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MULTIFRACTAL ANALYSIS AND PHASE TRANSITIONS FOR HYPERBOLIC AND PARABOLIC HORSESHOES

BY

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ABSTRACT

We effect a complete study of the thermodynamic formalism, the entropy spectrum of Birkhoff averages, and the ergodic optimization problem for a family of **parabolic horseshoes**. We consider a large class of potentials that are not necessarily regular, and we describe both the uniqueness of equilibrium measures and the occurrence of phase transitions for nonregular potentials in this class. Our approach consists in reducing the problems to the study of renewal shifts. We also describe applications of this approach to hyperbolic horseshoes as well as to noninvertible maps, both parabolic (with the Manneville–Pomeau map) and uniformly expanding. This allows us to recover in a unified manner several results scattered in the literature. For the family of hyperbolic horseshoes, we also describe the **dimension spectrum** of equilibrium measures of a class of potentials that are not necessarily regular. In particular, the dimension spectra need not be strictly convex.

1. Introduction

Our main objective is to effect a complete study of the thermodynamic formalism and of the multifractal analysis of entropy spectra of Birkhoff averages for a family of **parabolic horseshoes**. We also consider a large class of potentials that are not necessarily regular, although this class includes, for example, all Hölder continuous potentials. Roughly speaking, parabolic horseshoes are invariant sets topologically equivalent to hyperbolic horseshoes, and thus to (finite) Markov shifts, although they lack hyperbolicity (or at least uniform hyperbolicity) at one or more points. We shall consider the model case of parabolic horseshoes for which the hyperbolicity breaks down at a single point. We emphasize that even in this particular situation our results are substantially different from those in the case of hyperbolic horseshoes.

We recall that in the case of (uniformly) hyperbolic horseshoes the thermodynamic formalism is well-known (and in fact very well-behaved) for several classes of **sufficiently regular** potentials. For example, the topological pressure is analytic in the class of Hölder continuous functions, and thus there are no phase transitions for this class. In another direction, the multifractal analysis of *conformal* hyperbolic horseshoes is also well-established, and provides a detailed study of the complexity of the level sets of invariant local quantities obtained from a given dynamical system, such as Birkhoff averages, Lyapunov exponents, pointwise dimensions, and local entropies. The conformality means that the dynamics acts conformally both in the stable and unstable direction, such as in saddle-type horseshoes on surfaces. In particular, it was shown for several families of conformal dynamical systems and Hölder continuous potentials that the associated multifractal spectra (such as entropy spectra and dimension spectra) are real analytic and strictly convex. We refer to [17] for a detailed discussion and for a list of references.

The good behavior exhibited by hyperbolic horseshoes, in terms both of the thermodynamic formalism and of the multifractal analysis, may break down when we make the horseshoe nonhyperbolic (or at least nonuniformly hyperbolic, here understood in the sense of the existence of an invariant measure supported on the horseshoe having nonzero Lyapunov exponents). On the other hand, it may also break down when we consider more general classes of (nonregular) potentials. We discuss both situations in our paper and we show that indeed they occur.

We now describe our results in some detail. First, we obtain a complete description of the thermodynamic formalism for several families of maps, with emphasis on the case of parabolic horseshoes. In particular, for a potential ϕ in a certain class that includes the class of Hölder continuous functions, there exists a critical value $q_c \in (0, \infty]$ such that the pressure function $q \mapsto P(q\phi)$ is real analytic and strictly convex for $q \in (0, q_c)$, and linear for $q > q_c$ (see Theorem 14). Thus, for this class of potentials there are **phase transitions**. We also describe the equilibrium measures of the potentials $q\phi$. Furthermore, building on results of Takens and Verbitskiy [28] we are able to study the multifractal analysis of entropy spectra of Birkhoff averages. Namely, we show that there exists a critical value that separates the entropy spectrum into two parts with very different behavior, one real analytic and strictly convex, and the other linear (see Theorem 15). Finally, we also study the ergodic optimization problem for parabolic horseshoes (see Theorem 16).

We emphasize that even though our main results are formulated for parabolic horseshoes, including both the cases of regular and nonregular potentials, we also obtain new results in the classical case of **hyperbolic horseshoes** (for nonregular potentials). In addition, we describe applications of our approach to the case of noninvertible dynamics, both parabolic (with the Manneville–Pomeau map) and uniformly expanding (this allows us to recover in a unified manner several results scattered in the literature as well as to obtain new results;

the details are given in Section 6). In the case of hyperbolic horseshoes, we also describe the **dimension spectrum** of equilibrium measures of a class of potentials that are not necessarily regular (see Section 7). In particular, these spectra need not be strictly convex.

Our strategy in the proofs is to reduce the problems to the study of renewal shifts. The main idea is to consider dynamical systems that can be modeled by a full shift on two symbols and from this obtain a renewal shift (by removing the parabolic fixed point). We note that this has been accomplished before in some settings, namely in the work of Sarig [27] on the Manneville–Pomeau map, and in the work of Pesin and Zhang [20] for uniformly expanding maps of the interval (describing both potentials with a unique equilibrium measure and potentials exhibiting phase transitions). As it was the case in [20], the study of the thermodynamic formalism for the renewal shifts carried out by Sarig in [27] (see Section 2) is central to our analysis.

We also would like to comment on the relation of our work to results concerning the thermodynamic formalism for other classes of maps. Over the last years a great deal of attention has been given to the study of the thermodynamic formalism of one-dimensional real multimodal maps. Recent work by Bruin and Todd [5, 6] and by Pesin and Senti [18] describe potentials for which there is a unique equilibrium measure and such that the pressure function is real analytic in certain domains. Their proofs are based on the study of the so-called *induced maps*, which can be modeled by full-shifts on countable alphabets. In some cases their induced system and the one obtained from the renewal shift coincide [5]. We should stress that the lack of expansiveness considered here is milder than the one studied in [5, 6, 18]. Indeed, the maps studied in those papers have critical points. It should also be pointed out that their description of the thermodynamic formalism is not as complete as ours. An interesting feature relating both classes of systems is that the pressure function has the same type of phase transitions. The reason for this is that in both cases the dynamics can be divided into two parts: one which is **hyperbolic** and the other which is not (the parabolic fixed point or the post-critical set). The difference is that in a parabolic horseshoe there is only one invariant measure supported on the nonhyperbolic part of the dynamics (namely, the atomic measure supported on the parabolic fixed point), whereas in the multimodal case there can be plenty of them (see [8]). The common feature that causes both systems to have the same type of phase transitions is that the measures supported on the nonhyperbolic part of the system have zero entropy. The case of rational maps is better understood than the real case; see the works by Makarov and Smirnov [15] and by Przytycki and Rivera-Letelier [25]. In the latter, the authors give a complete description of the pressure function of certain natural potentials and the same type of phase transition is observed. The reason is again that the invariant measures supported on the nonhyperbolic part of the system have zero entropy.

2. Renewal shift

2.1. Preliminaries. Let $S = \{0, 1, 2, \dots\}$ be a countable alphabet. Consider the transition matrix $A = (a_{ij})_{i,j \in S}$ with $a_{0,0} = a_{0,n} = a_{n,n-1} = 1$ for each $n \geq 1$ and with all other entries equal to zero. The **renewal shift** is the (countable) Markov shift (Σ_R, σ) defined by the transition matrix A, that is, the shift map σ on the space

$$\Sigma_R = \bigl\{(x_i)_{i \geq 0} : x_i \in S \text{ and } a_{x_i x_{i+1}} = 1 \text{ for each } i \geq 0\bigr\}.$$

We equip Σ_R with the topology generated by the cylinders sets

$$C_{i_0 \cdots i_n} = \{x \in \Sigma_R : x_j = i_j \text{ for } 0 \leq j \leq n\}.$$

Given a function $\phi\colon \Sigma_R \to \mathbb{R}$, for each $n \geq 1$ we set

$$V_n(\phi) = \sup\left\{|\phi(x) - \phi(y)| : x, y \in \Sigma_R,\ x_i = y_i \text{ for } 0 \leq i \leq n-1\right\}.$$

We say that ϕ has **summable variation** if $\sum_{n=2}^{\infty} V_n(\phi) < \infty$. Clearly, if ϕ has summable variation then it is continuous. We say that ϕ is **weakly Hölder continuous** if there exist $B > 0$ and $\theta \in (0,1)$ such that $V_n(\phi) \leq B\theta^n$ for every $n \geq 2$. Clearly, any weakly Hölder continuous function has summable variation.

We recall the notion of topological pressure introduced by Sarig in [26]. Let $\phi\colon \Sigma_R \to \mathbb{R}$ be a function with summable variation. The **Gurevich pressure** of ϕ is defined by

$$P_G(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \phi(\sigma^i x)\right) 1_{C_{i_0}}(x),$$

where $1_{C_{i_0}}$ denotes the characteristic function of C_{i_0}. We note that $P_G(\phi)$ does not depend on the choice of i_0. The Gurevich pressure satisfies the following variational principle (see [26]). Let $\mathcal{M}_R$ be the set of σ-invariant probability measures on Σ_R, and let $h_\mu(\sigma)$ be the metric entropy with respect to the measure μ.

