ISRAEL JOURNAL OF MATHEMATICS **186** (2011), 197–220
DOI: 10.1007/s11856-011-0136-z

# THE HOLE PROBABILITY FOR GAUSSIAN ENTIRE FUNCTIONS

BY

Alon Nishry*

*School of Mathematical Sciences, Tel Aviv University*
*Tel Aviv 69978, Israel*
*e-mail: alonnish@post.tau.ac.il*

ABSTRACT

Consider the random entire function

$$f(z)=\sum_{n=0}^{\infty}\phi_n a_n z^n,$$

where the $\phi_n$ are independent standard complex Gaussian coefficients, and the $a_n$ are positive constants, which satisfy

$$\lim_{n\to\infty}\frac{\log a_n}{n}=-\infty.$$

We study the probability $P_H(r)$ that $f$ has no zeroes in the disk $\{|z|<r\}$ (hole probability). Assuming that the sequence $a_n$ is logarithmically concave, we prove that

$$\log P_H(r)=-S(r)+o\left(S(r)\right),$$

where

$$S(r)=2\cdot\sum_{n\,:\,a_n r^n\geq 1}\log\left(a_n r^n\right),$$

and $r$ tends to $\infty$ outside a (deterministic) exceptional set of finite logarithmic measure.

* Research supported by the Israel Science Foundation of the Israel Academy of Sciences and Humanities, grant 171/07.

Received November 5, 2009 and in revised form March 11, 2010

## 1. Introduction

Consider the random entire function

$$f(z)=\sum_{n=0}^{\infty}\phi_n a_n z^n, \tag{1.1}$$

where the $\phi_n$'s are independent standard complex Gaussian coefficients and the $a_n$'s are positive constants, such that

$$\lim_{n\to\infty}\frac{\log a_n}{n}=-\infty.$$

The latter condition guarantees that almost surely the series on the right-hand side of (1.1) has infinite radius of convergence. The probability $P_H(r)$ of the event that $f$ has no zeros in the disk $\{|z|\le r\}$ is called the hole probability. We are interested in the decay rate of the hole probability as $r$ grows to infinity.

This question was studied by Sodin and Tsirelson [ST3] for a special choice of the coefficients $a_n=\frac{1}{\sqrt{n!}}$ (see also the earlier paper [Sod], for an approach to the problem in a more general setting). Their work was continued in [Nis], where we gave more precise estimates for the hole probability. Since the technique in [Nis] was mostly independent of the special choice of the coefficients $a_n$, it led naturally to the generalizations in this paper. Here, we combine ideas introduced in [ST3] and [Nis] with the classical Wiman–Valiron theory of growth of power series.

To state the main result, we need to introduce two functions which depend on the coefficients $a_n$. The first $\mathcal{N}_1(r)$ is the set that contains the "significant" coefficients of $f(z)$, for the given value of $r$,

$$\mathcal{N}_1(r)=\{n\,:\,\log(a_n r^n)\ge 0\};$$

we also write

$$N_1(r)=\#\mathcal{N}_1(r).$$

The second function is

$$S(r)=\log\left(\prod_{n\in\mathcal{N}_1(r)}(a_n r^n)^2\right)=2\cdot\sum_{n\in\mathcal{N}_1(r)}\log(a_n r^n).$$

For the sake of simplicity of the presentation, we will assume that the coefficients $a_n$ are the restriction of a (real, positive) function $a(t)\in C^2((0,\infty))$ to the set of natural integers (it is clear though, that we can interpolate any such sequence with a smooth function). In order for $f(z)$ to be an entire function we require the following:

$$\lim_{t\to\infty} \frac{\log a(t)}{t} = -\infty.$$

In addition, we require that $a(t)$ is a log-concave function. We also use the notation

$$p_H(r) = -\log P_H(r) = \log^- P_H(r).$$

A measurable set $E \subset [1,\infty)$ has a finite logarithmic measure if

$$\int_E \frac{1}{t}\,dt < \infty.$$

The following is our main result

Theorem 1: *Suppose that $a(t)$ is a log-concave function. For $r \to \infty$ not belonging to a (deterministic) set of finite logarithmic measure,*

$$p_H(r) = S(r) + o\left(S(r)\right).$$

We do not know whether the log-concavity condition is essential for this result. If, in addition, we have some lower bound condition on the function $a(t)$ (i.e. the function $f$ does not grow too slowly), we can say more, for example we can prove [1]

Theorem 2: *Let $\alpha \geq 1$. If $a(t)$ is log-concave and $a(t) \geq \exp\left(-t\log^{\alpha} t\right)$, then there exist positive absolute constants $c_1$ and $c_2$, such that for any $\epsilon > 0$ and for $r$ not belonging to a set of finite logarithmic measure,*

$$S(r) - c_1\left(S(r)\right)^{0.9+\epsilon} \leq p_H(r) \leq S(r) + c_2\left(S(r)\right)^{0.5+\epsilon}.$$

*Remark:* If the coefficients $a_n$ are given in explicit form, then it is possible to prove results that are true for every value of $r$ that is large enough, using direct computations instead of Wiman–Valiron theory. As an example, one can take Mittag–Leffler coefficients ($\alpha > 0$)

$$a_n = \frac{1}{\Gamma\left(\alpha n + 1\right)};$$

in that case

$$p_H(r) = \frac{1}{2\alpha} r^{2/\alpha} + \mathcal{O}_{\alpha}\left(r^{9/5\alpha}\right).$$

[1] Recently we found a proof for this theorem, which does not require any regularity conditions on the coefficients $a_n$.

We do not reproduce these calculations here, since they are very similar to the general ones. See the paper [Nis] for the case $a_n = \frac{1}{\sqrt{n!}}$.

ACKNOWLEDGMENT. This work is based on part of my master's thesis, which was written in Tel Aviv University. I would like to thank my advisor Mikhail Sodin for his guidance and encouragement throughout my studies. I also thank Manjunath Krishnapur and the referee for numerous remarks on the preliminary version of this paper, that significantly improved the presentation.

## 2. Preliminaries

2.1. NOTATION. We denote by $r\mathbb{D}$ the disk $\{z : |z| < r\}$ and by $r\mathbb{T}$ its boundary $\{z : |z| = r\}$, with $r \geq 1$. The letters $c$ and $C$ denote positive absolute constants (which can change across lines). We use the notation $\log_n^m$ as a shortcut for the $n$ times iterated logarithm, taken to the $m$-th power (i.e. $\log_2^2(x) \equiv (\log\log(x))^2$ and $\log_1^m x$ is written as $\log^m x$).

