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SUPERTROPICAL MATRIX ALGEBRA II: SOLVING TROPICAL EQUATIONS

BY

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ABSTRACT

We continue the study of matrices over a supertropical algebra, proving the existence of a tangible adjoint of A, which provides the unique right (resp. left) quasi-inverse maximal with respect to the right (resp. left) quasi-identity matrix corresponding to A; this provides a unique maximal (tangible) solution to supertropical vector equations, via a version of Cramer's rule. We also describe various properties of this tangible adjoint, and use it to compute supertropical eigenvectors, thereby producing an example in which an $n \times n$ matrix has n distinct supertropical eigenvalues but their supertropical eigenvectors are tropically dependent.

1. Introduction

This paper is a continuation of [6]; here, we solve vector equations in supertropical algebra, using the tangible version $\widehat{\mathrm{adj}(A)}$ of the adjoint, which yields a version of Cramer's rule (Theorem 3.5 below). This solution is the unique maximal solution in a certain sense (Theorem 3.8). In §4 we compare $\mathrm{adj}(\mathrm{adj}(A))$ to A. These computational techniques using the adjoint are quite powerful; in Theorem 5.6, we apply them to compute supertropical eigenvectors and to refute

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the natural conjecture that the supertropical eigenvectors would be tropically independent when their supertropical eigenvalues are distinct.

Some of the parallels to classical matrix theory are quite unexpected, especially since their natural analogs in the max-plus algebra often fail. See [1] for some of the max-plus theory; related references are given in the bibliography of [6]. However, the supertropical algebra, which covers the max-plus algebra, is endowed with the "ghost surpassing" relation $\underset{\text{gs}}{\models}$ given in Definition 1.3, which specializes to equality on the "tangible" elements, and provides suitable analogs of these basic results from matrix theory.

The paper [2] was written independently of the earlier version of this paper, and contains some relevant results, especially an elegant meta-theorem about identities of matrix semirings described in Section 2.1 below. During the course of the current version of this paper, we indicate how the results of [2] interact with our results.

We recall that this work is in the environment of a **semiring with ghosts** [5], which is a triple $(R, \mathcal{G}_{\mathbb{0}}, \nu)$, where R is a semiring with zero element, $\mathbb{0}_R$ (often identified in the examples with $-\infty$, as indicated below), and $\mathcal{G}_{\mathbb{0}} = \mathcal{G} \cup \{\mathbb{0}_R\}$ is a semiring ideal, called the **ghost ideal**, together with an idempotent semiring projection

$$\nu : R \longrightarrow \mathcal{G} \cup \{\mathbb{0}_R\}$$

called the **ghost map**, i.e., which preserves multiplication as well as addition, defined as

$$\nu(a) = a + a.$$

We write a^ν for $\nu(a)$, called the **ν-value** of a. We write $a \geq_\nu b$, and say that a **dominates** b, if $a^\nu \geq b^\nu$. Likewise we say that a **strictly dominates** b, written $a >_\nu b$, if $a^\nu > b^\nu$. Two elements a and b in R are said to be **ν-matched** if they have the same ν-value, in which case we also write $a \cong_\nu b$.

1.1. Supertropical semirings.

Definition 1.1: A **supertropical semiring** is a semiring with ghosts that has the extra properties:

(i) $a + b = a^\nu \quad \text{if} \quad a^\nu = b^\nu$;

(ii) $a + b \in \{a, b\}, \ \forall a, b \in R$ s.t. $a^\nu \neq b^\nu$. (Equivalently, $\mathcal{G}_{\mathbb{0}}$ is ordered, via $a^\nu \leq b^\nu$ iff $a^\nu + b^\nu = b^\nu$.)

A **supertropical domain** (the focus of interest for us) is a commutative supertropical semiring $R = (R, \mathcal{G}_{\mathbb{0}}, \nu)$ in which the following two extra conditions are satisfied:

(i) $R \setminus \mathcal{G}_{\mathbb{0}}$ is a monoid $\mathcal{T}$ with respect to the semiring multiplication; the elements of $\mathcal{T}$ are called **tangible**.
(ii) The map $\nu_{\mathcal{T}} : \mathcal{T} \to \mathcal{G}$ (defined as the restriction from ν to $\mathcal{T}$) is onto; in other words, every element of $\mathcal{G}$ has the form a^ν for some $a \in \mathcal{T}$.

We write $\mathcal{T}_{\mathbb{0}}$ for $\mathcal{T} \cup \{\mathbb{0}_R\}$. Note that $\mathcal{T}_{\mathbb{0}}$ acts as the max-plus algebra, except in the case when $a^\nu = b^\nu$, in which case the ghost layer plays its role.

Definition 1.2: A **supertropical semifield** is a supertropical domain $(R, \mathcal{G}_{\mathbb{0}}, \nu)$ in which every tangible element is invertible; in other words, $\mathcal{T}$ is a multiplicative group. Thus, $\mathcal{G}$ is also a multiplicative group.

Recall from [5, Remark 3.12] that any supertropical domain R is ν-**cancellative**, in the sense that $ca^\nu = cb^\nu$ for $c \neq \mathbb{0}_R$ implies $a^\nu = b^\nu$, and in particular its ghost ideal $\mathcal{G}$ is cancellative as a multiplicative monoid. Since any commutative cancellative monoid has an Abelian group of fractions, one often can reduce from the case of a supertropical domain to that of a supertropical semifield. (More details are given in [5, Proposition 3.19 and Remark 3.20].)

1.2. The supertropical relation "ghost surpasses". The following relation, stronger than $\geq_\nu$, plays a key role in the theory, and especially in this paper.

Definition 1.3: We say $b = a +$ ghost if $b = a + c$ for c some ghost element. We define the relation $\underset{\text{gs}}{\models}$, called "ghost surpasses," on any semiring with ghosts R, by

$$b \underset{\text{gs}}{\models} a \quad \text{iff } b = a + \text{ghost}.$$

Note that $b \underset{\text{gs}}{\models} a$ implies $a + b \in \mathcal{G}_{\mathbb{0}}$. In a supertropical semiring, $b \underset{\text{gs}}{\models} a$ iff $b = a$ or b is a ghost $\geq_\nu a$. In particular, if $b \underset{\text{gs}}{\models} a$, then $b \geq_\nu a$. (The converse is false, since one could have tangible $b >_\nu a$.) In fact the relation $\underset{\text{gs}}{\models}$ is a partial order on R, and is not symmetric; for example $a^\nu \underset{\text{gs}}{\models} a$, for a tangible, but not vice versa; i.e., $a \underset{\text{gs}}{\not\models} a^\nu$.

Remark 1.4: In a supertropical domain, if a is tangible with $a \underset{\text{gs}}{\models} b$, then $a = b$. (Indeed, write $a = b + c$ with $c \in \mathcal{G}_{\mathbb{0}}$. Then $b \not\cong_\nu c$ since $a \in \mathcal{T}$, and likewise $a \neq c$, since a is tangible, so $a = b$.)

Thus, for tangible elements, the relation $\underset{\text{gs}}{\models}$ generalizes equality in the max-plus algebra, and seems to be the "correct" generalization to enable us to find analogs of theorems from classical linear algebra. This is the reason for our use of the symbol $\underset{\text{gs}}{\models}$, not to be confused with the usage in model theory. On the other hand, we have the following observation.

Lemma 1.5: *The relation $\underset{\text{gs}}{\models}$ is antisymmetric in any supertropical semiring.*

Proof. We need to show that if $a \underset{\text{gs}}{\models} b$ and $b \underset{\text{gs}}{\models} a$, then $a = b$. This holds by Remark 1.4 if a is tangible (and thus, by symmetry, if b is tangible). Hence, we may assume that $a, b \in \mathcal{G}_{\mathbb{0}}$, in which case

$$a = a^\nu = b^\nu = b. \qquad \blacksquare$$

1.3. The tangible retract function. Although in general, the map $\nu_{\mathcal{T}} : \mathcal{T} \to \mathcal{G}$ need not be 1:1 in a supertropical domain, $\nu_{\mathcal{T}}$ is onto by definition; we find it convenient to choose a "tangible retract" function $\hat{\nu} : R \to \mathcal{T}_{\mathbb{0}}$ restricting to the identity map on $\mathcal{T}_{\mathbb{0}}$, such that $\nu \circ \hat{\nu}$ restricts to the identity map on $\mathcal{G}_{\mathbb{0}}$. We write $\hat{b}$ for $\hat{\nu}(b)$; thus, $(\hat{b})^\nu = b$ for all $b \in \mathcal{G}_{\mathbb{0}}$. We retain the notation $\hat{\nu}$ when working with more complicated expressions.

We do not see any general way to define $\hat{\nu}$ on $\mathcal{G}$ other than applying the axiom of choice rather freely, although in special cases there are canonical definitions for $\hat{\nu}$ (such as when ν is 1:1 or a "lowest term" valuation on power series).

Proposition 1.6: *If F is a divisibly closed semifield, the map $\hat{\nu} : F \to \mathcal{T}$ can be defined such that its restriction $\hat{\nu}|_{\mathcal{G}} : \mathcal{G} \to \mathcal{T}$ is a multiplicative group homomorphism.*

Proof. This can be seen by using the general theory of ordered Abelian groups, but we present an easy direct proof for the reader's convenience. Consider all pairs $(M, \hat{\nu})$, where $M \subset (\mathcal{G}, \cdot)$ is a subgroup with a partial tangible retract function $\hat{\nu}_M : M \to \mathcal{T}_{\mathbb{0}}$ that is multiplicative. We order these pairs by saying

$$(M, \hat{\nu}_M) > (M', \hat{\nu}_{M'}) \quad \text{if } M \supset M'$$

and $\hat{\nu}_M$ restricts to $\hat{\nu}_{M'}$ on M', i.e., $\hat{\nu}_M|_{M'} = \hat{\nu}_{M'}$.

