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LUSIN TYPE AND COTYPE FOR LAGUERRE g-FUNCTIONS*

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ABSTRACT

We characterize Lusin type and cotype for a Banach space in terms of the L^p-boundedness of Littlewood–Paley g-functions associated with the Hermite and Laguerre expansions.

1. Introduction

The notions of martingale type and cotype for a Banach space B were introduced in the 1970's by G. Pisier ([19] and [20]) in connection with convexity and smoothness of the Banach space B. If $M = (M_n)_{n\in\mathbb{N}}$ is a martingale defined

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on some probability space and with values in B, the q-square function S_qM is defined by

$$S_qM = \left(\sum_{n=1}^{\infty} \|M_n - M_{n-1}\|_B^q\right)^{\frac{1}{q}}.$$

The Banach space B is said to be of martingale cotype q, $2 \leq q < \infty$, if for every bounded L^p-martingale $M = (M_n)_{n\in\mathbb{N}}$ on B we have

$$\|S_qM\|_{L^p} \leq C_p \sup_n \|M_n\|_{L^p_B},$$

for some $1 < p < \infty$. The Banach space B is said to be of martingale type q, $1 < q \leq 2$, if the reverse inequality holds for some $1 < p < \infty$. The martingale type or cotype properties do not depend on $1 < p < \infty$ for which the corresponding inequalities are satisfied.

It is a common fact that results in probability theory have parallels in harmonic analysis. In this line of thought Xu ([25]) defined the Lusin cotype and type properties for a Banach space B as follows. Let f be a function in $L^1(\mathbb{T}, B)$, where $\mathbb{T}$ denotes the one-dimensional torus and $L^1(\mathbb{T}, B)$ stands for the Bochner–Lebesgue space of strongly measurable B-valued functions such that the scalar function $\|f\|_B$ is integrable. Consider the generalized Littlewood–Paley g-function

$$g_q(f)(z) = \left(\int_0^1 (1-r)^q \left\|\frac{\partial P_r}{\partial r} * f(z)\right\|_B^q \frac{dr}{1-r}\right)^{\frac{1}{q}},$$

where $P_r(\theta)$ denotes the Poisson kernel. It is said that B has Lusin cotype q, $q \geq 2$, if for some $1 < p < \infty$ we have

$$\|g_q(f)\|_{L^p(\mathbb{T})} \leq C_p \|f\|_{L^p_B(\mathbb{T})},$$

and B has Lusin type q, $1 < q \leq 2$, if for some $1 < p < \infty$ the following inequality holds:

$$\|f\|_{L^p_B(\mathbb{T})} \leq C_p \left(\|\hat{f}(0)\|_B + \|g_q(f)\|_{L^p(\mathbb{T})}\right).$$

The Lusin cotype and type properties do not depend on $p \in (1, \infty)$; see [25]. Moreover, a Banach space B has Lusin cotype q (Lusin type q) if and only if B has martingale cotype q (martingale type q) ([25, Theorem 3.1]).

For the reader's convenience we recall that for scalar-valued functions and $1 < p < \infty$, the following double inequality is well-known:

$$\frac{1}{C_p}\|f\|_{L^p(\mathbb{T})} \leq |\hat{f}(0)| + \|g_2(f)\|_{L^p(\mathbb{T})} \leq C_p \|f\|_{L^p(\mathbb{T})}, \tag{1}$$

where C_p is a constant depending only on p. It is also well-known that for B-valued functions this double inequality holds if and only if B is isomorphic to a Hilbert space (see [10]).

Martínez, Torrea and Xu extended the results in [25] to subordinated Poisson semigroups $\{P_t\}_{t>0}$ of general symmetric diffusion markovian semigroups $\{T_t\}_{t>0}$; see [13]. Recall that a symmetric diffusion markovian semigroup is a collection of linear operators $\{T_t\}_{t\geq 0}$ defined on $L^p(\Omega, d\mu)$ satisfying: $T_0 = Id$, $T_{t+s} = T_t T_s$, $\lim_{t\to 0} T_t f =^{L^2} f$, for $f \in L^2(\Omega, d\mu)$, $T_t^* = T_t$ in L^2, $T_t f \geq 0$ if $f \geq 0$, and $T_t 1 = 1$. We also recall that the subordintated Poisson semigroup $\{P_t\}_{t>0}$ is defined as

$$P_t f = \frac{t}{2\sqrt{\pi}} \int_0^\infty u^{-\frac{3}{2}} e^{-\frac{t^2}{4u}} T_u f \, du, \quad t > 0. \tag{2}$$

The main purpose of this paper is to describe the Lusin cotype and the Lusin type of a Banach space in terms of Littlewood–Paley g-functions for Poisson semigroups associated to the Hermite and Laguerre differential operators; see (3), (5), Theorems 1 and 2. These semigroups are non-markovian. In fact, the Poisson semigroup associated to the Hermite operator does not send constants into constants; see [22]. In the Laguerre case, and for certain $\alpha > -1$, the Poisson semigroup is unbounded for some p in the range $1 < p < \infty$; see [12] and [3].

Let H be the Hermite differential operator

$$H = -\frac{1}{2}\Big(\frac{d^2}{dx^2} - x^2\Big), \quad x \in \mathbb{R}. \tag{3}$$

The heat semigroup $\{W_t^H\}_{t>0}$, generated by $-H$, has an integral representation; see (6). The subordinated Poisson semigroup $\{P_t^H\}_{t>0}$ can be defined by using formula (2), just by replacing T_u by W_u^H. Given a Banach space B and a B-valued function f defined on $\mathbb{R}$ we define the g-function $g_q^H(f)$, $1 < q < \infty$, by

$$g_q^H(f)(x) = \Big\{ \int_{-\infty}^\infty \Big\| t \frac{\partial}{\partial t} P_t^H(f)(x) \Big\|_B^q \frac{dt}{t} \Big\}^{1/q}. \tag{4}$$

Let L_α be the Laguerre differential operator

$$L_\alpha = \frac{1}{2}\Big(-\frac{d^2}{dy^2} + y^2 + \frac{1}{y^2}\Big(\alpha^2 - \frac{1}{4}\Big)\Big), \quad y \in (0,\infty) \text{ and } \quad \alpha > -1. \tag{5}$$

The heat semigroup $\{W_t^\alpha\}_{t>0}$, generated by $-L_\alpha$, also has an integral representation, see (9). The subordinated Poisson semigroup, $\{P_t^\alpha\}_{t>0}$, and the

g-function, g_q^α, are defined for functions defined in $(0,\infty)$, in a parallel way to the Hermite case; see (2) and (4).

We introduce the following notation. Let $\Omega_\alpha = (\frac{2}{2\alpha+3}, \frac{-2}{2\alpha+1})$ when $-1 < \alpha \leq -\frac{1}{2}$ and $\Omega_\alpha = (1,\infty)$ when $-\frac{1}{2} < \alpha$.

The main results of this paper are the following.

THEOREM 1: *Let B be a Banach space, $q \geq 2$ and $\alpha > -1$. The following assertions are equivalent.*

(i) *B has Lusin cotype q.*

(ii) *For every (or, equivalently, for some) $p \in \Omega_\alpha$ there exists $C_p > 0$ such that*

$$\|g_q^\alpha(f)\|_{L^p(0,\infty)} \leq C_p \|f\|_{L^p_B(0,\infty)}, \quad f \in L^p_B(0,\infty).$$

(iii) *For every (or, equivalently, for some) $1 < p < \infty$ there exists $C_p > 0$ such that*

$$\|g_q^H(f)\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p_B(\mathbb{R})}, \quad f \in L^p_B(\mathbb{R}).$$

THEOREM 2: *Let B be a Banach space, $1 < q \leq 2$ and $\alpha > -1$. The following assertions are equivalent.*

(i) *B has Lusin type q.*

(ii) *For every (or, equivalently, for some) $p \in \Omega_\alpha$ there exists $C_p > 0$ such that*

$$\|f\|_{L^p_B(0,\infty)} \leq C_p \|g_q^\alpha(f)\|_{L^p(0,\infty)}.$$

(iii) *For every (or, equivalently, for some) $1 < p < \infty$ there exists $C_p > 0$ such that*

$$\|f\|_{L^p_B(\mathbb{R})} \leq C_p \|g_q^H(f)\|_{L^p(\mathbb{R})}.$$

The investigations on Harmonic Analysis in the Laguerre and Hermite settings began with Muckenhoupt ([14] and [15]). In the last decade interest in this topic has reappeared. In general, the proofs of the boundedness of the operators in these settings are very technical and essentially they follow two patterns:

(1) By using some kind of spectral theorem, the operator is bounded in L^2. Then, in some sense trying to mimic the theory of Calderón–Zygmund, the kernel of the operator is analyzed. More precisely, the kernel is broken in the part close to the diagonal ("local part") and in the complementary part ("global part"). The local part behaves as a Calderón–Zygmund operator and the global part is controlled by a positive operator. This procedure goes back to Muckenhoupt (see [14] and [15]),

and has been used in an essential way in [3], [8], [12], [17], [18] and [21].

(2) The other common strategy is to use different formulae relating the Hermite polynomials in dimension n with Laguerre polynomials in dimension 1 and $\alpha = n/2 - 1$; see, for example, [23]. These formulae can be used in order to transfer results from the Hermite setting in dimension n to the one-dimensional Laguerre setting with $\alpha = n/2 - 1$. This was used systematically in [4], [6] and [7].

Our procedure inherits some ideas of (1) for the case of the Hermite operator. In particular, by using the results in [13], together with certain kernel estimates, some L^2 boundedness of the g_q^H-function can be obtained; see Lemma 1. Then some considerations about the kernel allow us to prove the L^p-estimates. However, in the Laguerre setting, our procedure to analyze L^p-boundedness properties of g_q^α-functions is completely different from (1) and (2). It relies on a new pointwise relation (see (10)) between the heat kernels W_t^α and W_t^H. This new relation had recently appeared in [2]. This pointwise identity is one-dimensional in both sides (Hermite and Laguerre) in contrast with the ideas exploited in [7], [9] and [5]. The identity can be transferred to a pointwise relation between g-functions. It can be also used backwards (from the Laguerre to the Hermite settings). One final consequence is Lemma 4.