In order to simplify some of the expressions in the paper, we will assume from now on that

$$a_0 = a(0) = 1.$$

2.2. RESULTS FROM WIMAN–VALIRON THEORY. Let $g(z)$ be a transcendental entire function given in the form

$$g(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We recall some of the results of Wiman–Valiron theory, taken from [Ha1] and [Ha2, Section 6.5]. Let $r \geq 0$; we denote by $M(r)$ the maximum of $g(z)$ inside $r\mathbb{D}$, by $\mu(r)$ the maximal term of $g(z)$,

$$\mu(r) = \max_n |a_n| r^n,$$

and by $\nu(r)$ the (maximal) index of the maximal term $\mu(r)$ (Hayman's survey uses the notation $N(r)$ for this function). For every transcendental entire function we have $\mu(r) \to \infty$ and $\nu(r) \to \infty$ as $r \to \infty$. We note that the maximal index and the maximal term are related to each other by a simple equation (see [Ha1, p. 318])

$$\log \mu(r) = \log \mu(1) + \int_1^r \frac{\nu(t)}{t}\, dt, \quad r \geq 1. \tag{2.1}$$

We will give the following simple example: Take $a_n = \frac{1}{n!}$, and so $g(z) = e^z$. The maximal term can be estimated using Stirling's approximation:

$$\begin{aligned}\frac{r^n}{n!} =& \frac{r^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}(1+o(1)) \Rightarrow\\ \nu(r) =& r + O(1),\\ \mu(r) =& \frac{e^r}{\sqrt{2\pi r}}(1+o(1)),\end{aligned}$$

so there is an asymptotic agreement with (2.1). Notice that we also have

$$\log M(r) = \log \mu(r) + o(\log \mu(r)).$$

Most of the statements in Wiman–Valiron theory include a positive decreasing function $b(m)$, which satisfies

$$\int_1^\infty b(m)\,dm < \infty.$$

Here, we always use the function

$$b(m) = \frac{1}{m\log^2 m}.$$

The following theorem ([Ha1, p. 322]) bounds from above the values of the terms away from the maximal term for values of $r$, outside a set of *finite logarithmic measure* (*FILM*).

Theorem I: *Set $n = k + \nu(r)$. If $r$ is outside a set of FILM then*

$$\frac{|a_n|r^n}{\mu(r)} \le \exp\left(-ck^2 \cdot b\left(|k| + \nu(r)\right)\right), \tag{2.2}$$

*with $c$ a positive absolute constant (notice that the exceptional set depends only on the $a_n$'s).*

The most famous result in this theory [Ha1, p. 333] gives an estimate for $M(r)$ in terms of $\mu(r)$ and $\nu(r)$:

Theorem II: *For all sufficiently large values of $r$, outside a set of FILM*

$$M(r) < \mu(r)\log^{1/2}\mu(r)\log_2^2\mu(r). \tag{2.3}$$

We will use the theorem above in the most basic way, claiming that $\log M(r) = \log\mu(r) + o(\log\mu(r))$ for large values of $r$ outside a set of FILM. We also borrow the following result from [Ha2, p. 360]:

THEOREM III: *Outside a set of FILM*

$$\nu(r) < \log \mu(r) \log_2^2 \mu(r). \tag{2.4}$$

From now on we will call $r$ *normal* if it satisfies both (2.2) and (2.4); this again holds outside a set of FILM.

2.3. THE FUNCTION $N_x(r)$. We use the following notation:

$$\mathcal{N}_x(r) = \{n \;:\; \log(a_n r^n) \geq (1-x)\log \mu(r)\}\,, \quad x \geq 0,$$

and

$$N_x(r) = \#\mathcal{N}_x(r).$$

Also

$$\mathcal{N}_{m,m+1}(r) = \mathcal{N}_{m+1}(r) \backslash \mathcal{N}_m(r)$$

and $N_{m,m+1}(r)$ is the size of $\mathcal{N}_{m,m+1}(r)$. Note that if $n \in \mathcal{N}_{m,m+1}(r)$ then

$$a_n r^n \leq \mu^{1-m}(r). \tag{2.5}$$

We also partition the "tail" indexes into a union of sets:

$$(\mathcal{N}_1(r))^c = \bigcup_{m=1}^{\infty} \mathcal{N}_{m,m+1}(r).$$

We will use the fact that $a(t)$ is a log-concave function to derive some properties of $N_x(r)$ and $N_{m,m+1}(r)$. We use the function

$$h(t) = \log a(t) + t \log r;$$

note that it is concave since $a(t)$ is log-concave. Now denote by $N_1'(r)$ the largest root of the equation $h(t) = 0$, we see that $N_1(r) = [N_1'(r)] + 1$, and in particular $N_1'(r) < N_1(r) \leq N_1'(r) + 1$. If we draw the line from the point $(\nu(r), \log \mu(r))$ to the point $(N_1'(r), 0)$, then it satisfies the following equation:

$$y(t) = \frac{\log \mu(r)}{N_1'(r) - \nu(r)} \cdot (N_1'(r) - t)\,. \tag{2.6}$$

It will be useful to keep in mind the following picture:

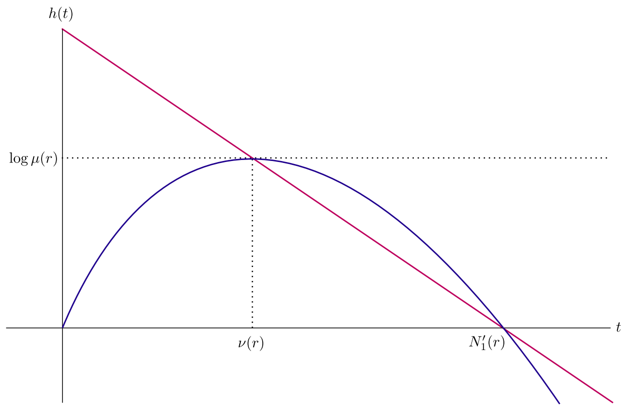


Figure 2.1. The graph of $h(t)$. The dashed line is $y(t)$.

The following lemma gives an estimate for the tail of $h(t)$.

Lemma 3: *For $t \geq N_1'(r)$ we have*

$$h(t) \leq \frac{\log \mu(r)}{N_1'(r) - \nu(r)} \cdot \left(N_1'(r) - t\right),$$

*and for $x \geq 1$ we have*

$$N_x(r) \leq x N_1(r).$$

*Proof.* Looking at the picture above, we see that for $t \geq N_1'(r)$ the function $h(t)$ lies under the line given by (2.6), and we get the first result. The second part follows from the log-concavity of $a(t)$, since

$$\begin{aligned} h\left(xN_1(r)\right) \leq & y\left(xN_1(r)\right) = \frac{\log \mu(r)}{N_1'(r) - \nu(r)} \cdot \left(N_1'(r) - xN_1(r)\right) \\ = & \frac{\log \mu(r)}{N_1'(r) - \nu(r)} \cdot \left(N_1'(r) - \nu(r) + \nu(r) - xN_1(r)\right) \\ = & \log \mu(r) - \log \mu(r) \cdot \left(\frac{xN_1(r) - \nu(r)}{N_1'(r) - \nu(r)}\right) \\ < & (1 - x) \log \mu(r). \end{aligned}$$

The last inequality is true since $N_1(r) > N_1'(r)$. ∎

It follows immediately from the previous lemma that for $m \geq 1$ we have

$$N_{m,m+1}(r) \leq m N_1(r). \tag{2.7}$$

We will now use Wiman–Valiron theory to find an upper bound for $N_1(r)$ in terms of $\log \mu(r)$.