Note that such pairs $(M, \hat{\nu})$ exist, since we could take a group $M = \langle a^\nu \rangle$ generated by a single element a^ν, and $\hat{\nu}(a^\nu) = a$.

By Zorn's lemma, there is a subgroup $M \subset (\mathcal{G}, \cdot)$ for which $(M, \hat{\nu}_M)$ is maximal. If $M \neq \mathcal{G}$, then take $a \in \mathcal{G} \setminus M$. Let

$$P = \{n \in \mathbb{Z} : a^n \in M\},$$

an ideal of $\mathbb{Z}$, and write $P = k\mathbb{Z}$ for some $k \geq 0$. If $k > 0$, choose $\hat{a}$ such that $\hat{a}^k = \widehat{a^k}$. (This is possible since F is divisibly closed.) If $k = 0$, choose $\hat{a}$ arbitrarily in $\mathcal{T}$ such that $(\hat{a})^\nu = a$. Define

$$\widehat{a^i b} := \widehat{a}^i \hat{b}$$

for each $b \in M$ and each $i > 0$. To see that this is well-defined, suppose $a^i b = a^j b'$ for $i \geq j$ and $b' \in M$. Then $a^{i-j} = b' b^{-1} \in M$, which by definition is $\widehat{a^{i-j}} = \hat{a}^{i-j}$ since k divides $i - j$, implying $\hat{a}^i \hat{b} = \hat{a}^j \widehat{b'}$. Thus, we could define $\hat{\nu}$ multiplicatively on the group generated by M and a, contradicting the maximality of M, so we must have $M = \mathcal{G}$. Then we put $\widehat{\mathbb{0}_F} = \mathbb{0}_F$. ∎

Remark 1.7: Whenever $a \not\cong_\nu b$, the retract map $\hat{\nu}$ must satisfy

$$\hat{\nu}(a + b) = \hat{a} + \hat{b}. \tag{1.1}$$

Indeed, we may assume that $a >_\nu b$, and thus $\hat{\nu}(a + b) = \hat{a} = \hat{a} + \hat{b}$.

But for $a \in \mathcal{G}$, we have $\hat{a} + \hat{a} = (\hat{a})^\nu = a$ which is not $\hat{a} = \widehat{a + a}$, so $\hat{\nu}$ is not a semiring homomorphism.

The following observation enables us to utilize $\hat{\nu}$ to make calculations paralleling those in the max-plus algebra.

Proposition 1.8: *If* $\sum_k a_k \widehat{b_{j,k}} \in \mathcal{G}_{\mathbb{0}}$ *for each* $1 \leq j \leq m$, *then*

$$\sum_k a_k \hat{\nu}\left(\sum_{j=1}^m b_{j,k} c_j\right) \in \mathcal{G}_{\mathbb{0}}$$

for any $c_j \in R$.

Proof. Otherwise, consider the single dominating term $a_{k_1}(\widehat{b_{j_1,k_1} c_{j_1}})$ of the left side. We are done unless $a_{k_1} \in \mathcal{T}$. But $a_{k_1}(\widehat{b_{j_1,k_1} c_{j_1}})$ dominates $a_k(\widehat{b_{j_1,k} c_{j_1}})$ for each k, implying $a_{k_1} \widehat{b_{j_1,k_1}} \in \mathcal{T}$ dominates each $a_k \widehat{b_{j_1,k}}$. Thus, there must be k_2

with $a_{k_1}\widehat{b_{j_1,k_1}} \cong_\nu a_{k_2}\widehat{b_{j_1,k_2}}$. But then

$$a_{k_1}(\widehat{b_{j_1,k_1}c_{j_1}}) \cong_\nu a_{k_2}(\widehat{b_{j_1,k_2}c_{j_1}})$$

implying that their sum is ghost. ∎

PROPOSITION 1.9: $\sum_k a_k \sum_j \widehat{b_{j,k}} \underset{\text{gs}}{\models} \sum_k a_k(\widehat{\sum_j b_{j,k}})$.

Proof. The two sides are ν-matched, so it remains to show that if the left side is tangible, then it equals the right side.

Suppose that $a_k\widehat{b_{j',k}}$ alone dominates the left side. Then $b_{j,k} <_\nu b_{j',k}$ for each $j \neq j'$, implying $\sum_j b_{j,k} = b_{j',k}$. Hence, the single dominating term in the sum at the right must also be $a_k\widehat{b_{j',k}}$. ∎

1.4. VECTORS. We also recall the definition of $R^{(n)}$ as the Cartesian product $\prod_{i=1}^n R$ of n copies of the supertropical semiring R, viewed as a module via componentwise multiplication, with zero element $\mathbb{0} = (\mathbb{0}_R)$ and ghost submodule $\mathcal{H}_\mathbb{0} = \mathcal{G}_\mathbb{0}^{(n)}$. Let $\mathcal{H} = \mathcal{H}_\mathbb{0} \setminus \{\mathbb{0}_R\}$. When R is a supertropical semifield, $R^{(n)}$ is called a **tropical vector space** over R. A vector $\neq \mathbb{0}$ is called **tangible** if all of its components are in $\mathcal{T}_\mathbb{0}$.

Our partial orders $\underset{\text{gs}}{\models}$ and $\geq_\nu$ on R, and the tangible retract function $\hat{\nu}: R \to \mathcal{T}_\mathbb{0}$, extend respectively to the partial orders $\underset{\text{gs}}{\models}$ and $\geq_\nu$ on $R^{(n)}$, and the tangible retract function $\hat{\nu} : R^{(n)} \to \mathcal{T}_\mathbb{0}^{(n)}$, by matching the corresponding components; note that vectors v, w satisfy $w \underset{\text{gs}}{\models} v$ iff $w = v +$ ghost. For example,

$$(\mathbb{1}_R^\nu, \mathbb{1}_R^\nu, \mathbb{1}_R^\nu) \underset{\text{gs}}{\models} (\mathbb{1}_R^\nu, \mathbb{1}_R^\nu, \mathbb{1}_R) \underset{\text{gs}}{\models} (\mathbb{1}_R^\nu, \mathbb{1}_R, \mathbb{1}_R).$$

Also, by checking components, we see that $\underset{\text{gs}}{\models}$ is antisymmetric for vectors.

LEMMA 1.10: *Suppose $v, w \in R^{(n)}$, with w tangible. Then $\underset{\text{gs}}{\models} w$ iff $v + w \in \mathcal{H}_\mathbb{0}$.*

Proof. ($\Rightarrow$) is obvious.

($\Leftarrow$) By assumption, each component w_i of w is in $\mathcal{T}_\mathbb{0}$, and $v_i + w_i \in \mathcal{H}_\mathbb{0}$ implies $v_i = w_i$ or v_i is ghost $\geq_\nu w_i$; thus $v_i \underset{\text{gs}}{\models} w_i$ for each i, implying $v \underset{\text{gs}}{\models} w$. ∎

1.5. THE ν-TOPOLOGY. We also need the following topology on R; cf. [5, Definition 3.22]:

Definition 1.11: Suppose $(R, \mathcal{G}_{\mathbb{0}}, \nu)$ is a supertropical domain. Viewing $\mathcal{G}$ as an ordered monoid with respect to $\geq_\nu$, we define the **ν-topology** on R, whose open sets have a base comprised of the **open intervals**

$$W_{\alpha,\beta} = \{a \in R : \alpha <_\nu a <_\nu \beta\}; \quad W_{\alpha,\beta;\mathcal{T}} = \{a \in \mathcal{T} : \alpha <_\nu a <_\nu \beta\}, \quad \alpha, \beta \in \mathcal{G}_{\mathbb{0}}.$$

This topology extends to the product topology on $R^{(n)}$ for any n.

Note that the tangible vectors in $R^{(n)}$ are a dense subset in the ν-topology. When we need to apply topological arguments, in order that multiplication be a continuous function, we assume that $\mathcal{T}$ is dense, in the sense that $W_{\alpha,\beta;\mathcal{T}} \neq \emptyset$ whenever $\alpha <_\nu \beta$.

1.6. The semiring of functions. Let $\mathrm{Fun}(R^{(n)}, R)$ (resp. $\mathrm{CFun}(R^{(n)}, R)$) denote the semiring of functions (resp. continuous functions) from $R^{(n)}$ to R; cf. [5, Definition 3.31]. We can also define our partial orders on $\mathrm{Fun}(R^{(n)}, R)$:

Definition 1.12: For $f, g \in \mathrm{Fun}(R^{(n)}, R)$, we write $f \geq_\nu g$ if $f(\mathbf{a}) \geq_\nu g(\mathbf{a})$ for all $\mathbf{a} = (a_1, \dots, a_n)$ in $R^{(n)}$.

The **ghost-surpassing identity** $f \underset{\text{gs}}{\models} g$ holds for $f, g \in \mathrm{Fun}(R^{(n)}, R)$, if $f(a_1, \dots, a_n) \underset{\text{gs}}{\models} g(a_1, \dots, a_n)$ for every $a_1, \dots, a_n \in R$.