The organization of the paper is as follows. Section 2 is devoted to the proof of Theorem 1. In order to not duplicate arguments, we present in the middle of the section two technical Lemmas (1 and 2) that will be used at the end of the section to prove the equivalence (i) $\Longleftrightarrow$ (iii) of Theorem 1. These two Lemmas together with Lemma 4 in the same section allow us to prove the equivalence with (ii). Section 3 is devoted to the proof of Theorem 2.

Throughout this paper C and c will always denote suitable positive constants that can change from one line to the other one. Also, if $1 \le p < \infty$, by p' we represent the exponent conjugate of p, that is, $p' = p/(p-1)$.

2. Proof of Theorem 1

The Hermite operator H (see (3)) is formally selfadjoint in $L^2(\mathbb{R}, dx)$. For every $n \in \mathbb{N}$, the Hermite function h_n is defined by

$$h_n(x) = (\sqrt{\pi} 2^n n!)^{-\frac{1}{2}} H_n(x) e^{-x^2/2}, \quad x \in \mathbb{R},$$

where H_n denotes the n-th Hermite polynomial ([23]). We have that $Hh_n = (n+1/2)h_n$, $n \in \mathbb{N}$. The heat semigroup $\{W_t^H\}_{t>0}$, generated by $-H$, has the integral representation $W_t^H(f)(x) = \int_{-\infty}^{+\infty} W_t^H(x,y)f(y)dy$, where (see [24, (1.1.11)])

$$\begin{aligned} W_t^H(x,y) &= \sum_{n=0}^{\infty} e^{-(n+1/2)t} h_n(x)h_n(y) \\ &= \frac{e^{-t/2}}{\sqrt{\pi}}(1-e^{-2t})^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(x^2+y^2)+\frac{2e^{-t}}{1-e^{-2t}}xy\Big) \\ &= \frac{e^{-t/2}}{\sqrt{\pi}}(1-e^{-2t})^{-\frac{1}{2}} \exp\Big(-\frac{|x-e^{-t}y|^2+|y-e^{-t}x|^2}{2(1-e^{-2t})}\Big), \\ &\qquad t\in(0,\infty),\ \ x,y\in\mathbb{R}. \end{aligned} \tag{6}$$

Given B a Banach space and a B-valued function f defined on $\mathbb{R}$, we define $g_{q,loc}^H$ the "local" part of the Hermite square function g_q^H (see (4) and comments after Theorem 2) as follows:

$$g_{q,\text{loc}}^H(f)(x) = \left\{\int_0^\infty \left\| t\int_{x/2}^{2x} \frac{\partial}{\partial t} P_t^H(x,y)f(y)dy\right\|_B^q \frac{dt}{t}\right\}^{1/q}, \quad x\in(0,\infty),$$

where $1<q<\infty$.

For technical reasons that will become clear later, we introduce $\{W_t^{H-I/2}\}_{t>0}$, the heat semigroup generated by $-(H-I/2)$. Clearly, $W_t^{H-I/2}(f)(x) = \int_{-\infty}^{\infty} W_t^{H-I/2}(x,y)f(y)dy$, where $W_t^{H-I/2}(x,y) = e^{t/2}W_t^H(x,y)$, $t>0$, $x,y\in\mathbb{R}$. The Poisson semigroup $\{P_t^{H-I/2}\}_{t>0}$ can be defined by the subordination formula (see (2)) and also the g-function, $g_q^{H-I/2}$ (see (4) and comments after Theorem 2). The corresponding local part is defined as

$$g_{q,\text{loc}}^{H-I/2}(f)(x) = \left\{\int_0^\infty \left\| t\int_{x/2}^{2x} \frac{\partial}{\partial t} P_t^{H-I/2}(x,y)f(y)dy\right\|_B^q \frac{dt}{t}\right\}^{1/q}, \quad x\in(0,\infty).$$

Lemma 1: *Let B be a Banach space and $1<p,q<\infty$. The following assertions are equivalent.*

(a) $g_q^{H-I/2}$ *is bounded from $L_B^p(\mathbb{R})$ into $L^p(\mathbb{R})$.*
(b) $g_{q,\text{loc}}^{H-I/2}$ *is bounded from $L_B^p(0,\infty)$ into $L^p(0,\infty)$.*
(c) g_q^H *is bounded from $L_B^p(\mathbb{R})$ into $L^p(\mathbb{R})$.*
(d) $g_{q,\text{loc}}^H$ *is bounded from $L_B^p(0,\infty)$ into $L^p(0,\infty)$.*

Proof. We shall prove (a) $\Longleftrightarrow$ (b), (c) $\Longleftrightarrow$ (d) and (b) $\Longleftrightarrow$ (d). Consider the operator T defined by

$$f \longrightarrow Tf = \left(t\frac{\partial}{\partial t}\int_{-\infty}^{\infty} P_t^{H-I/2}(x,y)f(y)dy\right)_{t>0}.$$

Then $g_q^{H-I/2}$ is bounded from $L_B^p(\mathbb{R})$ into $L^p(\mathbb{R})$ if and only if T is bounded from $L_B^p(\mathbb{R})$ into $L^p_{L^q_B[(0,\infty),dt/t]}(\mathbb{R})$. Since $W_v^{H-I/2}(x,y) = W_v^{H-I/2}(-x,-y)$, $x,y\in\mathbb{R}$ and $v\in(0,\infty)$, according to [1, Proposition 3.3], T is bounded from $L_B^p(\mathbb{R})$ into $L^p_{L^q_B[(0,\infty),dt/t]}(\mathbb{R})$ if and only if T_+ is bounded from $L_B^p(0,\infty)$ into $L^p_{L^q_B[(0,\infty),dt/t]}(0,\infty)$, where T_+ is defined as T but acting on functions vanishing on $(-\infty,0)$. Equivalently, $g_{q,+}^{H-I/2}$ is bounded from $L_B^p(0,\infty)$ into $L^p(0,\infty)$, where $g_{q,+}^{H-I/2}$ is defined as $g_q^{H-I/2}$ but for functions vanishing on $(-\infty,0)$.

On the other hand, since

$$\int_0^\infty e^{-t^2/(4u)}u^{-3/2}\Big(1-\frac{t^2}{2u}\Big)du = 0, \quad t>0, \tag{7}$$

by using Minkowski's inequality and subordination formula (2) we have

$$\begin{aligned}
&\Big\|t\frac{\partial}{\partial t}P_t^{H-I/2}(x,y)\Big\|_{L^q((0,\infty),dt/t)}\\
&\quad=\Big\{\int_0^\infty t^{q-1}\Big|\frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1}{u^{3/2}}\Big(1-\frac{t^2}{2u}\Big)e^{-t^2/(4u)}\\
&\qquad\qquad\times\big(W_u^{H-I/2}(x,y)-e^{-(x^2+y^2)/2}\big)du\Big|^q dt\Big\}^{1/q}\\
&\quad\le C\Big(\int_0^1 \frac{1}{u^{3/2}}W_u^{H-I/2}(x,y)\Big\{\int_0^\infty t^{q-1}\Big|1-\frac{t^2}{2u}\Big|^q e^{-qt^2/(4u)}dt\Big\}^{1/q}du\\
&\qquad+e^{-(x^2+y^2)/2}\Big\{\int_0^\infty t^{q-1}\Big|\int_0^1\frac{1}{u^{3/2}}\Big(1-\frac{t^2}{2u}\Big)e^{-t^2/(4u)}du\Big|^q dt\Big\}^{1/q}\\
&\qquad+\int_1^\infty \frac{1}{u^{3/2}}\big|W_u^{H-I/2}(x,y)-e^{-(x^2+y^2)/2}\big|\\
&\qquad\qquad\times\Big\{\int_0^\infty t^{q-1}\Big|1-\frac{t^2}{2u}\Big|^q e^{-qt^2/(4u)}dt\Big\}^{1/q}du\Big)\\
&\quad\le C\Big(\int_0^1\frac{1}{u}W_u^{H-I/2}(x,y)du+\int_1^\infty\frac{1}{u}|W_u^{H-I/2}(x,y)-e^{-(x^2+y^2)/2}|du\\
&\qquad+e^{-(x^2+y^2)/2}\Big\{\int_0^\infty t^{q-1}\Big|\int_0^1\frac{1}{u^{3/2}}\Big(1-\frac{t^2}{2u}\Big)e^{-t^2/(4u)}du\Big|^q dt\Big\}^{1/q}\Big).
\end{aligned}$$

We claim that

$$\left\{\int_0^\infty t^{q-1}\Big|\int_0^1 \frac{1}{u^{3/2}}\Big(1-\frac{t^2}{2u}\Big)e^{-t^2/(4u)}du\Big|^q dt\right\}^{1/q} \leq C<\infty. \tag{8}$$

To prove the claim we first observe that

$$\int_1^\infty \Big|\int_0^1 \frac{e^{-t^2/(4u)}}{u^{3/2}}\Big(1-\frac{t^2}{2u}\Big)du\Big|^q t^{q-1}dt \leq C\int_1^\infty \frac{dt}{t^{q+1}}\Big(\int_0^1 \frac{1}{\sqrt{u}}du\Big)^q<\infty.$$

On the other hand, by using (7) we have

$$\int_0^1 \Big|\int_0^1 \frac{e^{-t^2/(4u)}}{u^{3/2}}\Big(1-\frac{t^2}{2u}\Big)du\Big|^q t^{q-1}dt$$
$$=\int_0^1 \Big|\int_1^\infty \frac{e^{-t^2/(4u)}}{u^{3/2}}\Big(1-\frac{t^2}{2u}\Big)du\Big|^q t^{q-1}dt<\infty.$$

