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ON LOW DISCREPANCY SEQUENCES AND LOW DISCREPANCY ERGODIC TRANSFORMATIONS OF THE MULTIDIMENSIONAL UNIT CUBE

BY

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ABSTRACT

In this paper we describe a third class of low discrepancy sequences. Using a lattice $\Gamma \subset \mathbb{R}^s$, we construct Kronecker-like and van der Corput-like ergodic transformations $T_{1,\Gamma}$ and $T_{2,\Gamma}$ of $[0,1)^s$. We prove that for admissible lattices Γ, $(T^n_{\nu,\Gamma}(x))_{n\geq 0}$ is a low discrepancy sequence for all $x \in [0,1)^s$ and $\nu \in \{1,2\}$. We also prove that for an arbitrary polyhedron $P \subset [0,1)^s$, for almost all lattices $\Gamma \in \mathbb{L}_s = SL(s,\mathbb{R})/SL(s,\mathbb{Z})$ (in the sense of the invariant measure on $\mathbb{L}_s$), the following asymptotic formula

$$\#\{0 \leq n < N : T^n_{\nu,\Gamma}(x) \in P\} = N\mathrm{vol}P + O((\ln N)^{s+\varepsilon}), \quad N \to \infty$$

holds with arbitrary small $\varepsilon > 0$, for all $x \in [0,1)^s$, and $\nu \in \{1,2\}$.

1. Preliminaries

1.1. Let $(\beta_n)_{n\geq 0}$ be an infinite sequence of points in an s-dimensional unit cube $[0,1)^s$. The sequence $(\beta_n)_{n\geq 0}$ is said to be **uniformly distributed** in $[0,1)^s$ if for every box $\mathcal{O} = [0,y_1) \times \cdots \times [0,y_s) \subseteq [0,1)^s$

(1.1)
$$\Delta(\mathcal{O},(\beta_n)_{n=0}^{N-1}) = \#\{0 \leq n < N : \beta_n \in \mathcal{O}\} - Ny_1 \ddots y_s = o(N), \quad N \to \infty.$$

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We define the L_∞ and L_p **discrepancy** of a N-point set $(\beta_{n,N})_{n=0}^{N-1}$ as

$$D(N) = D((\beta_{n,N})_{n=0}^{N-1}) = \sup_{0<y_1,\ldots,y_s\leq 1} \Big| \frac{1}{N}\Delta(\mathcal{O}, (\beta_{n,N})_{n=0}^{N-1})\Big|, \tag{1.2}$$

$$D_p(N) = D_p((\beta_{n,N})_{n=0}^{N-1}) = \Big(\int_{[0,1]^s} \Big| \frac{1}{N}\Delta(\mathcal{O}, (\beta_{n,N})_{n=0}^{N-1})\Big|^p dy_1 \cdots dy_s\Big)^{1/p}. \tag{1.3}$$

It is known that a sequence $(\beta_n)_{n\geq 0}$ is uniformly distributed if and only if $D((\beta_n)_{n=0}^{N-1}) \to 0$ for $N \to \infty$.

In 1954, Roth proved that there exists a constant $C_1 > 0$, such that

$$ND_p((\beta_{n,N})_{n=0}^{N-1}) > C_1(\ln N)^{\frac{s-1}{2}} \quad \text{and} \quad \overline{\lim} \frac{ND_p((\beta_n)_{n=0}^{N-1})}{(\ln N)^{s/2}} > 0 \tag{1.4}$$

for all N-point sets $(\beta_{n,N})_{n=0}^{N-1}$ and all sequences $(\beta_n)_{n\geq 0}$ with $p = 2$. In 1977, W. Schmidt proved (1.4) for all $p > 1$ with $C_1 = C_1(p)$. According to the well-known conjecture (see, for example, [BC, p. 283] and [Ni, p. 32]), there exists a constant $C_2 > 0$, such that

$$ND((\beta_{n,N})_{n=0}^{N-1}) > C_2(\ln N)^{s-1} \quad \text{and} \quad \overline{\lim} \frac{ND((\beta_n)_{n=0}^{N-1})}{(\ln N)^{s}} > 0$$

for all N-point sets $(\beta_{n,N})_{n=0}^{N-1}$ and all sequences $(\beta_n)_{n\geq 0}$. In 1972, W. Schmidt proved this conjecture for $s = 1$. See known results for $s > 1$ in [Ba].

Definition 1: A sequence $(\beta_n)_{n\geq 0}$ is of **low discrepancy** (abbreviated l.d.s.) if $D((\beta_n)_{n=0}^{N-1}) = O(N^{-1}(\ln N)^s)$ for $N \to \infty$.

Definition 2: A sequence of point sets $((\beta_{n,N})_{n=0}^{N-1})_{N=1}^{\infty}$ is of low discrepancy (abbreviated l.d.p.s.) if $D((\beta_{n,N})_{n=0}^{N-1}) = O(N^{-1}(\ln N)^{s-1})$ for $N \to \infty$.

In this paper, by ergodic transformations we refer only to ergodic transformations of $[0,1)^s$ with respect to Lebesgue measure.

Definition 3: An ergodic transformation T is of low discrepancy (abbreviated l.d.e.t.) if $(T^n(x))_{n\geq 0}$ is l.d.s. for all $x \in [0,1)^s$. An ergodic transformation T is of L_p low discrepancy if

$$\Big(\int_{[0,1)^s} (D_p((T^n(x))_{n=0}^{N-1}))^p dx \Big)^{1/p} = O(N^{-1}(\ln N)^{s/2})$$

for $N \to \infty$, with $p > 1$.

Up to now, there were known three classes of multidimensional low discrepancy point sets in $[0,1)^s$:

a. Halton - Hammersley (1960),
b. Sobol (1967) - Faure (1981) - Niederreiter(1987),
c. Frolov (1976) - Skriganov (1994).

Analogues of the first two constructions were designed by the same authors to construct low discrepancy sequences. Our goal here is to perform the same for the third construction. The main points of our constructions are as follows:

Using a lattice $\Gamma \subset \mathbb{R}^s$, we construct Kronecker-like and van der Corput-like ergodic transformations $T_{1,\Gamma}$ and $T_{2,\Gamma}$ of $[0,1)^s$. We prove that for admissible lattices Γ, $T_{1,\Gamma}$ is an L_p low discrepancy ergodic transformation for all $p > 1$, and $T_{\nu,\Gamma}$ is an l.d.e.t. ($\nu \in \{1,2\}$). We also prove that for an arbitrary polyhedron $P \subset [0,1)^s$ and for almost all lattices $\Gamma \in \mathbb{L}_s = SL(s,\mathbb{R})/SL(s,\mathbb{Z})$ (in the sense of invariant measure on $\mathbb{L}_s$), one has the bound

$$\#\{0 \le n < N : T^n_{\nu,\Gamma}(x) \in P\} - N\mathrm{vol}P = O((\ln N)^{s+\varepsilon}) \tag{1.5}$$

as $N \to \infty$ for all $x \in [0,1)^s$, and $\nu \in 21,2\}$.

1.2. Brief review of low discrepancy sequences. (for a complete review see [BC], [DrTi], [Ma] and [Ni]).

1.2.1. *The 1-dimensional case.* Let α be a real number with bounded partial quotients. In 1922, Ostrowski, Behnke, Hecke, Hardy and Littlewood proved that the sequence $\{\alpha n + \beta\}_{n\ge 0}$ has low discrepancy (see[DrTi, pp. 155, 156, 735, 759, 1398]). Hence the orbit of the ergodic transformation $T_\alpha : x \to x + \alpha$ (mod 1) is l.d.s. for all $x \in [0,1)$. The second class of l.d.s. $(\psi_q(n))_{n\ge 0}$ was proposed by van der Corput; see [DrTi][pp. 1891]). There $q \ge 2$ is an integer,

$$n = \sum_{i\ge 0} e_{i,q}(n) q^i, \quad \text{with } e_{i,q}(n) \in \{0,1,\dots,q-1\} \tag{1.6}$$

be the q-expansion of the integer n, and

$$\psi_q(n) = \sum_{i\ge 0} e_{i,q}(n) q^{-i-1} \tag{1.7}$$

the radical inverse function. The low discrepancy property is valid for the more general case of the sequence $(T^n_q(x))_{n\ge 0}$, where T_q is von Neumann–Kakutani's ergodic adding machine:

Let $x = .x_1x_2\ldots$, $x' = .x'_1x'_2\ldots$ be the q-expansions of the numbers x and $x' \in [0,1)$, $T_q(x) = x'$ is defined by

$$x'_k = \begin{cases} 0, & k = 1,2,\ldots,i-1, \\ x_i + 1, & k = i, \\ x_k, & \text{otherwise}, \end{cases} \tag{1.8}$$

where $x_j = q-1$ for $j = 1,2,\ldots,i-1$, and $x_i \neq q-1, k = 1,2,\ldots$. Given a transformation T, we define

$$T^n(x) = T(T^{n-1}(x)), \quad n = 1,2,\ldots, \quad T^0(x) = x.$$

A detailed description of the ergodic adding machine is given in [Fr, pp. 75–83] and in [Pe, pp. 208–212]. As is known, the sequence $(T_q^n(x))_{n\geq 0}$ coincides for $x = 0$ with the van der Corput sequence (see, for example, [P]). We shall say that $(T_q(x)^n)_{n\geq 0}$ is the van der Corput sequence for all $x \in [0,1)$. Other examples of l.d.s. may be found in [Bo] and [Nin].

1.2.2. *The multidimensional case* ($s \geq 2$). The existence of multidimensional l.d.s. was discovered by Halton in 1960: $(\psi_{q_1}(n),\ldots,\psi_{q_s}(n))_{n\geq 0}$, where $q_1,\ldots,q_s \geq 2$ are pairwise coprime integers, and ψ_q is defined in (1.7) (see [DrTi, p. 729]). A similar result is true for the case of the ergodic transformation $T(x) = (T_{q_1}(x_1),\ldots,T_{q_s}(x_s))$ of $[0,1)^s$, with $x = (x_1,\ldots,x_s)$ and $T_{q_i}(x_i)$ defined in (1.8). This is a first class of l.d.e.t. In 1960, Hammersley proved that $(\psi_{q_1}(n),\ldots,\psi_{q_s}(n),\frac{n}{N})_{n=0}^{N-1}$ is an $(s+1)$-dimensional l.d.p.s. (see [DrTi, p. 730]). The second class of l.d.s. and l.d.p.s. was introduced in 1967 by Sobol (the so-called (t,m,s) point set, and (t,s)-sequences); see [So1] and [So2]. Generalizations of Sobol's approach were obtained by Faure (1981) and by Niederreiter (1987); see [Ni]. See also l.d.s. construction for $s = 2$ in [Mo].

Let $\alpha = (\alpha_1,\ldots,\alpha_s) \in \mathbb{R}^s$, $x = (x_1,\ldots,x_s) \in [0,1)^s$ ($s \geq 2$), and $T_\alpha : x \to (x_1+\alpha_1,\ldots,x_s+\alpha_s)$ be Kronecker's transformation of $[0,1)^s$. In 1994, J. Beck proved that for almost all α, $D((T_\alpha^n(x))_{n=0}^{N-1}) = O(N^{-1}(\ln N)^{s+\varepsilon})$ uniformly on $x \in [0,1)^s$ for arbitrary small $\varepsilon > 0$, as $N \to \infty$ [Be1]. He also proved that for almost all α we have $D((T_\alpha^n(x))_{n=0}^{N_k-1}) > N_k^{-1}(\ln N_k)^s \ln\ln N_k$ uniformly on $x \in [0,1)^s$ for some subsequence $N_k \to \infty$ [Be1]. Hence, Kronecker's sequence $(\{\alpha_1 n + x_1\},\ldots,\{\alpha_s n + x_s\})_{n\geq 0}$ is "almost" of low discrepancy for almost all $\alpha \in \mathbb{R}^s$. According to Littlewood's conjecture,

$$\varliminf_{n\to\infty} n\langle\langle n\alpha_1\rangle\rangle \cdots \langle\langle n\alpha_s\rangle\rangle = 0 \tag{1.9}$$

for $s \geq 2$ and all $(\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$, where $\langle\langle x \rangle\rangle = \min(\{x\}, 1-\{x\})$. Repeating the proof of [Skr1, Theorem 1.1] (see Theorem A below), we obtain that, if (1.9) is false, then T_α is a l.d.e.t. It is not hard to prove that, if (1.9) is true, then for all $\alpha \in \mathbb{R}^s$, T_α is not a L_2 l.d.e.t. In this paper, we generalize Kronecker's transformation to obtain L_p low discrepancy property for all $p > 1$.