Proposition 1.13: *Suppose $f, g \in \mathrm{CFun}(R^{(n)}, R)$.*

(i) If $f(\mathbf{a}) \geq_\nu g(\mathbf{a})$ for all $\mathbf{a}$ in a dense subset of $R^{(n)}$, then $f \geq_\nu g$.
(ii) If $f(\mathbf{a}) \underset{\text{gs}}{\models} g(\mathbf{a})$ for all $\mathbf{a}$ in a dense subset $\mathcal{S}$ of $R^{(n)}$, then $f \underset{\text{gs}}{\models} g$.

Proof.

(i) Otherwise, we have $f(\mathbf{a}) <_\nu g(\mathbf{a})$ for some $\mathbf{a} \in R^{(n)}$, so this inequality holds for some open interval $W_{\mathbf{a}}$ containing $\mathbf{a}$.
(ii) We are done by (i) unless there exists $\mathbf{a}$ such that $f(\mathbf{a}) \in \mathcal{T}$ and $f(\mathbf{a}) >_\nu g(\mathbf{a})$. But then this inequality holds for some open interval $W_{\mathbf{a}}$ containing $\mathbf{a}$, implying $f(\mathbf{a}') \in \mathcal{G}_{\mathbb{0}}$ for all $\mathbf{a}' \in \mathcal{S} \cap W_{\mathbf{a}}$. We conclude that $f(W_{\mathbf{a}}) \subseteq \mathcal{G}_{\mathbb{0}}$, contrary to $\mathbf{a} \in W_{\mathbf{a}}$. ∎

Several examples of ghost surpassing identities are given in [6]; as we shall see, many of these can be obtained via a powerful new technique of [2].

1.7. Identities of semirings with symmetry. Any commutative semiring with ghosts is a semiring with symmetry in the sense of [2, Definition 4.1], where their map τ is taken to be the identity map, and their S^o is the ghost ideal $\mathcal{G}_{\mathbb{0}}$. Furthermore, they define a relation $a \succeq^o b$ when $a = b + c$ for some $c \in S^o$; this clearly specializes to our relation $a \underset{\text{gs}}{\models} b$.

Akian, Gaubert and Guterman [2, Theorem 4.21] then proved their **strong transfer principle**, which we rephrase slightly:

Theorem 1.14: *Suppose $p^+, p^-, q^+, q^- \in \mathbb{N}[\xi_1, \dots, \xi_m]$ are polynomials in commuting indeterminates $\xi_1, \dots, \xi_m$, and let $p = p^+ - p^-$ and $q = q^+ - q^-$ in the free commutative ring $\mathbb{Z}[\xi_1, \dots, \xi_m]$. If $p = q$, and if no monomials appear in both q^+ and q^-, then $p^+ + p^- \succeq^o q^+ + q^-$ is an identity for all commutative semirings.*

2. Matrices and adjoints

In this section, we accumulate basic information about matrices and their adjoints. We write $M_n(R)$ for the semiring of $n \times n$ matrices, whose multiplicative identity is denoted as I, and we define the **supertropical determinant** $|A|$ of $A = (a_{i,j})$ to be the permanent as in [3, 4, 6, 7]; i.e.,

$$|A| = \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

A permutation $\sigma \in S_n$ **attains** $|A|$ if $|A| \cong_\nu a_{\sigma(1),1} \cdots a_{\sigma(n),n}$, where $A = (a_{i,j})$.

A matrix A is defined to be **nonsingular** if $|A| \in \mathcal{T}$ is invertible; A is defined to be **singular** if $|A| \in \mathcal{G}_{\mathbb{0}}$. Thus, over a supertropical semifield, every matrix is either singular or nonsingular.

Definition 2.1: The **minor** $A_{i,j}$ is obtained by deleting the i row and j column of A. The **adjoint** matrix $\operatorname{adj}(A)$ of a matrix $A = (a_{i,j})$ is defined as the transpose of the matrix $(a'_{i,j})$, where $a'_{i,j} = |A_{i,j}|$. The **tangible adjoint** matrix $\widehat{\operatorname{adj}(A)}$ of A is defined as the transpose of the matrix $(\widehat{a'_{i,j}})$.

Note that $\widehat{\operatorname{adj}(A)}$ depends on the choice of the tangible retract function $\hat{\nu}$.

Viewing matrices as n^2-dimensional vectors, we can introduce the product topology, as well as our relations $\geq_\nu$ and $\underset{\text{gs}}{\models}$, to matrices (by comparing the corresponding entries).

Lemma 2.2: *The function* det : $A \mapsto |A|$ *is a continuous function from* $R^{(n^2)}$ *to* R*, and the adjoint is a continuous function from* $R^{(n^2)}$ *to* $R^{(n^2)}$*.*

Proof. Clear, because the determinant is defined in terms of addition and multiplication, which are continuous functions in the ν-topology over $R^{(n^2)}$. ∎

Remark 2.3: We can reformulate [6, Theorem 3.5] as

$$|AB| \underset{\text{gs}}{\models} |A|\,|B|,$$

for any $A, B \in M_n(R)$, and [6, Proposition 4.8] as $\operatorname{adj}(AB) \underset{\text{gs}}{\models} \operatorname{adj}(B)\operatorname{adj}(A)$.

2.1. Ghost-surpassing identities of matrices. Suppose $P^+=(p^+_{i,j}), P^-=(p^-_{i,j}), Q^+ = (q^+_{i,j})$, and $Q^- = (q^-_{i,j})$ are matrix expressions whose respective (i,j) entries $p^+_{i,j}, p^-_{i,j}, q^+_{i,j}$, and $q^-_{i,j} \in \mathbb{N}[\xi_1, \dots, \xi_m]$ are semiring polynomials in the entries of $x_1, \dots, x_\ell$ (in other words, only involving addition and multiplication, but not negation). In particular, when x_i are $n \times n$ matrices, we set $m = \ell n$.

Formally set $P(x_1, \dots, x_\ell) = P^+ - P^-$ and $Q(x_1, \dots, x_\ell) = Q^+ - Q^-$. We say Q is **admissible** if the monomials of $q^+_{i,j}$ and $q^-_{i,j}$ are distinct, for each pair (i,j).

Theorem 1.14 provides the following metatheorem for matrices:

Theorem 2.4: *Suppose* $P = Q$ *is a matrix identity of* $M_n(\mathbb{Z})$*, with* Q *admissible. (In other words,* $P(A_1, \dots, A_\ell) = Q(A_1, \dots, A_\ell)$ *for all matrices* $A_1, \dots, A_\ell$*.) Then for any commutative semiring with ghosts* $(R, \mathcal{G}_{\mathbb{0}}, \nu)$*, the matrix semiring with ghosts* $M_n(R)$ *satisfies the ghost-surpassing matrix identity* $P^+ + P^- \underset{\text{gs}}{\models} Q^+ + Q^-$*.*

The proof is standard: It is enough to check for substitutions to "generic matrices" in which each indeterminate x_k is specialized to a matrix $(\xi^k_{i,j})$ whose entries are commuting indeterminates. Then the proposed ghost-surpassing identity $P^+ + P^- \underset{\text{gs}}{\models} Q^+ + Q^-$ can be expressed in terms of n^2 ghost-surpassing identities in the commuting indeterminates $\xi^k_{i,j}$, one for each matrix entry.

Remark 2.5: Define the **characteristic polynomial** f_A of A as $|A + \lambda I|$. If $f_A = \sum_{i=0}^n \alpha_i \lambda^i$, define the **tangible characteristic polynomial** $\widehat{f_A}$ of A as $\sum_{i=0}^n \widehat{\alpha}_i \lambda^i$. Here are some results from [6], which are reproved as easy applications of Theorem 2.4, for any semiring with ghosts $(R, \mathcal{G}_{\mathbb{0}}, \nu)$:

(i) $|AB| \underset{\text{gs}}{\models} |A|\,|B|$.

(ii) Any matrix A satisfies its tangible characteristic polynomial $\widehat{f_A}$; i.e., $\widehat{f_A}(A) \underset{\text{gs}}{\models} (\mathbb{0})$.

(iii) Notation as above, let $\widetilde{f_A} = \sum_{i=1}^{n} \alpha_i \lambda^{i-1}$; then $\widetilde{f_A}(A) \underset{\text{gs}}{\models} \operatorname{adj}(A)$.

In order to apply Theorem 2.4, one needs to observe that in each of these expressions the $q^+_{i,j}$ and $q^-_{i,j}$ are distinct. This is true in (i) and (iii) because of the standard formulas for the determinant and adjoint. (We can describe the left side of (iii) by applying Newton's formula for computing the coefficients of the characteristic polynomial of A in terms of traces of powers of A.)

Now (ii) is obtained by multiplying (iii) by A, noting that $A \operatorname{adj}(A) \underset{\text{gs}}{\models} I$ by [6, Remark 4.14].

2.2. Quasi-identity matrices and quasi-inverses. Recall the following definition from [6]:

Definition 2.6: A **quasi-identity matrix** is a nonsingular, multiplicatively idempotent matrix equal to the identity matrix I on the diagonal, and whose off-diagonal entries are in $\mathcal{G}_{\mathbb{0}}$.

Remark 2.7: Any quasi-identity matrix I' ghost surpasses I; i.e., $I' \underset{\text{gs}}{\models} I$.

Quasi-identities seem to be the key to the supertropical matrix theory. Note, however, that the product of quasi-identities is not necessarily a quasi-identity. For example, take

$$B_1 = \begin{pmatrix} 0 & 10^\nu \\ -\infty & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & -\infty \\ 10^\nu & 0 \end{pmatrix}.$$

Then B_1 and B_2 are quasi-identities, but

$$B_1 B_2 = \begin{pmatrix} 20^\nu & 10^\nu \\ 10^\nu & 0 \end{pmatrix}$$

is a singular matrix. Accordingly, we start with a given matrix A. Most of the following theorem is contained in [6, Theorem 4.12].

