpha\beta}\frac{d^2x^\alpha}{ds^2}\frac{dx^\beta}{\partial s^2}}ds.$$

In what follows, we will compute the two terms on the right-hand side of (4.11) separately.

Now we fix a point of the indicatrix $p = X_{s_0}$ and choose a special system of orthogonal coordinates in the auxiliary Euclidean space in the following way. We consider the center O of the indicatrix F_0 to be the origin of the system of coordinates, the coordinates $x^1, \dots, x^n$ in a hyperplane parallel to the tangent space of the indicatrix F_0 at the point p and the coordinate axis x^{n+1} to be parallel to the Euclidean normal to F_0 at p. In the neighborhood of the point p, the indicatrix F_0 can be described by

$$x^{n+1} = f(x^1, \dots, x^n).$$

It follows from the choice of the axis x^{n+1}, from (3.6) and (3.5), that

$$(4.12) \qquad f_i(p) = 0, \quad F_i(p) = 0, \ \forall i = 1, \dots, n, \quad F_{n+1}(p)\frac{dx^{n+1}}{ds}(s_0) = 1.$$

From (4.12), we get $F_{n+1}(p) \neq 0$ and (3.10) reduces to

$$(4.13) \qquad \frac{d^2x^{n+1}}{ds^2}(s_0) = 0.$$

Therefore,

$$(4.14) \qquad F_{\alpha\beta}\frac{d^2x^\alpha}{ds^2}\frac{d^2x^\beta}{ds^2}(s_0) = F_{ij}\frac{d^2x^i}{ds^2}\frac{d^2x^j}{ds^2}(s_0).$$

Moreover, at the point p, equation (3.7) reduces to

$$(4.15) \qquad F_{ij}(p) + F_{n+1}(p)f_{ij}(p) = 0.$$

Hence

$$\begin{aligned} F_{ij}\frac{d^2x^i}{ds^2}\frac{d^2x^j}{ds^2}(s_0) =& - F_{n+1}(p)f_{ij}(p)\frac{d^2x^i}{ds^2}\frac{d^2x^j}{ds^2}(s_0) \\ \leq& k_2\left|\frac{d^2X}{ds^2}(s_0)\right|^2_E F_{n+1}(p) \\ =& k_2\left\|\frac{d^2X}{ds^2}(s_0)\right\|^2_M r^2(\tau_1)F_{n+1}(p) \\ =& k_2k^2_M(s_0)r^2(\tau_1)F_{n+1}(p), \end{aligned}$$

where $\tau_1 = e(s_0)$. Since the point p was arbitrary, we finally obtain

$$\sqrt{F_{\alpha\beta}\frac{d^2x^\alpha}{ds^2}\frac{d^2x^\beta}{ds^2}} \leq \sqrt{k_2}\, k_M\, r(e)\, \sqrt{F_{n+1}}. \tag{4.16}$$

We will now estimate $\sqrt{F_{\alpha\beta}x^\alpha x^\beta}$. We consider another orthogonal system of coordinates for the auxiliary Euclidean space E^{n+1} as follows. We take the origin to be the center of the indicatrix F_0 and we assume the indicatrix is not a sphere. We fix a generic point $p = X_s(s_0)$ such that the vector $X_s(s_0)$ and the normal of F_0 are not parallel and hence generate a plane Π. We fix the axis $\tilde{x}^{n+1}$ to be in the direction of $X_s(s_0)$ and the axis $\tilde{x}^1$ to be orthogonal to $X_s(s_0)$ in the plane Π. Now we choose the axes $\tilde{x}^2, \ldots, \tilde{x}^n$ in a hyperplane parallel to the tangent space of F_0 at the point and orthogonal to the plane Π.

In a neighborhood of the point p, the indicatrix F_0 is a graph of a function

$$\tilde{x}^{n+1} = \tilde{f}(\tilde{x}^1, \ldots, \tilde{x}^n), \quad \text{where } \begin{cases} \tilde{x}^1 = \cos\alpha x^1 \mp \sin\alpha x^{n+1} \\ \tilde{x}^{n+1} = \pm\sin\alpha x^1 + \cos\alpha x^{n+1}, \\ \tilde{x}^i = x^i, \quad i = 2, \ldots, n, \end{cases}$$

α being the angle between $\frac{\partial X}{\partial s}(s_0)$ and the unit normal to F_0 in the auxiliary euclidean space E^{n+1}.

