MULTILINEAR COMMUTATORS
IN RESIDUALLY FINITE GROUPS

BY

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ABSTRACT

The following result is proved. Let w be a multilinear commutator and n a positive integer. Suppose that G is a residually finite group in which every product of at most 896 w-values has order dividing n. Then the verbal subgroup $w(G)$ is locally finite.

1. Introduction

According to the solution of the Restricted Burnside Problem — the celebrated result of Zelmanov [25], [26] — every residually finite group of finite exponent is locally finite. The Lie-theoretic methods used in the solution happened to be very effective in the treatment of other problems in group theory. In [18] we used the methods to prove the following theorem.

THEOREM 1.1: *Let n be a prime-power and G a residually finite group satisfying the identity $[x,y]^n \equiv 1$. Then G' is locally finite.*

Note that in general a periodic residually finite group need not be locally finite. The corresponding examples have been constructed in [2, 7, 9, 10, 24]. Moreover, if the assumption that G is residually finite is dropped from the hypothesis of Theorem 1.1, the derived group need not even be periodic.

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Deryabina and Kozhevnikov showed that for sufficiently big odd integers n there exist groups G in which all commutators have order dividing n such that G' has elements of infinite order [4]. Independently, this was also proved by Adian [1].

In view of Theorem 1.1 we raised in [19] the next problem.

PROBLEM 1.2: *Let n be a positive integer and w a word. Assume that G is a residually finite group such that any w-value in G has order dividing n. Does it follow that the verbal subgroup $w(G)$ is locally finite?*

If w is a word in variables $x_1, \ldots, x_t$ we think of it primarily as a function of t variables defined on any given group G. The corresponding verbal subgroup $w(G)$ is the subgroup of G generated by the values of w. The word w is a commutator if the sum of the exponents of any variable involved in w is zero. According to the solution of the Restricted Burnside Problem the answer to the above question is positive if $w(x) = x$. In fact it is easy to see that the answer is positive whenever w is any non-commutator word. Indeed, suppose $w(x_1, \ldots, x_t)$ is such a word and that the sum of the exponents of x_i is $r \neq 0$. Now, given a residually finite group G, substitute the unit for all the variables except x_i and an arbitrary element $g \in G$ for x_i. We see that g^r is a w-value for all $g \in G$. Hence G satisfies the identity $x^{nr} = 1$ and therefore is locally finite by the result of Zelmanov.

Hence, Problem 1.2 is essentially about commutator words. In [19] the problem was solved positively in the case where n is a prime-power and w a multilinear commutator (outer commutator word). A word w is a multilinear commutator if it can be written as a multilinear Lie monomial. Particular examples of multilinear commutators are the derived words, defined by the equations

$$\delta_0(x) = x,$$

$$\delta_k(x_1, \ldots, x_{2^k}) = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1} \ldots, x_{2^k})],$$

and the lower central words

$$\gamma_1(x) = x,$$

$$\gamma_{k+1}(x_1, \ldots, x_{k+1}) = [\gamma_k(x_1, \ldots, x_k), x_{k+1}].$$

In the case that n is not a prime-power Problem 1.2 seems to be very hard. We mention a theorem obtained in [21].

THEOREM 1.3: *Let n be a positive integer that is not divisible by p^2q^2 for any distinct primes p and q. Let G be a residually finite group satisfying the identity $([x_1, x_2][x_3, x_4])^n \equiv 1$. Then G' is locally finite.*

Some further progress has been made in [22] where the following theorem was proved.

THEOREM 1.4: *For any positive integer n there exists t depending only on n such that if w is a multilinear commutator and G is a residually finite group in which every product of t values of w has order dividing n, then $w(G)$ is locally finite.*

More recently, it was shown that in the case that $w = [x, y]$ the number t in the above theorem can be taken to be 68 independently of n [23]. The purpose of the present article is to prove a similar result for arbitrary multilinear commutators. Thus, we improve Theorem 1.4 as follows.

THEOREM 1.5: *Let n be a positive integer and w a multilinear commutator. Let G be a residually finite group in which every product of 896 w-values has order dividing n. Then $w(G)$ is locally finite.*

The constant 896 in the theorem comes from the famous results of Nikolov and Segal on commutator width of finite groups. In the course of proving Theorem 1.5 we need to consider subgroups of a finite soluble group that can be generated by 4 w-values. In the paper of Segal [17] it was shown that every element in the derived group of a finite soluble d-generated group is a product of at most $72d^2 + 46d$ commutators. A better bound can be obtained working through the proofs given in [15]. It follows that every element of the derived group of a finite soluble d-generated group is a product of at most

$$\min\{d(6d^2 + 3d + 4), 8d(3d + 2)\}$$

commutators. In the case that $d = 4$ this is 448. We apply this result in the situation where each commutator is a product of 2 w-values. So the constant 896 comes about.

Thus, the theorem of Nikolov and Segal plays an important role in the proof of Theorem 1.5. The proof also relies on the classification of finite simple groups as well as on the Lie-theoretic techniques that Zelmanov created in his solution of the Restricted Burnside Problem. We also use a recent result, essentially due

to Flavell, Guest and Guralnick, that an element a of a finite group G belongs to $F_h(G)$ if and only if every 4 conjugates of a generate a soluble subgroup of Fitting height at most h [6]. It is the necessity to use this result that accounts for the difference between the constants 896 in Theorem 1.5 and 68 in the case of simple commutators $[x, y]$ [23]. When dealing with simple commutators $[x, y]$ it is enough to bound the Fitting height of some 2-generated subgroups while, as was already mentioned, the case of general multilinear commutators requires considering 4-generated subgroups. It should be said that the case of arbitrary multilinear commutators differs from the case of simple commutators $[x, y]$ in several ways. Probably the most significant difference occurs when reducing the problem to questions about finite soluble groups. In the case of commutators $[x, y]$ this was performed with a relatively short argument as in [23, Proposition 2.3]. In the present paper the short argument was of no help at all. Instead, we use a rather intricate Proposition 2.5 in the next section.