Theorem 2.8: *Suppose $A = (a_{i,j})$, with $|A|$ invertible. Define*

$$A^{\nabla} = \frac{\mathbb{1}_R}{|A|}\operatorname{adj}(A) \quad \text{and} \quad A^{\widehat{\nabla}} = \frac{\mathbb{1}_R}{|A|}\widehat{\operatorname{adj}(A)};$$

$$I_A = AA^{\nabla} = AA^{\widehat{\nabla}}; \quad I'_A = A^{\nabla}A = A^{\widehat{\nabla}}A.$$

Then $AA^{\nabla} = AA^{\widehat{\nabla}} = I_A$ and $A^{\nabla}A = A^{\widehat{\nabla}}A = I'_A$ are quasi-identities.

Proof. Starting with [6, Theorem 4.12], it remains to show that $AA^{\nabla} = AA^{\widehat{\nabla}}$. Their ν-values are the same, so we need only check that the diagonal entries of $AA^{\widehat{\nabla}}$ are tangible (which is a fortiori, since this is true for the diagonal entries of AA^{∇}), and that the off-diagonal entries of $AA^{\widehat{\nabla}}$ are ghost, which holds because of [6, Remark 4.5]. ∎

The fact that $I_A^2 = I_A$, proved in [6, Theorem 4.12] by means of Hall's Marriage Theorem, is a key ingredient of the theory.

Inspired by Theorem 2.8, when $|A|$ is invertible, we say that A is **quasi-invertible** and call A^{∇} the **canonical two-sided quasi-inverse** of A and define the **right quasi-identity matrix of** A to be the matrix

$$I_A = AA^{\nabla} = AA^{\widehat{\nabla}},$$

and the **left quasi-identity matrix of** A to be the matrix

$$I'_A = A^{\nabla}A = A^{\widehat{\nabla}}A.$$

(The **left tangible quasi-inverse** $A^{\widehat{\nabla}}$ is introduced here since it plays a role in solving equations, in §3.) Over a supertropical semifield, a matrix is quasi-invertible iff it is nonsingular.

Remark 2.9: If $C \underset{\text{gs}}{\models} A$, then $BC \underset{\text{gs}}{\models} BA$. In particular, $BI' \underset{\text{gs}}{\models} B$ for any quasi-identity matrix I'; cf. Remark 2.7. By symmetry, we also have $I'B \underset{\text{gs}}{\models} B$.

Remark 2.10: If $A \cong_{\nu} B$ are quasi-invertible, then $I_A = I_B$. (Indeed, the diagonal of each is the identity matrix I, and the off-diagonal entries are clearly ν-matched and thus, being ghosts, are equal.)

Example 2.11: For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have $\operatorname{adj}(A) = \left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right)$; hence

$$A\operatorname{adj}(A) = \begin{pmatrix} |A| & (ab)^{\nu} \\ (cd)^{\nu} & |A| \end{pmatrix} \quad \text{whereas } \operatorname{adj}(A)A = \begin{pmatrix} |A| & (bd)^{\nu} \\ (ac)^{\nu} & |A| \end{pmatrix}.$$

Thus, the left and right quasi-identities of a quasi-invertible matrix can be quite different. This enigma will only be resolved in Corollary 4.5 below.

LEMMA 2.12: *The quasi-invertible matrices are dense in $M_n(R)$.*

Proof. Given any matrix, we take some permutation σ attaining $|A|$, and let α be a tangible element of ν-value slightly greater than $\mathbb{1}_R^\nu$. Replacing $a_{\sigma(i),i}$ by $\alpha\widehat{a_{\sigma(i),i}}$ for each $1 \leq i \leq n$ gives us a matrix close to A whose determinant is $\alpha^n\widehat{|A|} \in \mathcal{T}$, as desired. ∎

Lemma 2.12 shows us that much of the matrix theory can be developed by looking merely at the quasi-invertible matrices. For example, Remark 2.3 could be verified by checking only the quasi-invertible matrices. Along these lines, we have:

PROPOSITION 2.13: *If $f, g \in \mathrm{CFun}(R^{(n^2)}, R^{(m)})$ and $f(A) \underset{\mathrm{gs}}{\models} g(A)$ for all quasi-invertible matrices A, then $f \underset{\mathrm{gs}}{\models} g$.*

Proof. Combine Lemma 2.12 with Proposition 1.13 (ii). ∎

In view of [6, Proposition 4.17], every quasi-identity matrix I_A is its own left and right quasi-inverse as well as its own left and right quasi-identity matrix, and $I_A = {I_A}^\nabla$. In order to obtain the best results, we need to modify our notion of adjoint.

Remark 2.14: Define

$$A^{\overline{\nabla}} = A^\nabla I_A = A^\nabla A A^\nabla.$$

Then $AA^{\overline{\nabla}} = AA^\nabla I_A = (I_A)^2 = I_A$, so $A^{\overline{\nabla}}$ is a right quasi-inverse of A. By Remark 2.9,

$$A^{\overline{\nabla}} = A^\nabla I_A \underset{\mathrm{gs}}{\models} A^\nabla \underset{\mathrm{gs}}{\models} A^{\widehat{\nabla}}.$$

In fact, $A^{\overline{\nabla}}$ is the "maximal" right quasi-inverse of A, in the following sense:

LEMMA 2.15: *If $AB = I_A$, then $A^{\overline{\nabla}} \underset{\mathrm{gs}}{\models} B$.*

Proof. $A^{\overline{\nabla}} = A^\nabla I_A = A^\nabla(AB) = (A^\nabla A)B = I'_A B \underset{\mathrm{gs}}{\models} B$. ∎

By symmetry $A^{\overline{\nabla}}$ is also the "maximal" left quasi-inverse of A (although the corresponding left and right quasi-identities I_A and I'_A may differ!). From this point of view, $A^{\overline{\nabla}}$ is the "correct" supertropical version of the adjoint. (The distinction between A^{∇} and $A^{\overline{\nabla}}$ would not arise in classical matrix algebra.)

The same sort of reasoning as with Lemma 2.15 shows that I_A is maximal with respect to the following property:

Remark 2.16: (i) If $A^{\widehat{\nabla}}B = A^{\widehat{\nabla}}$, then $I_A = AA^{\widehat{\nabla}} = AA^{\widehat{\nabla}}B = I_AB \underset{\mathrm{gs}}{\models} B$.

(ii) If $A^{\overline{\nabla}}B = A^{\overline{\nabla}}$, then $I_A = AA^{\overline{\nabla}} = AA^{\overline{\nabla}}B = I_AB \underset{\mathrm{gs}}{\models} B$.

To proceed further, we need a result from [6] that relies on the Hall Marriage Theorem from graph theory, applied to the digraph of the matrix A (which we recall is the graph whose edges are indexed and weighted by the entries of A).

LEMMA 2.17: $|A|\,\mathrm{adj}(A) \ \geq_\nu \ \mathrm{adj}(A)\,A\,\mathrm{adj}(A)$, *for any matrix A.*

Proof. The (i,j)-entry of $\mathrm{adj}(A)A\,\mathrm{adj}(A)$ is the sum of terms of the form $a'_{k,i}\,a_{k,\ell}\,a'_{j,\ell}$, each of which we write out as a product of entries of A, thereby corresponding to a digraph (having multiple edges, each corresponding to one of the entries in this product) with in-degree 2 at every vertex except j, and out-degree 2 at every vertex except i. Hence, by [6, Lemma 3.16(iv)], we can take out an n-multicycle that has ν-value at most $|A|$, leaving at most $a'_{i,j}$, so $|A|\,\mathrm{adj}(A) \geq_\nu \mathrm{adj}(A)A\,\mathrm{adj}(A)$, as desired. ∎

THEOREM 2.18: *For any quasi-invertible matrix A,*

$$A^{\overline{\nabla}} \cong_\nu A^{\nabla} \cong_\nu A^{\widehat{\nabla}}.$$

Proof. By Lemma 2.17, $A^{\nabla} \geq_\nu A^{\nabla}AA^{\nabla} = A^{\overline{\nabla}}$, so we are done by Remark 2.14. ∎

Recall that the relation

$$A^{\overline{\nabla}} \underset{\mathrm{gs}}{\models} A^{\nabla} \underset{\mathrm{gs}}{\models} A^{\widehat{\nabla}}$$

holds for any matrix A.

As with [5], [6], we present our examples in logarithmic notation (i.e., $-\infty$ is the additive identity and 0 is the multiplicative identity); we often write $-$ for $-\infty$.

Example 2.19: In logarithmic notation, for

$$A = \begin{pmatrix} 0 & a & - \\ - & 0 & b \\ - & - & 0 \end{pmatrix}, \quad \text{we have} \quad A^{\nabla} = \begin{pmatrix} 0 & a & ab \\ - & 0 & b \\ - & - & 0 \end{pmatrix},$$

$$I_A = \begin{pmatrix} 0 & a^{\nu} & ab^{\nu} \\ - & 0 & b^{\nu} \\ - & - & 0 \end{pmatrix}, \quad \text{and} \quad A^{\overline{\nabla}} = \begin{pmatrix} 0 & a^{\nu} & ab^{\nu} \\ - & 0 & b^{\nu} \\ - & - & 0 \end{pmatrix}.$$

Remark 2.20: Here is an example where $A^{\overline{\nabla}}$ can be tangible off the diagonal: In Example 2.11, take $a = d = -\infty$, and b, c tangible. $I_A = I$, so $A^{\overline{\nabla}} = A^{\nabla}$, a tangible matrix.