Proposition 1: *If $\phi\colon \Sigma_R \to \mathbb{R}$ has summable variation and $\sup\phi < \infty$, then*

$$P_G(\phi) = \sup\left\{h_\mu(\sigma) + \int_{\Sigma_R} \phi\, d\mu : \mu \in \mathcal{M}_R\right\}. \tag{1}$$

A measure $\mu \in \mathcal{M}_R$ at which the supremum in (1) is attained is called an **equilibrium measure** for ϕ. We note that for arbitrary countable shifts we must add the integrability assumption $-\int_{\Sigma_R} \phi\, d\mu < \infty$ in (1), with the single purpose of avoiding expressions of the type $\infty - \infty$. In the particular case of the renewal shift, any measure $\mu \in \mathcal{M}_R$ has entropy $h_\mu(\sigma) \le \log 2$, and thus the integrability assumption is not needed.

2.2. Induced system and class $\mathcal{R}$. The **induced system** (Σ_I, σ) is defined as the full-shift on the new alphabet $\{C_{0n(n-1)(n-2)\cdots 1} : n \ge 1\}$. The first return map to the cylinder C_0 is defined by

$$r(x) = 1_{C_0}(x) \inf\{n \ge 1 : \sigma^n x \in C_0\}.$$

Given a function $\phi\colon \Sigma_R \to \mathbb{R}$ with summable variation we define a new function $\overline{\phi}\colon \Sigma_I \to \mathbb{R}$ by

$$\overline{\phi}(x) = \sum_{k=0}^{r(x)-1} (\phi \circ \sigma^k \circ \pi)(x),$$

where $\pi\colon \Sigma_I \to C_0$ is defined by $\pi(C_{a_0}C_{a_1}\cdots) = (a_0a_1\cdots)$. We now describe the class of functions that we will consider in the thermodynamic formalism. Let $\mathcal{R}$ be the class of functions $\phi\colon \Sigma_R \to \mathbb{R}$ such that:

1. ϕ has summable variation and is bounded from above;
2. ϕ has finite Gurevich pressure;
3. the induced map $\overline{\phi}$ is weakly Hölder continuous.

We observe that $\mathcal{R}$ includes the class of Hölder continuous functions. Nevertheless, there are non-Hölder continuous functions that belong to $\mathcal{R}$. Indeed, Sarig [27] constructed examples of functions in $\mathcal{R}$ that are not Hölder continuous. In fact, he constructed examples of functions exhibiting all possible modes of recurrence. We refer to his paper for explicit examples. We note that the first to describe the thermodynamic formalism for some non-Hölder potentials for the renewal shift was Hofbauer [9].

Sarig described the thermodynamic formalism for the class $\mathcal{R}$. Set

$$M = M(\phi) = \sup\left\{ \int_{\Sigma_R} \phi \, d\mu : \mu \in \mathcal{M}_R \right\}. \tag{2}$$

Proposition 2 ([27]; see also [10]): *Let (Σ_R, σ) be the renewal shift. For each $\phi \in \mathcal{R}$ there exists $q_c \in (0, +\infty]$ such that:*

1. *The function $q \mapsto P_G(q\phi)$ is strictly convex and real analytic in $(0, q_c)$ and linear in $(q_c, +\infty)$, with $P_G(q\phi) = Mq$. At q_c, the function is continuous but not analytic.*
2. *For $q \in (0, q_c)$ there exists a unique equilibrium measure μ_q for $q\phi$, while for $q > q_c$ there is no equilibrium measure for $q\phi$.*

We note that the potential $q_c\phi$ can have an equilibrium measure (the so-called positive recurrent case), an infinite σ-finite "equilibrium" measure (null recurrent case), or none of the above (transient case). To help determine whether q_c is finite or infinite, we set

$$A_n = \exp \sup \left\{ \sum_{i=0}^{n-1} \phi(\sigma^i(x)) : x \in C_{0(n-1)\cdots 0} \right\}.$$

For each $q \in \mathbb{R}$, let $R(q)$ be the radius of convergence of the series

$$F_q(\xi) = \sum_{n=1}^{\infty} A_n^q \xi^q.$$

If $F_q(R(q))$ is infinite for every q, then $q_c = \infty$. If there exists $q > 0$ such that $F_q(R(q)) < 1$, then $q_c < \infty$. We emphasize that it might happen that neither of the two alternatives holds.

We emphasize that in Proposition 2 we only discuss the behavior of the pressure function $q \mapsto P_G(q\phi)$ for $q > 0$. Under an additional assumption on ϕ we are able to consider an arbitrary $q \in \mathbb{R}$. Set

$$m = m(\phi) = \inf\left\{ \int_{\Sigma_R} \phi \, d\mu : \mu \in \mathcal{M}_R \right\}. \tag{3}$$

Proposition 3: *Let (Σ_R, σ) be the renewal shift. For each bounded $\phi \in \mathcal{R}$ there exist $q_c^+ \in (0, +\infty]$ and $q_c^- \in [-\infty, 0)$ such that:*

1. *$q \mapsto P_G(q\phi)$ is strictly convex and real analytic in (q_c^-, q_c^+).*
2. *$P_G(q\phi) = mq$ for $q < q_c^-$, and $P_G(q\phi) = Mq$ for $q > q_c^+$.*
3. *At q_c^- and q_c^+, the function $q \mapsto P_G(q\phi)$ is continuous but not analytic.*

4. *For each* $q \in (q_c^-, q_c^+)$ *there is a unique equilibrium measure* μ_q *for* $q\phi$*.*
5. *For each* $q \notin [q_c^-, q_c^+]$ *there is no equilibrium measure for* $q\phi$*.*
6. *The critical values* q_c^+ *and* q_c^- *are never simultaneously finite.*

Proof. By Proposition 1 we readily obtain the following.

Lemma 1: *If* $\phi \in \mathcal{R}$ *is bounded, then* $-\phi \in \mathcal{R}$*.*

Thus, we can apply Proposition 2 to ϕ and $-\phi$ to obtain q_c^- and q_c^+. Statements 1–5 are also direct consequences of Proposition 2. It remains to prove Statement 6. Assume on the contrary that $-\infty < q_c^- < q_c^+ < \infty$. For each $n \geq 1$, let $p_n = \overline{0n(n-1)\cdots 1} \in \Sigma_R$ be the periodic point of period $n+1$ in the cylinder set C_0. We consider the invariant measure

$$\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\sigma^k p_n}, \tag{4}$$

where $\delta_{\sigma^k p_n}$ is the atomic measure supported at $\sigma^k p_n$. Let also

$$R_n = \{x \in \Sigma_R : r(x) = n\}.$$

By the discriminant theorem in [27] and following [10], since $q_c^+ < \infty$ we have

$$\begin{aligned} M &= \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \phi(\sigma^i x)\right) 1_{R_n}(x) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(n \int_{\Sigma_R} \phi \, d\nu_n\right) 1_{R_n}(x) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \exp\left(n \int_{\Sigma_R} \phi \, d\nu_n\right) = \limsup_{n\to\infty} \int_{\Sigma_R} \phi \, d\nu_n. \end{aligned}$$

Similarly, since $-\infty < q_c^-$ we have

$$\begin{aligned} -m &= \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} -\phi(\sigma^i x)\right) 1_{R_n}(x) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \exp\left(n \int_{\Sigma_R} -\phi \, d\nu_n\right) \\ &= -\limsup_{n\to\infty} \int_{\Sigma_R} \phi \, d\nu_n = -M. \end{aligned}$$

Therefore, $M = m$. Hence, there exist $q_1 < 0$ and $q_2 > 0$ with $P_G(q_1\phi) = Mq_1$ and $P_G(q_2\phi) = Mq_2$. But then the pressure function would not be convex, since $P_G(0) = \log 2$. This contradiction proves Statement 6. ∎

We emphasize that in view of Statement 6 at least one of the critical values q_c^- and q_c^+ is not finite.

As the following example shows, it is possible that $\phi \in \mathcal{R}$ but $-\phi \notin \mathcal{R}$. Let $\phi \colon \Sigma_R \to \mathbb{R}$ be the locally constant function such that

$$\phi|_{C_n} = -(n+1) \quad \text{for each } n \geq 0.$$

Clearly, ϕ is weakly Hölder. In particular, ϕ has summable variation. Moreover, $\phi \leq 0$ and $P_G(\phi)$ is finite. Also, $\overline{\phi}$ is weakly Hölder. Therefore, $\phi \in \mathcal{R}$. We now show that $P_G(-\phi) = \infty$, which implies that $-\phi \notin \mathcal{R}$. For each $n \geq 1$, let ν_n be the σ-invariant measure in (4). By the variational principle in Proposition 1 we have

$$P_G(-\phi) = \sup \left\{ h_\mu(\sigma) - \int_{\Sigma_R} \phi \, d\mu : \mu \in \mathcal{M}_R \right\} \geq - \int_{\Sigma_R} \phi \, d\nu_n.$$

Since

$$- \int_{\Sigma_R} \phi \, d\nu_n = \frac{1}{n} \sum_{i=0}^{n-1} \phi|_{C_i} = \frac{1}{n} \sum_{i=0}^{n-1} (i+1) = \frac{n+1}{2},$$

letting $n \to \infty$ we obtain $P_G(-\phi) = \infty$. Therefore, $-\phi \notin \mathcal{R}$. We emphasize that the potential ϕ is not bounded from below.