1.3. Lattice nets. In this subsection we consider l.d.p.s. and l.d.s. based on lattices in $\mathbb{R}^s$. Let K be a totally real algebraic number field of degree s, and σ be the canonical embedding of K in the Euclidean space $\mathbb{R}^s$, $\sigma : K \ni \xi \to \sigma(\xi) = (\sigma_1(\xi), \dots, \sigma_s(\xi)) \in \mathbb{R}^s$, where $\{\sigma_j\}_{j=1}^s$ are s distinct embeddings of K in the field $\mathbb{R}$ of real numbers. Let $\lambda \in K$ be an algebraic integer, $\lambda_i = \sigma_i(\lambda)$ $(i = 1, \dots, s)$; $f(x)$ be the minimal polynomial of λ; λ is of degree s over $\mathbb{Q}$; $E = (\lambda_i^{j-1})_{i,j=1}^s$; $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_s)$; and $H = E\Lambda E^{-1}$ be the companion matrix of $f(x)$.

In 1976, Frolov introduced the point set $Fr(s,t) = \frac{1}{t}E\mathbb{Z}^s \cap [0,1)^s$ with the best possible estimate for the order of magnitude of the integration error as $t \to \infty$ on the Sobolev and Korobov class functions (see [Fr1], [Fr2]). In 1980, Frolov [Fr3] proved that $Fr(s,t)$ is a L_2 low discrepancy point set (i.e., $D_2(Fr(s,t)) = O(t^{-1}(\ln t)^{(s-1)/2})$ for $t \to \infty$); see also [By1], [By2], [Do], [Lv], [Tm, Chap. 4, Sec. 4] for investigations on Frolov's net.

In 1994, Skriganov [Skr1] proved that $Fr(s,t)$ is an l.d.p.s. He also proved the following more general result:

Let $\mathcal{O} \subset \mathbb{R}^s$ be a compact region, vol$\mathcal{O}$ the volume of $\mathcal{O}$, $t\mathcal{O}$ the dilatation of $\mathcal{O}$ by a factor $t > 0$, and let $t\mathcal{O} + X$ be the translation of $t\mathcal{O}$ by a vector $X \in \mathbb{R}^s$. Let $\Gamma \subset \mathbb{R}^s$ be a lattice, i.e., a discrete subgroup of $\mathbb{R}^s$ with a compact fundamental set $\mathbb{F}(\Gamma) = \mathbb{R}^s/\Gamma$, $\det\Gamma$=vol$\mathbb{F}(\Gamma)$. Let

$$N(\mathcal{O}, \Gamma) = \mathrm{card}(\mathcal{O} \cap \Gamma) = \sum_{\gamma \in \Gamma} \chi(\mathcal{O}, \gamma)$$

be the number of points of the lattice Γ lying inside the region $\mathcal{O}$, where we denote by $\chi(\mathcal{O}, X)$, $X \in \mathbb{R}^s$, the characteristic function of $\mathcal{O}$. We define the errors $\mathcal{R}(\mathcal{O} + X, \Gamma)$, $R(\mathcal{O}, \Gamma)$ and $R_p(\mathcal{O}, \Gamma)$ by setting

$$N(\mathcal{O} + X, \Gamma) = \frac{\mathrm{vol}\mathcal{O}}{\det\Gamma} + \mathcal{R}(\mathcal{O} + X, \Gamma), \tag{1.10}$$

$$R(\mathcal{O}, \Gamma) = \sup_{X \in \mathbb{R}^s} |\mathcal{R}(\mathcal{O} + X, \Gamma)|, \tag{1.11}$$

and

$$R_p(\mathcal{O},\Gamma) = \Big(\frac{1}{\det\Gamma}\int_{\mathbb{F}(\Gamma)} |\mathcal{R}(\mathcal{O}+X,\Gamma)|^p dX\Big)^{1/p}, \tag{1.12}$$

where $p > 0$ is a real number.

Definition 4: A lattice $\Gamma \subset \mathbb{R}^s$ is **admissible** if

$$\mathrm{Nm}\Gamma = \inf_{\gamma\in\Gamma\setminus\{0\}} |\mathrm{Nm}\gamma| > 0, \tag{1.13}$$

where $\mathrm{Nm}x = x_1 x_2 \cdots x_s$, $x = (x_1,\dots,x_s)$.

For example, $\Gamma = E\mathbb{Z}^s$ (in Frolov's net) is an admissible lattice. The set of all admissible lattices $\Gamma \subset \mathbb{R}^s$ is dense in $\mathbb{L}_s$, but its invariant measure on $\mathbb{L}_s$ is equal to zero. Let $\mathbb{K}^s = [-\frac{1}{2},\frac{1}{2}]^s$, $T = (t_1,\dots,t_s)$ and $T \centerdot \mathcal{O} = \{(t_1x_1,\dots,t_sx_s) : (x_1,\dots,x_s) \in \mathcal{O}\}$. It is easy to show the following estimate for the inadmissible lattice $\Gamma = \mathbb{Z}^s$:

$$\overline{\lim_{t\to\infty}}\, R(T \centerdot \mathbb{K}^s,\Gamma)/t^{s-1} > 0, \quad \text{with } T = (t,\dots,t). \tag{1.14}$$

Theorem A. (see [Skr1, Theorem 1.1]): *If $\Gamma \subset \mathbb{R}^s$ is an admissible lattice, then for all $T \subset \mathbb{R}^s$, one has the bounds*

$$R(T \centerdot \mathbb{K}^s,\Gamma) < C(\Gamma)(\ln(2+|\mathrm{Nm}T|))^{s-1}, \tag{1.15}$$

and

$$R_p(T \centerdot \mathbb{K}^s,\Gamma) < C_p(\Gamma)(\ln(2+|\mathrm{Nm}T|))^{(s-1)/2}, \quad p > 0. \tag{1.16}$$

The constants in (1.15) *and* (1.16) *depend upon the lattice* Γ *only by means of the invariants* $\det\Gamma$ *and* $\mathrm{Nm}\Gamma$*.*

In [L1] we proposed the following two constructions of l.d.s. $(\beta_{\nu,n})_{n\geq 0}$, $(\nu = 1,2)$ based on Frolov's and Skriganov's nets. In the first construction, we consider the sequence of lattices $(q^{-n}E\mathbb{Z}^s)_{n\geq 0}$, where $q = |\lambda_1\cdots\lambda_s| > 1$. Let $d_1,\dots,d_q$, be a complete residue system of $\mathbb{Z}^s (\mathrm{mod}\ H\mathbb{Z}^s)$; $e_{i,q}(n)$ are defined in (1.6); $a_1,\dots,a_m \in \mathbb{Z}^s$ such that

$$E[0,1)^s \subset \bigcup_{j=1}^m ([0,1)^s + a_j) \quad \text{and} \quad E[0,1)^s \cap ([0,1)^s + a_j) \neq \emptyset;$$

$$v_n = \sum_{i\geq 0} H^{-i-1} d_{e_{i,q}(n)}, \quad w_{mn+i} = v_n + a_i;\ i = 1,\dots,m,\ n \geq 0. \tag{1.17}$$

Now let $u(n) \geq 0$ be an increasing sequence of integers such that $w_k \in E[0,1)^s$ if and only if $k = u(n)$ for any integer $n \geq 0$. Next, $\beta_{1,n} = E^{-1} w_{u(n)}$, $n = 0, 1, \ldots$. In the twist construction, we consider the following sequence of lattices:

$$\Gamma_n = \{((q_{1,1} \cdots q_{1,n})^{-1}\gamma_1, \ldots, (q_{s,1} \cdots q_{s,n})^{-1}\gamma_s) : (\gamma_1, \ldots, \gamma_s) \in \Gamma\}, \tag{1.18}$$

where $\Gamma \subset \mathbb{R}^s$ is an admissible lattice, and $q_{i,j} \in \{f_1, \ldots, f_k\}$ with integers $f_i \geq 2$ $(i = 1, \ldots, k)$. Let $B_n = \Gamma_n \cap [0,1)^s$. Similarly to the first construction (1.17), we enumerate the set $B_{n+1} \setminus B_n$ by a sequence $(\beta_{2,m})_{m \in (\#B_n, \#B_{n+1}]}$. In [L1], we noted that from [Skr1, Corollary 2.1] it follows that $(\beta_{\nu,n})_{n \geq 0}$ is an l.d.s. $(\nu = 1, 2)$. In this paper, we generalize the twist construction (1.18) to obtain a second class (after Halton's class) of low discrepancy ergodic transformations of $[0,1)^s$. In a similar way, we can also generalize the first construction (1.17) (for the case of $|\lambda_i| > 1$ $(i = 1, \ldots, s)$) to obtain an l.d.e.t. of $[0,1)^s$.

1.4. Lattice point problem for compact polyhedron. For every region $\mathcal{O} \in \mathbb{R}^s$ and lattice $\Gamma \subset \mathbb{R}^s$, one has the bound

$$R(t\mathcal{O}, \Gamma) = O(t^{s-1}), \quad \text{for } t \to \infty; \tag{1.19}$$

see [GL] and [Kr]. Let $P \subset \mathbb{R}^s$ be a compact polyhedron. In [SkSt, Theorem 2], Skriganov and Starkov obtained the following extremely small upper bound on the error $R(tP, \Gamma)$:

Theorem B: *Let $P \subset \mathbb{R}^s$ be an arbitrary compact polyhedron, and $\Gamma \in \mathbb{L}_s$ be an arbitrary lattice. Then for μ_s almost all rotations $U \in SO(s)$ one has the bound*

$$R(tP, U\Gamma) = R(tU^{-1}P, \Gamma) = O(\ln^{s-1+\varepsilon} t) \tag{1.20}$$

with arbitrary small $\varepsilon > 0$, where μ_s is the Haar measure on the group $SO(s)$.

This theorem improves the previous result of Skriganov [Skr2, Theorem 2.2]. In [SkSt, p. 1471], it is noted that the bound $O(\ln^{s-1+\varepsilon} t)$ in (1.20) could be replaced by

$$R(tP, U\Gamma) = O((\ln t)^{s-1} \varphi(\ln \ln t)), \tag{1.21}$$

where $\varphi(\cdot)$ is an arbitrary positive monotone increasing function with $\int_1^\infty \frac{dt}{\varphi(t)} < \infty$. For example, $\varphi(t) = t^{1+\varepsilon}$, $\varphi(t) = t(\ln t)^{1+\varepsilon}$. The O-constant in (1.20) depends on the polyhedron P. In this paper, we modify the proof of Theorem B, and obtain that the O-constant in (1.20) only depends on vectors orthogonal to the $(s-1)$-dimensional faces of P. In this way, we give the estimate (1.5) for the sequences $(T^n_{\nu,U\Gamma}(x))_{n\geq 0}, (\nu=1,2)$.

Now we describe the structure of the paper. In §2, we construct Kronecker-like and van der Corput-like transformations $T_{1,\Gamma}$ and $T_{2,\Gamma}$. We also state all the theorems on discrepancy estimates of the sequence $(T^n_{\nu,\Gamma}(x))_{n\geq 0}, (\nu=1,2)$. Then we state the results on the lattice point problem in a polyhedron. The proofs of the theorems are given in §3 and §4. In the Appendix, we obtain a modification of Theorem B.