2. Some useful results on finite groups

All groups considered in this and the next sections are finite. We use the expression "$\{a, b, c, \dots\}$-bounded" to mean "bounded from above by some function depending only on $a, b, c, \dots$". Recall that the Fitting subgroup $F(G)$ of a group G is the product of all normal nilpotent subgroups of G. The Fitting series of G can be defined by the rules: $F_0(G) = 1$, $F_1(G) = F(G)$, $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$ for $i = 1, 2, \dots$. If G is a finite soluble group, then the minimal number $h = h(G)$ such that $F_h(G) = G$ is called the Fitting height of G. We will require the following proposition.

PROPOSITION 2.1: *Let G be a group and $a \in G$. Suppose that every subgroup of G that can be generated by four conjugates of a is soluble with Fitting height at most h. Then $a \in F_h(G)$.*

Essentially, the above proposition is due to Flavell, Guest and Guralnick. All the tools and arguments that are used in the proof of Proposition 2.1 can be found in [6]. However, since the proposition was not stated in [6] explicitly, we outline the proof here.

The fact of crucial importance is that if every four conjugates of an element a in a group G generate a soluble subgroup, then a belongs to the soluble radical of G. This was established independently in [6] and also in Gordeev, Grunewald,

Kunyavskii and Plotkin [8]. The proof uses the classification of finite simple groups. Therefore, it is sufficient to prove Proposition 2.1 under the additional hypothesis that G is soluble. The key rôle in the soluble case is played by the following lemma, due to Flavell, Guest and Guralnick [6].

LEMMA 2.2: *Let G be a soluble group that possesses an element a such that $G = \langle a^G \rangle$. Let k be a field. Let V be a nontrivial irreducible kG-module. Then $dim\, C_V(a) \leq \frac{3}{4} dim\, V$.*

The particular case of the above lemma where a has order 3 was established earlier in Al-Roqi and Flavell [3]. This enabled the authors to prove that if G is a soluble group containing an element a of order 3 such that $G = \langle a^G \rangle$, then there exist four conjugates of a that generate a subgroup with the same Fitting height as G. Using Lemma 2.2 in place of the particular case of 3-elements in the Al-Roqi and Flavell arguments, we see that the assumption that a is of order 3 can be dropped and so we obtain the following lemma.

LEMMA 2.3: *Let G be a soluble group that possesses an element a such that $G = \langle a^G \rangle$. Then there exist four conjugates of a that generate a subgroup with the same Fitting height as G.*

Now the proof of Proposition 2.1 becomes very easy.

Proof. As was mentioned above we can assume that G is soluble. Set $H_1 = \langle a^G \rangle$, $H_2 = \langle a^{H_1} \rangle$, $H_3 = \langle a^{H_2} \rangle$, etc. Let $H = \bigcap_i H_i$. Then H is the smallest subnormal subgroup of G containing a and it is clear that $H = \langle a^H \rangle$. It follows that the Fitting height of H is at most h. Since H is subnormal, we conclude that $a \in F_h(G)$. ∎

We call an element a of G a δ_k-commutator if it is a value of the word δ_k in G. A well-known corollary of the Hall–Higman theory [12] says that the Fitting height of a finite soluble group of exponent n is bounded by a number depending only on n. We will denote the number by $h(n)$.

LEMMA 2.4: *Let $k, n \geq 1$ and G be a soluble group in which every product of 896 δ_k-commutators has order dividing n. Then $h(G) \leq h(n) + k + 1$.*

Proof. Let $a = a_1 \in G$ be a δ_k-commutator and a_2, a_3, a_4 some conjugates of a. Put $H = \langle a_1, a_2, a_3, a_4 \rangle$ and $h = h(n)$. We know from [15] that every element of H' is a product of 448 commutators of the form $[x, a_i]$ for suitable $x \in G$.

Each commutator of the form $[x, a_i]$ is a product of 2 δ_k-commutators so every element of H' is a product of 896 δ_k-commutators. Hence, H' is of exponent n and so the Fitting height of H' is at most h. It follows that the Fitting height of H is at most $h + 1$. We now deduce from Proposition 2.1 that every δ_k-commutator of G is contained in $F_{h+1}(G)$. Therefore $G^{(k)} \leq F_{h+1}(G)$ and the lemma follows. ∎

In what follows $X_k(G)$ denotes the set of all δ_k-commutators obtained using elements of the group G. Note that if P is a subgroup of G, in general $X_k(G) \cap P \neq X_k(P)$.

Let G be a finite group and k a positive integer. We will associate with G a triple of numerical parameters $n_k(G) = (\lambda, \mu, \nu)$ where the parameters λ, μ, ν are defined as follows.