Lemma 4: *For large normal values of $r$, we have*

$$N_1(r) < C \log \mu(r) \log_2^2 \mu(r), \tag{2.8}$$

*with $C > 1$ some positive absolute constant.*

*Proof.* We use Theorem I with $n = k + \nu(r)$ and $k > 0$ and get

$$\log a_n r^n \leq \log \mu(r) - ck^2 \left(n \log^2 n\right)^{-1}.$$

We now put $n = \lfloor C \log \mu(r) \log_2^2 \mu(r) \rfloor$, with some $C > 1$, to be selected later. Notice that using (2.4) we have

$$k = n - \nu(r) \geq (C-1) \log \mu(r) \log_2^2 \mu(r).$$

We also note that for $r$ large enough

$$\log^2 n \leq 2 \cdot \log_2^2 \mu(r).$$

We now have the following inequality:

$$\log a_n r^n \leq \log \mu(r) - \frac{(C-1)^2}{C \cdot 2} \cdot \log \mu(r) \leq 0,$$

with a suitable choice of the constant $C$. ∎

We will also use the following lower bound for $N_1(r)$:

Lemma 5: *We have*

$$N_1(r) \geq \nu(r) \geq \frac{\log \mu(r) - \log \mu(1)}{\log r}.$$

*Proof.* The left inequality follows from the fact that $h(t)$ is concave. The right inequality follows from (2.1). We remark that as a conclusion we see that $N_1(r) \to \infty$ as $r \to \infty$. ∎

2.4. Properties of $S(r)$. Looking again at Figure 2.1, we clearly have

$$S(r) = 2 \cdot \sum_{n \in \mathcal{N}_1(r)} \log\left(a_n r^n\right) \geq N_1'(r) \cdot \max_{n \in \mathcal{N}_1(r)} \log\left(a_n r^n\right) = N_1'(r) \cdot \log \mu(r),$$

or

$$S(r) \geq (N_1(r) - 1) \cdot \log \mu(r). \tag{2.9}$$

Notice that similarly we also have

$$S(r) \le 2N_1(r)\log\mu(r) \le C\log^2\mu(r)\log_2^2\mu(r). \tag{2.10}$$

2.5. Gaussian distributions. We frequently use the fact that if $a$ has a $N_{\mathbb{C}}(0,1)$ distribution, we have

$$\mathbb{P}\left(|a| \ge \lambda\right) = \exp(-\lambda^2), \tag{2.11}$$

and for $\lambda \le 1$,

$$\mathbb{P}\left(|a| \le \lambda\right) \in \left[\frac{\lambda^2}{2}, \lambda^2\right]. \tag{2.12}$$

## 3. Upper bound for $p_H(r)$

In this section we prove the following.

Proposition 6: *For normal values of* $r$, *we have*

$$p_H(r) \le S(r) + C\cdot N_1(r)\log N_1(r),$$

*with* $C$ *some positive absolute constant.*

*Remark:* We note that $r$ is assumed to be large. Later we will analyze the error term.

The simplest case where $f(z)$ has no zeros inside $r\mathbb{D}$ is when the constant term dominates all the others. We therefore study the event $\Omega_r$, which is the intersection of the events (i), (ii) and (iii), where

$$\begin{aligned}
&\text{(i)}: && |\phi_0| \ge \sqrt{N_1(r)} + 3,\\
&\text{(ii)}: && \bigcap_{n\in\mathcal{N}_1(r)} \text{(ii)}_n,\\
&\text{(iii)}: && \bigcap_{m\in\{1,2,\dots\}} \text{(iii)}_{m,m+1},\\
&\text{(iii)}_{m,m+1}: && \bigcap_{n\in\mathcal{N}_{m,m+1}(r)} \text{(iii)}_{m,m+1,n},
\end{aligned}$$

and

$$\begin{aligned}
&\text{(ii)}_n: && |\phi_n| \le \tfrac{(a_n r^n)^{-1}}{(N_1(r))^{1/2}},\\
&\text{(iii)}_{m,m+1,n}: && |\phi_n| \le \tfrac{\mu(r)^{m-1}}{N_{m,m+1}(r)\cdot m^2}.
\end{aligned}$$

Lemma 7: *If* $\Omega_r$ *holds, then* $f$ *has no zeros inside* $r\mathbb{D}$.

*Proof.* To see that $f(z)$ has no zeros inside $r\mathbb{D}$ we note that

$$|f(z)| \geq |\phi_0| - \sum_{n=1}^{\infty} |\phi_n| a_n r^n. \tag{3.1}$$

First we estimate the sum over the terms in $\mathcal{N}_1(r) \setminus \{0\}$

$$\sum_{n\in\mathcal{N}_1(r)\setminus\{0\}} |\phi_n| a_n r^n \leq \sum_{n\in\mathcal{N}_1(r)} N_1(r)^{-\frac{1}{2}} = N_1(r)^{\frac{1}{2}}.$$

Now the tail is bounded by (using (2.5))

$$\begin{aligned}\sum_{n\in\mathcal{N}_1^c(r)} |\phi_n| a_n r^n &= \sum_{m=1}^{\infty} \left[ \sum_{n\in\mathcal{N}_{m,m+1}(r)} |\phi_n| a_n r^n \right] \\ &\leq \sum_{m=1}^{\infty} \left[ \sum_{n\in\mathcal{N}_{m,m+1}(r)} \left(N_{m,m+1}(r)\right)^{-1} m^{-2} \right]\end{aligned}$$

and we have

$$\sum_{n\in\mathcal{N}_1^c(r)} |\phi_n| a_n r^n \quad \leq \quad \sum_{m=1}^{\infty} \frac{1}{m^2} < 2. \tag{3.2}$$

From (3.1)

$$|f(z)| > \sqrt{N_1(r)} + 3 - N_1(r)^{1/2} - 2 = 1,$$

we have that $f(z) \neq 0$ inside $r\mathbb{D}$. ∎

Lemma 8: *The probability of the event $\Omega_r$ is bounded from below by*

$$\log \mathbb{P}(\Omega_r) \geq -S(r) - C \cdot N_1(r) \log N_1(r),$$

*for normal values of $r$ which are large enough.*

*Proof.* In the calculations we use the estimates (2.11) and (2.12). First we have

$$\mathbb{P}((\mathrm{i})) \geq \exp(-N_1(r) - 2C\sqrt{N_1(r)}).$$

For the second part, since $a_n r^n \geq 1$,

$$\mathbb{P}\left((\mathrm{ii})_n\right) \geq \frac{(a_n r^n)^{-2}}{2N_1(r)}$$

and so

$$\begin{aligned}\mathbb{P}\left((\mathrm{ii})\right) \geq & \prod_{n\in\mathcal{N}_1(r)\setminus\{0\}} \frac{(a_n r^n)^{-2}}{2N_1(r)}\\ \geq & \left(\prod_{n\in\mathcal{N}_1(r)} \frac{1}{(a_n r^n)^2}\right) \exp\left(-N_1(r)\log N_1(r) + C\cdot N_1(r)\right)\\ \geq & \exp\left(-S(r) - C\cdot N_1(r)\log N_1(r)\right).\end{aligned}$$