2. Statement of the results

2.1. Kronecker-like transformation. Let $\Gamma\subset\mathbb{R}^{s+1}$ be an arbitrary lattice, $\tilde\Gamma=l\Gamma$ with

$$\#\{(\tilde\Gamma+u)\cap[0,1)^{s+1}\}\leq 1 \tag{2.1}$$

for all $u\in\mathbb{R}^{s+1}$. For example, let

$$l=\left[\frac{\sqrt{s+1}}{\lambda}\right]+1,\quad \text{where } \lambda=\min_{\gamma\in\Gamma\setminus\{0\}}\|\gamma\|,$$

and $\|\centerdot\|$ is the Euclidean norm in $\mathbb{R}^{s+1}$. Then $\min_{\gamma\in\tilde\Gamma\setminus\{0\}}\|\gamma\|\geq 1$, and (2.1) follows.

Let $x\in[0,1)^s$, $(x,0)\in\mathbb{R}^{s+1}$, $M_x=M_{x,1}\cup M_{x,2}$, where

$$M_{x,1}=((x,0)+\tilde\Gamma)\cap[0,1)^s\times(0,+\infty), \tag{2.2}$$

and

$$M_{x,2}=((x,0)+\tilde\Gamma)\cap[0,1)^s\times(-\infty,0].$$

By (2.1), there are no two points $(u_1,u_{1,s+1})$ and $(u_2,u_{2,s+1})$ in M_x with $u_{1,s+1}=u_{2,s+1}$, where $u_i\in\mathbb{R}^s$ and $u_{i,s+1}\in\mathbb{R}$, $i=1,2$. Let $M_{x,1}\neq\emptyset$, using (2.1) we obtain that there exists a unique $(y_1,y_{s+1})\in M_{x,1}$ with

$$y_{s+1}=\min\{v>0:\exists y\in[0,1)^s,\ \text{such that}\quad (y,v)\in M_{x,1}\}. \tag{2.3}$$

Now let

$$T_{1,\Gamma}(x)=\begin{cases} y_1, & \text{if} M_{x,1}\neq\emptyset,\\ x, & \text{otherwise.}\end{cases} \tag{2.4}$$

It is easy to see that here a Kronecker transformation for

$$\Gamma = \{(n\alpha_1 - m_1, \dots, n\alpha_s - m_s, n)) : (m_1, \dots, m_s, n) \in \mathbb{Z}^{s+1}\}$$

and $l = 1$ is obtained. Recall that the dual lattice $\Gamma^\perp$ consists of all vectors $\gamma^\perp \in \mathbb{R}^{s+1}$ such that the inner product $\langle \gamma^\perp, \gamma \rangle$ belongs to $\mathbb{Z}$ for each $\gamma \in \Gamma$.

THEOREM 2.1: *Let $\Gamma \subset \mathbb{R}^{s+1}$ be an arbitrary lattice. Then for all $x \in [0,1)^s$, the sequence $(T^n_{1,\Gamma}(x))_{n \geq 0}$ is uniformly distributed in $[0,1)^s$ if and only if*

$$\nexists \gamma^\perp = (\gamma^\perp_1, \dots, \gamma^\perp_{s+1}) \in \Gamma^\perp \setminus \{0\} \quad \text{with } \gamma^\perp_{s+1} = 0. \tag{2.5}$$

If (2.5) is valid, then $T_{1,\Gamma}$ is a invertible ergodic transformation of $[0,1)^s$.

We will prove that for the lattices Γ considered in the following theorems we have $\#M_{x,1} = \#M_{x,2} = \infty$ for all $x \in [0,1)^s$. Applying (2.1), we get that $M_{x,1} \cup M_{x,2}$ can be enumerated by a sequence $(z_{x,k}, z_{s+1}(x,k))_{k=-\infty}^{+\infty}$ in the following way:

$$\begin{aligned} &z_{x,0} = x, \quad z_{s+1}(x,0) = 0, \quad z_{x,k} \in [0,1)^s \quad \text{and} \\ &z_{s+1}(x,k) < z_{s+1}(x,k+1) \in \mathbb{R}, \qquad \text{for } k \in \mathbb{Z}. \end{aligned} \tag{2.6}$$

THEOREM 2.2: *Let $\Gamma \subset \mathbb{R}^{s+1}$ be an admissible lattice. Then*

$$\mathrm{D}(T^n_{1,\Gamma}(x)^{M+N}_{n=M+1}) = O(N^{-1}(\ln N)^s), \quad N \to \infty \tag{2.7}$$

$$\mathrm{D}((T^n_{1,\Gamma}(x), z_{s+1}(x,n)/z_{s+1}(x,N))^{N-1}_{n=0}) = O(N^{-1}(\ln N)^s), \quad N \to \infty, \tag{2.8}$$

uniformly on $x \in [0,1)^s$ and $M \in \mathbb{Z}$, where the O-constant depends only on the invariants $\det \Gamma$ *and* $\mathrm{Nm}\Gamma$.

Remark 1: We showed in §1.2.2 the connection between the s-dimensional Halton's l.d.s. and the $(s+1)$-dimensional Hammersley's l.d.p.s. We obtain a similar connection in (2.7) and (2.8).

THEOREM 2.3: *Let $\Gamma \in \mathbb{L}_{s+1} = SL(s+1,\mathbb{R})/SL(s+1,\mathbb{Z})$ be an arbitrary lattice. Then for μ_{s+1} almost all rotations $U \in SO(s+1)$*

$$\mathrm{D}((T^n_{1,U\Gamma}(x))^{N-1}_{n=0}) = O(N^{-1}(\ln N)^{s+\varepsilon})$$

with arbitrary small $\varepsilon > 0$.

THEOREM 2.4: *Let $\Gamma \subset \mathbb{L}_{s+1}$ be an admissible lattice. Then $T_{1,\Gamma}$ is an L_p l.d.e.t.:*

$$\left(\int_{[0,1)^s} (\mathrm{D}_p((T^n_{1,\Gamma}(x))_{n=0}^{N-1}))^p dx \right)^{1/p} = O(N^{-1}(\ln N)^{s/2}). \tag{2.9}$$

In particular, for each $N > 0$ there exists a point $x_N \in [0,1)^s$ with

$$\mathrm{D}_p((T^n_{1,\Gamma}(x_N))_{n=0}^{N-1}) = O(N^{-1}(\ln N)^{s/2}), \tag{2.10}$$

where the O-constant in (2.9) *and* (2.10) *depends only on $p > 1$, $\det\Gamma$, and $\mathrm{Nm}\Gamma$, for $N \to \infty$.*

2.2. VAN DER CORPUT-LIKE TRANSFORMATION. Let $q \geq 2$ be an integer, and $(\mathbf{e_i} = (0, \dots, 1, \dots, 0))_{i=1}^s$ be the standard basis of the lattice $\mathbb{Z}^s$. Using von Neumann–Kakutani's ergodic adding machine T_{q^s} to the base q^s, we construct an ergodic transformation $T_{q,s}$ of $[0,1)^s$ in the following way: Let

$$x = (x_1, \dots, x_s) = \sum_{i=1}^{s} x_i \mathbf{e_i} \in [0,1)^s, \tag{2.11}$$

$$x_i = 0.x_{i,1}x_{i,2}\dots = \sum_{j=1}^{\infty} x_{i,j}/q^j \in [0,1), \quad x_{i,j} \in \{0, \dots, q-1\}$$

be the canonical q-expansion of x_i (i.e., there is no integer j_0 such that $x_{i,j} = q-1$ for all $j > j_0$), $i = 1, \dots, s$. We define the sequences $x'_{i,j} \in \{0, \dots, q-1\}$ $(i = 1, \dots, s, j = 1, 2, \dots)$ by setting

$$\sum_{j=1}^{\infty} \sum_{i=1}^{s} \frac{x'_{i,j} q^{i-1}}{q^{sj}} := T_{q^s} \Big(\sum_{j=1}^{\infty} \sum_{i=1}^{s} \frac{x_{i,j} q^{i-1}}{q^{sj}} \Big). \tag{2.12}$$

Now we put

$$T_{q,s}(x) = x', \quad \text{where } x' = (x'_1, \dots, x'_s) \text{ and } x'_i = \sum_{j=1}^{\infty} \frac{x'_{i,j}}{q^j}. \tag{2.13}$$

Let H be an arbitrary $s \times s$ nonsingular matrix with real coefficients, $\Gamma = H\mathbb{Z}^s$, $\mathbb{F} = H[0,1)^s$ be a fundamental set of Γ, $\mathrm{vol}\mathbb{F} = |\det \mathbb{F}| \neq 0$, $\mathbf{h_i} = H\mathbf{e_i}$ $(i = 1, \dots, s)$, and

$$r_1 = 3 + \Big[\max_{1 \leq i \leq s} \log_q \|\mathbf{h_i}\| \Big]. \tag{2.14}$$

Hence $q^{-r_1}\|\mathbf{h_i}\| \le q^{-2} \le 1/4$. Now let
(2.15)

$$G(a) = q^{-r_1}(\mathbb{F}+a)\cap[0,1)^s \text{ and } \{a_1,\dots,a_{r_2}\} = \{a\in\Gamma = H\mathbb{Z}^s : G(a)\neq\emptyset\}.$$

It is easy to see that

$$\bigcup_{i=1}^{r_2} G(a_i) = [0,1)^s \quad \text{and} \quad G(a_i)\cap G(a_j)=\emptyset \quad \text{for } i\neq j. \tag{2.16}$$

Bearing in mind that $(\mathbf{h}_1,\dots,\mathbf{h}_s)$ is a basis of the lattice Γ, we obtain from (2.14)–(2.16) that there exists $i\in[1,r_2]$ such that $G(a_i)\subset[0,1)^s$. We enumerate the set $\{a_1,\dots,a_{r_2}\}$ in such a way that

$$G(a_1)\subset[0,1)^s. \tag{2.17}$$

By (2.16), we obtain that for every $x\in[0,1)^s$ there exists a unique $i(x)\in[1,r_2]$ and $\tilde{x}\in[0,1)^s$, such that

$$x = q^{-r_1}(a_{i(x)} + H\tilde{x}). \tag{2.18}$$

Now let

$$k(x) = \{i(x) < j \le r_2 : q^{-r_1}(a_j + H\tilde{x})\in[0,1)^s\},$$

and

$$T_{2,\Gamma}(x) = \begin{cases} q^{-r_1}(a_m + H\tilde{x}), & k(x)\neq\emptyset \text{ and } m=\min k(x)\\ q^{-r_1}(a_1 + HT_{q,s}(\tilde{x})), & \text{otherwise.}\end{cases} \tag{2.19}$$

By (2.15)–(2.19), $T_{2,\Gamma}(x)\in[0,1)^s$ for all $x\in[0,1)^s$. Hence the transformation $T_{2,\Gamma}:[0,1)^s\to[0,1)^s$ is well-defined.

Theorem 2.5: *Let $\Gamma\subset\mathbb{R}^s$ be an arbitrary lattice. Then*

$$\mathrm{D}((T^n_{2,\Gamma}(x))_{n=0}^{N-1}) = O(N^{-1/s}\ln N) \tag{2.20}$$

uniformly on $x\in[0,1)^s$ for $N\to\infty$, and $T_{2,\Gamma}$ is an ergodic transformation of $[0,1)^s$.

We note that the bad estimate (2.20) cannot be improved essentialy because of the lower bound (1.14) for the inadmissible lattice $\Gamma=\mathbb{Z}^s$.

Theorem 2.6: *Let $\Gamma\subset\mathbb{R}^s$ be an admissible lattice. Then*

$$\mathrm{D}((T^n_{2,\Gamma}(x))_{n=0}^{N-1}) = O(N^{-1}(\ln N)^s) \tag{2.21}$$

uniformly on $x \in [0,1)^s$ *for* $N \to \infty$, *and*

$$\mathrm{D}((T^n_{2,\Gamma}(x))_{n=0}^{N_k-1}) = O(N_k^{-1}(\ln N_k)^{s-1}) \tag{2.22}$$

for some subsequence $N_k \to \infty$.

THEOREM 2.7: *Let* $\Gamma \subset \mathbb{R}^s$ *be an arbitrary lattice. Then for* μ_s *almost all rotations* $U \in SO(s)$

$$\mathrm{D}((T^n_{2,U\Gamma}(x))_{n=0}^{N-1}) = O(N^{-1}(\ln N)^{s+\varepsilon}) \tag{2.23}$$

for arbitrarily small $\varepsilon > 0$, *uniformly on* $x \in [0,1)^s$.