2.3. Embedding in a finite full shift. Let now (Σ_2^+, σ) be the one-sided full-shift on the alphabet $\{0,1\}$. There exists a topological conjugacy between the renewal shift (Σ_R, σ) and $(\Sigma_2^+ \setminus \bigcup_{i=0}^\infty \sigma^{-i}(\overline{0}), \sigma)$, where $\overline{0} = (000\cdots)$. Indeed, denote by $(0\cdots01)_n$ the cylinder $C_{0\cdots01}$ with n zeros, and consider the alphabet $\{(0\cdots01)_n : n \geq 1\} \cup \{C_1\}$. The possible transitions on this alphabet are

$$(0\cdots01)_n \to (0\cdots01)_{n-1}, \ C_1 \to C_1, \ \text{and } C_1 \to (0\cdots01)_n \text{ for } n \geq 1.$$

Note that this is simply a recoding of $(\Sigma_2^+ \setminus \bigcup_{i=0}^\infty \sigma^{-i}(\overline{0}), \sigma)$.

3. Hyperbolic approximations

Let $f \colon M \to M$ be a continuous map of the compact metric space M. We denote by $P(\phi)$ the classical topological pressure of a continuous function $\phi \colon M \to \mathbb{R}$. We recall that it satisfies the variational principle: for each continuous function

$\phi\colon M \to \mathbb{R}$,

$$P(\phi) = \sup\left\{h_\mu(f) + \int_M \phi\, d\mu : \mu \in \mathcal{M}\right\} \\ = \sup\left\{h_\mu(f) + \int_M \phi\, d\mu : \mu \in \mathcal{M}_E\right\}, \tag{5}$$

where $\mathcal{M}$ is the set of f-invariant probability measures on M, and $\mathcal{M}_E \subset \mathcal{M}$ is the subset of ergodic measures.

Let $\mathcal{M}_H \subset \mathcal{M}$ be a subset satisfying the following property:

H. For every $\mu \in \mathcal{M}_E \setminus \mathcal{M}_H$ there exists a sequence $(\mu_n)_{n\geq 1} \subset \mathcal{M}_H$ such that $\mu_n \to \mu$ (in the weak* topology) and $h_{\mu_n}(f) \to h_\mu(f)$ as $n \to \infty$.

We note that Property H is an assumption only on the system and not on the potential.

Theorem 4 (Hyperbolic variational principle): *If Property H holds, then for every continuous function $\phi\colon M \to \mathbb{R}$,*

$$P(\phi) = \sup\left\{h_\mu(f) + \int_M \phi\, d\mu : \mu \in \mathcal{M}_H\right\}. \tag{6}$$

Proof. By the variational principle it is clear that

$$P(\phi) \geq \sup\left\{h_\mu(f) + \int_M \phi\, d\mu : \mu \in \mathcal{M}_H\right\}.$$

To prove the reverse inequality, let $\mu \in \mathcal{M}_E \setminus \mathcal{M}_H$. By Property H there exists a sequence $(\mu_n)_{n\geq 1} \subset \mathcal{M}_H$ such that $\mu_n \to \mu$ and $h_{\mu_n}(f) \to h_\mu(f)$ as $n \to \infty$. Therefore,

$$h_{\mu_n}(f) + \int_M \phi\, d\mu_n \to h_\mu(f) + \int_M \phi\, d\mu.$$

Hence, by (5),

$$P(\phi) \leq \sup\left\{h_\mu(f) + \int_M \phi\, d\mu : \mu \in \mathcal{M}_H\right\}.$$

This completes the proof. ∎

Let now $f\colon M \to M$ be a $C^{1+\varepsilon}$ diffeomorphism of the compact manifold M. We say that a measure $\mu \in \mathcal{M}$ is **hyperbolic** if μ-almost every point $x \in M$ has nonzero Lyapunov exponents, i.e., for μ-almost every $x \in M$ and every nonzero $v \in T_xM$,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|d_x f^n v\| \neq 0.$$

When $\mathcal{M}_H = \{\mu \in \mathcal{M} : \mu \text{ is hyperbolic}\}$, i.e., the set of hyperbolic invariant measures, Property H essentially means that the nonhyperbolic parts of the dynamics can be arbitrarily approximated by their hyperbolic parts. We now show that for $C^{1+\varepsilon}$ diffeomorphisms satisfying Property H when $\mathcal{M}_H$ is the set of hyperbolic measures, the topological pressure can be approximated by the topological pressure on hyperbolic horseshoes.

THEOREM 5 (Approximation property): *Let $f\colon M \to M$ be a $C^{1+\varepsilon}$ diffeomorphism of a compact manifold. If Property H holds with respect to the set $\mathcal{M}_H$ of hyperbolic f-invariant probability measures on M, then*

$$P(\phi) = \sup\{P_\Lambda(\phi) : \Lambda \subset M \text{ is a hyperbolic horseshoe}\},$$

where P_Λ is the topological pressure computed with respect to $f|\Lambda$.

Proof. Since Property H holds, by Theorem 4 we have the identity in (6). In [12], Katok proved that for each hyperbolic measure $\mu \in \mathcal{M}_H$ there exists a sequence of invariant measures μ_n supported on hyperbolic horseshoes Λ_n such that

$$h_{\mu_n}(f) \to h_\mu(f) \quad \text{and} \quad \int_M \phi\, d\mu_n \to \int_M \phi\, d\mu$$

as $n \to \infty$. Therefore, the classical variational principle in (5) implies that

$$h_\mu(f) + \int_M \phi\, d\mu \le \sup_\Lambda \left\{ h_\nu(f) + \int_M \phi\, d\nu : \operatorname{supp}\nu \subset \Lambda \right\} = \sup_\Lambda P_\Lambda(\phi),$$

where the supremum is taken over all hyperbolic horseshoes $\Lambda \subset M$. Here $\operatorname{supp}\nu$ denotes the support of the measure. It follows from (6) that $P(\phi) \le \sup_\Lambda P_\Lambda(\phi)$. On the other hand, it is clear that $P_\Lambda(\phi) \le P(\phi)$, and we obtain the desired identity. ∎

As the following example shows, Property H is essential in Theorem 4. Let $f\colon M \to M$ be a $C^{1+\varepsilon}$ diffeomorphism of a sphere. We assume that there exists an f-invariant set $\Lambda \subset M$ topologically conjugated to the full-shift on two symbols. We also assume that for some open set $U \subset M$ containing Λ there is a parabolic fixed point $p \notin U$ which is the only f-invariant set that intersects $M \setminus U$. Let $\phi\colon M \to \mathbb{R}$ be a Hölder continuous function such that

$$\phi|\Lambda = -\log 2, \quad \phi(p) = 1, \quad \text{and} \quad \phi|(M \setminus \{p\}) < 1.$$

Let again $\mathcal{M}_H$ be the set of hyperbolic f-invariant probability measures on M. Note that if $\mu \in \mathcal{M}_H$ then $\operatorname{supp}\mu \subset \Lambda$. Moreover, $h_{top}(f) = \log 2$, and thus

$h_\mu(f) \le \log 2$ for every $\mu \in \mathcal{M}$. This implies that

$$\sup\left\{h_\mu(f) + \int_M \phi\,d\mu : \mu \in \mathcal{M}_H\right\} \le \log 2 - \log 2 = 0.$$

On the other hand, for the atomic measure $\delta_p \in \mathcal{M}$ supported at p we have

$$h_{\delta_p}(f) + \int_M \phi\,d\delta_p = 1.$$

Therefore,

$$\begin{aligned} P(\phi) &= \sup\left\{h_\mu(f) + \int_M \phi\,d\mu : \mu \in \mathcal{M}\right\} \ge 1 > 0 \\ &\ge \sup\left\{h_\mu(f) + \int_M \phi\,d\mu : \mu \in \mathcal{M}_H\right\}. \end{aligned}$$

This shows that the hyperbolic variational principle in (5) does not hold for the diffeomorphism f. On the other hand, the set of hyperbolic measures is not dense in $\mathcal{M}$. Indeed, for example, for every sequence $(\mu_n)_{n\ge 1} \subset \mathcal{M}_H$,

$$\limsup_{n\to\infty} \int_M \phi\,d\mu_n \le -\log 2 < 1 = \int_M \phi\,d\delta_p.$$

In particular, Property H does not hold.

4. Symbolic models

4.1. The model and its thermodynamic formalism. Let $f\colon M \to M$ be a $C^{1+\varepsilon}$ transformation of the smooth manifold M. We assume that there is a compact f-invariant set $X \subset M$ such that:

1. $\|d_x f\| > 1$ for every $x \in X \setminus \{p\}$ and $f(p) = p$;
2. there is a topological semiconjugacy $g\colon \Sigma_2^+ \to X$ between the two dynamics (Σ_2^+, σ) and (f, X).

Consider the *coding* map

$$\chi\colon \Sigma_R \to X \setminus \bigcup_{i=0}^{\infty} f^{-i}(p)$$

defined by $\chi = g|\Sigma_R$. We say that a continuous function $\phi\colon X \to \mathbb{R}$ is in the class $\mathcal{R}_X$ if the composition $\varphi := \phi\circ\chi\colon \Sigma_R \to \mathbb{R}$ is in the class $\mathcal{R}$ (see Section 2.2 for the definition).

We note that since ϕ is bounded, we have $-\varphi \in \mathcal{R}$. In view of Proposition 3, there is at most one critical value $q_c \in \mathbb{R}$ at which the pressure function undergoes a phase transition. For simplicity of the exposition, *we always assume in what follows that $q_c^- = -\infty$, or equivalently that $q_c^+ = q_c$.* We remark that all statements and results in the rest of the paper are valid, with obvious modifications, under the alternative assumption $q_c^+ = +\infty$.