These last two estimates give the claim. Hence, one can write, for every $x,y\in(0,\infty)$,

$$\Big|\Big|t\frac{\partial}{\partial t}P_t^{H-I/2}(x,y)\Big|\Big|_{L^q((0,\infty),dt/t)}$$
$$\leq C\Big(\int_0^1 \frac{1}{u}\frac{1}{(1-e^{-2u})^{1/2}}\exp\Big(-\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})}\Big)du$$
$$+\int_1^\infty \frac{1}{u}\Big|\frac{1}{(1-e^{-2u})^{1/2}}$$
$$\times\exp\Big(-\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})}\Big)-e^{-(x^2+y^2)/2}\Big|du+e^{-(x^2+y^2)/2}\Big).$$

We make the change of variable $u=\log\frac{1+w}{1-w}$, and since $\log\frac{1+w}{1-w}\sim w$, as $w\to 0$, we get, when $x,y\in(0,\infty)$, $x\neq y$,

$$\int_0^1 \frac{1}{u}\frac{1}{(1-e^{-2u})^{1/2}}\exp\Big(-\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})}\Big)du$$
$$\leq C\int_0^{(e-1)/(e+1)}\frac{1}{w^{3/2}}\exp\Big(-\frac{1}{4}(\frac{1}{w}(x-y)^2+w(x+y)^2)\Big)dw\leq\frac{C}{|x-y|}.$$

On the other hand, the mean value theorem leads to

$$\begin{aligned}
&\int_1^\infty \frac{1}{u}\Big|\frac{1}{(1-e^{-2u})^{1/2}}\exp\Big(-\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})}\Big)-e^{-(x^2+y^2)/2}\Big|du\\
&\quad\le C\Big(\int_1^\infty \frac{1}{u}\Big|\exp\Big(-\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})}\Big)-e^{-(x^2+y^2)/2}\Big|du\\
&\qquad\quad+e^{-(x^2+y^2)/2}\int_1^\infty \frac{1}{u}\Big|(1-e^{-2u})^{-1/2}-1\Big|du\Big)\\
&\quad\le C\Big(e^{-c|x-y|^2}\int_1^\infty \frac{1}{u}\Big|\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})}-\frac{x^2+y^2}{2}\Big|du\\
&\qquad\quad+e^{-(x^2+y^2)/2}\Big)\\
&\quad\le C\Big((x-y)^2e^{-c|x-y|^2}+e^{-(x^2+y^2)/2}\Big)\le C\begin{cases}\frac{1}{y}, & y>2x,\\ \frac{1}{x}, & y<\frac{x}{2}.\end{cases}
\end{aligned}$$

Combining the above estimates we conclude that

$$\Big\|t\frac{\partial}{\partial t}P_t^{H-I/2}(x,y)\Big\|_{L^q((0,\infty),dt/t)}\le C\begin{cases}\frac{1}{y}, & y>2x,\\ \frac{1}{x}, & y<\frac{x}{2}.\end{cases}$$

Therefore

$$|g_q^{H-I/2}(f)(x)-g_{q,\mathrm{loc}}^{H-I/2}(f)(x)|\le C\Big(\frac{1}{x}\int_0^x\|f(y)\|_B dy+\int_x^\infty\frac{1}{y}\|f(y)\|_B dy\Big),\quad x\in(0,\infty).$$

Hence well-known properties of Hardy operators ([16]) allow us to conclude that $g_q^{H-I/2}$ is bounded from $L^p_B(0,\infty)$ into $L^p(0,\infty)$ if and only if $g_{q,\mathrm{loc}}^{H-I/2}$ is bounded from $L^p_B(0,\infty)$ into $L^p(0,\infty)$. This ends the proof of (a) $\Longleftrightarrow$ (b). The proof of (c) $\Longleftrightarrow$ (d) can be built analogously.

Finally, we shall prove (b) $\Longleftrightarrow$ (d). By using (7) and Minkowski's inequality we have

$$\begin{aligned}
&\Big\{ \int_0^\infty \Big\| t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} [P_t^H(x,y) - P_t^{H-I/2}(x,y)]f(y)dy\Big\|_B^q \frac{dt}{t}\Big\}^{1/q} \\
&\leq C\int_{\frac{x}{2}}^{2x} \|f(y)\|_B \Big\{ \int_0^\infty t^{q-1} \\
&\times \Big| \int_0^\infty \frac{e^{-\frac{t^2}{4u}}}{u^{3/2}}\Big(1-\frac{t^2}{2u}\Big)[W_u^H(x,y) - W_u^{H-I/2}(x,y) + e^{-(x^2+y^2)/2}]du\Big|^q dt\Big\}^{1/q} dy \\
&\leq C\Big(\int_{\frac{x}{2}}^{2x} \|f(y)\|_B \Big\{ \int_0^\infty t^{q-1}\Big(\int_0^\infty \frac{e^{-\frac{t^2}{8u}}}{u^{3/2}} \\
&\qquad \times |W_u^H(x,y) - W_u^{H-I/2}(x,y) + \chi_{(1,\infty)}(u)e^{-(x^2+y^2)/2}|du\Big)^q dt\Big\}^{1/q} dy \\
&\qquad + \int_{\frac{x}{2}}^{2x} \|f(y)\|_B \Big\{ \int_0^\infty t^{q-1}\Big| \int_0^1 \frac{e^{-\frac{t^2}{8u}}}{u^{3/2}}\Big(1-\frac{t^2}{2u}\Big)du\Big|^q dt\Big\}^{1/q} \\
&\qquad \times e^{-(x^2+y^2)/2}dy\Big) \\
&= C\Big(\int_{\frac{x}{2}}^{2x} \|f(y)\|_B K_1(x,y)dy + \int_{\frac{x}{2}}^{2x} \|f(y)\|_B K_2(x,y)dy\Big),
\end{aligned}$$

with, for $x, y \in (0,\infty)$,

$$\begin{aligned}
&K_1(x,y) = \Big\{ \int_0^\infty t^{q-1} \\
&\quad \times \Big(\int_0^\infty \frac{e^{-\frac{t^2}{8u}}}{u^{3/2}}|W_u^H(x,y) - W_u^{H-I/2}(x,y) + \chi_{(1,\infty)}(u)e^{-\frac{x^2+y^2}{2}}|du\Big)^q dt\Big\}^{1/q}
\end{aligned}$$

and

$$K_2(x,y) = e^{-(x^2+y^2)/2}\Big\{ \int_0^\infty t^{q-1}\Big| \int_0^1 \frac{e^{-t^2/(4u)}}{u^{3/2}}\Big(1-\frac{t^2}{2u}\Big)du\Big|^q dt\Big\}^{1/q}.$$

By using Minkowski's inequality we get

$$\begin{aligned}
K_1(x,y) \leq &\int_0^\infty \frac{1}{u^{3/2}}\Big|W_u^H(x,y) - W_u^{H-I/2}(x,y) + \chi_{(1,\infty)}(u)e^{-\frac{x^2+y^2}{2}}\Big| \\
&\quad\times \Big\{\int_0^\infty t^{q-1}e^{-\frac{t^2 q}{8u}}dt\Big\}^{1/q} du \\
\leq &C\int_0^\infty \frac{1}{u}\Big|W_u^H(x,y) - W_u^{H-I/2}(x,y) + \chi_{(1,\infty)}(u)e^{-(x^2+y^2)/2}\Big|du \\
\leq &C\Big(\int_0^1 \frac{1}{u}|W_u^H(x,y) - W_u^{H-I/2}(x,y)|du + \int_1^\infty \frac{1}{u}|W_u^H(x,y)|du \\
&\quad + \int_1^\infty \frac{1}{u}|W_u^{H-I/2}(x,y) - e^{-(x^2+y^2)/2}|du\Big) \\
= &K_1^1(x,y) + K_1^2(x,y) + K_1^3(x,y), \quad x,y \in (0,\infty).
\end{aligned}$$

The change of variables $u = \log(1+w)/(1-w)$ leads, when $x/2 < y < 2x$, to

$$\begin{aligned}
K_1^1(x,y) \leq &C\int_0^1 \frac{1}{u}\frac{1-e^{-u/2}}{(1-e^{-2u})^{1/2}}\exp\Big(-\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})}\Big)du \\
\leq &C\int_0^1 \exp\Big(-\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})}\Big)\frac{du}{\sqrt{u}} \\
\leq &C\int_0^{(e-1)/(e+1)} \exp\Big(-\frac{1}{4}(\frac{1}{w}(x-y)^2 + w(x+y)^2)\Big) \\
&\quad\times \frac{1}{\Big(\log\frac{1+w}{1-w}\Big)^{1/2}}\frac{dw}{1-w^2} \\
\leq &C\int_0^{(e-1)/(e+1)} \exp\Big(-\frac{1}{4}(\frac{1}{w}(x-y)^2 + w(x+y)^2)\Big)\frac{1}{\sqrt{w}}dw \\
\leq &\frac{C}{\sqrt{y}}\int_0^{(e-1)/(e+1)} \frac{e^{-(x-y)^2/(4w)}}{w^{3/4}}dw \leq \frac{C}{\sqrt{y}}\frac{1}{\sqrt{|x-y|}} \leq \frac{C}{y}\frac{\sqrt{y}}{\sqrt{|x-y|}},
\end{aligned}$$

where in the sixth inequality we have used [21, Lemma 1.1].