THEOREM 2.8: *Let* $\Gamma \subset \mathbb{R}^s$ *be an admissible lattice. Then*

$$\left(\int_{[0,1)^s} (\mathrm{D}_p(T^n_{2,\Gamma}(x))_{n=0}^{N-1})^p dx \right)^{1/p} = O(N^{-1}(\ln N)^{(s+1)/2}), \tag{2.24}$$

where the O-constant depends only on $p > 1$, $\det \Gamma$ *and* $\mathrm{Nm}\Gamma$.

We will prove in a forthcoming paper a more precise bound in (2.24): $O(N^{-1}(\ln N)^{s/2})$.

2.3. POLYHEDRAL DISCREPANCY. In this paper, by a polyhedron we mean any bounded and nonempty intersection of finitely many closed or open half-spaces. We will use the assertions of this section in the $(s+1)$-dimensional and s-dimensional cases. For this reason, we consider here the case of the n-dimensional space $\mathbb{R}^n$. Let $P \subset \mathbb{R}^n$ be a polyhedron, and let

$$f = \{P = P^n \supset P^{n-1} \supset \cdots \supset P^0,\ \dim P^j = j,\ n \geq j \geq 0\} \tag{2.25}$$

be a flag of faces P^j $\in$Face(P), $0 \leq j \leq n$, of P. Let $V_{\mathfrak{f}} \in SO(n)$ be an orthogonal matrix associated with the flag (2.25) by the relation

$$V_{\mathfrak{f}}^{'} = [\mathbf{l}_n, \mathbf{l}_{n-1}, \ldots, \mathbf{l}_1]. \tag{2.26}$$

The columns of the transposed matrix $V_{\mathfrak{f}}^{'}$ in (2.26) are orthonormal vectors $\mathbf{l}_j, n \geq j \geq 1$, defined by the following rule: the unit vector $\mathbf{l}_j$ is the external normal to the face P^{j-1} parallel to the face P^j. Thus, coordinates of the vector $\mathbf{l}_j$ form the j-th row of the matrix $V_{\mathfrak{f}}$. For a given lattice $\Gamma \in \mathbb{L}_n$ and a flag of faces $\mathfrak{f}$ of P, we define a lattice $\Gamma_{\mathfrak{f}} \in \mathbb{L}_n$ by

$$\Gamma_{\mathfrak{f}} = V_{\mathfrak{f}} \Gamma. \tag{2.27}$$

It is easy to see that

$$\Gamma_{\mathfrak{f}}^{\perp} = V_{\mathfrak{f}} \Gamma^{\perp}. \tag{2.28}$$

Let Face$(P_1, n-1)$ be the finite set of all $(n-1)$-dimensional faces of P_1, and $\mathcal{V}(P_1)$ be the set of all convex polyhedra $P \subset [0,1)^n$ such that each face $P^{n-1} \in$Face$(P, n-1)$ is parallel to some face $P_1^{n-1} \in$Face$(P_1, n-1)$, and let (see (1.11))

$$\bar{R}(P_1, \Gamma) = \sup_{P \in \mathcal{V}(P_1)} R(P, \Gamma). \tag{2.29}$$

We will prove in the Appendix the following modification of Theorem B [SkSt, Theorem 2].

Proposition 2.1: *Let $P_1 \subset \mathbb{R}^n$ be a polyhedron, and let*

$$\Gamma \in \mathbb{L}_n = SL(n, \mathbb{R})/SL(n, \mathbb{Z})$$

be an arbitrary lattice. Then for μ_n almost all rotations $U \in SO(n)$

$$\bar{R}(P_1, t^{-1} U \Gamma) = O((\ln t)^{n-1+\varepsilon}), \tag{2.30}$$

for arbitrarily small $\varepsilon > 0$.

An **algebraic polyhedron** is a polyhedron $P_{\mathfrak{k}} \subset \mathbb{R}^n$ all of whose vertices have coordinates belonging to a real algebraic number field $\mathfrak{k}$. In [Skr2, p. 10], Skriganov gives an explicit construction of a matrix $A_{\mathfrak{k}} \in SL(n, \mathbb{R})$ depending only on the field $\mathfrak{k}$, such that the following theorem holds.

Theorem C ([Skr2, Theorem 2.3]): *With the above notations, one has the bound*

$$R(tP_{\mathfrak{k}}, A_{\mathfrak{k}} \mathbb{Z}^n) = O(t^{\varepsilon}) \tag{2.31}$$

for arbitrarily small $\varepsilon > 0$.

We will prove in the Appendix the following modification of Theorems A and C.

Proposition 2.2: *Let $P_1 \in \mathbb{R}^n$ be a polyhedron with a set of flags denoted by $Flag(P_1)$. Let $\Gamma \subset \mathbb{L}_n$ be an arbitrary lattice. Then the following assertions are valid:*

1. *If each of the lattices* $\Gamma_{\mathfrak{f}}$ *(see (2.27)) is admissible with* $\mathfrak{f} \in Flag(P_1)$, *then*

$$\bar{R}(P_1, t^{-1}\Gamma) = O((\ln t)^{n-1}), \quad t \to \infty. \tag{2.32}$$

2. *If the polyhedron* P_k *is algebraic, then*

$$\bar{R}(P_{\mathfrak{k}}, t^{-1}A_{\mathfrak{k}}\mathbb{Z}^n) = O(t^{\varepsilon}), \quad t \to \infty \tag{2.33}$$

with arbitrarily small $\varepsilon > 0$.

We define the *polyhedral discrepancy* of the sequence $\beta_m \in [0,1)^s$ $(m = 0, 1, \ldots)$ as follows:

$$D(P_1, (\beta_m)_{n=0}^{N-1}) = \sup_{P \in \mathcal{V}(P_1)} \left| \frac{1}{N} \#\{0 \le m < N : \beta_m \in P\} - \mathrm{vol} P \right|. \tag{2.34}$$

Let $P \subset \mathbb{R}^s$ be a polyhedron, $P(1) = P \times [0,1] \subset \mathbb{R}^{s+1}$ and $P(2) = P$.

Theorem 2.9: *Let* $\Gamma_\nu \in \mathbb{L}_{s+\nu-1}$ *be an arbitrary lattice* $(\nu = 1, 2)$. *Then the following assertions are valid:*

1. *If the lattice* Γ_ν *and each of the lattices* $\Gamma_{\nu,\mathfrak{f}}$ *are admissible, with* $\mathfrak{f} \in Flag(P_\nu)$, *then*

$$\mathrm{D}(P(\nu), (T^n_{\nu,\Gamma_\nu}(x))_{n=0}^{N-1}) = O(N^{-1}(\ln N)^s) \tag{2.35}$$

as $N \to \infty$, *with* $\nu \in \{1, 2\}$.

2. *For* $\mu_{s+\nu-1}$ *almost all rotations* $U \in SO(s+\nu-1)$, *one has the bound*

$$\mathrm{D}(P(\nu), (T^n_{\nu,\Gamma_\nu}(x))_{n=0}^{N-1}) = O(N^{-1}(\ln N)^{s+\varepsilon}), \tag{2.36}$$

for arbitrarily small $\varepsilon > 0$, *as* $N \to \infty$, *with* $\nu \in \{1, 2\}$.

3. *If the polyhedron* P *is algebraic, then*

$$\mathrm{D}(P(\nu), (T^n_{\nu, A_{\mathfrak{k}}\mathbb{Z}^{s+\nu-1}}(x))_{n=0}^{N-1}) = O(N^{\varepsilon - 1})$$

for arbitrarily small $\varepsilon > 0$, *as* $N \to \infty$, *with* $\nu \in \{1, 2\}$.

Remark 2: Using the estimate (1.21) instead of (1.20), we can precisely determine the corresponding estimates in Proposition 2.1, Theorem 2.3, Theorem 2.7 and Theorem 2.9. For example, the bound $O((\ln N)^{s+\varepsilon})$ in (2.36) could be replaced by $O((\ln N)^s \varphi(\ln \ln N))$.

2.4. Related questions. In this subsection we discus randomness of l.d.s., of the $\mathbb{Z}^d$ action by automorphisms of the s-torus, and randomness in the lattice point problem.

Definition: Let $(\Theta, \mathcal{F}, \mu)$ be a probability space, $d \geq 1$, $N_i \geq 1, n_i$ be integers $(i = 1, \dots, d)$, $n = (n_1, \dots, n_d)$, $f(n, \theta)$ be real numbers, $\theta \in \Theta$, $G(N, \theta) \subset \mathbb{Z}^d$ with $N = (N_1, \dots, N_d)$. We will say that the set $((f(n, \theta)_{n \in G(N,\theta)})$ satisfies the central limit theorem (abbreviated CLT) if there exists a function σ with $\sigma(N) > 0$, such that

$$\lim_{k \to \infty} \mu \left\{ \theta \in \Theta : \frac{1}{\sigma(N_k)} \sum_{n \in G(N_k, \theta)} f(n, \theta) \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \tag{2.37}$$

for each sequence $N_k = (N_{1,k}, \dots, N_{d,k})$ with $\max(N_{1,k}, \dots, N_{d,k}) \to \infty$ for $k \to \infty$.

Below Θ is a bounded domain in the Euclidean space, and μ is the Lebesgue measure.

2.4.1. *Beck's approach.* In 1992, J. Beck [Be2] discovered a very surprising phenomenon of randomness of the sequence $\{n\alpha\}_{n \geq 1}$, where α is a quadratic irrational. Beck (see, for example, [Be2], [Be3]) considered two cases: local and global:

Global case (Lattice points in tilted rectangles). In this case the error $\mathcal{R}([0, \gamma) \times [0, N-1] + (\beta, 0), (n\alpha + m, n)_{m,n \in \mathbb{Z}^2})$ (see (1.10)), coincides with the local discrepancy $\Delta([0, \gamma), \{n\alpha + \beta\}_{n=0}^{N-1})$ (see (1.1)). In [Be2] Beck proved CLT (2.37) with $\theta = (\beta, \gamma, \delta) \in \Theta = [0,1]^3$, $d = 1$, $G(N, \theta) = [0, [N\delta]]$, $\sigma(N) = \sigma_\alpha \ln(N)$ for some $\sigma_\alpha > 0$, and

$$f(n, \theta) = \Delta([0, \gamma), \{n\alpha + \beta\}).$$

Local case (Lattice points in tilted hyperbola-segment). The CLT (2.37) is true for $\theta \in \Theta = [0, 1]$, $d = 1$, $G(N, \theta) = [1, N]$, $\sigma(N) = \sigma_{c,\Gamma}(\ln(N))^{1/2}$ with $\Gamma = (m + n\alpha, m - n\alpha)_{m,n \in \mathbb{Z}^2}$ for some $\sigma_{c,\Gamma} > 0$, and

$$f(n, \theta) = \mathcal{R}(\{-c \leq (x + \theta)y \leq c : x \in [n, n+1), \ y \geq 1\}, \ \Gamma).$$

In [Be3], Beck notes that for almost all α the above CLT is false in the global case, and is true for the local case. In [LM], we transport CLT for the global case to the case of van der Corput's sequence (the case of the ergodic adding machine). According to [Be3], the generalization of these results to the simultaneous case for Kronecker's lattice is very difficult because of the problems connected to Littlewood's conjecture (1.9). In [L2], we generalize Beck's approach to the simultaneous case using an admissible lattice $\Gamma \subset \mathbb{R}^{s+1}$:

Global case. We obtain CLT (2.37) with $\theta = (\gamma_1, \dots, \gamma_s, \delta, x) \in \Theta = [0,1]^{2s+1}$, $d = 1$, $G(N, \theta) = [0, [N\delta]]$, $\sigma(N) = \sigma_\Gamma(\ln(N))^{s/2}$ for some $\sigma_\Gamma > 0$, and

$$f(n, \theta) = \Delta([0, \gamma_1) \times \cdots \times [0, \gamma_s), T^n_{1,\Gamma}(x)).$$