If G is of odd order, we set $\lambda = \mu = \nu = 0$. Suppose that G is of even order and choose a Sylow 2-subgroup P in G. If the derived length $dl(P)$ of P is at most $k + 1$, we define $\lambda = dl(P) - 1$. Put $\mu = 2$ if $X_\lambda(P)$ contains elements of order greater than two and $\mu = 1$ otherwise. We let $\nu = \mu$ if $X_\lambda(P) \not\subseteq Z(P)$ and $\nu = 0$ if $X_\lambda(P) \subseteq Z(P)$.

If the derived length of P is at least $k + 2$, we define $\lambda = k$. Then μ will denote the number with the property that 2^μ is the maximum of orders of elements in $X_k(P)$. Finally, we let 2^ν be the maximum of orders of commutators $[a, b]$, where $b \in P$ and a is an involution in a cyclic subgroup generated by some element from $X_k(P)$.

The set of all possible triples $n_k(G)$ is naturally endowed with the lexicographical order. Following the terminology used by Hall and Higman [12] we call a group G monolithic if it has a unique minimal normal subgroup which is non-abelian simple. In the modern literature such groups are very often called "almost simple".

PROPOSITION 2.5: *Let $k \geq 1$ and G be a group of even order such that G has no nontrivial normal soluble subgroups. Then G possesses a normal subgroup L such that L is residually monolithic and $n_k(G/L) < n_k(G)$.*

Proof. Let M be a minimal normal subgroup of G. We know that $M \cong S_1 \times S_2 \times \cdots \times S_r$, where $S_1, S_2, \ldots, S_r$ are isomorphic simple groups. The group G acts on M by permuting the simple factors so we obtain a representation of G by permutations of the set $\{S_1, S_2, \ldots, S_r\}$. Let L_M be the kernel of

the representation. We want to show that $n_k(G/L_M) < n_k(G)$. Suppose this is not true and $n_k(G/L_M) = n_k(G)$. Let P be a Sylow 2-subgroup of G and assume first that the derived length of P is at least $k+2$. Suppose further that $\nu(G) \neq 0$. Let $q = 2^\nu$. Since $\nu(G/L_M) = \nu(G)$, there exist an involution a in a cyclic subgroup generated by an element from $X_k(P)$ and b in P such that $[a,b]$ is of order q modulo L_M. Then $[a,b]$ permutes regularly some q factors in $\{S_1, S_2, \ldots, S_r\}$. Without loss of generality we will assume that S_1 is one of those factors and $S_1, \ldots, S_q$ is the corresponding orbit under $[a,b]$.

Suppose that a takes S_1 outside the orbit $S_1, \ldots, S_q$.

Let $P_i = P \cap S_i$. Choose a nontrivial element $x \in P_1$ and set $y = [a,x], c = [a,b]^x$. Then $yc = [a,bx]$ is a commutator of the required form and therefore $(yc)^q = 1$. Write

$$1 = (yc)^q = y y^{c^{-1}} y^{c^{-2}} \cdots y^c.$$

The element $y y^{c^{-1}} y^{c^{-2}} \cdots y^c$ is a product of elements of the form x^{c^j}, each lying in a different $S_j \in \{S_1, \ldots, S_q\}$ and elements of the form x^{-ac^j} lying in other simple factors. Looking at it we conclude that $y y^{c^{-1}} y^{c^{-2}} \cdots y^c \neq 1$. But that means that the order of yc is divisible by $2q$, a contradiction.

Therefore, for every i the a-orbit of S_i is contained in $S_1, \ldots, S_q$. Suppose that $S_1{}^a = S_{i_1}, S_2{}^a = S_{i_2}, \ldots, S_q{}^a = S_{i_q}$. Again we look at the expression $1 = (yc)^q = y y^{c^{-1}} y^{c^{-2}} \cdots y^c$. As above, this is the product of elements of the form x^{c^j}, each lying in a different $S_j \in \{S_1, \ldots, S_q\}$, and elements of the form x^{-ac^j}, each lying in a different $S_j \in \{S_1, \ldots, S_q\}$ as well. Since both elements $x^{-a}, x^{c^{i_1}}$ lie in S_{i_1} and since $1 = (yc)^q$, it follows that $x^{-a} x^{c^{i_1}} = 1$ for every $x \in S_1$. From this and the fact that $c = [a,b]^x$ we deduce that $ax^{-1}[b,a]^{i_1}$ commutes with x. Taking into account that also $x^{-[b,a]^{i_1}}$ commutes with x (because $x^{-[b,a]^{i_1}}$ belongs to S_{q-i_1}), we conclude that $a[b,a]^{i_1}$ commutes with x for all $x \in S_1$. Thus, $a[b,a]^{i_1}$ commutes with S_1. Similarly, we arrive at the conclusion that $a[b,a]^{i_2}$ commutes with S_2, $a[b,a]^{i_3}$ commutes with S_3 and so on. Recall that $S_1^{[a,b]} = S_2, S_2^{[a,b]} = S_3, \ldots$, etc. Therefore all the elements

$$[a,b]a[b,a]^{i_2}[b,a],$$

$$[a,b]^2 a[a,b]^{i_3}[b,a]^2,$$

$$\vdots$$

$$[a,b]^{q-1} a[a,b]^{i_q}[b,a]^{q-1}$$

commute with S_1. Remembering that a is an involution we write

$$[a, b]a[b, a]^{i_2}[b, a] = [a, b]^{i_2+2}a,$$
$$[a, b]^2a[b, a]^{i_3}[b, a]^2 = [a, b]^{i_3+4}a,$$
$$\vdots$$
$$[a, b]^{q-1}a[b, a]^{i_q}[b, a]^{q-1} = [a, b]^{i_q+2(q-1)}a.$$

Therefore the elements $[a, b]^{i_1}a, [a, b]^{i_2+2}a, \ldots, [a, b]^{i_q+2(q-1)}a$ commute with S_1. Since $[a, b]$ permutes regularly $S_1, S_2, \ldots, S_q$, we deduce that

$$i_1 \equiv i_2 + 2 \equiv i_3 + 4 \equiv \cdots \equiv i_q + 2(q-1) \pmod{q}.$$

It follows now that $i_1 = i_{1+q/2}$ and so $S_1 = S_{1+q/2}$, a contradiction.