We handle the terms of (iii) separately for the first term and the rest. For $m = 1$, we have

$$|\phi_n| \leq \frac{1}{N_{1,2}(r)}$$

and so (using (2.7))

$$\begin{aligned}\mathbb{P}\left((\mathrm{iii})_{1,2}\right) \geq & \left(\frac{1}{2\cdot\left(N_{1,2}(r)\right)^2}\right)^{N_{1,2}(r)}\\ \geq & \exp\left(-C\cdot N_{1,2}(r)\log N_{1,2}(r)\right)\\ \geq & \exp\left(-C\cdot N_1(r)\log N_1(r)\right).\end{aligned}$$

For a fixed $m \geq 2$ and $n \in \mathcal{N}_{m,m+1}$, we have

$$\mathbb{P}\left((\mathrm{iii})_{m,m+1,n}\right) = 1 - \exp\left(-\frac{\mu(r)^{2(m-1)}}{\left(N_{m,m+1}(r)\right)^2\cdot m^4}\right).$$

We use the following inequality (for some positive sequence $\{A_n\}$):

$$\mathbb{P}\left(\forall n\,:\,|\phi_n| \leq A_n\right) = 1 - \mathbb{P}\left(\exists n\,:\,|\phi_n| > A_n\right) \geq 1 - \sum \mathbb{P}\left(|\phi_n| > A_n\right).$$

Using this inequality, we have

$$\mathbb{P}\left((\mathrm{iii})_{m\geq 2}\right) \geq 1 - \sum_{m=2}^{\infty} N_{m,m+1}(r)\cdot\exp\left(-\frac{\mu(r)^{2(m-1)}}{\left(N_{m,m+1}(r)\right)^2\cdot m^4}\right) = 1 - \Sigma_1. \tag{3.3}$$

Taking $r$ which is normal and large enough we now have (using (2.7) and Lemma 4)

$$\Sigma_1 \leq C\cdot N_1(r)\cdot\sum_{m=1}^{\infty} m\cdot\exp\left(-\frac{\mu(r)^{2m-1}}{m^6}\right);$$

the first term in the sum is clearly the dominant one, and so

$$\begin{aligned}\mathbb{P}\left((\mathrm{iii})\right) \geq & 1 - \exp\left(-c_1\mu(r) + C_2\log N_1(r)\right)\\ \geq & 1 - \exp\left(-c\mu(r)\right).\end{aligned} \tag{3.4}$$

For our purposes here it is sufficient that $\mathbb{P}\left((\text{iii})\right)$ is larger than some absolute constant.

Since the $\phi_n$ are independent, we find that

$$\mathbb{P}\left(\Omega_r\right) = \mathbb{P}\left((\text{i})\right)\mathbb{P}\left((\text{ii})\right)\mathbb{P}\left((\text{iii})\right) \geq \exp\left(-S(r) - C \cdot N_1(r)\log N_1(r)\right)$$

and the lemma is proved. ∎

Proposition 6 now follows from the previous lemmas.

## 4. Lower bound for $p_H(r)$

In this section we prove the following theorem.

Proposition 9 (Lower bound): *Let $\delta \in (0,1)$. For normal values of $r$, and for values of $\delta$ which satisfy $\delta^{-4} = o\left(N_1(r)\right)$, we have*

$$p_H(r) \geq S((1-\delta)\,r) - C_1 \cdot N_1(r)\log\log\mu(r) - C_2\delta^{-4}\log\mu(r)$$

*where $C_1$, $C_2$ are positive absolute constants.*

*Remark:* In principle it is possible to select $\delta^{-4} = cN_1(r)$, for some constant $c > 0$, but we note that in this case the error term will be of the same order of magnitude as the main term (using (2.10)).

Recall that for the lower bound we study the event in which $f$ does not vanish in $r\mathbb{D}$ (for large values of $r$). We define the deterministic counterpart of $f(z)$,

$$\psi(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and write $M(r) = \max_{|z|\leq r}|\psi(z)| = \sum_{n=0}^{\infty} a_n r^n$; we also set $\mathcal{M}(r) = \max_{|z|\leq r}|f(z)|$. We start by studying the deviations of $\log\mathcal{M}(r)$ from $\log M(r)$. Then we consider large deviations of the expression

$$\int_{r\mathbb{T}} \log|f(z)|\,dm,$$

where $m$ is the normalized angular measure on $r\mathbb{T}$. Finally, we use the fact that if $f(z) \neq 0$ in $r\mathbb{D}$, then $\log|f(z)|$ is a harmonic function inside $r\mathbb{D}$, to get the result.

4.1. Large deviations for $\log \mathcal{M}(r)$. We expect that $\log \mathcal{M}(r)$ will be very close to $\log M(r)$ with high probability, but we don't need this accuracy for the lower bound. In the next lemma we prove that the probability that $\log \mathcal{M}(r)$ will be large relative to $\log M(r)$ is very small.

Lemma 10: *Let $0 < \sigma \leq \frac{1}{2}$. Then*

$$\log \mathbb{P}\left(\frac{\log \mathcal{M}(r)}{\log M(r)} \geq 1+\sigma\right) \leq -c\mu^{2\sigma}(r),$$

*for normal values of $r$ which are large enough.*

*Proof.* We will construct an event with probability close to one, for which $\log \mathcal{M}(r)$ is bounded by $(1+\sigma)\log M(r)$. Denote by $\Omega_r$ the event which is the intersection between the events (i), (ii), where

$$\begin{aligned} &(\mathrm{i}): && \bigcap_{n\in\mathcal{N}_1(r)} (\mathrm{i})_n, \\ &(\mathrm{ii}): && \bigcap_{m\in\{1,2,\dots\}} (\mathrm{ii})_{m,m+1}, \\ &(\mathrm{ii})_{m,m+1}: && \bigcap_{n\in\mathcal{N}_{m,m+1}(r)} (\mathrm{ii})_{m,m+1,n}, \end{aligned}$$

where

$$\begin{aligned} &(\mathrm{i})_n && |\phi_n| \leq \mu^{\sigma}(r), \\ &(\mathrm{ii})_{m,m+1,n} && |\phi_n| \leq \tfrac{\mu(r)^{m-1}}{N_{m,m+1}(r)\cdot m^2}. \end{aligned}$$