Local case. We obtain CLT (2.37) with $\theta \in \Theta = \mathbb{F}(\Gamma) = \mathbb{R}^{s+1}/\Gamma$, $d = s$, $G(N, \theta) = [-N_1, N_1] \times \cdots \times [-N_s, N_s]$, $\sigma(N) = \sigma_{c,\Gamma}(\ln(2N_1) \cdots \ln(2N_s))^{1/2}$ with $c > 1$ for some $\sigma_{c,\Gamma} > 0$, and

$$f(n, \theta) = \mathcal{R}\Big(\Big\{ -c \le \prod_{i=1}^{s+1}(x_i + \theta_i) \le c : x_i \in [n_i, n_i + 1),\ i = 1, \dots, s,\ x_{s+1} \ge 1 \Big\}, \Gamma\Big).$$

It is easy to reformulate CLT for the local case in terms of the $\mathbb{Z}^{s-1}$ actions of automorphisms of the s-torus for the case where Γ is Frolov's lattice $E\mathbb{Z}^s$. In [L2], we prove the following two variants of this approach:

2.4.2. *Z^d actions of automorphisms of the s-torus.* Let B_i be a $s \times s$ matrix with integer entries, $|\det B_i| = 1$ $(i = 1, \dots, d)$, $s - 1 \ge d \ge 2$. Let none of the eigenvalues of the matrix B_i be roots of unity $(i = 1, \dots, d)$. Suppose $B_1^{n_1} \cdots B_d^{n_d}$ is the identity matrix if and only if $n_1 = \cdots = n_d = 0$. Let g be C^∞ on $\mathbb{R}^s$, have period 1 for each coordinate, and let $g(\theta B_1^{n_1} \cdots B_d^{n_d})$ be not cohomologous to a constant cocycle (see [KK] for the definitions). Then CLT (2.37) is true with $\theta \in \Theta = [0,1]^s$, $G(N, \theta) = [1, N_1] \times \cdots \times [1, N_d]$, $\sigma(N) = \sigma_{g,B_1,\dots,B_d}(N_1 N_2 \cdots N_d)^{1/2}$ for some $\sigma_{g,B_1,\dots,B_d} > 0$, and

$$f(n, \theta) = g(\theta B_1^{n_1} \cdots B_d^{n_d}).$$

2.4.3. *Salem-Zygmund CLT.* Let $m_k \ge 1$ be integers, $m_{k+1}/m_k \ge c > 1$ for $k = 1, 2, \dots$. And let a_k be reals, $A_M = (a_1^2 + \cdots + a_M^2)^{1/2}$, $|A_M| \to \infty$, for $M \to \infty$, and $\max_{1 \le m \le M} |a_m|/A_M \to 0$ for $M \to \infty$. Then $(a_k \cos(2\pi\theta n_k))_{1 \le k \le M}$ satisfies the CLT (2.37) with $\sigma(M) = \sqrt{\pi/2}A_M$, and $\theta \in \Theta = [0, 1)$ [SZ]. In [L2], we prove the following multidimensional variant of Salem-Zygmund theorem:

Let B_i be an invertible $s \times s$ matrix with integer entries, $b_i = \det B_i$, $(i = 1, \dots, d)$, and let $b_1, \dots, b_d$ be pairwise coprime. Let none of the eigenvalues of the matrix B_i be roots of unity $(i = 1, \dots, s)$. Suppose $B_1^{n_1} \cdots B_d^{n_d}$ is the identity matrix if and only if $n_1 = \cdots = n_d = 0$. Let $h \in \mathbb{Z}^s$, $|h| > 0$, $N = (N_1, \ldots, N_d)$, $G(N) = [1, N_1] \times \cdots \times [1, N_d]$, $N_0 = \max(N_1, \ldots, N_d)$, a_n be reals, $A_N = (\sum_{n \in G(N)} a_N^2)^{1/2}$, $A_N \to \infty$ for $N_0 \to \infty$, and $\max_{n \in G(N)} |a_n|/A_N \to 0$ for $N_0 \to \infty$. Then CLT (2.37) is true with $\theta \in \Theta = [0,1]^s$, $G(N, \theta) = G(N)$, $\sigma(N) = \sigma_{B_1,\ldots,B_d} A_N$ for some $\sigma_{B_1,\ldots,B_d} > 0$, and

$$f(n, \theta) = a_n \cos(2\pi \langle \theta B_1^{n_1} \cdots B_d^{n_d}, h \rangle).$$

The main tool in the proofs of the CLT of Sections 2.4.2 and 2.4.3 is the S-unit theorem (see, for example, [ESS]).

2.4.4. *Non-Archimedean case.* Let F_q be the finite field with q elements, $F_q((x^{-1}))$ be the field of formal Laurent series. In [L3], we generalize the l.d.s. constructions mentioned in Sections 1.1.a and 1.1.b to the S-integers (adelic) case, obtained from admissible lattices in $\mathbb{R}^s$ and in $(F_q((x^{-1})))^s$. In [L2], we obtain probabilistic results similar to subsections (2.4.1)–(2.4.3) for this setting.

In [L2], we also get CLT with order of magnitude of standard deviation equal to $(\ln N)^{s/2}$ (instead of $N^{1/2}$ as usual) for all multidimensional l.d.s. mentioned in this paper. In [LM], we discuss this probabilistic phenomenon in the 1-dimensional case.

3. Proofs of Theorems 2.1–2.4 and Theorem 2.9 (case $\nu = 1$).

Proof of Theorem 2.1. First we will show that Theorem 2.1 follows easily from Lemmas 3.1 and 3.2.

Lemma 3.1: *Let $\Gamma \in \mathbb{R}^{s+1}$ be an arbitrary lattice, and $\#M_{x,\nu} = \infty$ for all $x \in [0,1)^s$ and $\nu = 1, 2$. Then $T_{1,\Gamma}$ is an invertible transformation, and $T_{1,\Gamma}^n(x) = z(x, n)$ for $n \in \mathbb{Z}$.*

Lemma 3.2: *Let $\Gamma \in \mathbb{R}^{s+1}$ satisfy (2.5), $0 < y_1, \ldots, y_s \leq 1$, and $z, z_1 \in \mathbb{R}$, $z > 0$. Then uniformly on z_1, we have*

$$R([0, y_1) \times \cdots \times [0, y_s) \times [z_1, z_1 + z], \tilde{\Gamma}) = o(z), \quad \text{for } z \to \infty. \tag{3.1}$$

Consider the case that (2.5) is valid. By (2.2) and Lemma 3.2, we have $\#M_{x,\nu} = \infty$ for all $x \in [0,1)^s$ and $\nu = 1, 2$. Applying Lemma 3.1, (2.2) and

(2.6), we get

$$(3.2)\qquad \bigcup_{n=M}^{M+N-1} (T^n_{1,\Gamma}(x), z_{s+1}(x,n)) = \bigcup_{n=M}^{M+N-1} (z(x,n), z_{s+1}(x,n)) = (\tilde{\Gamma} + (x,0)) \cap [0,1)^s \times [z_{s+1}(x,M), z_{s+1}(x, M+N-1)].$$

Let $M_1 = 0$ and $M_2 = -N+1$. Using (1.10) and Lemma 3.2, we obtain

$$(3.3)\qquad N = \#\{(\tilde{\Gamma} + (x,0)) \cap [0,1)^s \times [z_{s+1}(x,M_j), z_{s+1}(x,M_j+N-1)]\} = (\det\tilde{\Gamma})^{-1}(z_{s+1}(x,M_j+N-1) - z_{s+1}(x,M_j))(1+o(1)),$$

and

(3.4)

$$\#\{M_j \le n \le M_j + N - 1 : T^n_{1,\Gamma}(x) \in \prod_{i=1}^{s}[0,y_i)\} = \#\{(\tilde{\Gamma} + (x,0)) \cap [0,y_i) \times \cdots \times [0,y_s)[z_{s+1}(x,M_j), z_{s+1}(x,M_j+N-1)]\} = (\det\tilde{\Gamma})^{-1} y_1, \dots y_s(z_{s+1}(x,M_j+N-1) - z_{s+1}(x,M_j))(1+o(1)), \quad j = 1,2.$$

Now bearing in mind (3.2), we have from (3.3) and (3.4)

$$\#\left\{M_j \le n \le M_j + N - 1 : T^n_{1,\Gamma}(x) \in \prod_{i=1}^{s}[0,y_i) \subset [0,1)^s\right\} = Ny_1 \cdots y_s + o(N)$$

for all $y_i \in (0,1]$ $(i = 1,\dots,s)$, $j \in \{1,2\}$, and $x \in [0,1)^s$. Hence, the sequence $(T^{an}_{1,\Gamma}(x))_{n=0}^{N-1}$ is uniformly distributed in $[0,1)^s$ as $N \to \infty$ for all $x \in [0,1)^s$ and $a \in \{-1,1\}$. Thus $T_{1,\Gamma}$ is an invertible ergodic transformation of $[0,1)^s$.

Now let (2.5) be not valid. Hence there exists $(\gamma^{\perp}_{0,1},\dots,\gamma^{\perp}_{0,s+1}) \in \Gamma^{\perp} \setminus \{0\}$ with $\gamma^{\perp}_{0,s+1} = 0$. Let $\gamma^{\perp}_{0,j} \neq 0$ for $j \in [1,s]$, $f(x) = \exp(2\pi i(x_1\gamma^{\perp}_{0,1} + \cdots + x_s\gamma^{\perp}_{0,s}))$, and let $\sigma_1 := \int_{[0,1]^s} f(x)dx$. It is easy to see that

$$|\sigma_1| \le \left|\int_{[0,1]} \exp(2\pi i(x_j\gamma^{\perp}_{0,j})dx_j\right| = \left|\frac{\sin(\pi\gamma^{\perp}_{0,j})}{\pi\gamma^{\perp}_{0,j}}\right| = \beta < 1.$$

Consider the following statement:

$$(3.5)\qquad \exists w_n = w_n(x) \ge 0 \quad \text{with} \quad (T^n_{1,\Gamma}(x), w_n) \in \tilde{\Gamma} + (x,0).$$

It is evident that (3.5) is true for $n=0$ ($w_0=0$). Suppose that (3.5) is true for $n=k\geq 0$. By (2.4) and (2.2), we have

$$
(3.6)\quad \exists v\geq 0 \quad \text{with} \quad (T_{1,\Gamma}^{k+1}(x),v)\in\tilde{\Gamma}+(T_{1,\Gamma}^{k}(x),0) \\
=\tilde{\Gamma}+(T_{1,\Gamma}^{k}(x),w_k)-(0,w_k)=\tilde{\Gamma}+(x,0)-(0,w_k).
$$

Hence $(T_{1,\Gamma}^{k+1}(x),v+w_k)\in\tilde{\Gamma}+(x,0)$. Now by induction, we obtain that (3.5) is true for all $n\geq 0$. Therefore, $T_{1,\Gamma}^{n}(x)=x+(u_{1,n},\dots,u_{s,n})$ for some $(u_{1,n},\dots,u_{s+1,n})\in\tilde{\Gamma}$. Bearing in mind that

$$
u_{1,n}\gamma_{0,1}^{\perp}+\cdots+u_{s,n}\gamma_{0,s}^{\perp}=u_{1,n}\gamma_{0,1}^{\perp}+\cdots+u_{s,n}\gamma_{0,s}^{\perp}+u_{s+1,n}\gamma_{0,s+1}^{\perp}\in\mathbb{Z},
$$

we get

$$
\begin{aligned}
f(T_{1,\Gamma}^{n}(x))&=\exp(2\pi i((x_1+u_{1,n})\gamma_{0,1}^{\perp}+\cdots+(x_s+u_{s,n})\gamma_{0,s}^{\perp}))\\
&=\exp(2\pi i(x_1\gamma_{0,1}^{\perp}+\cdots+x_s\gamma_{0,s}^{\perp}))=\sigma_2(x).
\end{aligned}
$$

Thus

$$
\left|\frac{1}{N}\sum_{0\leq n<N}f(T_{1,\Gamma}^{n}(x))-\int_{[0,1]^s}f(x)dx\right|=|\sigma_2(x)-\sigma_1|\geq 1-\beta>0.
$$