Applying Proposition 2 we can describe the thermodynamic formalism of potentials in the class $\mathcal{R}_X$.

THEOREM 6: *For each $\phi \in \mathcal{R}_X$, there exists $q_c \in (0, \infty]$ such that:*

1. *The pressure function $q \mapsto P(q\phi)$ is strictly convex and real analytic in $(-\infty, q_c)$. Moreover, for each $q \in (-\infty, q_c)$ there exists a unique equilibrium measure μ_q for $q\phi$ (different from the atomic measure δ_p supported at p).*
2. *The pressure function is linear in (q_c, ∞), with $P(q\phi) = Mq$, and δ_p is the equilibrium measure for every $q\phi$ when $q > q_c$.*

Proof. Let $\mathcal{M}_X$ be the set of f-invariant probability measures on X. Note that $m \mapsto m \circ \chi^{-1}$ is a bijection between the sets $\mathcal{M}_R$ (see Section 2.1) and $\mathcal{M}_X \setminus \{\delta_p\}$. Thus, the statement in the theorem will follow from Proposition 3 after showing that

$$P(\phi) = \sup \left\{ h_\mu(f) + \int_X \phi \, d\mu : \mu \in \mathcal{M}_X \setminus \{\delta_p\} \right\}. \tag{7}$$

Notice that this is the identity in (6) when $\mathcal{M}_H = \mathcal{M}_X \setminus \{\delta_p\}$. Thus, in view of Theorem 4, to establish (7) it is sufficient to show that Property H in Section 3 holds for this set of measures.

For each $n \geq 1$, let $p_n = \overline{0n(n-1)\cdots 1} \in \Sigma_R$ be the periodic point of period $n+1$ in the cylinder set C_0. Let also $x_n = \chi(p_n) \in X$ and define the measure

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x_n},$$

where $\delta_{f^k x_n}$ is the atomic measure supported at $f^k x_n$. We note that $\mu_n \in \mathcal{M}_H = \mathcal{M}_X \setminus \{\delta_p\}$ for each n. Moreover, if $1_{\chi(C_m)}$ denotes the characteristic function of the set $\chi(C_m)$, then for each $m \in S$ and each sufficiently large $n \in S$ we have

$$\int_X 1_{\chi(C_m)} d\mu_n = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

Therefore, any accumulation point ν of the sequence of measures $(\mu_n)_{n \geq 1}$ is such that $\nu(\chi(C_m)) = 0$ for every $m \in S$. Since the sets $\chi(C_m)$ form a cover of

$X \setminus \{p\}$ we have that $\nu(X \setminus \{p\}) = 0$. Hence, $\nu \notin \mathcal{M}_X \setminus \{\delta_p\} = \mathcal{M}_H$, and the sequence $(\mu_n)_{n\geq 1}$ does not converge in $\mathcal{M}_H$. But since $\mathcal{M}_X$ is compact (recall that the space X is compact) we have that

$$\lim_{n\to\infty} \mu_n = \delta_p.$$

Moreover, since $h_{\mu_n}(f) = h_{\delta_p}(f) = 0$ for every $n \geq 1$, Property H holds and we can apply Theorem 4 to obtain equality (7). By the bijection between the spaces $\mathcal{M}_R$ and $\mathcal{M}_X \setminus \{\delta_p\}$, it follows from Proposition 1 that the functions P_G and P coincide, i.e., that $P_G(\phi \circ \chi) = P(\phi)$ for every $\phi \in \mathcal{R}_X$. The desired statement is now an immediate consequence of Proposition 3. ∎

We also discuss the differentiability of the function $q \mapsto P(q\phi)$ at $q = q_c$.

Theorem 7: *Let $q_c < \infty$. The function $q \mapsto P(q\phi)$ is differentiable at q_c if and only if δ_p is the only equilibrium measure for $q_c\phi$.*

Proof. By the continuity of the pressure function we have $P(q_c\phi) = Mq_c$, and thus (see Theorem 6) δ_p is an equilibrium measure for $q_c\phi$. Let us assume that there exists a measure $\mu \in \mathcal{M}_X$ with entropy $h_\mu(f) > 0$, which is also an equilibrium measure for $q_c\phi$. We note that if ν were an equilibrium measure for $q_c\phi$, different from δ_p but with zero entropy, then there would exist an optimal measure for the symbolic representation of ϕ in the renewal shift. But this would contradict Theorem 1.1 in [10]. By Theorem 6, we have

$$q_c M = q_c \int_X \phi \, d\delta_p = P(q_c\phi) = h_\mu(f) + q_c \int_X \phi \, d\mu.$$

Thus, since $q_c > 0$ we obtain

$$M = \int_X \phi \, d\delta_p > \int_X \phi \, d\mu.$$

Furthermore, since for $q > q_c$ the pressure function is linear we have

$$\lim_{q\to q_c^+} \frac{\partial P(q\phi)}{\partial q} = M.$$

Take now $q < q_c$. We obtain

$$P(q_c\phi) - P(q\phi) \leq h_\mu(f) + q_c \int_X \phi \, d\mu - h_\mu(f) - q \int_X \phi \, d\mu,$$

and hence

$$\frac{P(q_c\phi) - P(q\phi)}{q_c - q} \leq \int_X \phi \, d\mu < M.$$

Therefore,

$$\lim_{q\to q_c^-} \frac{\partial P(q\phi)}{\partial q} < M = \lim_{q\to q_c^+} \frac{\partial P(q\phi)}{\partial q},$$

and the pressure function is not differentiable at q_c.

Now we assume that $q \mapsto P(q\phi)$ is not differentiable at q_c. We will produce an equilibrium measure for $q_c\phi$ with positive entropy. Since for $q \in (0, q_c)$ the map is differentiable (it is analytic), this is equivalent to

$$\lim_{q\to q_c^-} \frac{\partial P(q\phi)}{\partial q} < M.$$

By the formula for the derivative of the pressure (which we can use in the interval $(0, q_c)$), this is the same as

$$\lim_{q\to q_c^-} \frac{\partial P(q\phi)}{\partial q} = \lim_{q\to q_c^-} \int_X \phi\, d\mu_q < M, \tag{8}$$

where μ_q is the equilibrium measure for $q\phi$. Let μ be any accumulation point of the family $\{\mu_q\}_{q\geq 0}$ when $q \to q_c^-$. Since the entropy map $\nu \mapsto h_\nu(f)$ is upper semi-continuous (notice that the map f is expansive on $X \setminus \{p\}$) we have that

$$\lim_{q\to q_c^-} \left(h_{\mu_q}(f) + q \int_X \phi\, d\mu_q \right) \leq h_\mu(f) + q_c \int_X \phi\, d\mu. \tag{9}$$

Furthermore, since the pressure function is continuous,

$$\lim_{q\to q_c^-} \left(h_{\mu_q}(f) + q \int_X \phi\, d\mu_q \right) = \lim_{q\to q_c^-} P(q\phi) = P(q_c\phi).$$

Combined with (9) this shows that μ is an equilibrium measure for $q_c\phi$, and

$$P(q_c\phi) = Mq_c = h_\mu(f) + q_c \int_X \phi\, d\mu. \tag{10}$$

By (8) we have that

$$\int_X \phi\, d\mu = \lim_{q\to q_c^-} \int_X \phi\, d\mu_q < M,$$

which together with (10) implies that $h_\mu(f) > 0$. ∎

We note that the statement in Theorem 7 was proved by Urbański in [29] for some Manneville–Pomeau type maps with the potential $-t\log|f'|$. Some of our arguments follow his proof.

COROLLARY 8: *Let $q_c < \infty$. If there exists an equilibrium measure μ for $q_c\phi$ with positive entropy, then*

$$A := \int_X \phi\, d\mu = \lim_{q \to q_c^-} \frac{\partial P(q\phi)}{\partial q} < M.$$

Proof. By the proof of Theorem 7, the measure μ is an accumulation point of the family $\{\mu_q\}_{q \geq 0}$ when $q \to q_c^-$, where μ_q is the equilibrium measure for $q\phi$. It follows from (8) that

$$\lim_{q \to q_c^-} \frac{\partial P(q\phi)}{\partial q} = \lim_{q \to q_c^-} \int_X \phi\, d\mu_q = \int_X \phi\, d\mu < M,$$

as desired. ∎

4.2. MULTIFRACTAL ANALYSIS. Let $f \colon X \to X$ be as in Section 4.1. We now discuss the multifractal analysis of the Birkhoff averages of a function $\phi \in \mathcal{R}_X$. We recall that we are assuming that $q_c^- = -\infty$ (see Section 4.1). For each $\alpha \in \mathbb{R}$ we consider the level set

$$J_\alpha = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) = \alpha \right\},$$

and the *irregular* set

$$J' = \left\{ x \in X : \text{the limit } \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) \text{ does not exist} \right\}. \tag{11}$$

The associated **multifractal decomposition** is the disjoint union

$$X = \left(\bigcup_{\alpha \in \mathbb{R}} J_\alpha \right) \cup J'.$$

The **entropy spectrum** of the function ϕ is defined by

$$\mathcal{E}(\alpha) = h(f|J_\alpha), \tag{12}$$

where $h(f|Z)$ denotes the topological entropy of f on the set Z (we note that Z need not be compact nor invariant; see, for example, [17, Chapter 4] for the definition of topological entropy in this general situation). We describe the entropy spectrum in this setting. We recall the constants M in (2) and m in (3) (with R replaced by X).