It follows from the definition of the axis that

$$\tilde{f}_i(p) = 0, \quad i = 2, \ldots, n, \quad \tilde{f}_1(p) = \pm\tan\alpha \neq 0. \tag{4.17}$$

It follows from (3.6) that

$$\tilde{F}_i(p) = 0,\ \forall i = 2, \ldots, n, \quad \tilde{F}_1(p) = -\tilde{F}_{n+1}(p)\tan\alpha. \tag{4.18}$$

From (3.3), along the curve we have $F_{\alpha\beta}(X_s)dx^\alpha/ds = 0$. But in the coordinates $\tilde{x}$, at the point p, we have $\tilde{x}^{n+1}$ in the direction of $X_s(s_0)$. Hence,

$$\tilde{F}_{n+1\alpha}(p) = 0, \quad \forall\alpha,$$

and using (3.7) we have

$$\tilde{F}_{ij}(p) + \tilde{f}_{ij}\tilde{F}_{n+1}(p) = 0 \quad \forall i, j = 1, \ldots, n. \tag{4.19}$$

Therefore,

$$\begin{aligned}\tilde{F}_{\alpha\beta}\tilde{x}^{\alpha}\tilde{x}^{\beta}(p) =&\tilde{F}_{ij}\tilde{x}^{i}\tilde{x}^{j}(p)\\ \leq& -\tilde{F}_{n+1}f_{ij}\tilde{x}^{i}\tilde{x}^{j}(p)\\ \leq&\tilde{F}_{n+1}(p)\frac{k_2\left\|\tilde{x}(p)\right\|_E^2}{\cos^3\alpha}\\ =&\tilde{F}_{n+1}(p)\frac{k_2\left\|x(p)\right\|_M^2 r_0^2}{\cos^3\alpha},\end{aligned}\tag{4.20}$$

where $r_0 = r(\tilde{x}/|\tilde{x}|_E(p))$.

Now we consider the curve L in the Minkowski space in the system of coordinates $\tilde{x}^1, \ldots, \tilde{x}^{n+1}$. Since at the point $p = X_s(s_0)$ the coordinate $\tilde{x}^{n+1}$ is in the direction of X_s, we have

$$\frac{d\tilde{x}^i}{ds}(s_0) = 0,\ \forall i = 1, \ldots, n, \quad \frac{d\tilde{x}^{n+1}}{ds}(s_0) = r_1 = r(X_\sigma(s_0)).$$

Therefore, we conclude from (3.8) that

$$\tilde{F}_{n+1}(p) = \frac{1}{r_1}.\tag{4.21}$$

Moreover, it follows from the relation of the coordinates $\tilde{x}$ and the coordinates x that $\frac{dx^{n+1}}{ds}(s_0) = r_1\cos\alpha$. Therefore, from (4.12) we get

$$F_{n+1}(p) = \frac{1}{r_1\cos\alpha}.\tag{4.22}$$

Substituting (4.22) and (4.21) into (4.16) and (4.20), respectively, we get

$$\sqrt{F_{\alpha\beta}\frac{d^2x^\alpha}{ds^2}\frac{d^2x^\beta}{ds^2}} \leq \sqrt{\frac{k_2}{r_1\cos\alpha}}\ r_0 k_M,\tag{4.23}$$

$$\sqrt{\tilde{F}_{\alpha\beta}\tilde{x}^\alpha\tilde{x}^\beta} = \sqrt{F_{\alpha\beta}x^\alpha x^\beta} \leq \sqrt{\frac{k_2}{r_1\cos^3\alpha}}\ r_0\left\|x\right\|_M \leq \sqrt{\frac{k_2}{r_1\cos^3\alpha}}\ r_0 R,\tag{4.24}$$

where we have used the assumption of the theorem $\|x\|_M \leq R$. Therefore, from (4.23) and (4.24), we conclude that at the generic point p we have

$$\sqrt{F_{\alpha\beta}x^\alpha x^\beta}\ \sqrt{F_{\alpha\beta}\frac{d^2x^\alpha}{ds^2}\frac{d^2x^\beta}{ds^2}} = \frac{k_2 r_0^2}{r_1\cos^2\alpha}Rk_M \leq \left(\frac{k_2}{k_1}\right)^4 Rk_M,\tag{4.25}$$

where, in the last inequality, from (3.13) of Lemma 1 we used that

$$\frac{1}{\cos\alpha} \leq \frac{k_2}{k_1},$$

and from (3.11)

$$r_0 \leq \frac{1}{k_1}, \quad r_1 \geq \frac{1}{k_2}.$$

It follows from (4.10), (4.11) and (4.25) that

$$\ell_M(L) \leq \left(\frac{k_2}{k_1}\right)^4 R \int_L k_M ds = \left(\frac{k_2}{k_1}\right)^4 R\omega_M(L),$$

which concludes the proof. ∎

We finally observe that it is possible to generalize these results to the case of a nonsymmetric indicatrix of the Minkowski space.

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