Using Weyl's criterion (see [DrTi, p. 3]), we have that for all $x\in[0,1)^s$ the sequence $(T_{1,\Gamma}^{n}(x))_{n\geq 0}$ is not uniformly distributed. Theorem 2.1 is proved. ∎

Proof of Lemma 3.1. By (2.6) and (2.2), we have

$$
z_{s+1}(x,k+1)=\min\{y>z_{s+1}(x,k):\exists w\in[0,1)^s \quad \text{with } (w,y)\in\tilde{\Gamma}+(x,0)\}.
$$

Let $k\in\mathbb{Z}$, $\mathbf{0}=(0,\dots,0)\in\mathbb{Z}^s$, and $x'=z_{x,k}\in[0,1)^s$. Then

$$
\begin{aligned}
\tilde{\Gamma}+(x',0)&=\tilde{\Gamma}+(z_{x,k},z_{s+1}(x,k)-z_{s+1}(x,k))\\
&=\tilde{\Gamma}+(z_{x,k},z_{s+1}(x,k))-(\mathbf{0},z_{s+1}(x,k))\\
&=\tilde{\Gamma}+(x,0)-(\mathbf{0},z_{s+1}(x,k)).
\end{aligned}
$$

Hence

$$
z_{s+1}(x',1)=\min\{y>0:\exists w\in[0,1)^s \\
\text{with } (w,y)\in\tilde{\Gamma}+(x',0)=\tilde{\Gamma}+(x,0)-(\mathbf{0},z_{s+1}(x,k))\}.
$$

Thus

$$z_{s+1}(x',1) + z_{s+1}(x,k) = \min\{y > z_{s+1}(x,k) : \exists w \in [0,1)^s \text{ with } (w,y) \in \tilde{\Gamma} + (x,0)\} = z_{s+1}(x,k+1),$$

and

$$(z_{x,k+1}, z_{s+1}(x',1) + z_{s+1}(x,k)) \in \tilde{\Gamma} + (x,0), \quad (z_{x,k+1}, z_{s+1}(x',1)) \in \tilde{\Gamma} + (x',0).$$

By (2.6), and (2.1) - (2.4), we get

$$z_{x',1} = z_{x,k+1} \quad \text{and} \quad T_{1,\Gamma}(z_{x,k}) = z_{x,k+1}, \quad \text{forall } k \in \mathbb{Z}.$$

Bearing in mind that $z_{x,0} = x$, we obtain the assertion of Lemma 3.1. ∎

Proof of Lemma 3.2. We will use the notation and the approach from the Appendix (Poisson's summation formula, etc.). Let $0 < \tau < \frac{1}{4}\min(y_1, \dots, y_s)$,

$$\mathcal{O}_\tau^+ = \prod_{i=1}^{s} [-\tau, y_i + \tau) \times [z_1 - \tau, z_1 + z + \tau]$$

and

$$\mathcal{O}_\tau^- = \prod_{i=1}^{s} [\tau, y_i - \tau) \times [z_1 + \tau, z_1 + z - \tau].$$

We see that

(3.7)

$$\begin{aligned}\mathrm{vol}\mathcal{O}_\tau^+ - \mathrm{vol}\mathcal{O}_\tau^- =&(2\tau + z)\prod_{i=1}^{s}(2\tau + y_i) - (z - 2\tau)\prod_{i=1}^{s}(y_i - 2\tau)\\ =&zy_1\cdots y_s\Big((1+\frac{2\tau}{z})\prod_{i=1}^{s}(1+\frac{2\tau}{y_i}) - (1-\frac{2\tau}{z})\prod_{i=1}^{s}(1-\frac{2\tau}{y_i})\Big)\\ =&O(z\tau(\frac{1}{z} + \frac{1}{y_1} + \cdots + \frac{1}{y_s})) = O(z\tau), \quad \text{for } z \to \infty.\end{aligned}$$

Now it is easy to compute the Fourier transform of the characteristic function $\chi(\mathcal{O}, \gamma)$ of the region $\mathcal{O} = [0, y_1] \times \cdots \times [0, y_s] \times [z_1, z_1 + z]$ (see, for example, [Skr1, (7.13)]):

$$\hat{\chi}(\mathcal{O}, \gamma) = \frac{\sin(\pi z \gamma_{s+1})}{\pi \gamma_{s+1}} \prod_{j=1}^{s} \frac{\sin(\pi y_j \gamma_j)}{\pi \gamma_j} e^{-\langle 2\pi i \gamma, x' \rangle} \tag{3.8}$$

with $\gamma \in \tilde{\Gamma}^{\perp} \setminus \{0\}$, and $x' = (\frac{1}{2}y_1, \dots, \frac{1}{2}y_s, z_1 + \frac{1}{2}z) \in \mathbb{R}^{s+1}$. Here we define $\frac{\sin(\pi y \gamma)}{\pi\gamma} = y$ for $\gamma = 0$. Taking into account that $|\sin u| \leq |u|$ and that $\gamma_{s+1} \neq 0$ for all $\gamma = (\gamma_1, \dots, \gamma_{s+1}) \in \tilde{\Gamma}^{\perp} \setminus \{0\}$, we obtain

$$|\hat{\chi}(\mathcal{O}, \gamma)| \leq \min\Big(z, \frac{1}{\pi|\gamma_{s+1}|}\Big) \quad \text{with } \gamma \in \tilde{\Gamma}^{\perp} \setminus \{0\}. \tag{3.9}$$

Consider the functions $\omega(X)$ and $\mathcal{R}^{\pm}_{\tau}(\mathcal{O}, X)$ defined in the Appendix (see (A.14)–(A.16)):

$$\mathcal{R}^{\pm}_{\tau}(\mathcal{O}, X) = (\det \tilde{\Gamma})^{-1} \sum_{y \in \tilde{\Gamma}^{\perp} \setminus \{0\}} \hat{\chi}(\mathcal{O}^{\pm}_{\tau}, \gamma)\hat{\omega}(\tau\gamma)e^{-2\pi i \langle \gamma, X\rangle}, \tag{3.10}$$

where

$$|\hat{\omega}(Y)| < C_A(1 + \|Y\|)^{-A} \tag{3.11}$$

with arbitrarily large $A > 0$.

According to (1.10), and Lemma A.3 (see Appendix), we have the following inequality for the left-hand side of (3.1):

$$R(\mathcal{O}, \Gamma) \leq (\det \tilde{\Gamma})^{-1}(\mathrm{vol}\mathcal{O}^{+}_{\tau} - \mathrm{vol}\mathcal{O}^{-}_{\tau}) + \sup_{X \in \mathbb{R}^n} (|\mathcal{R}^{+}_{\tau}(\mathcal{O}, X)| + |\mathcal{R}^{-}_{\tau}(\mathcal{O}, X)|). \tag{3.12}$$

By (3.10), we get

$$|\mathcal{R}^{\pm}_{\tau}(\mathcal{O}, X)| \leq \sigma_1 + \sigma_2, \tag{3.13}$$

where

$$\sigma_1 = (\det \tilde{\Gamma})^{-1} \sum_{\gamma \in \tilde{\Gamma}^{\perp} \setminus \{0\},\ \|\gamma\| \leq \frac{1}{8}\tau^{-2s}} |\hat{\chi}(\mathcal{O}^{\pm}_{\tau}, \gamma)\hat{\omega}(\tau\gamma)|, \tag{3.14}$$

and

$$\sigma_2 = (\det \tilde{\Gamma})^{-1} \sum_{\gamma \in \tilde{\Gamma}^{\perp} \setminus \{0\},\ \|\gamma\| > \frac{1}{8}\tau^{-2s}} |\hat{\chi}(\mathcal{O}^{\pm}_{\tau}, \gamma)\hat{\omega}(\tau\gamma)|. \tag{3.15}$$

Bearing in mind (3.9), (3.11) and taking into account that $\gamma_{s+1} \neq 0$ for all $\gamma = (\gamma_1, \dots, \gamma_{s+1}) \in \tilde{\Gamma}^{\perp} \setminus \{0\}$, we obtain

$$\sigma_1 \leq (\pi \det \tilde{\Gamma})^{-1} C_A \sum_{\gamma \in \tilde{\Gamma} \setminus \{0\},\ \|\gamma\| \leq \frac{1}{8}\tau^{-2s}} |\gamma_{s+1}|^{-1} = C(A, \tau, \tilde{\Gamma}). \tag{3.16}$$

Using (A.39) with $n = s+1$ and $A = s+4$, we derive from (3.9) and (3.11)

$$\begin{aligned}\sigma_2 \leq & z(\det \tilde{\Gamma})^{-1} \sum_{\gamma \in \tilde{\Gamma}^{\perp},\ \|\gamma\| > \frac{1}{8}\tau^{-2s}} |\hat{\omega}(\tau\gamma)| \\ \leq & zC_{s+4}(\tilde{\Gamma})\tau^{-s-4}(\det \tilde{\Gamma})^{-1} \sum_{\gamma \in \tilde{\Gamma}^{\perp}, \|\gamma\| > \frac{1}{8}\tau^{-2s}} \|\gamma\|^{-s-4} \\ \leq & zC_1(\tilde{\Gamma})\tau^{-s-4}(\tau^{-2s})^{s+1-s-4} = z\tau^{5s-4}C_1(\tilde{\Gamma}) \leq z\tau C_1(\tilde{\Gamma}).\end{aligned} \tag{3.17}$$

Substituting (3.7), (3.13), (3.16) and (3.17) into (3.12), we obtain $R(\mathcal{O}, \tilde{\Gamma}) = O(z\tau)$ for $z \to \infty$. Taking into account that $\tau > 0$ is arbitrarily small, we obtain

$$R(\mathcal{O}, \tilde{\Gamma}) = o(z) \quad \text{for } z \to \infty \tag{3.18}$$

with an arbitrary closed box $\mathcal{O}$ uniformly on z_1. Now (3.1) follows from (3.18). Lemma 3.2 is proved. ∎

Proofs of Theorems 2.2, 2.3 and 2.9 (case $\nu = 1$).

Proof of Theorem 2.3. Let $x \in [0,1)^s$, $0 < y_1, \ldots, y_s \leq 1$, $M, N \in \mathbb{Z}$, $N \geq 1$, $P_1 = [0,1)^s \times [0,1]$. According to Proposition 2.1 there exists a set $E_{P_1,\tilde{\Gamma}_1} \in SO(s+1)$ with $\mu_{s+1}(SO(s+1) \setminus E_{P_1,\tilde{\Gamma}_1}) = 0$ and a function $C(\varepsilon)$, such that for all $U \in E_{P_1,\tilde{\Gamma_1}}$

$$\begin{aligned}\Big| \#\Big\{ \prod_{i=1}^{s} [0, y_i) \times [M_1, M_1 + N_1] \cap (U\tilde{\Gamma}_1 + (x,0)) \Big\} \\ - (\det \tilde{\Gamma}_1)^{-1} N_1 y_1 \cdots y_s \Big| \leq C(\varepsilon)(\ln N_1)^{s+\varepsilon}, \quad N_1 = 1, 2, \ldots\end{aligned} \tag{3.19}$$

uniformly on $x \in [0,1)^s$, $M_1 \in \mathbb{R}$ and $y_i \in (0,1]$, $i = 1, \ldots, s$.

We fix $U \in E_{P_1,\tilde{\Gamma}_1}$. Let $\tilde{\Gamma} = U\tilde{\Gamma}_1$. Applying (3.19), (2.2) (2.6) and Lemma 3.1, we obtain that for all $x \in [0,1)^s$ we have $\# M_{x,\nu} = \infty$ $(\nu = 1, 2)$ and $T^n_{1,\Gamma}(x) = z(x, n)$ for $n \in \mathbb{Z}$. Now let

$$v_x(M, N) = [z_{s+1}(x, M), z_{s+1}(x, M + N - 1)],$$
$$|v_x(M, N)| = z_{s+1}(x, M + N - 1) - z_{s+1}(x, M).$$

By the definition (2.6) of the sequence $z_{s+1}(x, k)$, we obtain

$$N = \#\{[0,1)^s \times v_x(M, N) \cap (\tilde{\Gamma} + (x, 0))\}. \tag{3.20}$$

Applying (3.19) with $y_i = 1$ $(i = 1, \dots, s)$, and $[M_1, M_1 + N_1] = v_x(M, N)$, we get

$$|(\det \tilde{\Gamma})^{-1} |v_x(M, N)| - N| \leq C(\varepsilon)(\ln |v_x(M, N)|)^{s+\varepsilon}.$$

Hence,

$$|(\det \tilde{\Gamma})^{-1} |v_x(M, N)| - N|) \leq 2C(\varepsilon)(\ln N)^{s+\varepsilon} \tag{3.21}$$

for $N > N(\varepsilon, \tilde{\Gamma})$.