We notice that

$$\mathbb{P}\left((\mathrm{i})_n^c\right) = \exp\left(-\mu^{2\sigma}(r)\right).$$

In the proof of Lemma 8, we showed (see (3.4))

$$\mathbb{P}\left((\mathrm{ii})\right) \geq 1-\exp\left(-c\mu(r)\right).$$

Therefore the probability that $\Omega_r$ does not occur is bounded by

$$\mathbb{P}\left(\Omega_r^c\right) \leq \exp\left(-c\mu(r)\right) + N_2(r)\cdot\mathbb{P}\left((\mathrm{i})_n^c\right).$$

Using Lemmas 3 and 4 we have, for $r$ large enough,

$$\mathbb{P}\left(\Omega_r^c\right) \leq \exp\left(-c\mu^{2\sigma}(r)\right).$$

It is now sufficient to prove that for functions satisfying the above inequalities, we have the aforementioned upper bound. Indeed

$$|f(z)| \le \sum_{n\in\mathcal{N}_2(r)} |\phi_n| a_n r^n + \sum_{n\in\mathcal{N}_2^c(r)} |\phi_n| a_n r^n;$$

in (3.2) we already found that some absolute constant is an upper bound for the second summand. The first summand is bounded by

$$\sum_{n\in\mathcal{N}_2(r)} |\phi_n| a_n r^n \le \mu^\sigma(r) \cdot \sum_{n=0}^{\infty} a_n r^n = \mu^\sigma(r) \cdot M(r),$$

and so, for $r$ large enough (since $M(r) \ge \mu(r) + 1$),

$$|f(z)| \le \mu^\sigma(r) \cdot M(r) + C \le M^{1+\sigma}(r). \qquad \blacksquare$$

In the next lemma we prove that the probability that log $\mathcal{M}(r)$ will be small is also very small.

Lemma 11: *We have*

$$\log \mathbb{P}\left(\log \mathcal{M}(r) \le 0\right) \le -S(r).$$

*Proof.* Suppose that $\log|f(z)| \le 0$ in $r\mathbb{D}$; using Cauchy's estimate for the coefficients of $f(z)$ we can get an estimate as to the probability of this event. We have

$$|\phi_n| a_n r^n \le \mathcal{M}(r) \le 1;$$

therefore for $n \in \mathcal{N}_1(r)$ we have

$$\mathbb{P}\left(|\phi_n| \le (a_n r^n)^{-1}\right) \le (a_n r^n)^{-2},$$

and so

$$\mathbb{P}\left(\log \mathcal{M}(r) \le 0\right) \le \prod_{n\in\mathcal{N}_1(r)} (a_n r^n)^{-2} = \exp\left(-S(r)\right). \qquad \blacksquare$$

4.2. Discretization of the logarithmic integral. In this section $N \ge 1$ and $\delta \in (0,1)$ are fixed, $\kappa = 1-\delta$ and the points $\{z_j\}_{j=0}^{N-1}$ are equally distributed on $\kappa r\mathbb{T}$, that is

$$z_j = \kappa r \exp\left(\frac{2\pi i j}{N}\right).$$

Also $m$ is the normalized angular measure on $r\mathbb{T}$. Under these conditions we have

LEMMA 12: *For normal values of* $r$, *and outside an exceptional set of probability at most* $2 \cdot \exp(-S(\kappa r))$, *we have*

$$\left| \frac{1}{N} \sum_{j=1}^{N} \log |f(z_j)| - \int_{r\mathbb{T}} \log |f| \, dm \right| \leq \frac{C}{\delta^4 N} \log \mu(r). \tag{4.1}$$

*Proof.* Denote by $P_j(z) = P(z, z_j)$ the Poisson kernel for the disk $r\mathbb{D}$, $|z| = r$, $|z_j| < r$. Since $\log |f|$ is a harmonic function we have

$$\begin{aligned} \frac{1}{N} \sum_{j=1}^{N} \log |f(z_j)| &= \int_{r\mathbb{T}} \left( \frac{1}{N} \sum_{j=1}^{N} P_j \right) \log |f| \, dm \\ &= \int_{r\mathbb{T}} \log |f| \, dm + \int_{r\mathbb{T}} \left( \frac{1}{N} \sum_{j=1}^{N} P_j - 1 \right) \log |f| \, dm. \end{aligned}$$

The last expression can be estimated by

$$\int_{r\mathbb{T}} \left( \frac{1}{N} \sum_{j=1}^{N} P_j - 1 \right) \log |f| \, d\mu \leq \max_{z \in r\mathbb{T}} \left| \frac{1}{N} \sum_{j=1}^{N} P_j - 1 \right| \cdot \int_{r\mathbb{T}} |\log |f|| \, dm. \tag{4.2}$$

For the first factor in the RHS of (4.2), we start with

$$\int_{\kappa r\mathbb{T}} P(z, \omega) \, dm(\omega) = 1,$$

and then split the circle $\kappa r\mathbb{T}$ into a union of $N$ disjoint arcs $I_j$ of equal angular measure $\mu(I_j) = \frac{1}{N}$ centered at the $z_j$'s. Then

$$1 = \frac{1}{N} \sum_{j=1}^{N} P(z, z_j) + \sum_{j=1}^{N} \int_{I_j} \left( P(z, \omega) - P(z, z_j) \right) \, dm(\omega),$$

and

$$\begin{aligned} |P(z, \omega) - P(z, z_j)| &\leq \max_{\omega \in I_j} |\omega - z_j| \cdot \max_{z, \omega} |\nabla_\omega P(z, \omega)| \\ &\leq \frac{2\pi r}{N} \cdot \frac{Cr}{(r - |\omega|)^2} \leq \frac{C}{\delta^2 N}. \end{aligned} \tag{4.3}$$

For the second factor on the RHS of (4.2), using Lemma 11, we may suppose that there is a point $a \in \kappa r\mathbb{T}$ such that $\log |f(a)| \geq 0$ (discarding an exceptional event of probability at most $\exp(-S(\kappa r))$. Then we have

$$0 \le \int_{r\mathbb{T}} P(z,a) \log|f(z)|\, dm(z),$$

and hence

$$\int_{r\mathbb{T}} P(z,a) \log^{-}|f(z)|\, dm(z) \le \int_{r\mathbb{T}} P(z,a) \log^{+}|f(z)|\, dm(z).$$

For $|z| = r$ and $|a| = \kappa r$ we have

$$\frac{\delta}{2} \le \frac{1-(1-\delta)}{1+(1-\delta)} \le P(z,a) \le \frac{1+(1-\delta)}{1-(1-\delta)} \le \frac{2}{\delta}.$$