Observe that if $u \geq 1$ and $x/2 < y < 2x$, then $|x-ye^{-u}|^2 + |y-xe^{-u}|^2 \geq cy^2$. Hence for $x/2 < y < 2x$ we have

$$\begin{aligned}
K_1^2(x,y) \leq &C\int_1^\infty \frac{1}{u}\Big(\frac{e^{-u}}{1-e^{-2u}}\Big)^{1/2}\exp\Big(-\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})}\Big)du \\
\leq &Ce^{-cx^2}\int_1^\infty e^{-u/2}du \leq \frac{C}{y}.
\end{aligned}$$

Finally, for K_1^3 one has

$$\begin{aligned}
K_1^3(x,y) \leq & C\int_1^\infty \frac{1}{u}\Big|\frac{1}{(1-e^{-2u})^{1/2}}\exp\Big(-\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})}\Big) \\
& \qquad - e^{-(x^2+y^2)/2}\Big|du \\
\leq & C\Big(\int_1^\infty \frac{1}{u}\Big|\exp\Big(-\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})}\Big) - e^{-(x^2+y^2)/2}\Big|du \\
& \qquad + e^{-(x^2+y^2)/2}\int_1^\infty \frac{1}{u}\Big|\frac{1}{(1-e^{-2u})^{1/2}} - 1\Big|du\Big) \\
\leq & C\Big(e^{-cy^2}\int_1^\infty \Big|\frac{|x-ye^{-u}|^2+|y-xe^{-u}|^2}{2(1-e^{-2u})} - \frac{x^2+y^2}{2}\Big|du \\
& + e^{-y^2/2}\int_1^\infty e^{-u}du\Big) \\
\leq & Cy^2e^{-cy^2} \leq \frac{C}{y}, \quad \frac{x}{2} < y < 2x.
\end{aligned}$$

By combining the above estimates we obtain

$$K_1(x,y) \leq C\frac{1}{y}\Big(1+\Big(\frac{y}{|x-y|}\Big)^{1/2}\Big), \quad \frac{x}{2} < y < 2x.$$

Therefore, the operator

$$f \longrightarrow \int_{\frac{x}{2}}^{2x} K_1(x,y)f(y)dy$$

is bounded from $L^p(0,\infty)$ into itself.

On the other hand, using (8) we get

$$K_2(x,y) \leq Ce^{-\frac{x^2+y^2}{2}} \leq \frac{C}{x+y}, \quad x,y \in (0,\infty).$$

Hence the operator

$$f \longrightarrow \int_{\frac{x}{2}}^{2x} K_2(x,y)f(y)dy$$

is bounded from $L^p(0,\infty)$ into itself.

In order to finish the proof of (b) $\Longleftrightarrow$ (d) it is enough to observe that

$$\begin{aligned}
&|g_{q,\mathrm{loc}}^H(f)(x) - g_{q,\mathrm{loc}}^{H-I/2}(f)(x)| \\
&\quad \leq \Big\{\int_0^\infty \Big\|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}[P_t^H(x,y) - P_t^{H-I/2}(x,y)]f(y)dy\Big\|_B^q \frac{dt}{t}\Big\}^{1/q}.
\end{aligned}$$

∎

Lemma 2: *Let B be a Banach space and $1 < q < \infty$. If $g^H_{q,\mathrm{loc}}$ is a bounded operator from $L^p_B(0,\infty)$ into $L^p(0,\infty)$, for some $1 < p < \infty$, then it is also bounded for every $1 < p < \infty$.*

Proof. It is well-known that the g-function g^H_q can be analyzed from the point of view of vector-valued Calderón–Zygmund theory. Hence the lemma follows by using the equivalence (c) $\Longleftrightarrow$ (d) established in Lemma 1. ∎

Now, we shall deal with the Laguerre setting. The operator L_α (see (5)) is formally selfadjoint with respect to the Lebesgue measure on $(0,\infty)$. For every $n \in \mathbb{N}$ the Laguerre function φ^α_n defined by

$$\varphi^\alpha_n(y) = \left(\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}\right)^{\frac{1}{2}} e^{-\frac{y^2}{2}} y^\alpha L^\alpha_n(y^2)(2y)^{\frac{1}{2}}, \quad y \in (0,\infty),$$

where L^α_n denotes the Laguerre polynomial of order α ([23, p. 100]), is an eigenfunction of L_α. In fact

$$L_\alpha \varphi^\alpha_n = (2n+\alpha+1)\varphi^\alpha_n, \quad n \in \mathbb{N}.$$

The system $\{\varphi^\alpha_n\}_{n\in\mathbb{N}}$ is complete and orthonormal in $L^2((0,\infty),dx)$. The heat semigroup $\{W^\alpha_t\}_{t>0}$ generated by $-L_\alpha$ has an integral representation $W^\alpha_t(f)(x) = \int_0^\infty W^\alpha_t(x,y)f(y)dy$. By using Mehler's formula ([24, (1.1.47)]) we can write

$$\begin{aligned} W^\alpha_t(x,y) &= \sum_{n=0}^\infty e^{-(2n+1+\alpha)t}\varphi^\alpha_n(x)\varphi^\alpha_n(y) \\ &= 2(xy)^{\frac{1}{2}} \frac{e^{-t}}{1-e^{-2t}} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) \exp\Big(-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}\Big), \\ & \qquad t,x,y \in (0,\infty). \end{aligned} \tag{9}$$

Here, I_α denotes the modified Bessel function of the first kind and order α.

As we said in the introduction, the following identity that can be established with (6) and (9) will be our fulcrum between the Hermite and the Laguerre settings,

$$\begin{aligned} & W^\alpha_t(x,y) - W^H_t(x,y) \\ &\quad = \Big\{\sqrt{2\pi}\Big(\frac{2xye^{-t}}{1-e^{-2t}}\Big)^{1/2} I_\alpha\Big(\frac{2xye^{-t}}{1-e^{-2t}}\Big) e^{-\frac{2xye^{-t}}{1-e^{-2t}}} - 1\Big\} W^H_t(x,y). \end{aligned} \tag{10}$$

We shall also state a lemma for further reference; see [2].

LEMMA 3: *There exists $C > 0$ such that*

(i) $W_t^\alpha(x,y) \leq C y^{\alpha+1/2} x^{-\alpha-3/2}, t > 0, 0 < y < x/2;$
(ii) $W_t^\alpha(x,y) \leq C x^{\alpha+1/2} y^{-\alpha-3/2}, t > 0, y > 2x;$
(iii) $\left| W_t^\alpha(x,y) - W_t^H(x,y) \right| \leq C/y, t > 0, 0 < x/2 < y < 2x.$

We shall consider the "local" part of the square function g_q^α (see (4)) defined by

$$g_{q,\mathrm{loc}}^\alpha(f)(x) = \left\{ \int_0^\infty \left\| t\frac{\partial}{\partial t} \int_{x/2}^{2x} P_t^\alpha(x,y) f(y) dy \right\|_B^q \frac{dt}{t} \right\}^{1/q}.$$

LEMMA 4: *Let B be a Banach space, $\alpha > -1$, $1 < q < \infty$, and $p \in \Omega_\alpha$. The following assertions are equivalent:*

(a) g_q^α *is bounded from* $L_B^p(0,\infty)$ *into* $L^p(0,\infty)$.
(b) $g_{q,\mathrm{loc}}^H$ *is bounded from* $L_B^p(0,\infty)$ *into* $L^p(0,\infty)$.

Proof. We start by quoting estimates for Bessel's function I_α that will be used throughout the paper (see [11, Ch. 5]):

$$I_\alpha(z) \sim \frac{1}{2^\alpha \Gamma(\alpha+1)} z^\alpha, \quad \text{as } z \to 0, \tag{11}$$

$$e^{-z}\sqrt{z} I_\alpha(z) = \frac{1}{\sqrt{\pi}} \Big(1 + O\Big(\frac{1}{|z|}\Big)\Big), \quad \text{as } z \to \infty, \tag{12}$$

$$\frac{d}{dz}(z^{-\alpha} I_\alpha(z)) = z^{-\alpha} I_{\alpha+1}(z), \quad z \in (0,\infty). \tag{13}$$

We consider the following operators that can be seen as "global" (far from the diagonal) versions of the g_q^α,

$$g_{q,\mathrm{glob},+}^\alpha(f)(x) = \left\{ \int_0^\infty \left\| t\frac{\partial}{\partial t} \int_{2x}^\infty P_t^\alpha(x,y) f(y) dy \right\|_B^q \frac{dt}{t} \right\}^{1/q}$$

and

$$g_{q,\mathrm{glob},-}^\alpha(f)(x) = \left\{ \int_0^\infty \left\| t\frac{\partial}{\partial t} \int_0^{x/2} P_t^\alpha(x,y) f(y) dy \right\|_B^q \frac{dt}{t} \right\}^{1/q}.$$

By using Minkowski's inequality we have

$$g_{q,\mathrm{glob},+}^\alpha(f)(x) \leq \int_{2x}^\infty \|f(y)\|_B \left\{ \int_0^\infty t^{q-1} \left| \frac{\partial}{\partial t} P_t^\alpha(x,y) \right|^q dt \right\}^{1/q} dy, \quad x \in (0,\infty).$$

From subordination formula (2) we get

$$\frac{\partial}{\partial t}P_t^\alpha(x,y)=\frac{1}{\sqrt{4\pi}}\int_0^\infty \frac{1}{s^{3/2}}\Big(1-\frac{t^2}{2s}\Big)W_s^\alpha(x,y)e^{-t^2/4s}ds,\quad t,x,y\in(0,\infty).$$

Minkowski's inequality leads to

$$\begin{aligned}\Big\{\int_0^\infty & t^{q-1}\Big|\frac{\partial}{\partial t}P_t^\alpha(x,y)\Big|^q dt\Big\}^{1/q}\\ &\le C\int_0^\infty \frac{1}{s^{3/2}}W_s^\alpha(x,y)\Big\{\int_0^\infty t^{q-1}\Big|1-\frac{t^2}{2s}\Big|^q e^{-q\frac{t^2}{4s}}dt\Big\}^{1/q}ds\\ &\le C\int_0^\infty \frac{1}{s^{3/2}}W_s^\alpha(x,y)\Big\{\int_0^\infty t^{q-1}e^{-q\frac{t^2}{8s}}dt\Big\}^{1/q}ds\\ &\le C\int_0^\infty \frac{1}{s}W_s^\alpha(x,y)ds.\end{aligned}$$

To study the last integral we distinguish several different cases.