By (1.1) and (3.2) we have

$$\begin{aligned}
\Delta_1 =& N\left|\Delta\left(\prod_{i=1}^{s}[0, y_i), (T^n_{1,\tilde{\Gamma}}(x))_{n=M}^{M+N-1}\right)\right| \\
=& \left|\mathcal{R}\left(\prod_{i=1}^{s}[0, y_i) \times v_x(M, N), \tilde{\Gamma} + (x, 0)\right)\right. \\
& \left. + (\det \tilde{\Gamma})^{-1} |v_x(M, N)| y_1 \cdots y_s - N y_1 \cdots y_s\right| \\
\leq& \left|\mathcal{R}\left(\prod_{i=1}^{s}[0, y_i) \times v_x(M, N), \tilde{\Gamma} + (x, 0)\right)\right| \\
& + y_1 \cdots y_s |(\det \tilde{\Gamma})^{-1} |v_x(M, N)| - N|.
\end{aligned} \tag{3.22}$$

Using (3.19) and (3.21), we obtain

$$\Delta_1 \leq C(\varepsilon)(\ln |v_x(M, N)|)^{s+\varepsilon} + 2C(\varepsilon)(\ln N)^{s+\varepsilon} \leq 4C(\varepsilon)(\ln N)^{s+\varepsilon},$$

for $N > N_1(\varepsilon, \tilde{\Gamma})$. By (1.2), Theorem 2.3 is proved. ∎

Proof of Theorem 2.2. Let $\Gamma \subset \mathbb{R}^{s+1}$ be an admissible lattice. According to [Skr1, Lemma 3.1], the dual lattice is also admissible. By (1.13), we have that condition (2.5) is valid. Applying Lemmas 3.1 and 3.2, we obtain that for all $x \in [0, 1)^s$ we have $\#M_{x,\nu} = \infty$ $(\nu = 1, 2)$ and $T^n_{1,\Gamma}(x) = z(x, n)$ for all $n \in \mathbb{Z}$. Now, let $H_{j,a}$ be the hyperplane in $\mathbb{R}^{s+1}$ given by the equation $x_j = a$. We have for all $u \in \mathbb{R}^{s+1}$

$$\#\{H_{j,a} \cap (\Gamma + u)\} \leq 1. \tag{3.23}$$

Indeed, suppose that the hyperplane $H_{j,a}$ contains two different points $\gamma_1, \gamma_2 \in \Gamma + u$. Then the hyperplane $x_j = 0$ contains the nonzero point $\gamma_3 = \gamma_2 - \gamma_1 \in \Gamma$.

Thus $\mathrm{Nm}\gamma_3 = 0$ and so $\mathrm{Nm}\Gamma = 0$ (see (1.13)), i.e., the lattice is not admissible. Hence

$$\#\Big\{\Big(\prod_{i=1}^{s}[0,y_i] \times v_x(M,N) \setminus \prod_{i=1}^{s}[0,y_i) \times v_x(M,N)\Big) \cap (\tilde{\Gamma}+u)\Big\} \leq s. \tag{3.24}$$

Applying Theorem A, we obtain

$$\begin{aligned}\Big|\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i) \times v_x(M,N), \tilde{\Gamma}+(x,0)\Big)\Big| \\ &\leq s + |\mathcal{R}((y_1,\dots,y_s,|v_x(M,N)|) \centerdot \mathbb{K}^{s+1}+u, \tilde{\Gamma}+(x,0))| \\ &\leq s + C(\tilde{\Gamma})(\ln(|v_x(M,N)|+2))^{s-1},\end{aligned} \tag{3.25}$$

with $u = (\frac{1}{2}y_1,\dots,\frac{1}{2}y_s,\frac{1}{2}(z_{s+1}(x,M)+z_{s+1}(x,M+N-1)))$. For $y=(1,\dots,1)$, we get from (3.20) and (3.25)

$$|N - |v_x(u,N)|(\det\tilde{\Gamma})^{-1}| \leq s + C(\tilde{\Gamma})(\ln(|v_x(M,N)|+2))^{s-1}.$$

Thus

$$|v_x(M,N)| = N\det\tilde{\Gamma} + \theta C(\tilde{\Gamma})(\ln N)^{s-1} \tag{3.26}$$

with $|\theta| \leq 2$ for $N \geq N_0(\tilde{\Gamma})$. Now, by (3.22), (3.25), and (3.26), we obtain the assertion of Theorem 2.2. ∎

Using Propositions 2.1, 2.2 and Theorem A, we obtain in a similar way the assertion of Theorem 2.9 (case $\nu = 1$).

Proof of Theorem 2.4. By (1.3),

$$\int_{[0,1)^s}\Big(D_p((T^n_{1,\Gamma}(x))_{n=0}^{N-1})\Big)^p dx = \int_{[0,1]^s}\int_{[0,1)^s}|\Delta(\mathcal{O},(T^n_{1,\Gamma}(x))_{n=0}^{N-1})|^p dxdy$$

with $\mathcal{O} = [0,y_1)\times\cdots\times[0,y_s)$. Therefore to prove Theorem 2.4, it is sufficient to show that

$$\Big(\int_{[0,1)^s}|\Delta(\mathcal{O},((T^n_{1,\Gamma}(x))_{n=0}^{N-1})|^p dx)\Big)^{1/p} = O(N^{-1}(\ln N)^{s/2}) \tag{3.27}$$

for all $(y_1,\dots,y_s)\in[0,1]^s$. Let

$$u_x(N) = \begin{cases}[N\det\tilde{\Gamma}, z_{s+1}(x,N)], & \text{if} N\det\tilde{\Gamma} \leq z_{s+1}(x,N),\\ [z_{s+1}(x,N), N\det\tilde{\Gamma}], & \text{otherwise.}\end{cases} \tag{3.28}$$

Using (3.22) with $M = 0$, we obtain

$$\begin{aligned}\Delta_1 =&|N\Delta(\mathcal{O},(T^n_{1,\Gamma}(x))_{n=0}^{N-1})| \\ =&\Big|\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i)\times[0,z_{s+1}(x,N)],\tilde{\Gamma}+(x,0)\Big) \\ &+(\det\tilde{\Gamma})^{-1}z_{s+1}(x,N)y_1\cdots y_s - Ny_1\cdots y_s\Big|.\end{aligned} \tag{3.29}$$

For $y_i = 1$, $i = 1,\dots,s$, we get from (1.1) that $\Delta_1 = 0$, and

$$N = (\det\tilde{\Gamma})^{-1}z_{s+1}(x,N) + \mathcal{R}\Big([0,1)^s\times[0,z_{s+1}(x,N)],\tilde{\Gamma}+(x,0)\Big). \tag{3.30}$$

It follows from (1.10) that:

$$\begin{aligned}&\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i)\times[0,z_{s+1}(x,N)],\tilde{\Gamma}+(x,0)\Big) \\ &=\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i)\times[0,\det\tilde{\Gamma}N],\tilde{\Gamma}+(x,0)\Big) \\ &\quad+(-1)^{\theta(x)}\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i)\times u_x(N),\tilde{\Gamma}+(x,0)\Big),\end{aligned} \tag{3.31}$$

where $\theta(x) = 0$ if $N\det\tilde{\Gamma} \le z_{s+1}(x,N)$, and $\theta(x) = 1$ otherwise.

We derive from (3.29)–(3.31)

$$\begin{aligned}&|N-(\det\tilde{\Gamma})^{-1}z_{s+1}(x,N)| \\ &\quad\le|\mathcal{R}([0,1)^s\times[0,N\det\tilde{\Gamma}],\tilde{\Gamma}+(x,0))|+|\mathcal{R}([0,1)^s\times u_x(N),\tilde{\Gamma}+(x,0))|,\end{aligned}$$

and

$$\begin{aligned}\Delta_1\le&\Big|\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i)\times[0,N\det\tilde{\Gamma}],\tilde{\Gamma}+(x,0)\Big)\Big| \\ &+\Big|\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i)\times u_x(N),\tilde{\Gamma}+(x,0)\Big)\Big| \\ &+|\mathcal{R}([0,1)^s\times[0,N\det\tilde{\Gamma}],\tilde{\Gamma}+(x,0))| \\ &+|\mathcal{R}([0,1)^s\times u_x(N),\tilde{\Gamma}+(x,0))|.\end{aligned} \tag{3.32}$$

Similarly to (3.20)–(3.21), we have from Theorem A

$$|N-(\det\tilde{\Gamma})^{-1}z_{s+1}(x,N)|\le C_1(\tilde{\Gamma})(\ln N)^s. \tag{3.33}$$

Again applying Theorem A, we get from (3.28) and (3.33):

$$\Big|\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i)\times u_x(N),\tilde\Gamma+(x,0)\Big)\Big|\leq C_2(\tilde\Gamma)(\ln(\ln N))^s.$$

By (3.24) and (3.32), we have

$$\begin{aligned}\Delta_1\leq & 2s+2C_2(\tilde\Gamma)(\ln(\ln N))^s\\ &+\Big|\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i]\times[0,N\det\tilde\Gamma],\tilde\Gamma+(x,0)\Big)\Big|\\ &+|\mathcal{R}([0,1]^s\times[0,N\det\tilde\Gamma],\tilde\Gamma+(x,0))|.\end{aligned}\tag{3.34}$$

From (2.1), we obtain

$$\begin{aligned}&\Big|\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i]\times[0,N\det\tilde\Gamma],\tilde\Gamma+(x,0)\Big)\Big|\\ &\quad\leq 2+\Big|\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i]\times[0,N\det\tilde\Gamma],\tilde\Gamma+(x,u)\Big)\Big|\quad\text{with } u\in[0,1].\end{aligned}\tag{3.35}$$

Using Minkowski's inequality,

$$\begin{aligned}\Big(\int_{[0,1)^{s+1}}|f(x)+g(x)|^p dx\Big)^{1/p}\leq & \Big(\int_{[0,1)^{s+1}}|f(x)|^p dx\Big)^{1/p}\\ &+\Big(\int_{[0,1)^{s+1}}|g(x)|^p dx\Big)^{1/p},\end{aligned}\tag{3.36}$$

we get from (3.35)

$$\begin{aligned}\sigma:=&\Big(\int_{[0,1)^s}\Big|\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i]\times[0,N\det\tilde\Gamma],\tilde\Gamma+(x,0)\Big)\Big|^p dx\Big)^{1/p}\\ \leq & 2+\min_{0\leq u<1}\Big(\int_{[0,1)^s}\Big|\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i]\times[0,N\det\tilde\Gamma],\tilde\Gamma+(x,u)\Big)\Big|^p dx\Big)^{1/p}\\ \leq & 2+\Big(\int_{[0,1)^s}\int_{[0,1)}\Big|\mathcal{R}\Big(\prod_{i=1}^{s}[0,y_i]\times[0,N\det\tilde\Gamma],\tilde\Gamma+(x,u)\Big)\Big|^p dxdu\Big)^{1/p}.\end{aligned}$$

It is easy to see that there exists an integer $k \geq 1$ such that $\frac{1}{k}[0,1)^{s+1} \subset \tilde{\mathbb{F}}$, where $\tilde{\mathbb{F}}$ is a fundamental set of the lattice $\tilde{\Gamma}$. We derive from Theorem A that

$$\sigma \leq 2 + \left(\int_{k\tilde{\mathbb{F}}} \left| \mathcal{R}\left(\prod_{i=1}^{s} [0, y_i] \times [0, N \det \tilde{\Gamma}], \tilde{\Gamma} + v \right) \right|^p dv \right)^{1/p}$$
$$\leq C_2(p, \Gamma)(\ln N)^{s/2}.$$

Applying Minkowski's inequality to (3.34), we obtain

$$\left(\int_{[0,1)^s} |\Delta_1|^p dx \right)^{1/p} \leq 2s + 2C_2(\tilde{\Gamma})(\ln(\ln N))^s + 2C_2(p, \Gamma)(\ln N)^{s/2}.$$

By (3.29), assertion (3.27) and Theorem 2.4 are proved. ∎

4. Proofs of Theorems 2.5–2.8, and 2.9 (case $\nu = 2$).

First we will show that the set $(T^n_{2,\Gamma}(x))_{n=0}^{N-1}$ is the union of $O(\ln N)$ lattice nets. Next, applying Theorem A, Propositions 2.1 and 2.2, we obtain the assertions of the theorems.