By Lemma 10, outside a very small exceptional set (of the order $\exp(-\mu(r))$), we have $\log \mathcal{M}(r) \le 2 \cdot \log M(r)$, and we notice that from (2.10) it follows that $\mu(r)$ is much bigger than $S(\kappa r)$, so this exceptional set is indeed small. Therefore

$$\int_{r\mathbb{T}} \log^{+}|f|\, d\mu \le 2 \log M(r).$$

Now we have

$$\int_{r\mathbb{T}} \log^{-}|f|\, d\mu \le \frac{C}{\delta^2} \log M(r).$$

Finally, (and using (2.3))

$$\int_{r\mathbb{T}} |\log|f||\, d\mu \le \frac{C}{\delta^2} \log M(r) \le \frac{C}{\delta^2} \log \mu(r). \tag{4.4}$$

Combining (4.3) and (4.4) we get the result. ∎

4.3. Deviations for the logarithmic integral. We recall that if

$$f(z) = \sum_{n=0}^{\infty} \phi_n a_n z^n,$$

where $\phi_n$ are i.i.d. standard complex Gaussian random variables, then the vector $(f(z_1), \ldots, f(z_N))$ has a multivariate complex Gaussian distribution, with covariance matrix:

$$\Sigma_{ij} = \mathrm{Cov}(f(z_i), f(z_j)) = \mathbb{E}(f(z_i)\overline{f(z_j)}) = \sum a_k^2 \left(z_i \bar{z}_j\right)^k. \tag{4.5}$$

The density function of a multivariate complex Gaussian distribution is

$$\zeta \mapsto \frac{1}{\pi^N \det \Sigma} \exp(-\zeta^* \Sigma^{-1} \zeta).$$

We introduce the set ($\log_2 \mu(r) \equiv \log\log \mu(r)$)

$$\mathcal{A}' = \left\{ \zeta \in \mathbb{C}^N \ : \ \prod_{j=1}^{N} |\zeta_j| \leq \exp\left(2N \log_2 \mu(r) + C\delta^{-4} \log \mu(r)\right) \right\} \tag{4.6}$$

and denote by $\mathcal{B}$ the set where estimate (4.1) in Lemma 12 holds. We abuse notation by writing

$$\mathbb{P}\left(\mathcal{A}'\right) = \mathbb{P}\left((f(z_1), \ldots, f(z_N)) \in \mathcal{A}'\right).$$

Using this notation we get the simple

Lemma 13:

$$\mathbb{P}\left( \int_{r\mathbb{T}} \log |f(z)| \, dm \leq 2 \log_2 \mu(r) \right) \leq \mathbb{P}\left(\mathcal{A}'\right) + \mathbb{P}\left(\mathcal{B}^c\right).$$

*Proof.* We start by discarding the exceptional set in Lemma 12; this adds the term $\mathbb{P}\left(\mathcal{B}^c\right)$. Now we can assume that

$$\frac{1}{N} \sum_{j=1}^{N} \log |f(z_j)| \leq \int\limits_{r\mathbb{T}} \log |f| \, dm + \frac{C}{\delta^4} \cdot \frac{\log \mu(r)}{N},$$

or

$$\prod_{j=1}^{N} |f(z_j)| \leq \exp\left( N \cdot \int\limits_{r\mathbb{T}} \log |f| \, dm + \frac{C}{\delta^4} \log \mu(r) \right).$$

In terms of probabilities we can write

$$\mathbb{P}\left( \int_{r\mathbb{T}} \log |f(z)| \, dm \leq 2 \log_2 \mu(r) \right) \leq \mathbb{P}\left(\mathcal{B}^c\right) + \mathbb{P}\left(\mathcal{A}'\right). \qquad \blacksquare$$

Before we continue, we need two asymptotic estimates.

Lemma 14: *Let* $\Sigma$ *be the covariance matrix defined in* (4.5). *Choose* $N = N_1(r)$*; then we have the following estimate:*

$$\log\left(\det \Sigma\right) \geq S(\kappa r).$$

*Proof.* Notice that we can represent $\Sigma$ in the following form:

$$\Sigma = V \cdot V^*$$

where

$$V = \begin{pmatrix} a_0 & a_1 \cdot z_1 & \dots & a_n \cdot z_1^n & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ a_0 & a_1 \cdot z_N & \dots & a_n \cdot z_N^n & \dots \end{pmatrix}.$$

We observe that since $a(t)$ is a log-concave function, it follows that $n \mapsto a_n \cdot r^n$ is a unimodal sequence, and therefore $\mathcal{N}_1(r) = \{0, 1, \dots, N_1(r) - 1\}$. Therefore we can estimate the determinant of $\Sigma$ by projecting $V$ on the first $N_1(r)$ coordinates (let us denote this projection by $P$). Since $\det \Sigma$ is the square of the product of the singular values of $V$, and these values are only reduced by the projection, we have

$$\det \Sigma \geq (\det PV)^2 = \begin{vmatrix} a_0 & a_1 z_1 & \dots & a_{N-1} z_1^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_0 & a_1 z_N & \dots & a_{N-1} z_N^{N-1} \end{vmatrix}^2$$

and so

$$\begin{aligned} \det \Sigma \geq & \prod_{n \in \mathcal{N}_1(r)} a_n^2 \cdot \begin{vmatrix} 1 & z_1 & \dots & z_1^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & z_N & \dots & z_N^{N-1} \end{vmatrix}^2 \\ = & \prod_{n \in \mathcal{N}_1(r)} a_n^2 \cdot \prod_{1 \leq i \neq j \leq N} |z_i - z_j| \\ = & \Pi_1 \cdot \Pi_2. \end{aligned}$$

The $z_i$'s are the roots of the equation $z^N = (\kappa r)^N$. Denoting $z_1 = \kappa r$ we get

$$\prod_{i=2}^{N} (z_1 - z_i) = N \left(\kappa r\right)^{N-1},$$

and

$$\Pi_2 = \prod_{1 \leq i \neq j \leq N} |z_i - z_j| = \left( \prod_{i=2}^{N} |z_1 - z_i| \right)^N = \left(\kappa r\right)^{N(N-1)} N^N.$$

We now partition the product of the $\kappa r$'s in the following way:

$$\left(\kappa r\right)^{N(N-1)} = \prod_{n=0}^{N-1} \left(\kappa r\right)^{2n},$$

and get

$$\det \Sigma \geq \prod_{n=0}^{N-1} a_n^2 \left(\kappa r\right)^{2n} = \exp\left(S\left(\kappa r\right)\right). \qquad \blacksquare$$

We denote by $\mathcal{A}$ the following set (see (4.6) for the definition of the set $\mathcal{A}'$):

$$\mathcal{A} = \left\{\zeta \in \mathbb{C}^N \,:\, \zeta \in \mathcal{A}' \text{ and } |\zeta_j| \leq M^2(r), \quad 0 \leq j \leq N-1\right\}, \tag{4.7}$$

and by $I$ the following quantity:

$$I = \pi^{-N} \cdot \mathrm{vol}_{\mathbb{C}^N}(\mathcal{A}). \tag{4.8}$$

We use the following lemma (see [Nis, Lemma 11]) to estimate $I$:

Lemma 15: *Set $s > 0$, $t > 0$ and $N \in \mathbb{N}^+$, such that $\log\left(t^N/s\right) \geq N$. Denote by $\mathcal{C}_N$ the following set:*

$$\mathcal{C}_N = \mathcal{C}_N\left(t, s\right) = \left\{r = (r_1, \ldots, r_N) \,:\, 0 \leq r_j \leq t, \prod_1^N r_j \leq s\right\}.$$

*Then*

$$\mathrm{vol}_{\mathbb{R}^N}(\mathcal{C}_N) \leq \frac{s}{(N-1)!} \log^N\left(t^N/s\right).$$

Now we have an almost immediate

Corollary 16: *Suppose that $r$ is normal and large enough and that $\delta$ satisfies $\delta^{-4} = o\left(N_1(r)\right)$. Then we have*

$$\log I \leq C \cdot N_1(r) \log_2 \mu(r) + C\delta^{-4} \log \mu(r).$$

*Proof.* Set $N = N_1(r)$ and recall that

$$\mathcal{A} = \left\{ \zeta \,:\, \begin{array}{c} |\zeta_j| \leq M^2(r), \quad 0 \leq j \leq N-1 \\ \text{and} \\ \prod_{j=1}^N |\zeta_j| \leq \exp\left(2N \log\log \mu(r) + C\delta^{-4} \log \mu(r)\right) \end{array} \right\}.$$

To shorten the expressions above, we write

$$s = \exp\left(2N \log\log \mu(r) + C\delta^{-4} \log \mu(r)\right), \quad t = M^2(r).$$

We want to translate the integral $I$ into an integral in $\mathbb{R}^N$, using the change of variables $\zeta_j = r_j \cos(\theta_j) + i r_j \sin(\theta_j)$. Integrating out the variables $\theta_j$, we get

$I' = 2^N \int_{\mathcal{C}} \prod r_j \, dr$, where the new domain is

$$\mathcal{C} = \left\{ r = (r_1, \dots, r_N) \,:\, 0 \le r_j \le t, \prod_{j=1}^{N} r_j \le s \right\}.$$

We can find an explicit expression for this integral, but instead we will simplify it even more to

$$I' \le 2^N s \cdot \mathrm{vol}_{\mathbb{R}^N}(\mathcal{C}). \tag{4.9}$$

Now, in order to use the previous lemma, we have to check the condition $\log\left(t^N/s\right) \ge N$, or (where $C > 0$)

$$2N_1(r)\log M(r) - 2N_1(r)\log_2 \mu(r) - C\delta^{-4}\log\mu(r) \ge N_1(r),$$

which is satisfied under our assumptions, for $r$ large enough. After applying the lemma, we get (for $r$ large enough)

$$\begin{aligned} I' \le & \frac{N \cdot 2^N s^2}{N!} \log^N\left(t^N/s\right) \le \frac{s^2 e^{2N}}{N^N} \log^N\left(t^N/s\right) \\ = & \exp\left(2\log s + N\log_2 t + 2N - N\log_2 s\right) \\ \le & \exp\left(2\log s + N\log_2 t\right). \end{aligned}$$

Recalling the definitions of $s$ and $t$, we finally get

$$\begin{aligned} \log I' \le & 4N\log_2\mu(r) + C\delta^{-4}\log\mu(r) + N\log_2 M(r) + C \\ \le & C_1 N_1(r)\log_2\mu(r) + C_2\delta^{-4}\log\mu(r). \qquad \blacksquare \end{aligned}$$

We now continue to estimate probabilities of the events $\mathcal{A}$ and $\mathcal{A}'$ introduced in (4.7) and (4.6).

Lemma 17: *With $r$ and $\delta$ satisfying the conditions of Corollary 16, we have the following estimates:*

$$\mathbb{P}\left(\mathcal{A}'\backslash\mathcal{A}\right) \le \exp\left(-c\mu(r)\right)$$

*and*

$$\mathbb{P}\left(\mathcal{A}\right) \le \exp\left(-S(\kappa r) + C_1 N_1(r)\log_2\mu(r) + C_2\delta^{-4}\log\mu(r)\right).$$

*Proof.* If $\zeta \in \mathcal{A}'\backslash\mathcal{A}$, then for some $j$ we have $|f(z_j)| = |\zeta_j| > M^2(r)$. Using Lemma 10, with the choice $\sigma = \frac{1}{2}$, we see that this event has a probability at most $\exp(-c\mu(r))$. For the second estimate we need to bound from above the integral

$$\int_{\mathcal{A}} \frac{1}{\pi^N \det\Sigma} \exp\left(-\zeta^*\Sigma^{-1}\zeta\right) \, d\zeta.$$

Discarding the exponential function and using Lemma 14 and Corollary 16, we get

$$\mathbb{P}(\mathcal{A}) \le \frac{\mathrm{vol}_{\mathbb{C}^N}(\mathcal{A})}{\pi^N \det\Sigma} \le \exp\left(-S(\kappa r) + C_1 N_1(r) \log_2 \mu(r) + C_2 \delta^{-4} \log \mu(r)\right). \qquad \blacksquare$$

4.4. Lower bound for $p_H$. We collect all the previous results into the proof of Proposition 9

*Proof.* Suppose that $f(z)$ has no zeros inside $r\mathbb{D}$; then

$$\int_{r\mathbb{T}} \log|f(z)| \, dm = \log|f(0)|.$$

We can use the fact that $\log|f(0)|$ cannot be too large; in fact

$$\mathbb{P}(\log|f(0)| \ge 2\log_2 \mu(r)) = \mathbb{P}\left(|\phi_0| \ge \log^2 \mu(r)\right) \le \exp\left(-\log^4 \mu(r)\right).$$

Now combining Lemma 13 and Lemma 17, we see that the probability of the event $\{f(z) \ne 0 \text{ in } r\mathbb{D}\}$ does not exceed

$$\begin{aligned} \exp\left(-\log^4 \mu(r)\right) &+ \exp(-\mu(r)) \\ &+ 2\exp(-S(\kappa r)) \\ &+ \exp\left(-S(\kappa r) + C_1 N_1(r) \log_2 \mu(r) + C_2 \delta^{-4} \log \mu(r)\right). \end{aligned}$$

Since by (2.10) the functions $\mu(r)$ and $\log^4 \mu(r)$ are much bigger than $S(\kappa r)$, we have the required estimate

$$p_H(r) \ge S(\kappa r) - C_1 N_1(r) \log_2 \mu(r) - C_2 \delta^{-4} \log \mu(r). \qquad \blacksquare \tag{4.10}$$

## 5. Proofs of Theorems 1 and 2

In this section we prove Theorem 1 using the lower and upper bound estimates from the previous sections. We also estimate the size of the error terms, for functions with sufficient growth rate, and prove Theorem 2.

5.1. Proof of Theorem 1. By Proposition 6 the error term for the upper bound is (using (2.9))

$$N_1(r)\log N_1(r) \le C\cdot N_1(r)\log_2\mu(r) = o\left(S(r)\right),$$

so it is indeed small.