Thus, if $\nu(G) \neq 0$, then $\nu(G/L_M) < \nu(G)$. Suppose that $\nu(G) = 0$, that is, every involution a in a cyclic subgroup generated by an element from $X_k(P)$ is central in P. Keeping the above notation we remark that since a centralizes P_i, it follows that a normalizes S_i for every i. Therefore $a \in L_M$ and we conclude that $\mu(G/L_M) < \mu(G)$.

So $n_k(G/L_M) < n_k(G)$ whenever the derived length of P is at least $k+2$. We will now assume that the derived length of P is at most $k+1$ and so $X_\lambda(P)$ is contained in a normal abelian subgroup of P. Choose $d \in X_\lambda(P)$ and $x \in P_i$. Since $[x, d, d] = 1$, it is easy to see that d normalizes $S_iS_i^d$. It follows that $d^2 \in L_M$. Thus, $\mu(G/L_M) = 1$. If $\mu(G) = 2$, then $n_k(G/L_M) < n_k(G)$ so suppose that $\mu(G) = 1$. If $\nu(G) = 0$, it follows that $d \in Z(P)$ and so d centralizes P_i whence we deduce that $d \in L_M$, in which case $\lambda(G/L_M) < \lambda(G)$ and we are done.

It remains to deal with the case where $\nu(G) = 1$ and $\nu(G/L_M) = 1$. In other words, we have to deal with the case where the elements of $X_\lambda(P)$ are involutions generating a normal abelian subgroup of P. Moreover, there exist $d \in X_\lambda(P)$ and $b \in P$ such that $[d, b] \notin L_M$. It is clear that for some i we have $S_i^d \neq S_i^{[d,b]}$. Without loss of generality we assume that $S_1^d = S_2$ and $S_1^{[d,b]} = S_3$. Now choose $x \in P_1$ and write

$$1 = [d, bx]^2 = ([d, x][d, b]^x)^2 = [d, x][d, x]^{[d,b]^x}.$$

This shows that $[d, x]$ commutes with $[d, b]^x$, which easily leads to a contradiction in view of the assumption that $S_1^d = S_2$ and $S_1^{[d,b]} = S_3$.

Thus, we have shown that $n_k(G/L_M) < n_k(G)$. Let now L be the intersection of all the subgroups L_M, where M ranges through the minimal normal

subgroups of G. It follows that $n_k(G/L) < n_k(G)$, so the proof of the proposition will be complete once it is shown that L is residually monolithic. If T is the product of the minimal normal subgroups of G, it is clear that T is the product of pairwise commuting simple groups $S_1, S_2, \ldots, S_t$ and that L is the intersection of the normalizers of S_i. Since G has no nontrivial normal soluble subgroups, it follows that $C_G(T) = 1$ and therefore any element of L induces a nontrivial automorphism of some of the S_i. Let ρ_i be the natural homomorphism of L into the group of automorphisms of S_i. It is easy to see that the image of ρ_i is monolithic and that the intersection of the kernels of all ρ_i is trivial. Hence L is residually monolithic.   ∎

The next lemma is given without proof as it is precisely Lemma 3.2 from [20]. The proof is based on Lie-theoretic techniques created by Zelmanov.

LEMMA 2.6: *Let G be a group in which every δ_k-commutator is of order dividing n. Let H be a nilpotent subgroup of G generated by a set of δ_k-commutators. Assume that H is in fact m-generated for some $m \geq 1$. Then the order of H is $\{k, m, n\}$-bounded.*

The lemma that follows partially explains why Proposition 2.5 is important for the proof of Theorem 1.5.

LEMMA 2.7: *There exist $\{k, n\}$-bounded numbers λ_0, μ_0, ν_0 with the property that if G is a group in which every δ_k-commutator is of order dividing n, then $n_k(G) \leq (\lambda_0, \mu_0, \nu_0)$.*

Proof. Suppose that G is a group of even order in which every δ_k-commutator is of order dividing n and let P be a Sylow 2-subgroup of G. It suffices to show that there exists a $\{k, n\}$-bounded number ν_0 such that if $b \in P$ and a is the involution in a cyclic subgroup generated by some element $d \in X_k(P)$, then the order of $[a, b]$ is at most 2^{ν_0}. It is clear that $[a, b] \in \langle d, d^b \rangle$. By Lemma 2.6 the order of $\langle d, d^b \rangle$ is $\{k, n\}$-bounded, so the result follows.   ∎

3. Bounding the order of a finite group

The purpose of this section is to find some sufficient conditions for a finite group to have bounded order. Recall that all groups considered in this section are finite.