(This is the only way of getting such an example. Looking into the computations of the proof of Lemma 2.17, one sees that when the determinant of A is attained by a product of terms including a diagonal entry, then the computation of any off-diagonal entry of $A^{\overline{\nabla}}$ yields two matching terms containing $|A|$, and thus $A^{\overline{\nabla}}$ is ghost off the diagonal.)

Remark 2.21: Although our discussion in this section has focused on nonsingular matrices, one could define more generally

$$A^{\nabla} = \frac{\mathbb{1}_R}{\widehat{|A|}} \operatorname{adj}(A)$$

whenever $|A|^{\nu}$ is invertible in $\mathcal{G}$. Some computational results are available in this situation, such as AA^{∇} being idempotent, but the diagonal is no longer tangible.

3. Solving equations

We are ready to turn to one of the main features of this paper. Our objective in this section is to solve matrix equations over supertropical domains. We look for tangible solutions, since any large ghost vector would be a solution. There is an extensive theory of solving equations over the max-plus algebra [1], but the supertropical theory has a different flavor, relying mostly on standard tools from classical matrix theory. We work in $R^{(n)}$, with $\mathcal{H}_{\mathbb{0}} = \mathcal{G}_{\mathbb{0}}^{(n)}$.

In general, although the matrix equation $Ax = v$ need not be solvable, we shall see in Theorems 3.5 and 3.8 that

$$Ax \underset{\text{gs}}{\models} v, \quad v = (v_1, \dots, v_n),$$

always has a tangible solution for $x = (x_1, \dots, x_n)$, and the unique maximal tangible solution can be computed explicitly, for any $n \times n$ quasi-invertible matrix A and tangible vector $v \in R^{(n)}$ over a supertropical domain $R = (R, \mathcal{G}_{\mathbb{0}}, \nu)$. (These results are somewhat stronger than those in [2, Theorems 6.4 and 6.6], which deal with a weaker relation.)

Example 3.1: In logarithmic notation, let

$$A = \begin{pmatrix} 0 & 10 \\ - & 0 \end{pmatrix} \quad \text{and} \quad v = (0, 0).$$

We first look for a tangible solution $x = (x_1, x_2)$ of the equation $Ax + v \in \mathcal{G}_{\mathbb{0}}^{(2)}$, that is, a tangible solution of the equations

$$x_1 + 10x_2 + 0 \in \mathcal{G}_{\mathbb{0}}, \quad x_2 + 0 \in \mathcal{G}_{\mathbb{0}},$$

which requires $x_2 = 0$ and thus $x_1 = 10$.

But this unique tangible solution fails to satisfy the matrix equation $Ax = v$, which thus has no tangible solutions!

In view of this example, we turn instead to the equation $Ax \underset{\text{gs}}{\models} v$, which we solve in its entirety, and obtain a condition when it gives us a solution to $Ax = v$. (When v is tangible, we have seen that the equation $Ax \underset{\text{gs}}{\models} v$ is equivalent to $Ax + v \in \mathcal{H}_{\mathbb{0}} = \mathcal{G}_{\mathbb{0}}^{(n)}$.) First we dispose of a trivial situation.

Remark 3.2: When A is a singular matrix over a supertropical semifield, then its rows are tropically dependent, and thus $Ax \in \mathcal{H}_{\mathbb{0}}$ for some tangible vector x by [7, Theorem 2.10] which could be taken with $x_k = \widehat{|A_{i,k}|}$ for some i (see [7, proof of Lemma 2.8]). Accordingly, for any given vector v, under the mild assumption that $\widehat{|A_{i,k}|} \neq \mathbb{0}_R$ for each k, the matrix equation $Ax \underset{\text{gs}}{\models} v$ has the tangible vector solution cx for any fixed large tangible constant c.

Here is one case in which we can compute the tangible solution to $Ax \underset{\text{gs}}{\models} (\mathbb{0})$. We say that two multicycles are **disjoint** if they have no common edges.

Proposition 3.3: *Suppose* $\mathbb{0}_R \neq |A| \in \mathcal{G}$, *and* $|A| = \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$, *is attained only by tangible terms* $a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ *whose corresponding multicycles are disjoint. Then taking* x *to be any column of* $\operatorname{adj}(A)$, *we have* $A\hat{x} \in \mathcal{H}_{\mathbb{0}}$.

Proof. Write x as the column vector $(a'_{i,1}, \dots, a'_{i,n})^{\mathrm{t}}$. Fix j and write $|A| = \sum_j a_{i,j} a'_{i,j}$, where $a'_{i,j} = |A_{i,j}|$, so the j component of $A\hat{x}$ is $\sum_{k=1}^n a_{j,k} \widehat{a'_{i,k}}$. By [6, Remark 4.5], this is ghost unless $i = j$. When $i = j$, we get some value $a = \sum_{k=1}^n a_{i,k} \widehat{a'_{i,k}}$, which is ghost unless it has a single dominating summand $a_{i,k} \widehat{a'_{i,k}}$. But $\sum_{k=1}^n a_{i,k} a'_{i,k} = |A|$ is ghost, and is dominated by $a_{i,k} a'_{i,k}$ alone, which thus must be ghost, and $|A| = a_{i,k} a'_{i,k}$. Since $|A| \neq \mathbb{0}_R$, we see that $a_{i,k} \in \mathcal{T}$, implying $a'_{i,k} \in \mathcal{G}$.

Let $J = \{\sigma \in S_n : \sigma \text{ attains } |A|\}$; i.e., $\sigma \in J$ iff $a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \cong_\nu |A|$. Taking

$$J_{i,k} = \{\sigma \in J : \sigma(i) = k\},$$

we have

$$a'_{i.k} = \sum_{\sigma \in J_{i,k}} a_{1,\sigma(1)} \cdots a_{i-1,\sigma(i-1)} a_{i+1,\sigma(i+1)} \cdots a_{n,\sigma(n)},$$

a ghost. By hypothesis, each summand is tangible, so $J_{i,k}$ has order at least 2. This shows J has two permutations with the common edge (i, k), contrary to hypothesis. ∎

Corollary 3.4: *Suppose* $\mathbb{0}_R \neq |A| \in \mathcal{G}$, *but every entry of* A *and of* $\operatorname{adj}(A)$ *is in* $\mathcal{T}_{\mathbb{0}}$. *Then taking* x *to be the* i *column of* $\operatorname{adj}(A)$, *we have* $Ax \in \mathcal{H}_{\mathbb{0}}$.

Proof. Otherwise, by the contrapositive of the proposition, two permutations $\sigma \neq \tau$ attain the determinant where $\sigma(i) = \tau(i) = k$ for suitable i, k, and thus $a'_{i.k} \in \mathcal{G}$, contrary to hypothesis. ∎

The same argument will be used in Theorem 5.6 in a more technical setting, when we consider eigenvalues. Accordingly, we assume that A is quasi-invertible (which is the same as nonsingular when R is a supertropical semifield). We start with the tropical analog of Cramer's rule.

Theorem 3.5: *If* A *is a quasi-invertible matrix and* v *is a tangible vector, then the equation* $Ax \underset{\mathrm{gs}}{\models} v$ *has the tangible vector solution* $x = \widehat{(A^\nabla v)}$.

Proof. The proposed solution $x = (x_1, \dots, x_n)$ satisfies $|A|x_k = \hat{\nu}\left(\sum_j a'_{j,k} v_j\right)$, for $v = (v_1, \dots, v_n)$. Thus,

$$|A|(Ax)_i = \sum_k \left(a_{i,k}\, \hat{\nu}\left(\sum_j a'_{j,k} v_j \right)\right), \tag{3.1}$$

which we want to show ghost-surpasses $|A|v_i$. For $j = i$, we see that $\sum_k a_{i,k}(\widehat{a'_{i,k} v_i})$ has the same ν-value as $\sum_k a_{i,k} a'_{i,k} v_i$, which is $|A|v_i \in \mathcal{T}_{\mathbb{0}}$, implying

$$\sum_k a_{i,k}(\widehat{a'_{i,k} v_i}) \underset{\text{gs}}{\models} |A|v_i.$$

Thus, we are done if $\sum_k a_{i,k}(\widehat{a'_{i,k} v_i})$ dominates $|A|(Ax)_i$, and we may assume that

$$|A|(Ax)_i = \sum_k a_{i,k} \hat{\nu}\left(\sum_{j \neq i} a'_{j,k} v_j \right),$$

which is ghost by Proposition 1.8 (since $\sum_k a_{i,k} \widehat{a'_{j,k}} \in \mathcal{G}_{\mathbb{0}}$ by [6, Remark 4.5]). Hence, by components, $|A|(Ax) \underset{\text{gs}}{\models} |A|v$, implying $Ax \underset{\text{gs}}{\models} v$. ∎

Note: Suppose that A is quasi-invertible, and $v \in R^{(n)}$.

(i) $A^{\widehat{\nabla}} v \cong_\nu A^{\nabla} v$, in view of Theorem 2.18.

(ii) When $v = I_A v$, we claim that we have the "true" solution $Ax = v$. Indeed,

$$v = I_A v = (AA^{\nabla})v \underset{\text{gs}}{\models} A(\widehat{A^{\nabla} v}) = Ax,$$

so $Ax = v$ since v is presumed tangible.