THEOREM 9 (Multifractal analysis): *For a function $\phi \in \mathcal{R}_X$ with $P(\phi) = 0$ the following properties hold:*

1. *If $q_c = \infty$, then $\mathcal{E}$ is strictly convex and real analytic.*
2. *If $q_c < \infty$, then $\mathcal{E}$ is strictly convex and real analytic on the interval (m, A), and linear with slope q_c on the interval (A, M), where*

$$A := \lim_{q \to q_c^-} \frac{\partial P(q\phi)}{\partial q}.$$

Proof. The proof can be obtained from work of Takens and Verbitskiy combined with Theorem 6. Namely, it is shown in [28, Theorem 6.2] that

$$\mathcal{E}(\alpha) = \inf\{P(q\phi) - q\alpha : q \in \mathbb{R}\},$$

for systems with the specification property (that we have for free because we assume the dynamics to be topologically conjugated to the full-shift), and for which the metric entropy is upper semi-continuous (that is also satisfied in our setting since our maps are expansive). Therefore, the result follows from the description of the function $q \mapsto P(q\phi)$ in Theorem 6. In particular, when $q_c < \infty$ the behavior of $\mathcal{E}$ changes at $q = A$: the result is clear for the interval (m, A) (see [1]), and for $\alpha \in (A, M)$ we prove the following statement.

Lemma 2: *For $q_c < \infty$ and $\alpha \in (A, M)$ we have*

$$\inf\{P(q\phi) - q\alpha : q \in \mathbb{R}\} = P(q_c\phi) - q_c\alpha. \tag{13}$$

Proof. Let $\alpha \in (A, M)$. Assume first that $q > q_c$. We have

$$-q(\alpha - M) > -q_c(\alpha - M), \text{ that is, } qM - q\alpha > q_cM - q_c\alpha.$$

Therefore,

$$P(q\phi) - q\alpha > P(q_c\phi) - q_c\alpha,$$

and

$$\inf\{P(q\phi) - q\alpha : q \geq q_c\} = P(q_c\phi) - q_c\alpha.$$

It remains to prove that

$$\inf\{P(q\phi) - q\alpha : q \leq q_c\} \geq P(q_c\phi) - q_c\alpha.$$

Let $q < q_c$ and note that $P(q_c\phi) - q_c\alpha \leq P(q\phi) - q\alpha$ is equivalent to

$$\frac{P(q_c\phi) - P(q\phi)}{q_c - q} \leq \alpha.$$

Assume by way of contradiction that

$$\beta := \frac{P(q_c\phi) - P(q\phi)}{q_c - q} > \alpha. \tag{14}$$

Since the pressure function $q \mapsto P(q\phi)$ is analytic on (q, q_c), by the mean value theorem there exists $q^* \in (q, q_c)$ such that $\partial P(q^*\phi)/\partial q = \beta$. It follows from (14) that

$$A = \lim_{q \to q_c^-} \frac{\partial P(q\phi)}{\partial q} < \alpha < \beta = \frac{\partial P(q^*\phi)}{\partial q}.$$

On the other hand, the pressure function is convex, and $q \mapsto \partial P(q\phi)/\partial q$ is an increasing function (strictly increasing on the interval $(0, q_c)$), that is,

$$\frac{\partial P(q^*\phi)}{\partial q} < \lim_{q \to q_c^-} \frac{\partial P(q\phi)}{\partial q}.$$

This contradiction establishes the identity in (13). ∎

This completes the proof of the theorem. ∎

We recall that for any (uniformly) hyperbolic dynamical system and any Hölder potential, the entropy spectrum is strictly convex and real analytic. Theorem 9 shows that the behavior of the entropy spectrum can be very different in our setting.

We note that it is possible to have $q_c < \infty$ with $A = M$. This happens when the pressure function $q \mapsto P(q\phi)$ is differentiable (see Theorem 7). In this case the interval $[A, M]$ is degenerate and consists of a single point. On the other hand, if the pressure function is not differentiable at q_c, then the interval (A, M) is nondegenerate.

We now consider the irregular set J' in (11).

Theorem 10: *The set J' has full topological entropy, i.e., $h(f|J') = h(f)$.*

Proof. For each $n \in \mathbb{N}$, set

$$\Delta_n = \{(x_i)_{i \geq 0} \in \Sigma_R : x_i \in \{0, \ldots, n\}\}.$$

Then $\sigma|\Delta_n$ is a finite Markov shift. Setting $\Lambda_n = \chi(\Delta_n)$, by the approximation property of the topological entropy (see [26]) we obtain

$$h(f) = \lim_{n \to \infty} h(f|\Lambda_n).$$

It follows from work of Barreira and Schmeling in [3] that each set $J' \cap \Lambda_n$ carries full topological entropy, i.e.,

$$h(f|(J' \cap \Lambda_n)) = h(f|\Lambda_n).$$

Therefore,

$$h(f|J') \geq \lim_{n\to\infty} h(f|(J' \cap \Lambda_n)) = \lim_{n\to\infty} h(f|\Lambda_n) = h(f),$$

and the desired result follows. ∎

A measure $\mu \in \mathcal{M}_X$ is called a **full measure** for the level set J_α if $\mu(J_\alpha) = 1$ and $\mathcal{E}(\alpha) = h_\mu(f)$.

Theorem 11 (Full measures): *Let $\phi \in \mathcal{R}_X$ be such that $P(\phi) = 0$.*

1. *If $q_c = \infty$, then for each $\alpha \in (m, M)$ there exists a full measure for J_α.*
2. *If $q_c < \infty$, then for each $\alpha \in (m, A)$ there exist a full measure for J_α and for each $\alpha \in (A, M)$ there are no full measures for J_α.*

Proof. Assume first that $q_c = \infty$ and $\alpha \in (m, M)$. In this case the result follows from classical techniques in multifractal analysis (see [17]) together with the conditional variational principle established in [28] (with a mistake corrected in [21]), i.e., the identity

$$\mathcal{E}(\alpha) = \sup\left\{h_\mu(f) : \mu \in \mathcal{M}_X \text{ and } \int_X \phi\, d\mu = \alpha\right\}.$$

Indeed, denote by μ_α the equilibrium measure of $q\phi$ where $q \in \mathbb{R}$ is chosen in such a way that $\partial P(q\phi)/\partial q = \alpha$. Then $\mathcal{E}(\alpha) = h_{\mu_\alpha}(f)$ (see [28] and [11]), and since the measure μ_α is ergodic we have $\mu_\alpha(J_\alpha) = 1$. The same argument can be used when $q_c < \infty$ and $\alpha \in (m, A)$.

Assume now that $q_c < \infty$ and $\alpha \in (A, M)$. We have

$$\begin{aligned}\mathcal{E}(\alpha) &= \sup\left\{h_\mu(f) : \mu \in \mathcal{M}_X \text{ and } \int_X \phi\, d\mu = \alpha\right\}\\ &= \inf\left\{P(q\phi) - q\alpha : q \in \mathbb{R}\right\} = P(q_c\phi) - q_c\alpha.\end{aligned}$$

The first two identities follow again from [28]. The last identity was established in Lemma 2.

Assume now that there exists a measure $\mu_\alpha \in \mathcal{M}_X$ such that $\mathcal{E}(\alpha) = h_{\mu_\alpha}(f)$ and $\int_X \phi\, d\mu_\alpha = \alpha$. Then

$$h_{\mu_\alpha}(f) + q_c \int_X \phi\, d\mu_\alpha = P(q_c\phi) - q_c\alpha + q_c\alpha = P(q_c\phi),$$

and μ_α is an equilibrium measure for $q_c\phi$. Note that $q_c\phi$ has at most two ergodic equilibrium measures: δ_p and possibly another measure, say μ, corresponding to the projection of the (unique) equilibrium measure of $q_c(\phi\circ\chi)$ on the renewal shift (note that it might happen that in the renewal shift there is no equilibrium measure). We have $\int_X \phi\, d\delta_p = M$ (see Theorem 6) and $\int_X \phi\, d\mu = A$ (see Corollary 8). This implies that $\mu_\alpha = c\delta_p + (1-c)\mu$ for some $c \in (0,1)$. Therefore, $\mu_\alpha(J_\alpha) = 0$, and μ_α is not a full measure. This means that there are no full measures for J_α. ∎

4.3. Ergodic optimization. We continue to consider $f\colon X \to X$ as above, and let $\phi \in \mathcal{R}_X$ with $q_c^- = -\infty$. Set

$$\alpha(\phi) = \sup\left\{ \int_X \phi\, d\mu : \mu \in \mathcal{M}_X \right\}.$$

A measure $m \in \mathcal{M}_X$ is said to be ϕ-**optimal** if $\int_X \phi\, dm = \alpha(\phi)$. A basic problem in ergodic optimization is to prove the existence of optimal measures and to describe their properties. This study has been carried out for Markov shifts of renewal type in [10]. Applying these results to our setting we obtain the following.

Theorem 12 (Ergodic optimization): *Let $\phi \in \mathcal{R}_X$.*

1. *If $q_c = \infty$, then any accumulation point of $\{\mu_q\}_{q \geq 0}$ as $q \to \infty$, where μ_q denotes the equilibrium measure for $q\phi$, is a ϕ-optimal measure.*
2. *If $q_c < \infty$, then $\alpha(\phi) = M$, where $q \mapsto P(q\phi) = Mq$ for $q > q_c$. Moreover, δ_p is a ϕ-optimal measure.*

Proof. Note that the space $\mathcal{M}_X$ is compact and hence every function $\phi \in \mathcal{R}_X$ has optimal measures. The first statement follows directly from Theorem 1.1 in [10] and from the symbolic model for f in terms of the renewal shift.