LEMMA 3.1: *Let k, m, n be positive integers and G a nilpotent group in which every δ_k-commutator is of order dividing n. Assume that G can be generated by m elements $g_1, g_2, \ldots, g_m$ such that each g_i and each commutator of the form $[g, x]$, where $g \in \{g_1, g_2, \ldots, g_m\}$ and $x \in G$, have order dividing n. Then the order of G is bounded by a function depending only on k, m, n.*

Proof. Since G is a nilpotent group, it is clear that any prime divisor of $|G|$ is a divisor of n. Hence, it is sufficient to bound the order of the Sylow p-subgroup of G for any prime p. We can pass to the quotient $G/O_{p'}(G)$. Thus, G can be assumed to be a p-group and n a p-power.

Suppose first that G is soluble with derived length $j \leq k$. If G is abelian, it is easy to see that $|G|$ is $\{k, m, n\}$-bounded. Arguing by induction on j, we assume that the index $[G : G^{(j-1)}]$ is $\{k, m, n\}$-bounded.

Suppose that $G^{(j-1)}$ is central. In this case the index $[G : Z(G)]$ is $\{k, m, n\}$-bounded and Schur's Theorem [16, p. 102] guarantees that so is $|G'|$. Since G can be generated by m elements of order dividing n, it follows that G has $\{k, m, n\}$-bounded order.

Let us see what happens if $G^{(j-1)}$ is not central. Consider the subgroup

$$A = \langle [g_1, G^{(j-1)}], [g_2, G^{(j-1)}], \ldots, [g_m, G^{(j-1)}] \rangle.$$

Clearly, A is normal in G. Applying the results of the previous paragraph to the quotient G/A, it follows that A has $\{k, m, n,\}$-bounded index in G. Now Schreier's Theorem says that A can be generated by a $\{k, m, n\}$-bounded number of elements. Since A is abelian and, by the hypothesis, the order of each commutator of the form $[g, x]$ divides n, it follows that A has exponent dividing n. Hence, $|A|$ and therefore $|G|$ is $\{k, m, n\}$-bounded.

Now consider the general case, that is, we do not assume anymore that G is soluble with derived length at most k. Applying the results of the previous paragraph to the quotient $G/G^{(k)}$, it follows that $[G : G^{(k)}]$ is $\{k, m, n\}$-bounded. It remains to show that $|G^{(k)}|$ is $\{k, m, n\}$-bounded. Let r be the minimal number of generators of $G^{(k)}$. Note that r is $\{k, m, n\}$-bounded. A well-known corollary of the Burnside Basis Theorem [13] says that if a p-group is r-generated, then any generating set contains a generating set of precisely r elements. Thus, $G^{(k)}$ can be generated by r δ_k-commutators. By Lemma 2.6 we conclude that the order of $G^{(k)}$ is $\{k, r, n\}$-bounded, as required.   ∎

In what follows, we denote by $\pi(G)$ the set of prime divisors of $|G|$.

LEMMA 3.2: *Let k, l, m, n be positive integers and G a group in which every δ_k-commutator is of order dividing n. Assume that G can be generated by m elements $g_1, g_2, \ldots, g_m$ such that each g_i and each commutator of the forms $[g, x]$ and $[g, x, y]$, where $g \in \{g_1, g_2, \ldots, g_m\}$ and $x, y \in G$, have order dividing n. Assume further that $|G/F(G)| = l$. Then the order of G is bounded by a function depending only on k, l, m, n.*

Proof. Let $F = F(G)$ be the Fitting subgroup of G. Suppose first that F is central. In this case the index $[G : Z(G)]$ is $\{k, l, m, n\}$-bounded and Schur's Theorem guarantees that so is $|G'|$. Since G can be generated by m elements of order dividing n, it follows that $|G|$ is $\{k, l, m, n\}$-bounded.

If F is not central, consider the subgroup

$$N = \langle [g_1, F], [g_2, F], \ldots, [g_m, F] \rangle.$$

It is easy to see that N is normal in G. Applying the results of the previous paragraph to the quotient G/N, it follows that the index $[G : N]$ is $\{k, l, m, n\}$-bounded. We will show that $|N|$, and therefore $|G|$, is $\{k, l, m, n\}$-bounded.

We know that N can be generated by a $\{k, l, m, n\}$-bounded number of elements. Let s be the minimal number of generators of N. Since N is nilpotent, $\pi(N)$ consists of prime divisors of n. Thus, it is sufficient to bound the order of the Sylow p-subgroup of N for every prime $p \in \pi(N)$. Let P be the Sylow p-subgroup of N and write $N = P \times O_{p'}(N)$. If $y_1, y_2, \ldots$ is the list of all elements of the form $[g_i, y]$, where $1 \leq i \leq m$ and $y \in F$, we write $b_1, b_2, \ldots$ for the corresponding projections of y_j in P. Then $P = \langle b_1, b_2, \ldots \rangle$. Since P is an s-generated p-group, the Burnside Basis Theorem shows that P is actually generated by s elements in the list $b_1, b_2, \ldots$. By the hypothesis, the order of each of them divides n. Each commutator of the form $[b_i, z]$ also has order dividing n. By Lemma 3.1 we conclude that P has $\{k, l, m, n\}$-bounded order. The proof is complete. ■

PROPOSITION 3.3: *Let k, m, n be positive integers and G a group in which every product of 896 δ_k-commutators is of order dividing n. Assume that G can be generated by m elements $g_1, g_2, \ldots, g_m$ such that each g_i and all commutators of the forms $[g, x]$ and $[g, x, y]$, where $g \in \{g_1, g_2, \ldots, g_m\}$, $x, y \in G$, have orders dividing n. Then the order of G is bounded by a function depending only on k, m, n.*

Proof. We use $n_k(G)$ to denote the triple of numerical parameters as in the previous section. According to Lemma 2.7 the number of all triples that can be realized as $n_k(G)$ is $\{k, n\}$-bounded. We therefore can use induction on $n_k(G)$. If $n_k(G) = (0, 0, 0)$, then G has odd order. By the Feit–Thompson Theorem [5], G is soluble. By Lemma 2.4, $h(G) \leq h(n) + k + 1$. Arguing by induction on $h(G)$ we can assume that $F(G)$ has $\{k, m, n\}$-bounded index in G. Now the result is immediate from Lemma 3.2. Hence, we assume that $n_k(G) > (0, 0, 0)$ and there exists a $\{k, m, n\}$-bounded number N_0 with the property that if L is a normal subgroup such that $n_k(G/L) < n_k(G)$, then the index of L is at most N_0.