(iii) From this point of view, the "good" vectors for solving the matrix equation $Ax = v$ are those tangible vectors $v = I_A w$ for some w, since then

$$v = I_A w = I_A^2 w = I_A v.$$

Let us turn to the question of uniqueness of our solution. Note that if A is a nonsingular matrix, then the only tangible solution to $Ax \in \mathcal{H}_{\mathbb{0}}$ is $x = (\mathbb{0}_R)$, in view of [6, Lemma 6.9]. On the other hand, we have the following example.

Example 3.6: In logarithmic notation, take

$$A = \begin{pmatrix} 5 & 0 \\ 5 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

The tangible solution for $Ax \underset{\text{gs}}{\models} v$ obtained from Theorem 3.5 is

$$x = \hat{\nu}\left(\begin{pmatrix} -5 & -6 \\ -1 & -1 \end{pmatrix} v\right) = \begin{pmatrix} 0 \\ 4 \end{pmatrix},$$

and indeed $Ax = \left(\begin{smallmatrix} 5 \\ 5^{\nu} \end{smallmatrix}\right) \underset{\text{gs}}{\models} v$. However, instead of x we could take $y = \left(\begin{smallmatrix} 0 \\ \alpha \end{smallmatrix}\right)$ for every tangible $\alpha <_{\nu} 4$ and get the equality $Ay = v$.

Note that these solutions exist despite the fact that $I_A v \neq v$. The supertropical solution is the limiting case of the other solutions, and would provide the "maximal" solution over the max-plus algebra.

In general, we do have uniqueness in the sense of the following theorem (3.8):

Proposition 3.7: *If $Ax \underset{\text{gs}}{\models} v$ and $Ay \underset{\text{gs}}{\models} v$ for tangible vectors x and y, then $A(\widehat{x+y}) \underset{\text{gs}}{\models} v$.*

Proof. This is clear unless some tangible component in $A(\widehat{x+y})$, say the i-component, has ν-value at least that of the corresponding component v_i in v. But then it comes from some dominating $a_{i,j}x_j$ or $a_{i,j}y_j$ with $a_{i,j}$ tangible. Say $a_{i,j}x_j \geq_{\nu} v_i$ dominates the i-component of $A(\widehat{x+y})$. But then $a_{i,j}x_j$ is tangible, so either $a_{i,j}x_j = v_i$ and we are done, or $a_{i,j}x_j >_{\nu} v_i$, and thus by hypothesis $a_{i,j'}x_{j'} = a_{i,j}x_j$ for some j', implying that the i-component of $A(\widehat{x+y})$ is $(a_{i,j}x_j)^{\nu}$, a ghost, so again we are done. ∎

It follows that taking the tangible retract of the sum of all tangible solutions x to $Ax \underset{\text{gs}}{\models} v$ gives us the dominating tangible solution. Actually, this can be obtained from the solution given in Theorem 3.5, as we see in the next result.

Theorem 3.8: *If $Ax \underset{\text{gs}}{\models} v$ for A quasi-invertible and a tangible vector x, then $x \leq_{\nu} \widehat{(A^{\nabla}v)}$.*

Proof. First we assume that $A = I_A = (a_{i,j})$ is a quasi-identity matrix. Since $A^{\nabla} = I_A^{\nabla} = I_A = A$, the equation $Ax \underset{\text{gs}}{\models} v$ has the tangible solution $y = \widehat{(A^{\nabla}v)} = \widehat{(Av)}$; i.e., for each i, $y_i = \widehat{a_{i,j}v_j}$ for suitable j (depending on i), and $y_i \geq_{\nu} a_{i,i}v_i = v_i$. Note that

$$Ay \cong_{\nu} AA^{\nabla}v \cong_{\nu} I_A v \cong_{\nu} A^{\nabla}v \cong_{\nu} y,$$

implying $y = \widehat{(Ay)}$. Thus, $y_i \geq_\nu a_{i,j}y_j$ for all i, j, and hence, since $a_{i,i} = \mathbb{1}_R$,

$$y_i \ = \ a_{i,i}y_i \ \geq_\nu \ \sum_j a_{i,j}y_j \ \geq_\nu \ v_i.$$

Suppose

$$Ax \underset{\text{gs}}{\models} v, \tag{3.2}$$

with $x = (x_1, \dots, x_n)$. We need to show that $y_i \geq_\nu x_i$ for each i.

If not, then, for some i, $x_i >_\nu y_i$; take such an $i_0 = i$ with x_{i_0}/y_{i_0} ν-maximal. (If some $y_i = \mathbb{0}_R$, we take i_0 such that x_{i_0} is ν-maximal for which $y_{i_0} = \mathbb{0}_R$.) Since by hypothesis $x_{i_0} \in \mathcal{T}_\mathbb{0}$, we must have

$$a_{i_0,i_0}x_{i_0} = x_{i_0} >_\nu y_{i_0} = a_{i_0,i_0}y_{i_0},$$

implying $a_{i_0,i_0}x_{i_0} >_\nu v_{i_0}$, and thus, in view of (3.2), $a_{i_0,i_0}x_{i_0} \leq_\nu a_{i_0,i_1}x_{i_1}$ for some $i_1 \neq i_0$. Then

$$a_{i_0,i_1}x_{i_1} \ \geq_\nu \ a_{i_0,i_0}x_{i_0} >_\nu \ y_{i_0} \ \geq_\nu \ a_{i_0,i_1}y_{i_1}. \tag{3.3}$$

Hence,

$$\frac{x_{i_1}}{y_{i_1}} \ \geq_\nu \ \frac{x_{i_0}}{y_{i_0}},$$

so by assumption

$$\frac{x_{i_1}}{y_{i_1}} \ \cong_\nu \ \frac{x_{i_0}}{y_{i_0}},$$

and the ends of Equation (3.3) are ν-matched. Inductively, by the same argument, for each $t \geq 0$ we get $i_{t+1} \neq i_t$ such that $y_{i_t} \cong_\nu a_{i_t,i_{t+1}}y_{i_{t+1}}$, and we consider the path obtained from the indices $i_0, i_1, \dots, i_t$ in the reduced digraph of A (cf. [6, Section 3.2]). For $t > n$ this must contain a cycle, so there are $s < t$ such that

$$y_{i_s} \ \cong_\nu \ y_{i_s}a_{i_s,i_{s+1}} \cdots a_{i_t,i_{t+1}}$$

Hence, $a_{i_s,i_{s+1}} \cdots a_{i_t,i_{t+1}} \cong_\nu \mathbb{1}_R$, contradicting the fact that A is a quasi-identity matrix (and thus cannot have a loopless cycle of weight $\cong_\nu \mathbb{1}_R$).

In general, suppose that $Ax \underset{\text{gs}}{\models} v$. Then $I'_A x = A^\nabla Ax \underset{\text{gs}}{\models} A^\nabla v$, implying by the previous case that

$$x \ \leq_\nu \ \hat{\nu}({I'_A}^\nabla A^\nabla v) \ = \ \hat{\nu}((I'_A A^\nabla)v) \ = \ \hat{\nu}((\overline{A^\nabla})v) \ = \ \hat{\nu}(A^\nabla v),$$

in view of Theorem 2.18. ∎

This theorem does not provide tangible solutions when $v = \mathbb{0}$, i.e., $Ax \in \mathcal{H}_{\mathbb{0}}$ for A quasi-invertible, since then $\hat{\nu}(\widehat{A^{\nabla}} v) = \mathbb{0}$ and we have no nontrivial solutions; in this sense, Proposition 3.4 is sharp.

4. Properties of the adjoint and tangible adjoint

Example 4.1: Let us compute $A^{\nabla\nabla} = (A^{\nabla})^{\nabla}$ for the triangular nonsingular matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ - & a_{2,2} & a_{2,3} \\ - & - & a_{3,3} \end{pmatrix}.$$

Then $|A| = a_{1,1}\, a_{2,2}\, a_{3,3}$ and

$$A^{\nabla} = \frac{\mathbb{1}_R}{|A|} \begin{pmatrix} a_{2,2}a_{3,3} & a_{1,2}a_{3,3} & a_{1,2}a_{2,3} + a_{1,3}a_{2,2} \\ - & a_{1,1}a_{3,3} & a_{1,1}a_{2,3} \\ - & - & a_{1,1}a_{2,2} \end{pmatrix}, \quad \text{so } |A^{\nabla}| = \frac{\mathbb{1}_R}{|A|},$$

and

$$\begin{aligned} A^{\nabla\nabla} &= \frac{\mathbb{1}_R}{|A^{\nabla}|} \operatorname{adj}(A^{\nabla}) = |A| \operatorname{adj}(A^{\nabla}) \\ &= \frac{\mathbb{1}_R}{|A|} \begin{pmatrix} a_{1,1}|A| & a_{1,1}a_{1,2}a_{2,3}{a_{3,3}}^{\nu} + a_{1,2}|A| & a_{1,3}|A| \\ - & a_{2,2}|A| & a_{2,3}|A| \\ - & - & a_{3,3}|A| \end{pmatrix}. \end{aligned}$$

Clearly $a_{1,1}a_{1,2}a_{2,3}{a_{3,3}}^{\nu} + a_{1,2}|A| \underset{\text{gs}}{\models} a_{1,2}|A|$, and thus $A^{\nabla\nabla} \underset{\text{gs}}{\models} A$.

For further reference, we note that

$$A^{\overline{\nabla}} = A^{\nabla} I_A = \frac{\mathbb{1}_R}{|A|} \begin{pmatrix} a_{2,2}a_{3,3} & a_{1,2}{a_{3,3}}^{\nu} & a_{1,2}{a_{2,3}}^{\nu} + a_{1,3}{a_{2,2}}^{\nu} \\ - & a_{1,1}a_{3,3} & a_{1,1}{a_{2,3}}^{\nu} \\ - & - & a_{1,1}a_{2,2} \end{pmatrix},$$

with $|A^{\nabla}| = \frac{\mathbb{1}_R}{|A|}$.