Let $0<2x<y<\infty$. According to (12) and (9) it follows that

$$\begin{aligned}\int_{0,\frac{2xye^{-s}}{1-e^{-2s}}\ge 1}^1 & \frac{1}{s}W_s^\alpha(x,y)ds\\ &\le C\int_0^1\Big(\frac{e^{-s}}{1-e^{-2s}}\Big)^{\frac12}\frac{1}{s}\exp\Big(-\frac12\frac{|x-ye^{-s}|^2+|y-xe^{-s}|^2}{1-e^{-2s}}\Big)ds\\ &\le C\int_0^1\Big(\frac{2xye^{-s}}{1-e^{-2s}}\Big)^{\alpha+1}\frac{e^{-cy^2/s}}{s^{3/2}}ds\\ &\le C(xy)^{\alpha+1}\int_0^1\frac{e^{-cy^2/s}}{s^{\alpha+5/2}}ds\\ &\le C\frac{(xy)^{\alpha+1}}{y^{2\alpha+3}}\le C\frac{x^{\alpha+1}}{y^{\alpha+2}},\end{aligned}$$

and

$$\int_{1,\frac{2xye^{-s}}{1-e^{-2s}}\ge 1}^\infty \frac{1}{s}W_s^\alpha(x,y)ds\le C(xy)^{\alpha+1}e^{-cy^2}\int_1^\infty e^{-s(\alpha+\frac32)}ds\le C\frac{x^{\alpha+1}}{y^{\alpha+2}}.$$

On the other hand, (11) implies that

$$\begin{aligned}\int_{0,\frac{2xye^{-s}}{1-e^{-2s}}\le 1}^1 \frac{1}{s}W_s^\alpha(x,y)ds &\le C(xy)^{\alpha+\frac12}\int_0^1\frac{1}{s^{\alpha+2}}e^{-c\frac{x^2+y^2}{s}}ds\le C\frac{(xy)^{\alpha+\frac12}}{(x^2+y^2)^{\alpha+1}}\\ &\le C\frac{x^{\alpha+\frac12}}{y^{\alpha+\frac32}}\end{aligned}$$

and

$$\begin{aligned}\int_{1,\frac{2xye^{-s}}{1-e^{-2s}}\leq 1}^{\infty}\frac{1}{s}W_s^{\alpha}(x,y)ds &\leq C(xy)^{\alpha+\frac{1}{2}}e^{-c(x^2+y^2)}\int_1^{\infty}e^{-s(\alpha+1)}ds\\ &\leq C\frac{(xy)^{\alpha+\frac{1}{2}}}{(x^2+y^2)^{\alpha+1}}\\ &\leq C\frac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}}.\end{aligned}$$

By combining the above estimates we conclude that

$$\left\{\int_0^{\infty}t^{q-1}\Big|\frac{\partial}{\partial t}P_t^{\alpha}(x,y)\Big|^q dt\right\}^{1/q}\leq C\frac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}},\quad 0<2x<y<\infty.$$

Hence since the Hardy-type operator $\mathcal{H}_{\alpha}^{\infty}$ defined by

$$\mathcal{H}_{\alpha}^{\infty}(g)(x)=x^{\alpha+\frac{1}{2}}\int_{2x}^{\infty}\frac{1}{y^{\alpha+\frac{3}{2}}}g(y)\,dy$$

is bounded from $L^p(0,\infty)$ into itself when $(\alpha+\frac{1}{2})p+1>0$ (see [3, Lemma 3.2]), $g_{q,\mathrm{glob},+}^{\alpha}$ defines a bounded operator from $L_B^p(0,\infty)$ into $L^p(0,\infty)$ provided that $(\alpha+\frac{1}{2})p+1>0$.

Analogously, it can be proved that

$$g_{q,\mathrm{glob},-}^{\alpha}(f)(x)\leq\frac{C}{x^{\alpha+\frac{3}{2}}}\int_0^{x/2}\|f(y)\|_B y^{\alpha+\frac{1}{2}}dy.$$

The Hardy type operator $\mathcal{H}_{\alpha}^{0}$ defined by

$$\mathcal{H}_{\alpha}^{0}g(x)=\frac{1}{x^{\alpha+\frac{3}{2}}}\int_0^{x}g(y)y^{\alpha+\frac{1}{2}}dy$$

is bounded from $L^p(0,\infty)$ into itself when $1<p(\alpha+\frac{3}{2})$ ([3, Lemma 3.1]). Therefore, $g_{q,\mathrm{glob},-}^{\alpha}$ defines a bounded operator from $L_B^p(0,\infty)$ into $L^p(0,\infty)$ provided that $1<p(\alpha+\frac{3}{2})$.

On the other hand, Minkowski's inequality and the subordination formula (2) give

$$\begin{aligned}
&\left|g^{\alpha}_{q,\mathrm{loc}}(f)(x)-\sqrt{2}g^{H}_{q,\mathrm{loc}}(f)(x)\right|\\
&\quad=\left|\left\{\int_0^\infty\left\|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}P_t^\alpha(x,y)f(y)dy\right\|_B^q\frac{dt}{t}\right\}^{\frac{1}{q}}\right.\\
&\qquad\left.-\sqrt{2}\left\{\int_0^\infty\left\|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}P_t^H(x,y)f(y)dy\right\|_B^q\frac{dt}{t}\right\}^{\frac{1}{q}}\right|\\
&\quad\le\left\{\int_0^\infty\left\|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}(P_t^\alpha(x,y)-\sqrt{2}P_t^H(x,y))f(y)dy\right\|_B^q\frac{dt}{t}\right\}^{\frac{1}{q}}\\
&\quad\le\int_{\frac{x}{2}}^{2x}\|f(y)\|_B\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}(P_t^\alpha(x,y)-\sqrt{2}P_t^H(x,y))\right|^q\frac{dt}{t}\right\}^{\frac{1}{q}}dy\\
&\quad\le C\int_{x/2}^{2x}\|f(y)\|_B\int_0^\infty\frac{1}{s}|W_s^\alpha(x,y)-\sqrt{2}W_s^H(x,y)|dsdy,\quad x\in(0,\infty).
\end{aligned}$$

We denote

$$M_\alpha(x,y)=\int_0^\infty\frac{1}{s}|W_s^\alpha(x,y)-\sqrt{2}W_s^H(x,y)|ds,\quad 0<\frac{x}{2}<y<2x.$$

To analyze M_α we distinguish the cases $\frac{2xye^{-s}}{1-e^{-2s}}\ge 1$, and $\frac{2xye^{-s}}{1-e^{-2s}}\le 1$.

By using (6), (10) and (12) we get

$$\begin{aligned}
&\int_{1,\frac{2xye^{-s}}{1-e^{-2s}}\ge 1}^\infty\frac{1}{s}|W_s^\alpha(x,y)-\sqrt{2}W_s^H(x,y)|\,ds\\
&\le C\int_{1,\frac{2xye^{-s}}{1-e^{-2s}}\ge 1}^\infty\frac{1}{s}\Big(\frac{1-e^{-2s}}{2xye^{-s}}\Big)^{1/4}\Big(\frac{e^{-s}}{1-e^{-2s}}\Big)^{1/2}\\
&\qquad\qquad\times\exp\Big(-\frac{|x-e^{-s}y|^2+|y-xe^{-s}|^2}{2(1-e^{-2s})}\Big)ds\\
&\le Ce^{-c(x^2+y^2)}\int_1^\infty e^{-\frac{s}{2}}ds\le\frac{C}{x},\quad 0<\frac{x}{2}<y<2x.
\end{aligned}$$

By using again (6), (10), (12) and making the change of variables $s = \log \frac{1+u}{1-u}$ we have

$$\begin{aligned}
&\int_{0,\frac{2xye^{-s}}{1-e^{-2s}}\geq 1}^{1} \frac{1}{s}|W_s^\alpha(x,y) - \sqrt{2}W_s^H(x,y)|\, ds \\
&\leq C\int_{0,\frac{2xye^{-s}}{1-e^{-2s}}\geq 1}^{1} \frac{1}{s}\Big(\frac{1-e^{-2s}}{2xye^{-s}}\Big)^{1/4}\Big(\frac{e^{-s}}{1-e^{-2s}}\Big)^{1/2} \\
&\qquad\qquad \times \exp\Big(-\frac{|x-e^{-s}y|^2 + |y-xe^{-s}|^2}{2(1-e^{-2s})}\Big)ds \\
&\leq C(xy)^{-1/4}\int_0^1 \frac{1}{s^{5/4}}\exp\Big(-\frac{|x-e^{-s}y|^2 + |y-xe^{-s}|^2}{2(1-e^{-2s})}\Big)ds \\
&\leq C(xy)^{-1/4}\int_0^{\frac{e-1}{e+1}} \frac{1}{(-\log\frac{1-u}{1+u})^{5/4}}\exp\Big(-\frac{1}{4}(\frac{1}{u}(x-y)^2 + u(x+y)^2)\Big)\frac{du}{1-u^2} \\
&\leq C(xy)^{-1/4}\int_0^{\frac{e-1}{e+1}} \frac{1}{u^{5/4}}e^{-(x-y)^2/(4u)}du \\
&\leq C\frac{1}{(xy)^{1/4}|x-y|^{1/2}} \leq C\Big(\frac{x}{|x-y|}\Big)^{1/2}\frac{1}{x}, \quad 0<\frac{x}{2}<y<2x.
\end{aligned}$$