Suppose first that G has no nontrivial normal soluble subgroups. Proposition 2.5 tells us that G possesses a normal subgroup L such that L is residually monolithic and $n_k(G/L) < n_k(G)$. It follows that the index of L in G is at most N_0. We deduce that L can be generated by r elements for some $\{k, m, n\}$-bounded number r.

A result of Jones [14] says that any infinite family of finite simple groups generates the variety of all groups. It follows that up to isomorphism there exist only finitely many monolithic groups in which every δ_k-commutator is of order dividing n. Let $N_1 = N_1(k, n)$ be the maximum of their orders. Then L is residually of order at most N_1. Since L is r-generated, the number of distinct normal subgroups of index at most N_1 in L is $\{r, N_1\}$-bounded [11, Theorem 7.2.9]. Therefore L has $\{k, m, n\}$-bounded order. We conclude that $|G|$ is $\{k, m, n\}$-bounded.

Now let us drop the assumption that G has no nontrivial normal soluble subgroups. Let S be the product of all normal soluble subgroups of G. The above paragraph shows that G/S has $\{k, m, n\}$-bounded order and we know from Lemma 2.4 that $h(S) \leq h(n) + k + 1$. Again we let $F = F(G)$ be the Fitting subgroup of G. Using induction on the Fitting height of S, we assume that F has $\{k, m, n\}$-bounded index in G, in which case the result is immediate from Lemma 3.2. ∎

4. Main results

We are now ready to prove Theorem 1.5 in the case where w is a δ_k-commutator.

THEOREM 4.1: *Let n be a positive integer and G a residually finite group in which every product of 896 δ_k-commutators has order dividing n. Then $G^{(k)}$ is locally finite.*

Proof. Let T be any finite subset of $G^{(k)}$. Clearly one can find finitely many δ_k-commutators $h_1, h_2, \ldots, h_m \in G$ such that T is contained in $H = \langle h_1, h_2, \ldots, h_m \rangle$. It is sufficient to prove that the subgroup H is finite. The order of each h_i divides n. Moreover, if $h \in \{h_1, h_2, \ldots, h_m\}$ and $x, y \in H$, then the commutator $[h, x]$ is a product of two δ_k-commutators and the commutator $[h, x, y]$ is a product of four δ_k-commutators. So the order of each of the commutators divides n. If Q is any finite quotient of G, by Proposition 3.3 the order of the image of H in Q is finite and $\{k, m, n\}$-bounded, so it follows that this order actually does not depend on Q. Since G is residually finite, we conclude that H is finite, as required. ∎

Theorem 1.5 easily follows from Theorem 4.1. For the reader's convenience we will reproduce here a couple of lemmas from [19]. We say that a multilinear commutator w has weight k if it depends on precisely k independent variables. It is clear that there are only k-boundedly many distinct multilinear commutators of weight k.

LEMMA 4.2: *Let G be a group and w a multilinear commutator of weight k. Then every δ_k-commutator in G is a w-value.*

Proof. The case $k = 1$ is quite obvious, so we assume that $k \geq 2$ and use induction on k. Write $w = [w_1, w_2]$, where w_1 and w_2 are multilinear commutators of weight k_1 and k_2, respectively, $k_1 + k_2 = k$, and the variables involved in one of w_1, w_2 do not occur in the other. Let k_0 be the maximum of k_1, k_2. By the induction hypothesis any δ_{k_0}-commutator in G is a w_1-value as well as a w_2-value. Since $w = [w_1, w_2]$, it follows that the set of w-values contains the set of elements of the form $[x, y]$, where x, y range independently through the set of δ_{k_0}-commutators. Hence any δ_{k_0+1}-commutator represents a w-value. It remains to remark that $k_0 + 1 \leq k$, so the lemma follows. ∎

Let w be a multilinear commutator of weight t. In the next lemma we shall require the concept of a subcommutator of weight $s \leq t$ of w. This can be defined by backward induction on s in the following way. The only subcommutator of w of weight t is w itself. If $s \leq t - 1$, a multilinear commutator v of

weight s is a subcommutator of w if and only if there exists a subcommutator u of weight $> s$ of w and a multilinear commutator v_1 such that either $u = [v, v_1]$ or $u = [v_1, v]$. It is quite obvious that if v is a subcommutator of w, then $w(G) \leq v(G)$ for any group G.

