ISRAEL JOURNAL OF MATHEMATICS **182** (2011), 149–156
DOI: 10.1007/s11856-011-0027-3

COMMUTATORS IN RESIDUALLY FINITE GROUPS

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ABSTRACT

The following result is proved. Let n be a positive integer and G a residually finite group in which every product of at most 68 commutators has order dividing n. Then G' is locally finite.

1. Introduction

Groups with commutators of finite order bounded by a given number have received a good deal of attention in recent years. The following question, originally due to MacDonald [15], was posed by Mazurov in [14, Question 13.34].

1.1: Let G be a group satisfying the identity $[x, y]^n \equiv 1$. Does it follow that G' is periodic?

It has been known for some time that G' is periodic if $n = 2$ [15] or $n = 3$ [9, 16] (in the former case G' has exponent 4). Some other partial positive results were obtained by Brandl [4].

In general, the answer to the above question is negative: Deryabina and Kozhevnikov showed that for sufficiently big odd integers n there exist counterexamples [5]. Their methods are based on Olshanskii's techniques [18]. Independently, Adian used the Novikov–Adian theory [1] to prove that the answer is negative for any odd $n > 1001$ [2]. In sharp contrast with the above negative results in the case that G is residually finite we have the following theorem.

* Supported by CNPq-Brazil.

Received September 2, 2009 and in revised form October 16, 2009

THEOREM 1.2: *Let n be a prime-power and G a residually finite group satisfying the identity $[x,y]^n \equiv 1$. Then G' is locally finite.*

Note that in general a periodic residually finite group need not be locally finite. The corresponding examples have been constructed in [3, 7, 8, 10, 27]. Theorem 1.2 was proved in [21] using the techniques developed by Zelmanov in his solution of the Restricted Burnside Problem [28, 29]. The natural question whether Theorem 1.2 remains valid if n is not a prime-power proved to be hard to deal with. We mention the following two results obtained in [22] and [25], respectively.

THEOREM 1.3: *Let n be a positive integer that is not divisible by p^2q^2 for any distinct primes p and q. Let G be a residually finite group satisfying the identity $([x_1,x_2][x_3,x_4])^n \equiv 1$. Then G' is locally finite.*

THEOREM 1.4: *For any positive integer n there exists t depending only on n such that if G is a residually finite group in which every product of t commutators has order dividing n, then G' is locally finite.*

The purpose of the present article is to improve the above result as follows.

THEOREM 1.5: *Let n be a positive integer and let G be a residually finite group in which every product of 68 commutators has order dividing n. Then G' is locally finite.*

The constant 68 in the theorem comes from the famous results of Nikolov and Segal on commutator width of finite groups. In the paper of Segal [20] it was shown that every element in the derived group of a finite soluble d-generated group is a product of at most $72d^2+46d$ commutators. A better bound can be obtained working through the proofs given in [17]. It follows that every element of the derived group of a finite soluble d-generated group is a product of at most

$$\min\{d(6d^2+3d+4), 8d(3d+2)\}$$

commutators. In the case that $d=2$ this is 68. The author would like to thank the referee for mentioning this improvement.

Thus, the theorem of Nikolov and Segal plays an important role in the proof of Theorem 1.5. The proof also relies on the classification of finite simple groups as well as on the Lie-theoretical techniques that Zelmanov created in his solution of the Restricted Burnside Problem. We also use our recent result that a finite group G is soluble with Fitting height at most h if and only if every pair of conjugate elements generates a subgroup with that property [24].

2. The proof

We use the expression "$\{a, b, c, \dots\}$-bounded" to mean "bounded from above by some function depending only on $a, b, c, \dots$". Recall that the Fitting subgroup $F(G)$ of a group G is the product of all normal nilpotent subgroups of G. The Fitting series of G can be defined by the rules: $F_0(G) = 1$, $F_1(G) = F(G)$, $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$, for $i = 1, 2, \dots$. If G is a finite soluble group, then the minimal number $h = h(G)$ such that $F_h(G) = G$ is called the Fitting height of G.

A well-known corollary of the Hall–Higman theory [12] says that the Fitting height of a finite soluble group of exponent n is bounded by a number depending only on n. We will denote the number by $h(n)$.

Lemma 2.1: *Let G be a finite soluble group in which every product of 68 commutators has order dividing n. Then $h(G) \leq h(n) + 1$.*

Proof. Assume the lemma is false. Set $h = h(n)$. By [24] G contains two conjugate elements a, b such that $h(\langle a, b\rangle) \geq h + 2$. Put $H = \langle a, b\rangle$. We know from [17] that every element of H' is a product of 68 commutators. Hence, H' is of exponent n and so the Fitting height of H' is at most h. But then the Fitting height of H is at most $h + 1$, a contradiction. ∎

The next lemma is given without proof as it is precisely Lemma 2.8 from [22].