On the other hand, by combining (6), (10) and (11), we obtain

$$\begin{aligned}
&\int_{0,\frac{2xye^{-s}}{1-e^{-2s}}\leq 1}^{1} \frac{1}{s}|W_s^\alpha(x,y) - \sqrt{2}W_s^H(x,y)|\, ds \\
&\leq C\Big(\int_{0,\frac{2xye^{-s}}{1-e^{-2s}}\leq 1}^{1} \frac{1}{s}\Big(\frac{e^{-s}}{1-e^{-2s}}\Big)^{1/2}\Big(\frac{2xye^{-s}}{1-e^{-2s}}\Big)^{\alpha+\frac{1}{2}} \\
&\qquad\qquad \times \exp\Big(-\frac{1}{2}\frac{1+e^{-2s}}{1-e^{-2s}}(x^2+y^2)\Big)ds \\
&\quad + \int_{0,\frac{2xye^{-s}}{1-e^{-2s}}\leq 1}^{1} \frac{1}{s}\Big(\frac{e^{-s}}{1-e^{-2s}}\Big)^{1/2}\exp\Big(-\frac{1}{2}\frac{1+e^{-2s}}{1-e^{-2s}}(x^2+y^2)\Big)ds\Big) \\
&\leq C\Big((xy)^{\alpha+\frac{1}{2}}\int_0^1 \frac{1}{s^{\alpha+2}}e^{-c(x^2+y^2)/s}ds + \int_0^1 \frac{1}{s^{3/2}}e^{-c(x^2+y^2)/s}ds\Big) \\
&\leq C\Big(\frac{(xy)^{\alpha+\frac{1}{2}}}{(x^2+y^2)^{\alpha+1}} + \frac{1}{(x^2+y^2)^{1/2}}\Big) \leq \frac{C}{x}, \quad 0<\frac{x}{2}<y<2x,
\end{aligned}$$

and

$$\int_{1,\frac{2xye^{-s}}{1-e^{-2s}}\leq 1}^{\infty} \frac{1}{s}|W_s^\alpha(x,y)-\sqrt{2}W_s^H(x,y)|ds$$
$$\leq C\Big((xy)^{\alpha+\frac{1}{2}}e^{-c(x^2+y^2)}\int_1^\infty e^{-s(\alpha+1)}ds+e^{-c(x^2+y^2)}\int_1^\infty e^{-\frac{s}{2}}ds\Big)$$
$$\leq C\Big(\frac{(xy)^{\alpha+\frac{1}{2}}}{(x^2+y^2)^{\alpha+1}}+\frac{1}{(x^2+y^2)^{1/2}}\Big)\leq \frac{C}{x},\quad 0<\frac{x}{2}<y<2x.$$

Hence we conclude that

$$M_\alpha(x,y)\leq C\frac{1}{x}\Big(1+\Big(\frac{x}{|x-y|}\Big)^{1/2}\Big),\quad 0<\frac{x}{2}<y<2x.$$

We observe that the operator $\mathfrak{M}_\alpha$,

$$\mathfrak{M}_\alpha(g)(x)=\int_{\frac{x}{2}}^{2x}\frac{1}{x}\Big(1+\Big(\frac{x}{|x-y|}\Big)^{1/2}\Big)g(y)\,dy,$$

is bounded from $L^p(0,\infty)$ into $L^p(0,\infty)$, for every $1<p<\infty$. As a consequence, $g^\alpha_{q,\mathrm{loc}}$ is bounded from $L^p_B(0,\infty)$ into $L^p(0,\infty)$, $1<p<\infty$, if and only if $g^H_{q,\mathrm{loc}}$ is bounded from $L^p_B(0,\infty)$ into $L^p(0,\infty)$, $1<p<\infty$. ∎

Proof of Theorem 1. It is easy to see that

(14) $$g_q^{2H-I}=g_q^{H-I/2},\quad 1<q<\infty.$$

Consider the operator $Uf(x)=e^{-x^2/2}f(x)$. It is clear that U defines an isometry from $L^2(\mathbb{R},e^{-x^2}dx)$ onto $L^2(\mathbb{R})$. If we denote by $\mathbb{L}$ the Ornstein–Uhlenbeck operator $\mathbb{L}=-d^2/dx^2+2xd/dx$ then, for every $q>1$, it can be checked that for $f=\sum_k c_kh_k$,

(15) $$g_q^{\mathbb{L}}(f)=U^{-1}g_q^{2H-I}(Uf),$$

where $g_q^{\mathbb{L}}$, $1<q<\infty$, denotes the g-function associated with the Poisson semigroup for the operator $\mathbb{L}$. By using identity (15) one immediately gets that for every $1<q<\infty$ the boundedness of $g_q^{\mathbb{L}}$ from $L^2_B(\mathbb{R},e^{-x^2}dx)$ into $L^2(\mathbb{R},e^{-x^2}dx)$ is equivalent to the boundedness of g_q^{2H-I} from $L^2_B(\mathbb{R})$ into $L^2(\mathbb{R})$.

If B has Lusin cotype q, according to [13, Theorem 5.2], $g_q^{\mathbb{L}}$ is bounded from $L^2_B(\mathbb{R},e^{-x^2}dx)$ into $L^2(\mathbb{R},e^{-x^2}dx)$. By the previous arguments, this implies that $g_q^{H-I/2}$ is bounded from $L^2_B(\mathbb{R})$ into $L^2(\mathbb{R})$. By using Lemma 1 and Lemma 2 we get (i)$\Longleftrightarrow$ (iii) of Theorem 1 and also the equivalence with the boundedness of $g^H_{q,\mathrm{loc}}$. Then Lemma 4 gives the equivalence of (i) with (ii). ∎

3. Proof of Theorem 2

The implication (i) $\Longrightarrow$ (ii) is contained in the following proposition.

Proposition 1: *Let B be a Banach space, $\alpha > -1$, $1 < q \leq 2$, and $p \in \Omega_\alpha$. If B has Lusin type q, then*

$$\|f\|_{L^p_B(0,\infty)} \leq C\|g^\alpha_q(f)\|_{L^p(0,\infty)}, \quad f \in L^p_B(0,\infty).$$

Proof. We claim that

(16)
$$\int_0^\infty f(x)h(x)dx = 4\int_0^\infty \int_0^\infty t\frac{\partial P^\alpha_t(f)(x)}{\partial t}t\frac{\partial P^\alpha_t(h)(x)}{\partial t}\frac{dt}{t}dx, \quad f,h \in L^2(0,\infty).$$

Indeed, assume $f = \sum_{n=0}^k a_n\varphi^\alpha_n$ and $h = \sum_{n=0}^k b_n\varphi^\alpha_n$, with $k \in \mathbb{N}$. Then $P^\alpha_t f = \sum_{n=1}^k e^{-t\sqrt{\lambda_{n,\alpha}}}a_n\varphi^\alpha_n$ and $P^\alpha_t h = \sum_{n=1}^k e^{-t\sqrt{\lambda_{n,\alpha}}}b_n\varphi^\alpha_n$, where $\lambda_{n,\alpha} = 2n+\alpha+1$, $n \in \mathbb{N}$. Hence

$$\begin{aligned}\int_0^\infty & t\frac{\partial P^\alpha_t(f)(x)}{\partial t}\frac{\partial P^\alpha_t(h)(x)}{\partial t}dt \\ &= \sum_{n,m=0}^k a_n b_m \varphi^\alpha_n(x)\varphi^\alpha_m(x)\int_0^\infty te^{-t(\sqrt{\lambda_{n,\alpha}}+\sqrt{\lambda_{m,\alpha}})}\sqrt{\lambda_{n,\alpha}\lambda_{m,\alpha}}\,dt \\ &= \sum_{n,m=0}^k \frac{a_n b_m\varphi^\alpha_n(x)\varphi^\alpha_m(x)}{(\sqrt{\lambda_{n,\alpha}}+\sqrt{\lambda_{m,\alpha}})^2}\sqrt{\lambda_{n,\alpha}\lambda_{m,\alpha}}, \quad x \in (0,\infty).\end{aligned}$$

By orthonormality we get

$$\int_0^\infty \int_0^\infty t\frac{\partial P^\alpha_t(f)(x)}{\partial t}\frac{\partial P^\alpha_t(h)(x)}{\partial t}dtdx = \frac{1}{4}\sum_{n=0}^k a_n b_n = \frac{1}{4}\int_0^\infty f(x)h(x)dx.$$

Since g^α_2 is bounded from $L^2(0,\infty)$ into itself (Theorem 1), Hölder's inequality implies that both members of the equality (16) define bounded bilinear mappings from $L^2(0,\infty) \times L^2(0,\infty)$ into $\mathbb{R}$. Then, as $\{\varphi^\alpha_n\}_{n\in\mathbb{N}}$ is a complete system in $L^2(0,\infty)$, we conclude that (16) holds for every $f, g \in L^2(0,\infty)$.

Suppose now that B has Lusin type q. By using [25, Corollary 2.6] it follows that the dual space B^* of B has Lusin cotype q'. Hence, according to Theorem 1, we have that for every $p \in \Omega_\alpha$,

$$\|g^\alpha_{q'}(f)\|_{L^p(0,\infty)} \leq C\|f\|_{L^p_{B^*}(0,\infty)}, \quad f \in L^p_{B^*}(0,\infty).$$

Then, by using (16) and duality arguments (as in [13, proof of Theorem 2.2]) we get

$$\|f\|_{L^p_B(0,\infty)} \leq C\|g^\alpha_q(f)\|_{L^p(0,\infty)}, \quad f \in L^p_B(0,\infty), \quad 1 < p < \infty. \qquad \blacksquare$$

In order to prove (ii) $\Longrightarrow$ (i) of Theorem 2 we follow some ideas developed in [13, section 3] (also see [25]). Assume that $p \in \Omega_\alpha$, $\alpha > -1$ and $1 < q < \infty$. Consider the operator Q_α defined for good enough functions h as follows:

$$Q_\alpha(h)(x) = \int_0^\infty t \int_0^\infty \frac{\partial}{\partial t}(P^\alpha_t(x,y))h(y,t)dy\frac{dt}{t}.$$

Lemma 5: *Let B be a Banach space, $\alpha > -1$, $1 < q < \infty$ and $p \in \Omega_\alpha$. Then*

$$\|g^\alpha_q(Q_\alpha h)\|_{L^p(0,\infty)} \leq C\|h\|_{L^p_A(0,\infty)},$$

where $A = L^q_B((0,\infty), dt/t)$.