For the second statement recall that there exists a bijection between the spaces $\mathcal{M}_R$ and $\mathcal{M}_X \setminus \{\delta_p\}$. Furthermore, for the symbolic representation of $f\colon X \setminus \{p\} \to X \setminus \{p\}$ there are no ϕ-optimal measures (see [10]). Together with the fact that $\mathcal{M}_X$ is compact, this readily implies the desired result. ∎

5. Parabolic horseshoes

In this section we study *parabolic horseshoes* and show that the corresponding versions of Theorems 6, 9 and 12, respectively concerning the thermodynamic formalism, multifractal analysis, and ergodic optimization, hold for this class of dynamical systems. The statements are obtained directly from the above theorems after an appropriate preparation.

Let $S \subset \mathbb{R}^2$ be a closed topological disk with smooth boundary and let $f\colon S \to \mathbb{R}^2$ be a $C^{1+\varepsilon}$ diffeomorphism. We assume that:

1. $f(S) \cap S$ consists of two disjoint topological disks R_0 and R_1;
2. $\overline{f(S) \setminus S}$ consists of three disjoint topological disks R_2, R_3, R_4.

We also assume that there exist one-dimensional transverse foliations W^u and W^s of $f(S) \cup S$ with connected smooth leaves such that:

3. if $x \in R_i$ then the sets $W^u(x) \cap R_i$ and $W^s(x) \cap R_i$ are connected;
4. for each $x, y \in R_i$ the set $W^u(x) \cap W^s(y)$ is a singleton, denoted $[x, y]$;
5. for every $x \in R_i$ the map $(W^s(x) \cap R_i) \times (W^u(x) \cap R_i) \to R_i$ defined by $(y, z) \mapsto [y, z]$ is a homeomorphism;
6. for every $x \in R_i$ we have

$$f(W^s(x) \cap R_i) \subset W^s(fx) \quad \text{and} \quad f^{-1}(W^u(x) \cap R_i) \subset W^u(f^{-1}x).$$

We denote by $d^u_x f$ and $d^s_x f$, respectively, the derivatives

$$d^u_x f\colon T_x W^u(x) \to T_{fx} W^u(fx) \quad \text{and} \quad d^s_x f\colon T_x W^s(x) \to T_{fx} W^s(fx),$$

and we assume that there exists a fixed point $p \in S$ such that:

7. $|d^s_x f| \le 1$ for $x \in S$, and $|d^s_x f| < 1$ for $x \in S \setminus W^u(p)$;
8. $|d^u_x f| \ge 1$ for $x \in S$, and $|d^u_x f| > 1$ for $x \in S \setminus W^s(p)$.

We note that Urbański and Wolf in [30] considered a related class of parabolic horseshoes although with very different purposes in mind. Conditions 1–6 are the same as those considered by them, while our Conditions 7–8 are more general. More precisely, we allow the fixed point p to have derivative equal to 1 simultaneously in the stable and unstable directions, and we make no assumption on the type of parabolic point: it is required in [30] that

$$f^{-1}(x) = x - \operatorname{sgn}(x) a |x|^{c+1} + o(|x|^{c+1})$$

for some constants $a, c > 0$, with x in some parametrization of $W^u(p)$.

Let X be the maximal invariant set of f contained in S. Note that the *parabolic horseshoe* (f, X) can be coded by a two-sided full-shift on two symbols (Σ_2, σ). Indeed, if $(x_n)_{n\in\mathbb{Z}} \in \Sigma_2$ then the set

$$\bigcap_{n\in\mathbb{Z}} f^{-n}(R_{x_n})$$

is a singleton. We now translate the problems for the two-sided full-shift into corresponding ones for the one-sided full-shift. Fortunately, there exists a standard procedure for this (see for example [1, Appendix A]). Two continuous functions $\phi, \gamma \in C(\Sigma_2)$ are said to be **cohomologous** if there exists a continuous function $\psi \in C(\Sigma_2)$ such that $\phi = \gamma + \psi \circ \sigma - \psi$. The following statement is due to Sinai (see [16, Proposition 1.2]) for Hölder continuous functions and to Coelho and Quas [7] for functions of summable variation.

Proposition 13: *If $\phi \in C(\Sigma_2)$ has summable variation, then there exists $\gamma \in C(\Sigma_2)$ cohomologous to ϕ such that $\gamma(x) = \gamma(y)$ whenever $x_i = y_i$ for all $i \geq 0$ (that is, γ depends only on the future coordinates).*

Furthermore, if the function ϕ has summable variation, then the same happens with γ. We note that γ can be canonically identified with a function $\varphi\colon \Sigma_2^+ \to \mathbb{R}$, and $P_{\Sigma_2}(\phi) = P_{\Sigma_2^+}(\varphi)$. Therefore, the results obtained for the renewal shift can be translated to the two-sided shift, and thus also to the parabolic horseshoes. Indeed, the class of potentials that we consider are the functions in $C(\Lambda)$ whose lift $\phi \in C(\Sigma_2)$ to the two-sided full-shift on two symbols is cohomologous to a function in $\mathcal{R}_X$. We note that from the above discussion the regularity assumptions are the same as those in the one-sided situation. For completeness we formulate the statements. Let $f\colon X \to X$ be a parabolic horseshoe as above

Theorem 14 (Thermodynamic formalism): *For each function $\phi \in \mathcal{R}_X$ there exists $q_c \in (0, \infty]$ such that:*

1. *The pressure function $q \mapsto P(q\phi)$ is strictly convex and real analytic on $(-\infty, q_c)$. Moreover, for each $q \in (-\infty, q_c)$ there exists a unique equilibrium measure μ_q for $q\phi$ (different from the atomic measure δ_p).*
2. *The pressure function is linear on (q_c, ∞), with $P(q\phi) = Mq$. Moreover, δ_p is the equilibrium measure for $q\phi$ for each $q > q_c$.*

Theorem 15 (Multifractal analysis): *Let $\phi \in \mathcal{R}_X$ be such that $P(\phi) = 0$.*

1. *If $q_c = \infty$, then $\mathcal{E}$ is strictly convex and real analytic.*
2. *If $q_c < \infty$, then $\mathcal{E}$ is strictly convex and real analytic on the interval (m, A), and linear with slope q_c on the interval (A, M), where $A = \lim_{q \to q_c^-} \partial P(q\phi)/\partial q$.*

Theorem 16 (Ergodic optimization): *Let $\phi \in \mathcal{R}_X$.*

1. *If* $q_c = \infty$, *then any accumulation point of* $\{\mu_q\}_{q\geq 0}$ *as* $q \to \infty$, *where* μ_q *denotes the equilibrium measure for* $q\phi$, *is a* ϕ*-optimal measure.*
2. *If* $q_c < \infty$, *then* $\alpha(\phi) = M$, *where* $P(q\phi) = Mq$ *for* $q > q_c$. *Moreover,* δ_p *is a* ϕ*-optimal measure.*

6. Further applications

We discuss here other classes of dynamical systems to which the results in Section 2 can also be applied. This also allows us to recover in a unified manner several results scattered in the literature.

6.1. Parabolic expanding maps. The Manneville–Pomeau map [23] is the interval map $T\colon [0,1] \to [0,1]$ defined by $T(x) = x + x^{1+\alpha} \pmod 1$, for some $\alpha > 0$. It has two branches and 0 is a parabolic fixed point. Furthermore, T can be modeled by the full-shift on two symbols, and removing 0 and its preimages the map can be modeled by the renewal shift. The description of the thermodynamic formalism for this map (the corresponding version of Theorem 6), was established by Sarig in [27]. The particular case when $\phi = -\log|f'|$ was considered earlier in [14, 24].

Theorem 17: *For the Manneville–Pomeau map, let* $\phi(x) = -\log|f'(x)|$. *Then* $q_c = 1$ *and:*

1. *For* $q \in (0,1)$ *the pressure function* $q \mapsto P(q\phi)$ *is strictly convex and real analytic, and there exists a unique equilibrium measure* μ_q *for* $q\phi$ *(different from the atomic measure* δ_0 *supported at* 0*).*
2. *For* $q > 1$ *the pressure function is identically zero, and* δ_0 *is the equilibrium measure for every* $q\phi$.

The multifractal analysis of the Lyapunov exponents (which corresponds to Theorem 9 for the potential $\phi = -\log|f'|$) was considered by Pollicott and Weiss [22] and later by Takens and Verbitskiy [28]. For each $\alpha \in \mathbb{R}$, set

$$J_\alpha = \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(f^i x)| = \alpha \right\},$$

and consider the entropy spectrum $\mathcal{E}$ in (12). We also set

$$A = \lim_{q\to 1^-} \frac{\partial P(-q\log|f'|)}{\partial q}.$$

The following is a combination of work in [22] and [28].

Theorem 18: *For the Manneville–Pomeau map, the entropy spectrum $\mathcal{E}$ is strictly convex and real analytic on the interval (m, A), and linear with slope -1 on the interval $(A, 0)$.*

We emphasize that Theorems 17 and 18 are also consequences of our work (by obtaining corresponding versions respectively of Theorems 6 and 9 in the present setting, simply by repeating the proofs of these theorems). On the other hand, the corresponding version of Theorem 12 concerning the ergodic optimization problem is new.