Consequently,

$$A^{\overline{\nabla\nabla}} = \frac{\mathbb{1}_R}{|A^{\nabla}|} \operatorname{adj}(A^{\nabla}) = \frac{\mathbb{1}_R}{|A|} \begin{pmatrix} a_{1,1}|A| & a_{1,1}a_{1,2}a_{2,3}{a_{3,3}}^{\nu} + a_{1,2}|A|^{\nu} & a_{1,3}|A|^{\nu} \\ - & a_{2,2}|A| & a_{2,3}|A|^{\nu} \\ - & - & a_{3,3}|A| \end{pmatrix},$$

which is not necessarily $A^{\nabla\nabla}$ (although they are ν-matched).

Remark 4.2: Although $A^{\nabla\nabla} \neq A$ in general, one does get $A^{\nabla\nabla} \underset{\mathrm{gs}}{\models} A$, as a consequence of Akian, Gaubert and Guterman [2, Theorem 4.21], quoted above as Theorem 2.4.

Here are some more computations with adjoints.

THEOREM 4.3: $\operatorname{adj}(A)\operatorname{adj}(\operatorname{adj}(A))\operatorname{adj}(A) \cong_\nu |A|^{n-1}\operatorname{adj}(A)$ *for any* $n \times n$ *matrix* A.

Proof. Another application of Hall's Marriage Theorem. Let $\operatorname{adj}(\operatorname{adj}(A)) = (a''_{i,j})$. Clearly

$$\operatorname{adj}(A)\operatorname{adj}(\operatorname{adj}(A))\operatorname{adj}(A) \geq_\nu |A|^{n-1}\operatorname{adj}(A),$$

by [6, Theorems 4.9(ii) and 4.12], so it suffices to prove $\leq_\nu$. But the (i,j) entry of the left side is a sum of elements of the form $a'_{i,k}a''_{\ell,k}a'_{\ell,j}$ which has in-degree n in all indices except i (which has in-degree $n-1$), and out-degree n in all indices except j (which has out-degree $n-1$), and thus by [6, Lemma 3.16(iv)] we can factor out $(n-1)$ n-multicycles, each of weight $\leq_\nu |A|$, and conclude that each summand $\leq_\nu |A|^{n-1}a'_{i,j}$. ∎

COROLLARY 4.4: *If A is a quasi-invertible matrix, then* $A^{\nabla}A^{\nabla\nabla}A^{\nabla} \cong_\nu A^{\nabla}$.

We are finally ready for the connection between left quasi-identities and right quasi-identities; the key is to switch from A to $A^{\widehat{\nabla}}$.

COROLLARY 4.5: *If A is a quasi-invertible matrix, then* $A^{\widehat{\nabla}\widehat{\nabla}}A^{\widehat{\nabla}} = I_A$. *In other words,* $I_A = I'_{A^{\widehat{\nabla}}} = I'_{A^{\overline{\nabla}}}$.

Proof. $I'_{A^{\widehat{\nabla}}} = A^{\widehat{\nabla}\widehat{\nabla}}A^{\widehat{\nabla}} \leq_\nu I_A$ by Corollary 4.4 and Remark 2.16, but $A^{\widehat{\nabla}\widehat{\nabla}}A^{\widehat{\nabla}} \geq_\nu AA^{\widehat{\nabla}} = I_A$ by Theorem 2.8. Hence the entries of I_A and $I'_{A^{\widehat{\nabla}}}$ have the same respective ν-values. We conclude by noting that both I_A and $I'_{A^{\widehat{\nabla}}}$ are tangible on the diagonal and ghost off the diagonal. ($I'_{A^{\widehat{\nabla}}} = I'_{A^{\overline{\nabla}}}$ by Remark 2.10.) ∎

COROLLARY 4.6: *By symmetry,* $I'_A = I_{A^{\widehat{\nabla}}} = I_{A^{\overline{\nabla}}}$.

At last we have resolved the enigma arising from Example 2.11: The left quasi-identity of a matrix corresponds to the right quasi-identity of its adjoint, and vice versa.

5. Application: Supertropical eigenvectors

Recall from [6] that a tangible vector v is a **supertropical eigenvector** of A, with **supertropical eigenvalue** $\beta \in \mathcal{T}_{\mathbb{0}}$, if

$$Av \underset{\text{gs}}{\models} \beta v,$$

i.e., if $Av = \beta v + \text{ghost}$.

In [6, Theorem 7.10] we showed that every root of the characteristic polynomial of A is a supertropical eigenvalue. However, the proof does not give much insight into the specific eigenvector. Here, we use the properties of the adjoint matrix to compute explicitly the supertropical eigenvectors; this method is expected to be a useful tool for developing linear algebra.

Recall the following observation from [6, Remark 7.9]:

Remark 5.1: If $\widehat{A}$ is a tangible matrix (i.e., all entries are in $\mathcal{T}_{\mathbb{0}}$), such that $\widehat{A} \cong_\nu A$, then every tangible supertropical eigenvector of $\widehat{A}$ is also a supertropical eigenvector of A with respect to the same supertropical eigenvalue.

In view of this remark, in the sequel, we may assume that all of the entries of our matrix A are tangible.

Definition 5.2: A polynomial is **quasi-tangible** if all of its coefficients except perhaps the constant term are tangible.

We also assume from now on that the essential part ${f_A}^{\text{es}}$*, cf.* [5, Definition 4.9]*, of the characteristic polynomial* f_A *is quasi-tangible.*

(The reason that we exclude the constant term from our hypothesis is that we want to permit $\mathbb{0}_R$ to be an eigenvalue.) We write $\beta_1, \dots, \beta_t$ for the distinct roots of ${f_A}^{\text{es}}$, written in order of descending ν-values. Thus, $\beta_\ell \in \mathcal{T}_{\mathbb{0}}$ for each $\ell \le t$, with $\beta_\ell \in \mathcal{T}$ for each $\ell < t$. Recall from [6, Theorem 7.10] that

$$(5.1) \qquad {f_A}^{\text{es}} = \lambda^n + \sum_{\ell=1}^{t} \alpha_\ell \lambda^{n-m_\ell},$$

where α_ℓ equals the maximal weight (with respect to ν-value) of an m_ℓ-multicycle in the digraph of A, which we denote as C_ℓ.

Remark 5.3: C_ℓ is unique for each $\ell < t$, since α_ℓ is assumed tangible.

Since β_ℓ is a tangible root of ${f_A}^{\mathrm{es}}$, we have

$$\beta_\ell^{n-m_{\ell-1}}\alpha_{\ell-1} = \beta_\ell^{n-m_\ell}\alpha_\ell, \tag{5.2}$$

implying

$$\beta_\ell^{m_\ell-m_{\ell-1}}\alpha_{\ell-1} = \alpha_\ell. \tag{5.3}$$

(For $\ell = t$, we only have this up to ν-values, and only when $\beta_t \neq \mathbb{0}_R$.) Hence, β_ℓ^ν equals the negative of the slope of the edge connecting $(m_\ell, \alpha_\ell^\nu)$ to $(m_{\ell-1}, \alpha_{\ell-1}^\nu)$ in the graph of the coefficients of f_A.

Here is an intuitive way of computing a supertropical eigenvector. Let

$$B_\ell = A + \beta_\ell I.$$

In [6, Proposition 7.8], we showed that B_ℓ is a singular matrix for every tangible root β_ℓ of f_A. Taking an arbitrary vector w and letting $v = \mathrm{adj}(B_\ell)w$, we have

$$Av + \beta_\ell v = (A + \beta_\ell I)(\mathrm{adj}(B_\ell)w) = B_\ell\,\mathrm{adj}(B_\ell)w$$

is ghost. If w can be chosen such that v is tangible, this implies by [6, Lemma 7.4] that v is a supertropical eigenvector. This is the motivation for the next result. First we make our discussion more explicit.

Remark 5.4: Write $B_\ell = (b_{i,j})$. Thus, $b_{i,j} = a_{i,j}$ for $i \neq j$, and $b_{i,i} = a_{i,i} + \beta_\ell$.

$$|B_\ell| = (\alpha_\ell \beta_\ell^{n-m_\ell})^\nu.$$

Indeed, the determinant of $B_\ell = A + \beta_\ell I$ comes from the n-multicycles of maximal weight. Since β_ℓ is a tangible root of f_A, there are two dominating contributions: One comes from $n - m_\ell$ entries of β_ℓ along the diagonal, where the remaining m_ℓ entries must come from the dominating m_ℓ-multicycle C_ℓ in the digraph of A. (Note that for $\ell = t$ this contribution might not be unique.) The other dominating term comes from $n - m_{\ell-1}$ entries of β_ℓ along the diagonal, where the remaining $m_{\ell-1}$ entries must come from the dominating $m_{\ell-1}$-multicycle $C_{\ell-1}$ (in the digraph of A), and we also have

$$|B_\ell| = (\alpha_{\ell-1}\beta_\ell^{n-m_{\ell-1}})^\nu$$

(which follows from Equation (5.2)).