Proof. Let h be in the dense set of compactly supported and continuous B-valued functions defined on $(0,\infty)\times(0,\infty)$. By using the semigroup property we have

$$\begin{aligned}
&s\frac{\partial}{\partial s}\int_0^\infty P^\alpha_s(x,y)Q_\alpha(h)(y)dy\\
&\qquad=s\frac{\partial}{\partial s}\int_0^\infty P^\alpha_s(x,y)\int_0^\infty t\int_0^\infty \frac{\partial}{\partial t}(P^\alpha_t(y,z))h(z,t)\,dz\frac{dt}{t}dy\\
&\qquad=\int_0^\infty st\int_0^\infty h(z,t)\frac{\partial}{\partial s}\frac{\partial}{\partial t}\int_0^\infty P^\alpha_t(y,z)P^\alpha_s(x,y)\,dydz\frac{dt}{t}\\
&\qquad=\int_0^\infty st\int_0^\infty h(z,t)\frac{\partial^2}{\partial u^2}P^\alpha_u(x,z)_{|u=t+s}dz\frac{dt}{t}\\
&\qquad=\int_0^\infty\int_0^\infty h(z,t)M^\alpha(x,s;z,t)\frac{dt}{t}dz, \quad x,s\in(0,\infty),
\end{aligned}$$

where

$$M^\alpha(x,s;z,t) = st\frac{\partial^2}{\partial u^2}P^\alpha_u(x,z)_{|u=t+s}, \quad x,z,s,t \in (0,\infty). \tag{17}$$

In order to prove the lemma, it is enough to show that

$$T_\alpha(h)(x,s) = \int_0^\infty\int_0^\infty M^\alpha(x,s;z,t)h(z,t)\frac{dt}{t}dz$$

is bounded from $L^p_A(0,\infty)$ into itself.

By the subordination formula (2) we have

$$
\begin{aligned}
\frac{\partial^2}{\partial u^2}P_u^\alpha(x,z) =&\frac{\partial^2}{\partial u^2}\Big(\frac{u}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{u^2}{4v}}}{v^{\frac{3}{2}}}W_v^\alpha(x,z)dv\Big)\\
=&\frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{u}{v^{\frac{5}{2}}}\Big(\frac{u^2}{4v}-\frac{3}{2}\Big)e^{-\frac{u^2}{4v}}W_v^\alpha(x,z)dv, \quad u,x,z\in(0,\infty).
\end{aligned}\tag{18}
$$

Hence the estimates for the heat kernel W^α contained in Lemma 3 drive us to

$$
\Big|\frac{\partial^2}{\partial u^2}P_u^\alpha(x,z)\Big| \leq Cu^{-2}\begin{cases} z^{\alpha+\frac{1}{2}}x^{-\alpha-\frac{3}{2}}, & 0<z<x/2\\ x^{\alpha+\frac{1}{2}}z^{-\alpha-\frac{3}{2}}, & 2x<z<\infty\end{cases}, \quad u\in(0,\infty).
$$

Therefore

$$
|M^\alpha(x,s;z,t)| \leq C\frac{st}{(s+t)^2}\begin{cases} z^{\alpha+\frac{1}{2}}x^{-\alpha-\frac{3}{2}}, & 0<z<x/2\\ x^{\alpha+\frac{1}{2}}z^{-\alpha-\frac{3}{2}}, & 2x<z<\infty\end{cases}, \quad s,t\in(0,\infty).
$$

We split T_α in three parts as follows;

$$
\begin{aligned}
T_\alpha(h)(x,s) =&\Big(\int_0^{\frac{x}{2}}+\int_{\frac{x}{2}}^{2x}+\int_{2x}^\infty\Big)\int_0^\infty M^\alpha(x,s;z,t)h(z,t)\frac{dt}{t}dz\\
=&T_{\alpha,1}(h)(x,s)+T_{\alpha,2}(h)(x,s)+T_{\alpha,3}(h)(x,s).
\end{aligned}
$$

Then Minkowski's and Jensen's inequalities lead to

$$\begin{aligned}
&\|T_{\alpha,1}(h)\|_{L^p_A(0,\infty)}\\
&=\Big\{\int_0^\infty\Big\{\int_0^\infty\Big\|\int_0^{\frac{x}{2}}\int_0^\infty M^\alpha(x,s;z,t)h(z,t)\frac{dt}{t}dz\Big\|_B^q\frac{ds}{s}\Big\}^{\frac{p}{q}}dx\Big\}^{\frac{1}{p}}\\
&\le C\Big\{\int_0^\infty\Big\{\int_0^\infty\Big\{\int_0^{\frac{x}{2}}\int_0^\infty\frac{st}{(s+t)^2}\frac{z^{\alpha+\frac12}}{x^{\alpha+\frac32}}\|h(z,t)\|_B\frac{dt}{t}dz\Big\}^q\frac{ds}{s}\Big\}^{\frac{p}{q}}dx\Big\}^{\frac{1}{p}}\\
&\le C\Big\{\int_0^\infty\Big\{\int_0^\infty\Big\{\int_0^\infty\frac{st}{(s+t)^2}\\
&\qquad\times\Big[\frac{1}{x^{\alpha+\frac32}}\int_0^{\frac{x}{2}}z^{\alpha+\frac12}\|h(z,t)\|_Bdz\Big]\frac{dt}{t}\Big\}^q\frac{ds}{s}\Big\}^{\frac{p}{q}}dx\Big\}^{\frac{1}{p}}\\
&\le C\Big\{\int_0^\infty\Big\{\int_0^\infty\Big[\frac{1}{x^{\alpha+\frac32}}\int_0^{\frac{x}{2}}z^{\alpha+\frac12}\|h(z,t)\|_Bdz\Big]^q\frac{dt}{t}\Big\}^{\frac{p}{q}}dx\Big\}^{\frac{1}{p}}\\
&\le C\Big\{\int_0^\infty\Big\|\frac{1}{x^{\alpha+\frac32}}\int_0^{\frac{x}{2}}z^{\alpha+\frac12}\|h(z,t)\|_Bdz\Big\|^p_{L^q((0,\infty),\frac{dt}{t})}dx\Big\}^{\frac{1}{p}}\\
&\le C\Big\{\int_0^\infty\Big|\frac{1}{x^{\alpha+\frac32}}\int_0^{\frac{x}{2}}z^{\alpha+\frac12}\|h(z,\cdot)\|_{L^q_B((0,\infty),\frac{dt}{t})}dz\Big|^pdx\Big\}^{\frac{1}{p}}\\
&\le C\|h\|_{L^p_A(0,\infty)}.
\end{aligned}$$

In the last inequality we have taken into account that the Hardy-type operator $\mathcal{H}^0_\alpha$ defined by

$$\mathcal{H}^0_\alpha(g)(x)=\frac{1}{x^{\alpha+3/2}}\int_0^x y^{\alpha+1/2}g(y)dy$$

is bounded from $L^r(0,\infty)$ into itself when $1<r(\alpha+3/2)$ ([3, Lemma 3.1]).

In a similar way we obtain that

$$\|T_{\alpha,3}(h)\|_{L^p_A(0,\infty)}\le C\|h\|_{L^p_A(0,\infty)}.$$

Now we shall deal with $T_{\alpha,2}$. By invoking again Lemma 3 one has

$$W^\alpha_v(x,y)=W^H_v(x,y)+N_v(x,y),\quad x/2<y<2x \text{ and } v\in(0,\infty),$$

with $|N_v(x,y)|\le C/y,\quad x/2<y<2x$ and $v\in(0,\infty)$. Observe that the integral

$$N(x,s;z,t)=st\int_0^\infty\frac{u}{v^{\frac52}}\Big(\frac{u^2}{4v}-\frac32\Big)e^{-u^2/(4v)}N_v(x,z)\,dv_{|u=s+t}$$

satisfies

$$|N(x,s;z,t)| \leq C\frac{1}{(s+t)^2 z}, \quad x/2 < z < 2x, \text{ and } s,t \in (0,\infty).$$

Hence the operator

$$h \longrightarrow \int_{\frac{x}{2}}^{2x} \int_0^\infty N(x,s;z,t)h(z,t)\frac{dt}{t}dz$$

is bounded from $L^p_A(0,\infty)$ into itself. Then (see (17) and (18)) the operator T_α is bounded from $L^p_A(0,\infty)$ into itself if and only if the operator S_2 defined by

$$S_2(h)(x,s) = \int_{\frac{x}{2}}^{2x} \int_0^\infty M^H(x,s;z,t)h(z,t)\frac{dt}{t}dz,$$

with M^H given by

$$M^H(x,s;z,t) = st\frac{\partial^2}{\partial u^2}P_u^H(x,z)_{|u=t+s}, \quad x,z,t,s \in (0,\infty),$$

is bounded from $L^p_A(0,\infty)$ into itself.

We claim that

$$(19) \qquad |M^H(x,s;z,t)| \leq C\frac{st}{(s+t+|x-z|)^3}, \quad s,t,x,z \in (0,\infty).$$

To see the claim, we make the change of variable $v = \log\frac{1+w}{1-w}$ and get

$$\begin{aligned} M^H(x,s;z,t) =& \frac{st}{2\sqrt{\pi}} \int_0^\infty \frac{s+t}{v^{\frac{5}{2}}} \left(\frac{(s+t)^2}{4v} - \frac{3}{2}\right) e^{-(s+t)^2/(4v)} W_v^H(x,z)dv \\ =& \frac{st}{2\pi} \int_0^1 \frac{s+t}{\left(\log\frac{1+w}{1-w}\right)^{\frac{5}{2}}} \left(\frac{(s+t)^2}{4\log\frac{1+w}{1-w}} - \frac{3}{2}\right) \exp\left(\frac{-(s+t)^2}{4\log\frac{1+w}{1-w}}\right) \\ & \times \left(\frac{1-w^2}{4w}\right)^{\frac{1}{2}} \exp\Big(-\frac{1}{4}(\frac{1}{w}(x-z)^2 + w(x+z)^2)\Big)\frac{2dw}{1-w^2} \\ =& I_1(x,s;z,t) + I_2(x,s;z,t), \end{aligned}$$

where for I_1 and I_2 the integral is extended to $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$, respectively. Since $\log \frac{1+w}{1-w} \sim w$, as $w \to 0$, we can write

$$\begin{aligned}
|I_1(x,s;z,t)| \leq & Cst \int_0^{\frac{1}{2}} \frac{s+t}{w^3} \left(\frac{(s+t)^2}{w} + 1 \right) e^{-\frac{c(s+t)^2}{w}} \\
& \times \exp\Big(-\frac{1}{4}\Big(\frac{(x-z)^2}{w} + w(x+z)^2 \Big)\Big) dw \\
\leq & Cst \int_0^{\frac{1}{2}} \frac{1}{w^{\frac{5}{2}}} \exp\Big(-c\frac{(s+t)^2}{w} - \frac{1}{4}\Big(\frac{(x-z)^2}{w} + w(x+z)^2 \Big)\Big) dw \\
\leq & Cst \int_0^{\frac{1}{2}} \frac{1}{w^{\frac{5}{2}}} \exp\Big(-c\frac{(s+t+|x-z|)^2}{w} \Big) dw, \quad s,t,x,z \in (0,\infty).
\end{aligned}$$