Lemma 2.2: *Let G be a group with the identity $[x, y]^n \equiv 1$. Let H be a nilpotent subgroup of G generated by a set of commutators. Assume that H is m-generated for some $m \geq 1$. Then the order of H is $\{m, n\}$-bounded.*

Let G be a finite group. If G has odd order, put $n_2(G) = 0$. If G is of even order, choose a Sylow 2-subgroup P in G and define $n_2(G)$ as the maximum of the orders of $[x, y]$, where $x, y \in P$.

We call a group G monolithic if it has a unique minimal normal subgroup which is non-abelian simple.

PROPOSITION 2.3: *Let G be a finite group of even order. Assume that G has no non-trivial normal soluble subgroups. Then G possesses a normal subgroup L such that L is residually monolithic and $n_2(G/L) < n_2(G)$.*

Proof. Let M be a minimal normal subgroup of G. We know that $M \cong S_1 \times S_2 \times \cdots \times S_r$, where $S_1, S_2, \ldots, S_r$ are isomorphic simple groups. The group G acts on M by permuting the simple factors so we obtain a representation of G by permutations of the set $\{S_1, S_2, \ldots, S_r\}$. Let L_M be the kernel of the representation. We want to show that $n_2(G/L_M) < n_2(G)$. Suppose this is not true. Let P be a Sylow 2-subgroup of G and let $q = n_2(G)$. First we consider the case where $q \neq 1$. Since $n_2(G/L_M) = n_2(G)$, there exist a and b in P such that $[a, b]$ is of order q modulo L_M. Then $[a, b]$ permutes regularly some q factors in $\{S_1, S_2, \ldots, S_r\}$. Without loss of generality we will assume that S_1 is one of those factors and $S_1, \ldots, S_q$ is the corresponding orbit under $[a, b]$. Suppose that for every i both the a-orbit and the b-orbit of S_i are contained in $S_1, \ldots, S_q$. We have the natural representation of the group $\langle a, b\rangle$ by permutations of $S_1, \ldots, S_q$. The cycle $(1, 2, \ldots, q)$ corresponds to the commutator $[a, b]$. However this cycle does not belong to the alternating group A_q and so it cannot be a commutator in the symmetric group. A contradiction.

So either a or b takes some S_i outside $S_1, \ldots, S_q$. Suppose that a takes S_1 outside the orbit $S_1, \ldots, S_q$.

Let $P_i = P \cap S_i$. Choose a non-trivial element $x \in P_1$ and set $y = [a, x], c = [a, b]^x$. Then $yc = [a, bx]$ is a commutator in elements of P. Write

$$(yc)^q = yy^{c^{-1}}y^{c^{-2}} \cdots y^c.$$

The element $yy^{c^{-1}}y^{c^{-2}} \cdots y^c$ is a product of some non-trivial 2-elements (elements of the form x^{c^j}), each lying in a different S_j and some other elements in other simple factors. Looking at it we conclude that $yy^{c^{-1}}y^{c^{-2}} \cdots y^c \neq 1$. But that means that the order of yc is divisible by $2q$, a contradiction.

Now assume that $n_2(G) = 1$, that is, the Sylow 2-subgroups of G are abelian. If g is any 2-element of G, it is contained in a Sylow 2-subgroup that has non-trivial intersection with S_1. The Sylow subgroup is abelian so g must normalize S_1. Similarly, g must normalize S_i for every i. Hence $g \in L_M$ and G/L_M is of odd order.

Let now L be the intersection of all the subgroups L_M, where M ranges through the minimal normal subgroups of G. It follows that $n_2(G/L) < n_2(G)$, so the proof of the proposition will be complete once it is shown that L is residually monolithic. If T is the product of the minimal normal subgroups of G, it is clear that T is the product of pairwise commuting simple groups $S_1, S_2, \ldots, S_t$ and that L is the intersection of the normalizers of S_i. Since G has no non-trivial normal soluble subgroups, it follows that $C_G(T) = 1$ and therefore any element of L induces a non-trivial automorphism of some of the S_i. Let ρ_i be the natural homomorphism of L into the group of automorphisms of S_i. It is easy to see that the image of ρ_i is monolithic and that the intersection of the kernels of all ρ_i is trivial. Hence L is residually monolithic. ∎

Proposition 2.4: *Let m and n be positive integers and G a finite group in which every product of 68 commutators has order dividing n. Assume that G can be generated by m elements $g_1, g_2, \ldots, g_m$ each of which is of order dividing n. Then the order of G is bounded by a number depending only on m and n.*

Proof. Let S be the product of all normal soluble subgroups of G. We know from Lemma 2.1 that $h(S) \leq h(n) + 1$ so we will use induction on $h(S)$. Let $F = F(S)$. We assume that $S \neq 1$. Thus, by induction the order of G/F is $\{m, n\}$-bounded. Suppose first that $F \leq Z(G)$. In this case $|G : Z(G)|$ is $\{m, n\}$-bounded and Schur's Theorem [19, p. 102] guarantees that so is the order of G'. Since G is generated by m elements of order dividing n, the result follows.