For the lower bound (4.10), we start by selecting $\delta$ in the following way:

$$\delta = \left(N_1(r)\right)^{-1/5}. \tag{5.1}$$

Now by Proposition 9 the error term is

$$C_1 N_1(r)\log_2\mu(r) + C_2\delta^{-4}\log\mu(r)$$

and we see that the error term is asymptotically smaller than $S(r)$. What remains is to show that $S(\kappa r)$ is close to $S(r)$.

Lemma 18: *Set $r' = (1-\delta)r$. For normal values of $r$ which are large enough,*

$$S(r') \ge S(r) - C\left(N_1(r)\right)^{9/5}$$

*and*

$$\left(N_1(r)\right)^{9/5} = o\left(S(r)\right).$$

*Proof.* We notice that for $\delta < \frac{1}{2}$ we have

$$\log(1-\delta) \ge -\delta - \delta^2.$$

Then it follows that (note that $\mathcal{N}_1(r') \subset \mathcal{N}_1(r)$)

$$\begin{aligned}\frac{S(r)-S(r')}{2} &= \sum_{n\in\mathcal{N}_1(r)\setminus\mathcal{N}_1(r')}\log a_n r^n + \sum_{n\in\mathcal{N}_1(r')}\left(\log a_n r^n - \log a_n\left(r'\right)^n\right)\\ &\le \Sigma_1 + \Sigma_2.\end{aligned}$$

For the first sum we notice that if $n\in\mathcal{N}_1(r)\setminus\mathcal{N}_1(r')$, then

$$\begin{gathered}0 \ge \log a_n\left(r'\right)^n \ge \log a_n r^n - n\left(\delta+\delta^2\right)\\ \Downarrow\\ \log a_n r^n \le n\left(\delta+\delta^2\right) \le 2N_1(r)\delta \le 2\left(N_1(r)\right)^{4/5}\end{gathered}$$

and so

$$\Sigma_1 \le 2\left(N_1(r) - N_1(r')\right)\left(N_1(r)\right)^{4/5} \le 2\left(N_1(r)\right)^{9/5}.$$

For the second sum we have

$$\Sigma_2 \le \left(N_1(r)\right)^2\left(-\log(1-\delta)\right) \le 2\left(N_1(r)\right)^{9/5},$$

and overall we get the required estimate.

Now we will prove that $N_1(r) \leq C\sqrt{S(r)}\log S(r)$, which will give us the second claim. We start with

$$S(r) \overset{(2.10)}{\leq} C\log^2\mu(r)\log_2^2\mu(r) \leq C\log^2\mu(r)\log^2 S(r),$$

therefore

$$\sqrt{S(r)} \leq C\log\mu(r)\log S(r)$$

or

$$N_1(r) \overset{(2.9)}{\leq} 2\cdot\frac{S(r)}{\log\mu(r)} \leq C\sqrt{S(r)}\log S(r), \tag{5.2}$$

and so

$$(N_1(r))^{9/5} \leq C\,(S(r))^{9/10}\,(\log S(r))^{9/5} = o\,(S(r))\,.$$

This concludes the proof of Lemma 18 and Theorem 1. ∎

5.2. Proof of Theorem 2. We need the following

Lemma 19: *Let* $\gamma > 0$. *Suppose that* $\log\mu(r) \geq \exp(\log^\gamma r)$. *Then for values of* $r$ *which are normal and large enough*

$$(N_1(r))^{4/5}\log\mu(r) \leq C\,(S(r))^{9/10}\log^{1/\gamma+8/5} S(r).$$

*Proof.* We first note that from the assumption on $\log\mu(r)$, we have (for $r$ large enough)

$$\log_2^{8/5}\mu(r) \leq \frac{\log_2^{1/\gamma+8/5}\mu(r)}{\log^{9/10} r}.$$

Now (by Lemma 4, Lemma 5 and (2.9))

$$\begin{aligned}(N_1(r))^{4/5}\log\mu(r) \leq& C\log^{9/5}\mu(r)\log_2^{8/5}\mu(r)\\ \leq& C\left(\frac{\log^2\mu(r)}{\log r}\right)^{9/10}\log_2^{1/\gamma+8/5}\mu(r)\\ \leq& C\left(\frac{1}{2}\cdot N_1(r)\log\mu(r)\right)^{9/10}\log_2^{1/\gamma+8/5}\mu(r)\\ \leq& C\,(S(r))^{9/10}\log^{1/\gamma+8/5}S(r). \qquad ∎\end{aligned}$$

*Proof of Theorem 2.* Let $\alpha \geq 1$. We note that if $a(t) \geq \exp(-t\log^\alpha t)$, then for $r$ large enough we have

$$\log\mu(r) \geq \max_{n\in\mathbb{N}}[-n\log^\alpha n + n\log r] \geq c_1\exp\left(c_2\,(\log r)^{1/\alpha}\right),$$

for example by selecting $n$ in such a way that it will satisfy $\log^\alpha n \approxeq \frac{1}{2}\log r$. Finally, we see that $\log \mu(r)$ satisfies the condition in the previous lemma. ∎

We note that using our methods, Theorem 2 cannot be proved for arbitrary (log-concave) coefficients. The problem comes from the following error term in the lower bound:

$$\delta^{-4}\log\mu(r) = (N_1(r))^{4/5}\log\mu(r).$$

To see that we cannot bound it by an expression of the form $(S(r))^\alpha$ with $\alpha < 1$, we take $a(t) = \exp(-\exp(t))$. For this function we have

$$\begin{aligned} N_1(r) &= \log_2 r + \log_3 r + \mathcal{O}(1), \\ \log\mu(r) &= \log r \log_2 r + \mathcal{O}(\log r), \\ S(r) &= \Theta\left(\log r \log_2^2 r\right). \end{aligned}$$

We see that for every $\epsilon > 0$ (for $r$ large enough)

$$\frac{S(r)}{(N_1(r))^{4/5}\log\mu(r)} = \Theta\left(\log_2^{1/5} r\right) = o\left((S(r))^\epsilon\right).$$

## References


[Ha1] W. K. Hayman, *The local growth of power series: a survey of the Wiman–Valiron method*, Canadian Mathematical Bulletin **17** (1974), 317–358.

[Ha2] W. K. Hayman, *Subharmonic Functions*, Vol. 2, London Mathematical Society Monographs **20**, Academic Press, London, 1989.

[Nis] A. Nishry, *Asymptotics of the hole probability for zeros of random entire functions*, International Mathematics Research Notices IMRN 2010, (2010), no. 15, 2925–2946.

[Sod] M. Sodin, *Zeros of Gaussian analytic functions*, Mathematical Research Letters **7** (2000), 371–381.

[ST3] M. Sodin and B. Tsirelson, *Random complex zeroes. III. Decay of the hole probability*, Israel Journal of Mathematics **147** (2005), 371–379.