LEMMA 4.3: *Let w be a multilinear commutator, G a soluble group in which all w-values have finite order. Then the verbal subgroup $w(G)$ is locally finite.*

Proof. Let G be a counter-example whose derived length is as small as possible, and let T be the last nontrivial term of the derived series of G. Passing to the quotient over the subgroup generated by all normal locally finite subgroups of G, we can assume that G has no nontrivial normal locally finite subgroups. Since T is abelian, it follows that no w-value lies in $T \setminus \{1\}$. Let $s = s(w, G)$ be the smallest number such that any subcommutator of weight s of w has no values in $T \setminus \{1\}$. Obviously, $s \geq 2$ since $T \neq 1$. We can choose a subcommutator $v = [v_1, v_2]$ of weight $\geq s$ of w such that both v_1 and v_2 are subcommutators of weight $< s$, at least one of which having nontrivial values in $T \setminus \{1\}$. Let H_i be the subgroup of T generated by the v_i-values contained in T; $i = 1, 2$. By the choice of v at least one of these subgroups is nontrivial. Since v has no values in $T \setminus \{1\}$, it follows that $H_1 \leq C_G(v_2(G))$ and $H_2 \leq C_G(v_1(G))$. Taking into account that $w(G) \leq u(G)$ for any subcommutator u of w, we conclude that H_1 and H_2 centralize the verbal subgroup $w(G)$. Hence both subcommutators v_1 and v_2 have no nontrivial value in the image of T in $G/C_G(w(G))$. This shows that $s(w, G/C_G(w(G))) \leq s - 1$. The induction on $s(w, G)$ now shows that $w(G)/Z(w(G))$, the image of $w(G)$ in $G/C_G(w(G))$, is locally finite. Then, by Schur's Theorem, the derived group of $w(G)$ is locally finite. Because $w(G)$ is generated by elements of finite order, G must be locally finite. ∎

Theorem 1.5 is now immediate.

Proof. Indeed, suppose that G satisfies the hypothesis of Theorem 1.5. By Lemma 4.2 there exists $k \geq 1$ such that any δ_k-commutator is a w-value. Hence any product of 896 δ_k-commutators in G has order dividing n. Theorem 4.1 now tells us that $G^{(k)}$ is locally finite. It is straightforward from Lemma 4.3 that $w(G)/G^{(k)}$ is likewise locally finite, as required. ∎

In [20] we raised the following problem that generalizes the Restricted Burnside Problem.

PROBLEM 4.4: *Let $n \geq 1$ and w be a group-word. Consider the class of all groups G satisfying the identity $w^n \equiv 1$ and having the verbal subgroup $w(G)$ locally finite. Is that a a variety?*

Recall that a variety is a class of groups defined by equations. More precisely, if W is a set of words in $x_1, x_2, \ldots$, the class of all groups G such that $W(G) = 1$ is called the variety determined by W. By a well-known theorem of Birkhoff, varieties are precisely classes of groups closed with respect to taking quotients, subgroups and cartesian products of their members.

We do not know if Problem 1.2 and Problem 4.4 are equivalent. It is fairly easy to see that whenever the answer to Problem 4.4 is positive, so is the answer to Problem 1.2. We will show now that for words that are products of multilinear commutators on independent variables, the problems are equivalent indeed.

PROPOSITION 4.5: *Let C be a positive integer and w a multilinear commutator of weight k. The following statements are equivalent.*

1. *Every residually finite group G in which all products of C w-values are of order dividing n has $w(G)$ locally finite.*
2. *Let G be a finite group in which all products of C w-values are of order dividing n. Let $a_1, \ldots, a_m$ be w-values. Then the order of $\langle a_1, \ldots, a_m \rangle$ is $\{k, m, n\}$-bounded.*
3. *The class of all groups G in which $w(G)$ is locally finite and every product of C w-values has order dividing n is a variety.*

Proof. Suppose first that the first statement is correct but the second is false. Choose a family of finite groups $G_1, G_2, \ldots, G_i, \ldots$ in which all products of C w-values are of order dividing n with the property that for some m the groups G_i contain w-values $a_{i1}, \ldots, a_{im}$ such that

$$|\langle a_{i1}, \ldots, a_{im} \rangle| < |\langle a_{j1}, \ldots, a_{jm} \rangle|$$

whenever $i < j$. Let G be the Cartesian product of the groups G_i. It is clear that G is residually finite. The elements

$$b_1 = (a_{11}, a_{21}, \ldots, a_{i1}, \ldots)$$
$$b_2 = (a_{12}, a_{22}, \ldots, a_{i2}, \ldots)$$
$$\ldots$$
$$b_m = (a_{1m}, a_{2m}, \ldots, a_{im}, \ldots)$$

are w-values in G but $\langle b_1, \ldots, b_m \rangle$ is infinite. A contradiction. Therefore, the first statement implies the second.

Let us now show that the second statement implies the third. We assume that the second statement is correct and let $\mathfrak{X}$ denote the class of all groups G in which $w(G)$ is locally finite and every product of C w-values has order dividing n. It is easy to see that the class $\mathfrak{X}$ is closed to taking subgroups and quotients of its members. Hence, we only need to show that if D is a cartesian product of groups from $\mathfrak{X}$ then $D \in \mathfrak{X}$. Obviously, every product of C w-values in D has order dividing n so it remains only to prove that $w(D)$ is locally finite. In view of Lemma 4.3 it is sufficient to show that so is some term of the derived series of D. According to Lemma 4.2 every δ_k-commutator in G is a w-value. We wish to show that $D^{(k)}$ is locally finite. Let T be any finite subset of $D^{(k)}$. Clearly one can find finitely many δ_k-commutators $h_1, h_2, \ldots, h_m \in D$ such that $T \leq \langle h_1, h_2, \ldots, h_m \rangle = H$. It is sufficient to prove that the subgroup H is finite. Since H is generated by finitely many δ_k-commutators and since every commutator of δ_k-commutators is again a δ_k-commutator, we deduce that $H^{(k)}$ has finite index in H and so is generated by finitely many δ_k-commutators, too. Since the second statement is correct, it follows that the image of $H^{(k)}$ in any finite quotient of H is finite and has bounded order. Thus, it is sufficient to show that H is residually finite. However, this is immediate from the facts that H is finitely generated and every group G in $\mathfrak{X}$ has $G^{(k)}$ locally finite.