6.2. Uniformly expanding maps. Set $I = [0, 1]$ and let I^1, I^2 be closed intervals such that $I = I^1 \cup I^2$ and $\operatorname{int} I^1 \cap \operatorname{int} I^2 = \emptyset$. In [20], Pesin and Zhang considered a map $f\colon I \to I$ such that $f\colon I^i \to I$ is a C^1 diffeomorphism with $|f'| > 1$, for $i \in \{1, 2\}$. In particular, they proved a version of Theorem 6. In this setting, the corresponding version of Theorem 9 is new, although it also follows from work in [28]. The ergodic optimization problem for this type of dynamics has been studied, for example, by Bousch [4] and Jenkinson [11]. Nevertheless, the class of potentials that we consider is larger than the one considered by them. In particular, for the class $\mathcal{R}_X$ the ergodic optimization result corresponding to Theorem 12 is new.

6.3. Hyperbolic horseshoes. Let $S \subset \mathbb{R}^2$ be a closed topological disk with smooth boundary and let $f\colon S \to \mathbb{R}^2$ be a $C^{1+\varepsilon}$ diffeomorphism. We assume that f satisfies Conditions 1–6 for the parabolic horseshoes (see Section 5), and we replace Conditions 7–8 by the following:

7. there exists $\lambda < 1$ such that $|d_x^s f| \le \lambda$ for every $x \in S$;
8. there exists $\mu > 1$ such that $|d_x^u f| \ge \mu$ for every $x \in S$.

Let X be the maximal invariant set of f. The pair (f, X) is called a **hyperbolic horseshoe**. Clearly, the dynamics is conjugated to the two-sided full-shift on two symbols. In particular, the statement in Theorem 6 holds for any hyperbolic horseshoe. Recall that the class of potentials considered here is larger than the Hölder class and thus, in particular, uniqueness of equilibrium states together with phase transitions are new phenomena in this setting (as in the case of uniformly expanding maps, the thermodynamic formalism for hyperbolic horseshoes is well-known in the case of Hölder continuous potentials). Furthermore, the statements in Theorems 9 and 12 hold for any hyperbolic horseshoe. The multifractal analysis for hyperbolic horseshoes was studied in [1], but again only for Hölder continuous potentials.

7. Dimension spectra for hyperbolic horseshoes

This section is dedicated to the study of dimension spectra for hyperbolic horseshoes. We consider dimension spectra of equilibrium measures of a class of potentials that are not necessarily regular.

Let again $f\colon S \to \mathbb{R}^2$ be a $C^{1+\varepsilon}$ diffeomorphism on a closed topological disk $S \subset \mathbb{R}^2$ with smooth boundary, and consider a fixed point $p \in S$. Let also $X \subset S$ be a hyperbolic horseshoe (see Section 6.3 for the definition).

Let now μ be a Borel probability measure on X. The **pointwise dimension** of μ at the point $x \in X$ is defined by

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

whenever the limit exists, where $B(x,r)$ denotes the ball of radius r centered at x. It was shown in [2] that if μ is a hyperbolic f-invariant measure, then the pointwise dimension exists μ-almost everywhere. For each $\alpha \in \mathbb{R}$ we consider the level set

$$K_\alpha = \{x \in X : d_\mu(x) = \alpha\},$$

and the **irregular set**

$$K' = \left\{ x \in X : \text{the limit } \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ does not exist} \right\}.$$

The **dimension spectrum** of μ is defined by

$$D_\mu(\alpha) = \dim_H K_\alpha,$$

where $\dim_H$ denotes the Hausdorff dimension. We shall consider equilibrium measures of certain potentials in $\mathcal{R}_X$. Namely, we call the function $\log\phi \in \mathcal{R}_X$ **parabolic** if:

1. $\log\phi \circ \chi$ is weakly Hölder continuous;
2. $\log\phi(p) = 0$ and $\log\phi(x) < 0$ for every $x \in X \setminus \{p\}$;
3. $\log\phi$ has a unique equilibrium measure μ with positive entropy;
4. $q \mapsto P(q\log\phi)$ is C^1 in $(-\infty, 1)$;

5. $P(q\log\phi)=0$ for each $q\geq 1$.

The following is a multifractal analysis of the dimension spectra for equilibrium measures of parabolic functions.

Theorem 19: *If $\log\phi\in\mathcal{R}_X$ is parabolic, then there exist $B,U>0$ with $B<U$ such that:*

1. *For each $\alpha\in(0,B)$ we have $D_\mu(\alpha)=\alpha$, and there are no full measures for K_α.*
2. *The dimension spectrum is C^1 in (B,U), and for each $\alpha\in(B,U)$ there is a (noninvariant) full measure for K_α.*
3. *The irregular set has full Hausdorff dimension.*

Proof. For each $q\in\mathbb{R}$ we define

$$T^u(q)=\inf\left\{t: P(-t\log|d_x^u f|+q\log\phi)\leq 0\right\},$$
$$T^s(q)=\inf\left\{t: P(t\log|d_x^s f|+q\log\phi)\leq 0\right\}.$$

We set

$$T(q)=T^u(q)+T^s(q).$$

We note that parabolic functions are not Hölder continuous on X, and thus in general $T^u(q)$ and $T^s(q)$ are not the unique numbers such that

$$P(-T^u(q)\log|d_x^u f|+q\log\phi)=P(T^s(q)\log|d_x^s f|+q\log\phi)=0.$$

We also consider the family

$$\mathcal{H}=\{\Lambda\subset X:\Lambda \text{ hyperbolic horseshoe with } \log\phi|\Lambda \text{ Hölder continuous}\}.$$

It is well-known in the theory of multifractal analysis (see for example [17] for details) that for each $\Lambda\in\mathcal{H}$ and $q\in\mathbb{R}$ there is a unique number $T_\Lambda^u(q)$ such that

$$P_\Lambda(-T_\Lambda^u(q)\log|d_x^u f|+q\log\phi)=0,$$

where P_Λ denotes the topological pressure computed with respect to Λ.

Lemma 3: *For each $q\in\mathbb{R}$ we have*

$$T^u(q)=\sup\left\{T_\Lambda^u(q):\Lambda\in\mathcal{H}\right\}\quad \textit{and}\quad T^s(q)=\sup\left\{T_\Lambda^s(q):\Lambda\in\mathcal{H}\right\}.$$

Proof. Since $\bigcup_{\Lambda\in\mathcal{H}}\Lambda=X\setminus\{p\}$, we can show that for every $q,t\in\mathbb{R}$,

$$P(-t\log|d_x^u f|+q\log\phi)=\sup\left\{P_\Lambda(-t\log|d_x^u f|+q\log\phi):\Lambda\in\mathcal{H}\right\}. \tag{15}$$

This is a consequence of Theorem 5 with the family of measures

$$\mathcal{M}_H = \{\mu \in \mathcal{M} : \operatorname{supp} \mu \subset \Lambda \text{ for some } \Lambda \in \mathcal{H}\},$$

proceeding as in the proof of Theorem 6 to show that Property H holds. Therefore,

$$S := \sup \{T^u_\Lambda(q) : \Lambda \in \mathcal{H}\} \le T^u(q).$$

We claim that equality holds. Assume on the contrary that $S < T^u(q)$ and let $a \in (S, T^u(q))$. Since the function

$$t \mapsto P(-t \log |d^u_x f| + q \log \phi)$$

is decreasing we find that

$$P(-a \log |d^u_x f| + q \log \phi) > 0.$$

On the other hand, for every $\Lambda \in \mathcal{H}$ we have

$$P_\Lambda(-a \log |d^u_x f| + q \log \phi) < 0.$$

This contradicts (15), and thus $S = T^u(q)$. A similar argument establishes the identity for $T^s(q)$. ∎

Lemma 4: *The functions $T^u(q)$ and $T^s(q)$ are convex and decreasing.*

Proof. It was shown by Pesin and Weiss in [19] that for $\Lambda \in \mathcal{H}$ the functions $q \mapsto T^u_\Lambda(q)$ and $q \mapsto T^s_\Lambda(q)$ are real analytic, strictly decreasing, and strictly convex. The desired result thus follows immediately from Lemma 3. ∎

Lemma 5: *For $q \ge 1$ we have $T^u(q) = 0$.*

Proof. Fix $q > 1$. Since $P(q \log \phi) = 0$ we have

$$T^u(q) = \inf \{t : P(-t \log |d^u_x f| + q \log \phi) \le 0\} \le 0.$$

Assume by way of contradiction that $T^u(q) < 0$. This implies the existence of $t \in (T^u(q), 0)$ such that

$$P(-t \log |d^u_x f| + q \log \phi) = 0. \tag{16}$$

Since $P(q \log \phi) = 0$ and the function

$$t \mapsto P(-t \log |d^u_x f| + q \log \phi)$$

is decreasing and convex, we conclude that in fact the identity (16) holds for every $t \geq T^u(q)$. Let now $t^* > t_u$, where t_u is the unique root of the equation $P(-t\log|d_x^u f|) = 0$. Since $P(\chi+\psi) \leq P(\chi) + P(\psi)$, we obtain

$$\begin{aligned} P(-t^*\log|d_x^u f| + q\log\phi) &\leq P(-t^*\log|d_x^u f|) + P(q\log\phi) \\ &= P(-t^*\log|d_x^u f|) < 0. \end{aligned}$$

This contradiction establishes the desired statement. ∎

A similar result holds for $T^s(q)$. Thus, for every $q \geq 1$ we have

$$T(q) = T^u(q) + T^s(q) = 0.$$

We now consider the function

$$Q_q(t) = P(-t\log|d_x^u f| + q\log\phi).$$

Lemma 6: *For $q \leq 1$, the function $T^u(q)$ is strictly decreasing, of class C^1, and there is a unique nonatomic equilibrium measure for*

$$-T^u(q)\log|d_x^u f| + q\log\phi.$$

Proof. We note that the function $T^u(q)$ is convex, with $T^u(0) > 0$ and $T^u(1) = 0$. Therefore, $T^u(q)$ can fail to be strictly decreasing for $q < 1$ only in an interval of the form $(q^*, 1)$ for some $q^* > 0$. If this happens, then for every $q \in (q^*, 1)$ we have

$$T^u(q) = \inf\{t : Q_q(t) = 0\} = 0.$$

Furthermore, it follows from the hypotheses on ϕ that $A_q := P(q\log\phi) > 0$ for $q \in (q^*, 1)$. This implies that if $q \in (q^*, 1)$, then $Q_q(t) \in (A_q, +\infty)$ for $t \leq 0$, and $Q_q(t) \leq 0$ for $t > 0$. But then the pressure function Q_q would be neither convex nor continuous. This contradiction shows that $T^u(q)$ is strictly decreasing for $q \leq 1$.