Then by using [21, Lemma 1.1] we obtain

$$|I_1(x,s;z,t)| \leq C\frac{st}{(s+t+|x-z|)^3}, \quad s,t,x,z \in (0,\infty).$$

On the other hand, since $\log \frac{1+w}{1-w} \sim -\log(1-w)$, as $w \to 1^-$, we get

$$\begin{aligned}
|I_2(x,s;z,t)| \leq & Cst \int_{\frac{1}{2}}^1 \frac{1}{|\log(1-w)|^{\frac{3}{2}}(1-w)^{\frac{1}{2}}} \\
& \times \exp\Big(-\frac{c}{|\log(1-w)|}((s+t)^2 + (x-z)^2)\Big) dw \\
\leq & Cst \int_{\frac{1}{2}}^1 \frac{1}{|\log(1-w)w|^{\frac{3}{2}}(1-w)^{\frac{1}{2}}} \\
& \times \exp\Big(-\frac{c}{|\log(1-w)|}(s+t+|x-z|)^2\Big) dw \\
\leq & C\frac{st}{(s+t+|x-z|)^3}, \quad s,t,x,z \in (0,\infty).
\end{aligned}$$

The claim is proved. We now consider the operator

$$L(x,z)(\phi)(s) = \int_0^\infty M^H(x,s;z,t)\phi(t)\frac{dt}{t}, \quad \phi \in A.$$

Note that

$$S_2(h)(x,s) = \int_{\frac{x}{2}}^{2x} L(x,z)(h(z,\cdot))(s)\, dz.$$

We define the operator

$$\mathbb{S}_2(h)(x,s) = \int_0^\infty L(x,z)(h(z,\cdot))(s)\, dz.$$

Now we claim that $\mathbb{S}_2$ is an A-valued Calderón–Zygmund operator. By Hölder's inequality and (19) we get

$$\begin{aligned}\|\mathbb{S}_2(h)(x,s)\|_B =&\Big\| \int_0^\infty L(x,z)(h(z,\cdot))(s)dz\Big\|_B\\ =&\Big\| \int_0^\infty \int_0^\infty M^H(x,s;z,t)h(z,t)\frac{dt}{t}dz\Big\|_B\\ \leq& C\Big\{ \int_0^\infty \int_0^\infty \frac{st}{(s+t+|x-z|)^3}\|h(z,t)\|_B^q\frac{dt}{t}dz\Big\}^{\frac{1}{q}}\\ &\times \Big\{ \int_0^\infty \int_0^\infty \frac{st}{(s+t+|x-z|)^3}\frac{dt}{t}dz\Big\}^{\frac{1}{q'}}\\ \leq& C\Big\{ \int_0^\infty \int_0^\infty \frac{s}{(s+t+|x-z|)^3}\|h(z,t)\|_B^q dtdz\Big\}^{\frac{1}{q}}.\end{aligned}$$

Then

$$\begin{aligned}\|\mathbb{S}_2(h)\|^q_{L^q_A(0,\infty)} \leq& C\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{(s+t+|x-z|)^3}\|h(z,t)\|_B^q dt\,dz\,ds\,dx\\ \leq& C\int_0^\infty\int_0^\infty \|h(z,t)\|_B^q \int_0^\infty\int_0^\infty \frac{1}{(s+t+|x-z|)^3}ds\,dx\,dt\,dz \qquad (20)\\ \leq& C\|h\|^q_{L^q_A(0,\infty)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|L(x,z)\|_{A\to A} \leq& \Big\{ \int_0^\infty \Big\{ \int_0^\infty |M(x,s;z,t)|^{q'}\frac{ds}{s}\Big\}^{\frac{q}{q'}}\frac{dt}{t}\Big\}^{\frac{1}{q}}\\ \leq& C\Big\{ \int_0^\infty \Big\{ \int_0^\infty \frac{(st)^{q'}}{(s+t+|x-z|)^{3q'}}\frac{ds}{s}\Big\}^{\frac{q}{q'}}\frac{dt}{t}\Big\}^{\frac{1}{q}} \qquad (21)\\ \leq& C\Big\{ \int_0^\infty \Big\{ \int_0^\infty \frac{ds}{(s+t+|x-z|)^{2q'+1}}\Big\}^{\frac{q}{q'}}t^{q-1}dt\Big\}^{\frac{1}{q}}\\ \leq& C\Big\{ \int_0^\infty \frac{t^{q-1}}{(t+|x-z|)^{2q}}dt\Big\}^{\frac{1}{q}}\\ \leq& C\Big\{ \int_0^\infty \frac{dt}{(t+|x-z|)^{q+1}}\Big\}^{\frac{1}{q}} \leq \frac{C}{|x-z|},\quad x,z\in(0,\infty),\ x\neq z.\end{aligned}$$

Moreover,

$$\begin{aligned}
\Big|\frac{\partial}{\partial x}\Big[\exp\Big(-\frac{1}{4}(\frac{1}{w}(x-z)^2+w(x+z)^2)\Big)\Big]\Big| & \\
=&\frac{1}{2}\Big|\frac{1}{w}(x-z)+w(x+z)\Big|\exp\Big(-\frac{1}{4}(\frac{1}{w}(x-z)^2+w(x+z)^2)\Big) \\
\le& C\frac{1}{\sqrt{w}}\exp\Big(-c(\frac{1}{w}(x-z)^2+w(x+z)^2)\Big), \quad w\in(0,1),\ x,z\in(0,\infty). \qquad (22)
\end{aligned}$$

Proceeding as above we get

$$\Big|\frac{\partial}{\partial x}M^H(x,s;z,t)\Big| \le C\frac{st}{(s+t+|x-z|)^4}, \quad s,t,x,z\in(0,\infty).$$

Hence

$$\Big\|\frac{\partial}{\partial x}L(x,z)\Big\|_{A\to A} \le \frac{C}{|x-z|^2}, \quad x,z\in(0,\infty),\ x\neq z. \qquad (23)$$

Analogously,

$$\Big\|\frac{\partial}{\partial z}L(x,z)\Big\|_{A\to A} \le \frac{C}{|x-z|^2}, \quad x,z\in(0,\infty),\ x\neq z. \qquad (24)$$

Inequalities (20), (21), (23) and (24) allow us to use vector-valued Calderón–Zygmund's theory and therefore $\mathbb{S}_2$ is a bounded operator from $L^p_A(0,\infty)$ into itself. Moreover, from the size condition on $\|L(x,z)\|$ we deduce

$$\begin{aligned}
\Big\|\int_0^{\frac{x}{2}} L(x,z)(h(z,\cdot))(s)dz\Big\|_A \le& \int_0^{\frac{x}{2}} \|L(x,z)\|_{A\to A}\|h(z,\cdot)\|_A dz \\
\le& C\frac{1}{x}\int_0^x \|h(z,\cdot)\|_A dz
\end{aligned}$$

and

$$\begin{aligned}
\Big\|\int_{2x}^{\infty} L(x,z)(h(z,\cdot))(s)dz\Big\|_A \le& \int_{2x}^{\infty} \|L(x,z)\|_{A\to A}\|h(z,\cdot)\|_A dz \\
\le& C\int_{2x}^{\infty} \|h(z,\cdot)\|_A \frac{dz}{z}.
\end{aligned}$$

Then, well-known results about Hardy operators ([16]) imply that the operators

$$h \longrightarrow \int_0^{\frac{x}{2}} L(x,z)(h(z,\cdot))(s)dz$$

and

$$h \longrightarrow \int_{2x}^{\infty} L(x,z)(h(z,\cdot))(s)dz$$

are bounded from $L^p_A(0,\infty)$ into itself. Thus we have proved that S_2 and then T_α is bounded from $L^p_A(0,\infty)$ into itself and the proof of the Lemma is finished. ∎

Now we can prove (ii) $\Longrightarrow$ (i). Let $f \in L^{p'}_{B^*}(0,\infty)$. We choose $h \in L^p_{L^q_B((0,\infty),\frac{dt}{t})}(0,\infty)$ such that $\|h\|_{L^p_{L^q_B((0,\infty),dt/t)}(0,\infty)} = 1$ and

$$\|g^\alpha_{q'}(f)\|_{L^{p'}(0,\infty)} = \int_0^\infty \int_0^\infty \left\langle t\frac{\partial P^\alpha_t f}{\partial t}(x), h(x,t)\right\rangle \frac{dt}{t}dx.$$

We assume that f and h are smooth. Then it can be written as

$$\begin{aligned}\|g^\alpha_{q'}(f)\|_{L^{p'}(0,\infty)} =& \int_0^\infty \int_0^\infty \left\langle f(x), t\int_0^\infty \frac{\partial}{\partial t}(P^\alpha_t(x,y)))h(y,t)dy\right\rangle \frac{dt}{t}dx \\ =& \int_0^\infty \langle f(x), Q_\alpha(h)(x)\rangle dx \\ \leq& \|f\|_{L^{p'}_{B'}(0,\infty)}\|Q_\alpha(h)\|_{L^p_B(0,\infty)}.\end{aligned}$$

Hence, if (ii) holds, by using Lemma 5 we get

$$\|g^\alpha_{q'}(f)\|_{L^{p'}(0,\infty)} \leq C\|f\|_{L^{p'}_{B'}(0,\infty)}\|g^\alpha_q(Q_\alpha(h))\|_{L^p(0,\infty)} \leq C\|f\|_{L^p_{B'}(0,\infty)}.$$

According to Theorem 1 this proves that B^* has Lusin cotype q'. By using [25, Corollary 2.6] we conclude that B has Lusin type q.

Finally, (i)$\Leftrightarrow$ (iii) can be proved following similar arguments to these used previously to establish (i) $\Leftrightarrow$ (ii).

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