It remains to show that the third statement implies the first. Let $\mathfrak{X}$ have the same meaning as in the above paragraph and assume that $\mathfrak{X}$ is a variety. Let G be a residually finite group in which all products of C w-values are of order dividing n. Then any finite quotient of G belongs to the variety $\mathfrak{X}$. However, it is clear that if a group residually belongs to a certain variety, then it actually belongs to the variety. Thus, it follows that $w(G)$ is locally finite. ∎

The next corollaries are now immediate.

COROLLARY 4.6: *Let w be a multilinear commutator and n a positive integer. The class of all groups G in which $w(G)$ is locally finite and every product of 896 w-values has order dividing n is a variety.*

COROLLARY 4.7: *Let w be a multilinear commutator of weight k and n a positive integer. Let G be a finite group in which every product of 896 w-values has order dividing n. If $a_1, \ldots, a_m \in G$ are w-values, then the order of $\langle a_1, \ldots, a_m \rangle$ is $\{k, m, n\}$-bounded.*

References

[1] S. I. Adian, *On groups with periodic commutators*, Doklady Mathematics **62** (2000), 174–176.

[2] S. V. Aleshin, *Finite automata and the Burnside problem for periodic groups*, Mathematical Notes **11** (1972), 199–203.

[3] A. Al-Roqi and P. Flavell, *On the Fitting height of a soluble group that is generated by a conjugacy class of 3-elements*, The Bulletin of the London Mathematical Society **39** (2007), 973–981.

[4] G. S. Deryabina and P. A. Kozhevnikov, *The derived subgroup of a group with commutators of bounded order can be non-periodic*, Communications in Algebra **27** (1999), 4525–4530.

[5] W. Feit and J. Thompson, *Solvability of groups of odd order*, Pacific Journal of Mathematics **13** (1963), 773–1029.

[6] P. Flavell, S. Guest and R. Guralnick, *Characterizations of the solvable radical*, Proceedings of the American Mathematical Society **138** (2010), 1161–1170.

[7] E. S. Golod, *On nil-algebras and residually finite groups*, Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya **28** (1964), 273–276.

[8] N. Gordeev, F. Grunewald, B. Kunyavskii and E. Plotkin, *A description of Baer–Suzuki type of solvable radical of a finite group*, Journal of Pure and Applied Algebra **213** (2009), 250–258.

[9] R. I. Grigorchuk, *On the Burnside problem for periodic groups*, Functional Analysis and its Applications **14** (1980), 53–54.

[10] N. Gupta and S. Sidki, *On the Burnside problem for periodic groups*, Mathematische Zeitschrift **182** (1983), 385–386.

[11] M. Hall, *The Theory of Groups*, Macmillan, New York, 1959.

[12] P. Hall and G. Higman, *The p-length of a p-soluble group and reduction theorems for Burnside's problem*, Proceedings of the London Mathematical Society. Third Series **6** (1956), 1–42.

[13] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, 1967.

[14] G. A. Jones, *Varieties and simple groups*, Journal of the Australian Mathematical Society **17** (1974), 163–173.

[15] N. Nikolov and D. Segal, *On finitely generated profinite groups, I: strong completeness and uniform bounds*, Annals of Mathematics **165** (2007), 171–238.

[16] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups,* Part 1, Springer-Verlag, Berlin-New York, 1972.

[17] D. Segal, *Closed subgroups of profinite groups,* Proceedings of the London Mathematical Society. Third Series **81** (2000), 29–54.

[18] P. Shumyatsky, *Groups with commutators of bounded order,* Proceedings of the American Mathematical Society **127** (1999), 2583–2586.

[19] P. Shumyatsky, *Verbal subgroups in residually finite groups,* The Quarterly Journal of Mathematics **51** (2000), 523–528; doi:10.1093/qjmath/51.4.523.

[20] P. Shumyatsky, *On varieties arising from the solution of the Restricted Burnside Problem,* Journal of Pure and Applied Algebra **171** (2002), 67–74.

[21] P. Shumyatsky, *Commutators in residually finite groups*, Monatshefte für Mathematik **137** (2002), 157–165.

[22] P. Shumyatsky and J. C. Silva, *The Restricted Burnside Problem for multilinear commutators,* Mathematical Proceedings of the Cambridge Philosophical Society **146** (2009), 603–613.

[23] P. Shumyatsky, *Commutators in residually finite groups*, Israel Journal of Mathematics **182** (2011), 149–156.

[24] V. I. Sushchansky, *Periodic p-elements of permutations and the general Burnside problem,* Rossiĭskaya Akademiya Nauk **247** (1979), 447–461.

[25] E. Zelmanov, *The solution of the restricted Burnside problem for groups of odd exponent*, Math. USSR Izv. **36** (1991), 41–60.

[26] E. Zelmanov, *The solution of the restricted Burnside problem for 2-groups,* Sbornik Mathematics **182** (1991), 568–592.