Formally take $\alpha_0 = \mathbb{1}_R$. Applying induction to (5.2) yields

$$\alpha_\ell = \prod_{u=1}^{\ell} \beta_u^{m_u - m_{u-1}}, \tag{5.4}$$

and thus

$$|B_\ell| = (\beta_\ell^{n-m_{\ell-1}} \prod_{u=1}^{\ell-1} \beta_u^{m_u - m_{u-1}})^\nu. \tag{5.5}$$

We introduce some more notation: For any root β_ℓ of ${f_A}^{\mathrm{es}}$, let

$$J_\ell = \{\text{Vertices of } C_\ell\} \setminus \{\text{Vertices of } C_{\ell-1}\}. \tag{5.6}$$

(Note that this definition is well-defined even for $\ell = t$, since every n-multicycle contains all the vertices $\{1, \dots, n\}$ in the digraph of A.) Write $b'_{i,j}$ for the (i,j) minor of $B_\ell = A + \beta_\ell I$.

Lemma 5.5: *$a_{i,i} \leq_\nu \beta_\ell$ for any $i \in J_\ell$, and thus*

$$b'_{i,i} = \alpha_{\ell-1}\beta_\ell^{n-m_{\ell-1}-1},$$

which is tangible and has the same ν-value as $|B_\ell|/\beta_\ell$.

Proof. By definition, $C_{\ell-1}$ occurs in the digraph of the minor $A_{i,i}$, of weight $\alpha_{\ell-1}$, so $C_{\ell-1} \cup \{a_{i,i}\}$ is an $m_{\ell-1}+1$ multicycle of weight $\alpha_{\ell-1}a_{i,i}$, and the coefficient of $\lambda^{n-(m_{\ell-1}+1)}$ in f_A must have at least its ν-value. If $a_{i,i} >_\nu \beta_\ell$, then $C_{\ell-1} \cup \{a_{i,i}\}$ would produce the single dominant value for $f_A(\beta_\ell)$, contrary to hypothesis.

It follows that $b_{i,i} = a_{i,i} + \beta_\ell \cong_\nu \beta_\ell$. Remark 5.4 then implies

$$|B_\ell| \cong_\nu \alpha_{\ell-1}\beta_\ell^{n-m_{\ell-1}} \cong_\nu b'_{i,i}b_{i,i},$$

and we conclude that $b'_{i,i} \cong_\nu |B_\ell|/\beta_\ell$. Furthermore, the only terms which can contribute to $|B_\ell|$ are $\alpha_\ell\beta_\ell^{n-m_\ell}$ and $\alpha_{\ell-1}\beta_\ell^{n-m_{\ell-1}}$. But, by choice of i, $a_{i,i}$ cannot occur in $C_{\ell-1}$. Hence, the only contribution to $b'_{i,i}$ is $\alpha_{\ell-1}\beta_\ell^{n-m_{\ell-1}-1}$, as desired. ∎

Theorem 5.6: *For any root β_ℓ of f_A, and for any $i \in J_\ell$, taking v to be the i column of* $\operatorname{adj}(B_\ell)$*, we have $A\hat{v} \underset{\mathrm{gs}}{\models} \beta_\ell\hat{v}$. (In other words, $\hat{v}$ is a supertropical eigenvector of A.)*

Proof. In view of [6, Lemma 7.4], it suffices to prove that $A\hat{v} + \beta\hat{v} \in \mathcal{H}_{\mathbb{0}}$; i.e., that $B_\ell\hat{v} \in \mathcal{H}_{\mathbb{0}}$. Write $b'_{i,j}$ for the (i,j) minor of B_ℓ. By definition,

$$\hat{v} = (\widehat{b'_{i,1}}, \dots, \widehat{b'_{i,n}}).$$

In view of Proposition 1.8 and [6, Remark 4.5], the j component of $B_\ell\hat{v}$ is ghost unless $i = j$, and it suffices to prove that $\sum_{k=1}^n b_{i,k}\widehat{b'_{i,k}}$ is ghost. This is clear unless the right side has a single dominating summand $b_{i,k}\widehat{b'_{i,k}}$. But $\sum_{k=1}^n b_{i,k}b'_{i,k} = |A|$ is ghost, and is dominated by $b_{i,k}b'_{i,k}$ alone, which thus must be ghost. Furthermore, by Remark 5.4,

$$b_{i,k}b'_{i,k} \cong_\nu \alpha_\ell\beta_\ell^{n-m_\ell} \cong_\nu \alpha_{\ell-1}\beta_{\ell-1}^{n-m_{\ell-1}};$$

in other words, the two terms on the right side must occur in $b_{i,k}b'_{i,k}$ as the dominating terms. In particular, one summand of $b_{i,k}b'_{i,k}$ must consist of diagonal elements β_ℓ and the multicycle $C_{\ell-1}$. But, by choice of i, $a_{i,k}$ cannot occur in $C_{\ell-1}$; if $k \neq i$ then $b_{i,k} = a_{i,k}$ cannot occur in this summand, a contradiction.

Thus, $k = i$ and Lemma 5.5 shows that $a_{i,i}$ is part of C_ℓ, implying that

$$C_\ell = C_{\ell-1} \cup \{a_{i,i}\}.$$

But then $\beta_\ell \cong_\nu a_{i,i}$, implying $b_{i,i} = \beta_\ell + a_{i,i} \in \mathcal{G}$, and thus $b_{i,i}\widehat{b'_{i,i}} \in \mathcal{G}$, as desired. ∎

Here is a surprising example.

Example 5.7: A matrix A whose characteristic polynomial has distinct roots, but the supertropical eigenvectors are supertropically dependent. Let

$$A = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 1 & - & - \\ - & - & - & 9 \\ 9 & - & - & - \end{pmatrix}. \tag{5.7}$$

Notation as in Remark 5.4,

- $C_1 = (1)$, of weight 10,
- $C_2 = (1,2)$, of weight 19,
- $C_3 = (1,3,4)$, of weight 27,
- $C_4 = (1,3,4)(2)$, of weight 28.

Hence, the characteristic polynomial of A is

$$f_A = \lambda^4 + 10\lambda^3 + 19\lambda^2 + 27\lambda + 28,$$

whose roots are $10, 9, 8, 1$, which are the respective eigenvalues $\beta_1, \beta_2, \beta_3, \beta_4$ of A.

We also have $J_1 = \{1\}$, $J_2 = \{2\}$, $J_3 = \{3, 4\}$, and $J_4 = \{2\}$. (The pathology of this example is explained by the fact that $J_4 = J_2$, cf. (5.6).)

For each $1 \leq \ell \leq 4$, let us compute $B_\ell = A + \beta_\ell I$ and $\widehat{v_\ell}$, where v_ℓ is the column of $\mathrm{adj}(B_\ell)$ corresponding to the j column with $j \in J_\ell$.

$\beta_1 = 10$**:**

$$B_1 = A + \beta_1 I = \begin{pmatrix} 10^\nu & 10 & 9 & - \\ 9 & 10 & - & - \\ - & - & 10 & 9 \\ 9 & - & - & 10 \end{pmatrix},$$

$$\text{so } \widehat{v_1} = \begin{pmatrix} 30 \\ 29 \\ 28 \\ 29 \end{pmatrix}, \text{ the first column of } \widehat{\mathrm{adj}(B_1)}, \text{ and } A\widehat{v_1} = 10\widehat{v_1}.$$

$\beta_2 = 9$**:**

$$B_2 = A + \beta_2 I = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 9 & - & - \\ - & - & 9 & 9 \\ 9 & - & - & 9 \end{pmatrix},$$

$$\text{so } \widehat{v_2} = \begin{pmatrix} 28 \\ 28 \\ 28 \\ 28 \end{pmatrix}, \text{ the second column of } \widehat{\mathrm{adj}(B_2)}, \text{ and } A\widehat{v_2} = \begin{pmatrix} 38^\nu \\ 37 \\ 37 \\ 37 \end{pmatrix} \underset{\mathrm{gs}}{\models} 9\widehat{v_2}.$$

$\beta_3 = 8$**:**

$$B_3 = A + \beta_3 I = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 8 & - & - \\ - & - & 8 & 9 \\ 9 & - & - & 8 \end{pmatrix},$$

so $v_3 = \begin{pmatrix} 25 \\ 26 \\ 27 \\ 26 \end{pmatrix}$, the third column of $\widehat{\mathrm{adj}(B_3)}$. (Or we could use the fourth column, which is $\begin{pmatrix} 26 \\ 27 \\ 28 \\ 27 \end{pmatrix} = 1 \cdot v_3$, so we would obtain the same result.) $A\widehat{v_3} = \begin{pmatrix} 36^\nu \\ 34 \\ 35 \\ 34 \end{pmatrix} \underset{\text{gs}}{\models} 8\widehat{v_3}$.

$\beta_4 = 1$:

$$B_4 = A + \beta_4 I = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 1 & - & - \\ - & - & 1 & 9 \\ 9 & - & - & 1^\nu \end{pmatrix},$$

so $v_4 = \begin{pmatrix} 12 \\ 27 \\ 28 \\ 20 \end{pmatrix}$, the *second* column of $\widehat{\mathrm{adj}(B_4)}$, and $A\widehat{v_4} = \begin{pmatrix} 37^\nu \\ 28 \\ 29^\nu \\ 21^\nu \end{pmatrix} \underset{\text{gs}}{\models} 1\widehat{v_4}$.

Combining these four column vectors yields the matrix

$$V = \begin{pmatrix} 30 & 28 & 25 & 12 \\ 29 & 28 & 26 & 27 \\ 28 & 28 & 27 & 28 \\ 29 & 28 & 26 & 20 \end{pmatrix},$$

which is singular, having determinant $112^\nu = v_{1,1}v_{2,4}v_{3,3}v_{4,2} = v_{1,1}v_{2,2}v_{3,4}v_{4,3}$.

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