To show that there is a unique root of the equation $Q_q(t) = 0$, which then must be $T^u(q)$, we note that

$$Q_q(t) \leq P(-t\log|d_x^u f|) + P(q\log\phi).$$

Since $P(-t\log|d_x^u f|) \to -\infty$ as $t \to +\infty$, there exists $t_1 > 0$ such that $Q_q(t_1) < 0$. Furthermore, since $Q_q(0) > 0$, together with the fact that the pressure is decreasing and convex this implies the existence of a unique root.

Let now

$$t_c^q = \inf\{t : (-t\log|d_x^u f| + q\log\phi)\circ\chi \text{ has no equilibrium measure}\}.$$

The nonexistence of an equilibrium measure includes both the transient case (with no conformal measure), and the null recurrent case (with an infinite "equilibrium measure"). Note that $t_c^q > 0$. Indeed, when $t = 0$ the potential $q \log \phi \circ \chi$ is "well-behaved" for $q < 1$. On the other hand, if $t_1 > 0$ is such that $(-t_1 \log |d_x^u f| + q \log \phi) \circ \chi$ has no equilibrium measure, then

$$\begin{aligned} Q_q(t_1) &= P(-t_1 \log |d_x^u f| + q \log) \\ &= -t_1 \int_X \log |d_x^u f| \, d\delta_p + q \int_X \log \phi \, d\delta_p \\ &= -t_1 \log |d_p^u f| = -t_1 M < 0. \end{aligned}$$

Note that the constant M is independent of q. Since $q < 1$ we have $P(q \log \phi) > 0$ (ϕ is a parabolic function), and thus $T^u(q) \in (0, t_c^q)$. In particular, there is a nonatomic equilibrium measure for $-T^u(q) \log |d_x^u f| + q \log \phi$.

To show that the function $T^u(q)$ is of class C^1 we start by considering the function Q_q. Note that Q_q is differentiable in the interval $(0, t_c^q)$. Indeed, assume by way of contradiction that there exists $t_1 \in (0, t_c^q)$ for which

$$d^- := \lim_{t \to t_1^-} Q_q'(t) < \lim_{t \to t_1^+} Q_q'(t) =: d^+. \tag{17}$$

Recall that for each $t \in (0, t_c^q)$ the function $(-t \log |d_x^u f| + q \log \phi) \circ \chi$ defined in Σ_R has a unique equilibrium measure. It follows from (17) that the function $-t_1 \log |d_x^u f| + q \log \phi$ defined in X has two equilibrium measures. One is the projection of the measure for $(-t \log |d_x^u f| + q \log \phi) \circ \chi$, and the other is δ_p (which is the only measure that does not belong to the projection of $\mathcal{M}_R$). But if δ_p is an equilibrium measure for $-t_1 \log |d_x^u f| + q \log \phi$, then $t_c^q < t_1$. This contradiction shows that Q_q is differentiable on $(-\infty, 1)$. This implies that Q_q is in fact of class C^1 (see Theorem 4.2.11 and Remark 4.3.4 in [13]).

Note that if $q_1 < q_2$ then $t_c^{q_2} \le t_c^{q_1}$. This is a consequence of the monotonicity of the pressure together with the identity $Q_q(t) = Mt$ for $t_c^q < t$. Indeed, if $q_1 < q_2$ then $P(q_2 \log \phi) < P(q_1 \log \phi)$, and

$$P(-t \log |d_x^u f| + q_2 \log \phi) \le P(-t \log |d_x^u f| + q_1 \log \phi). \tag{18}$$

If $t \ge \max\{t_c^{q_2}, t_c^{q_1}\}$, then

$$P(-t \log |d_x^u f| + q_2 \log \phi) = P(-t \log |d_x^u f| + q_1 \log \phi) = Mt.$$

The result thus follows immediately from (18). Fix now $q_1 < 1$. Since the function

$$q \mapsto P(-t_c^{q_1} \log |d_x^u f| + q \log \phi)$$

is continuous, there exists $q_2 < q_1$ such that

$$P(-t_c^{q_1} \log |d_x^u f| + q_2 \log \phi) \le 0.$$

Since $t_c^{q_2} \le t_c^{q_1}$ we conclude that $T^u(q_2) \in (0, t_c^{q_1})$, and since the function

$$(t, q) \mapsto P(-t \log |d_x^u f| + q \log \phi)$$

is C^1 for $t \in (0, t_c^{q_1})$ and $q \in (q_2, q_1)$, it follows from the implicit function theorem that $q \mapsto T^u(q)$ is of class C^1. This completes the proof. ∎

Therefore,

$$T(q) = \begin{cases} C^1 \text{ and strictly decreasing} & \text{for } q \le 1, \\ 0 & \text{for } q > 1. \end{cases}$$

Set $B = \lim_{q \to 1^-} T'(q)$. The following is a consequence of the classical theory of multifractal analysis and of the approximation property.

Lemma 7: *Given $\alpha \in (B, 0)$, there exists a sequence $q_n \in \mathbb{R}$ with $q_n \to 1$ such that $-T'_{\Lambda_n}(q_n) = \alpha$, where $\Lambda_n \in \mathcal{H}$ is a hyperbolic horseshoe for each n.*

Let

$$K_{n,\alpha} := \left\{ x \in \Lambda_n : \lim_{m \to \infty} \frac{\sum_{i=0}^{m-1} \log \phi(f^i x)}{\sum_{i=0}^{m-1} \log |d_x^u f^i|} = \alpha \right\} \subset K_\alpha.$$

Lemma 8: *If $\alpha \in (0, B)$ then $\dim_H K_\alpha = \alpha$.*

Proof. With a sequence $q_n \to 1$ as in Lemma 7, standard arguments of multifractal analysis yield

$$\begin{aligned} \lim_{n \to \infty} \dim_H K_{n,\alpha} &= \lim_{n \to \infty} (T_{\Lambda_n}(q_n) + q_n \alpha) \\ &= T(1) + \alpha = \alpha \le \dim_H K_\alpha. \end{aligned}$$

We recall that if ν is a finite Borel measure with lower pointwise dimension $\underline{d}_\nu(x) \le d$, for some $d > 0$ and every $x \in Z$, then $\dim_H Z \le d$ (see for example Theorem 7.2 in [17]). In particular, for every $x \in K_\alpha$ we have $\underline{d}_\mu(x) = d_\mu(x) = \alpha$, and hence

$$D_\mu(\alpha) = \dim_H K_\alpha \le \alpha.$$

This yields the desired result. ∎

Lemma 9: *The dimension spectrum is C^1 in (B, U), and for each $\alpha \in (B, U)$ there is a full measure for K_α.*

Proof. This follows from the "classical" multifractal analysis (see for example [17] for an exposition). Since $T(q)$ is of class C^1 for $q < 1$, it is possible to prove that T and D_μ form a Legendre pair in this range, satisfying

$$D_\mu(-T'(q)) = T(q) - qT'(q).$$

Furthermore, we can obtain a (noninvariant) full measure μ_q for each K_α in the following manner. Given $\alpha \in (B, U)$, take $q < 1$ such that $\alpha = -T'(q)$. Let μ_q^u be the equilibrium measure for

$$-T^u(q) \log |d_x^u f| + q \log \phi$$

on the symbolic dynamics represented by one-sided sequences indexed by the nonnegative integers. Similarly, let μ_q^s be the equilibrium measure for

$$T^s(q) \log |d_x^s f| + q \log \phi$$

also on the symbolic dynamics but now represented by one-sided sequences indexed by the nonpositive integers. Then the product measure $\mu_q^s \times \mu_q^u$ (defined on the two-sided sequences indexed by all integers) induces a noninvariant measure μ_q on the horseshoe. One can easily verify that μ_q is a full measure for K_α (see [17] for details). This completes the proof. ∎

Lemma 10: *The irregular set has full Hausdorff measure.*

Proof. This follows from work of Barreira and Schmeling in [3], together with the fact that $\dim_H X = \sup\{\dim_H \Lambda : \Lambda \in \mathcal{H}\}$. ∎

This completes the proof of the theorem. ∎

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