If F is not central, consider the subgroup $K = [F, G]$. According to our previous conclusions G/K has $\{m, n\}$-bounded order. Hence, K can be generated by a bounded number of elements. Now it is sufficient to show that K has $\{m, n\}$-bounded order. However, this is immediate from Lemma 2.2. Therefore it is sufficient to deal with the case where $S = 1$.

Since $n_2(G)$ divides n, we use induction on $n_2(G)$. If $n_2(G) = 0$, the group G has odd order and so is soluble by the Feit–Thompson theorem [6]. This contradicts the assumption that $S = 1$. Thus, we assume that $n_2(G) \geq 1$. The induction hypothesis is that there exists a number N_0 with the property that if Q is a quotient of G such that $n_2(Q) < n_2(G)$, then Q has order at most N_0.

Proposition 2.3 tells us that G possesses a normal subgroup L such that L is residually monolithic and $n_2(G/L) < n_2(G)$. It follows that G/L is of order at most N_0. We conclude that the minimal number, say r, of generators for L is $\{m, n\}$-bounded.

A result of Jones [13] says that any infinite family of finite simple groups generates the variety of all groups. It follows that up to isomorphism there exist only finitely many monolithic groups in which every product of 68 commutators has order dividing n. Let N_1 be the maximum of their orders. Then L is residually of order at most N_1. Since L is r-generated, the number of distinct normal subgroups of index at most N_1 in L is $\{m, n\}$-bounded [11, Theorem 7.2.9]. Therefore L has $\{m, n\}$-bounded order. We conclude that $|G|$ is $\{m, n\}$-bounded, as required. ∎

In [23] we raised the following problem that generalizes the Restricted Burnside Problem.

Problem 2.5: Let $n \geq 1$ and w a group-word. Consider the class of all groups G satisfying the identity $w^n \equiv 1$ and having the verbal subgroup $w(G)$ locally finite. Is that a variety?

Recall that variety is a class of groups defined by equations. More precisely, if W is a set of words in $x_1, x_2, \ldots$, the class of all groups G such that $W(G) = 1$ is called the variety determined by W. By a well-known theorem of Birkhoff, varieties are precisely classes of groups closed with respect to taking quotients, subgroups and cartesian products of their members.

Proposition 2.4 allows us to solve Problem 2.5 positively in the case where w is the product of 68 commutators, that is,

$$w = [x_1, x_2][x_3, x_4] \cdots [x_{135}, x_{136}].$$

Other results in this direction can be found in [25, 26].

THEOREM 2.6: *Given a positive integer n, let $\mathfrak{X}$ denote the class of all groups G in which every product of 68 commutators has order dividing n and G' is locally finite. Then $\mathfrak{X}$ is a variety.*

Proof. We observe that the class $\mathfrak{X}$ is closed with respect to taking quotients and subgroups of its members. Hence, we only need to show that if D is a cartesian product of groups from $\mathfrak{X}$, then $D \in \mathfrak{X}$. Of course, every product of 68 commutators in D has order dividing n, so it remains only to show that D' is locally finite. Let T be any finite subset of D'. Clearly, there exist finitely many commutators $h_1, \ldots, h_m$ in D such that $T \leq \langle h_1, \ldots, h_m \rangle$. Thus, it is sufficient to show that the group $H = \langle h_1, \ldots, h_m \rangle$ is finite. Note that D' is residually locally finite. By Proposition 2.4 any locally finite image of H has finite $\{m, n\}$-bounded order. We conclude that H is finite, as required. ∎

The proof of Theorem 1.5 is now straightforward.

Proof. Let $\mathfrak{X}$ have the same meaning as in Theorem 2.6, and let G satisfy the hypothesis of Theorem 1.5. Then G residually belongs to $\mathfrak{X}$. Now, Theorem 2.6 tells us that $\mathfrak{X}$ is a variety. It follows that actually $G \in \mathfrak{X}$, that is, G' is locally finite. ∎

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