Let

$$B = (b_0, \dots, b_{q^s-1}), \quad \text{with } b_k = \sum_{i=1}^{s} j_i H \mathbf{e_i},\ b_0 = \mathbf{0}, \tag{4.1}$$
$$\text{where} \quad k = \sum_{i=1}^{s} j_i q^{i-1} \quad \text{and} \quad j_i \in \{0, 1, \dots, q-1\}$$

(see (2.11)–(2.14)). According to (2.15)–(2.18), for all $x \in [0,1)^s$, we have a unique expansion

$$x = q^{-r_1}\left(x_0 + \sum_{i=1}^{\infty} x_i / q^i \right), \tag{4.2}$$

where $x_0 \in B_0 = \{a_1, \dots, a_{r_2}\}$ and $x_i \in B$, $i = 1, 2, \dots$. We will use also the notation $x_0.x_1x_2\dots$ for the right-hand side of (4.2):

$$x = x_0.x_1x_2\dots = q^{-r_1}\left(x_0 + \sum_{i=1}^{\infty} x_i / q^i \right).$$

We set

$$a_i \prec a_j \quad \text{for } i < j, \quad b_i \prec b_j \quad \text{for } i < j\ (a_i \in B_0, b_i \in B), \tag{4.3}$$

and

$$x = x_0.x_1x_2\ldots \prec y = y_0.y_1y_2\ldots \tag{4.4}$$

if there exists an integer $n_0 \geq 1$ with $x_{n_0} \prec y_{n_0}$, and $x_j = y_j$ for $j < n_0$; or if $x_0 \prec y_0$.

In this section we use T to denote the transformation $T_{2,\Gamma}$. By induction we derive from (1.8), (2.12), (2.13) and (2.19) that

$$T(x) = u \quad \Longleftrightarrow \quad \{x \prec u, \text{ and } \nexists z \in [0,1)^s : x \prec z \prec u\}. \tag{4.5}$$

Let

$$\Gamma_i = q^{-i-r_1+1}\Gamma, \quad \mathbb{F}_i = q^{-i-r_1+1}\mathbb{F}, \quad \mathbb{F} = H[0,1)^s, \tag{4.6}$$

$$x = .x_1x_2\ldots = q^{-r_1}\sum_{i=1}^{\infty} x_i/q^i, \tag{4.7}$$

be the canonical q-expansion of $x \in \mathbb{F}_1$ with $x_i \in B$, $i = 1,2,\ldots$, and

$$T_{\mathbb{F}}(x) = q^{-r_1}H(T_{q,s}(H^{-1}q^{r_1}x)). \tag{4.8}$$

We prove the ergodicity of the transformation $T_{q,s} : [0,1)^s \to [0,1)^s$ (see (2.13)) in a way completely similar to the proof of the ergodicity of the transformation T_q (see [Fr, pp. 75–83] and [Pe, pp. 208–212]). The ergodicity of the transformation $T_{\mathbb{F}} : \mathbb{F}_1 \to \mathbb{F}_1$ follows from (4.8).

Let $x = x_0.x_1x_2\ldots$,

$$\bar{x}_k = .x_kx_{k+1}\ldots \in \mathbb{F}_1, \quad \text{with } x_{k+i} \in B, \ \ i = 0,1,\ldots$$

and

$$\tilde{x}_k^\nu = T_{\mathbb{F}}^\nu(\bar{x}_k), \quad \nu = 0,1. \tag{4.9}$$

We will use the notation

$$\tilde{x}_k^\nu = .x_{k,1}^\nu x_{k,2}^\nu \ldots := .x_k^\nu \ldots, \tag{4.10}$$

with $x_k^\nu = x_{k,1}^\nu$, $x_{k,i}^\nu \in B\,(i,k = 1,2\ldots)$, and

$$a_1.0\ldots 0bx_ix_{i+1}\ldots := q^{-r_1}\left(a_1 + b/q^{i-1} + \sum_{j=i}^{\infty} x_j/q^j\right), \tag{4.11}$$

$$a_1.0\ldots 0x_i^\nu\ldots := q^{-r_1}\left(a_1 + \sum_{j=i}^{\infty} x_{i,j}^\nu/q^j\right). \tag{4.12}$$

Let $x = x_0.x_1x_2\ldots \in [0,1)^s$,

$$P(i,x) = \{T^n(x),\ n \geq 0 : T^n(x) \prec a_1.0\ldots 0x_i^1\ldots\}, \quad i \geq 1. \tag{4.13}$$

From (4.3)–(4.5), we have by induction that

$$P(i,x) = \{u \in [0,1)^s : x \preceq u \prec a_1.0\ldots 0x_i^1\ldots\}. \tag{4.14}$$

Now let

(4.15)

$$Q(1,x) = \{u \in [0,1)^s : x = x_0.x_1x_2\ldots \preceq u \prec a_1.x_1^1\ldots\}$$

$$Q(i,x,j) = \{u \in [0,1)^s : a_1.0\ldots 0b_jx_ix_{i+1}\ldots . \preceq u \prec a_1.0\ldots 0b_{j+1}x_ix_{i+1}\ldots\}$$

for $i \geq 2,\ j \in [0, q^s-2]$ and

(4.16)

$$Q(i,x,q^s-1) = \{u \in [0,1)^s : a_1.0\ldots 0b_{q^s-1}x_ix_{i+1}\ldots \preceq u \prec a_1.0\ldots 0x_i^1\ldots\}.$$

From (4.14)–(4.16), we obtain by induction that

$$P(i,x) = Q(1,x) \bigcup_{2\leq k\leq i} \bigcup_{\substack{0\leq j<q^s,\\ x_{k-1}\prec b_j}} Q(k,x,j). \tag{4.17}$$

Lemma 4.1: *Let $x \in [0,1)^s,\ k \geq 2$ and $j \in [0, q^s-1]$. Then*

$$Q(k,x,j) = (\Gamma_{k-1} + q^{-k+2}.b_jx_kx_{k+1}) \cap [0,1)^s. \tag{4.18}$$

Proof. By (2.15)–(2.16) and (4.6) we have

$$G(a_i) = (q^{-r_i}a_i + \mathbb{F}_1) \cap [0,1)^s, \tag{4.19}$$

$$G(a_i) \cap G(a_j) = \emptyset \quad \text{for} i \neq j \quad \text{and} \quad \bigcup_{i=1}^{r_2} G(a_i) = [0,1)^s.$$

Let $u = u_0.u_1u_2\ldots$,

$$\hat{u}_k = (u_0, u_1, \ldots, u_k), \quad \mathbb{F}_{k+1,u} = \mathbb{F}_{k+1} + u_0.u_1, \ldots, u_k \tag{4.20}$$

with $u_0 \in B_0 = \{a_1, \ldots, a_{r_2}\}$, and $u_i \in B$, $i = 1, 2, \ldots$;

$$B_k = \{(u_0, u_1, \ldots, u_k) : u_0 \in B_0,\ \ u_i \in B,\ \ i = 1, \ldots, k\}, \tag{4.21}$$

and

$$\begin{aligned} G_1(\hat{u}_k) &= (q^{-r_1}(u_0 + \sum_{i=1}^{k} u_i/q^i) + \mathbb{F}_{k+1}) \cap [0,1)^s \\ &= (u_0.u_1\ldots u_k + \mathbb{F}_{k+1}) \cap [0,1)^s = \mathbb{F}_{k+1,u} \cap [0,1)^s. \end{aligned} \tag{4.22}$$

From (4.1), (4.2), (4.6) and (4.19), we obtain by induction

$$\bigcup_{\hat{u}_k \in B_k} G_1(\hat{u}_k) = [0,1)^s, \tag{4.23}$$

and

$$G_1(\hat{u}_k) \cap G_1(\hat{u}_k') = \emptyset \quad \text{for } \hat{u}_k \neq \hat{u}_k'.$$

We derive from (4.15) and (4.16) that

$$Q(k,x,j) = \bigcup_{\hat{u}_{k-2} \in B_{k-2}} [0,1)^s \cap \{u_0.u_1 \dots u_{k-2} + q^{-k+2}.b_j x_k x_{k+1} \dots\}, \tag{4.24}$$

with $k \geq 2$ and $j \in [0, q^s - 1]$. Bearing in mind that $\Gamma = H\mathbb{Z}^s$, $\Gamma_k = q^{-k-r_1+1}\Gamma$, we obtain from (2.11), (4.1) and (4.2)

$$u_0.u_1 \dots u_{k-2} \in \Gamma_{k-1}. \tag{4.25}$$

Hence, $Q(k,x,j)$ belongs to the right-hand side of (4.18). Let z belong to the right-hand side of (4.18). Then

$$z = \gamma + q^{-k+2}.b_j x_k x_{k+1} \dots \in [0,1)^s, \quad \text{with } \gamma \in \Gamma_{k-1}. \tag{4.26}$$

By (4.23) we have $z \in G_1(\hat{u}_{k-2})$ for some $\hat{u}_{k-2} \in B_{k-2}$. Using (4.6), (4.22) and (4.26), we get

$$z = u_0.u_1 \dots u_{k-2} + w, \quad \text{with } w \in \mathbb{F}_{k-1} = q^{-k-r_1+2}\mathbb{F},$$

and

$$\gamma - u_0.u_1 \dots u_{k-2} = w - q^{-k+2}.b_j x_k x_{k+1} \dots . \tag{4.27}$$

From (4.6), (4.7) and (4.25), we obtain

$$\gamma - u_0.u_1 \dots u_{k-2} \in \Gamma_{k-1}, \quad \text{and} \quad q^{-k+2}.b_j x_k \dots \in \mathbb{F}_{k-1}.$$

Taking into account that $\mathbb{F}_{k-1}$ is a fundamental set of the lattice Γ_{k-1}, we get from (4.27) that $\gamma = u_0.u_1 \dots u_{k-2}$.

By (4.24) and (4.26) we obtain (4.18). Lemma 4.1 is proved. ∎

Proof of Theorem 2.5. By Lemma 4.1, (1.20) and (1.10), we get

$$\#Q(k,x,q^s-1) \to \infty, \quad \text{for all } x \in [0,1)^s \text{ as } k \to \infty.$$

According to (2.11) and (2.18), for arbitrary $x \in [0,1)^s$ and all $n_0 > 0$ there exists $n_1 > n_0$ such that $x_{n_1} \neq b_{q^s-1}$. Hence, for $i = n_1 + 1$ the interior sum in (4.17) is not empty. Thus

$$\#P(i,x) \to \infty \quad \text{for all } x \in [0,1)^s \text{ as } i \to \infty.$$

Define the integer $m_0 = m_0(n,x)$ by the inequality

$$\#P(m_0,x) \le N < \#P(m_0+1,x). \tag{4.28}$$

We derive from (4.13) and (4.28) that

$$T^k(x) = a_1.0\dots 0x^1_{m_0} \quad \text{with}\, k = \#P(m_0,x),\ x_{m_0} \ne b_{q^s-1}, \tag{4.29}$$

and

$$T^N(x) = z = z_0.z_1\dots z_{m_0}x_{m_0+1}x_{m_0+2}\dots \tag{4.30}$$

for some $(z_0,z_1,\dots,z_{m_0}) \in B_{m_0}$, with $x_{m_0} \prec z_{m_0}$. Hence by (4.5)

$$\begin{aligned}\{T^n(x): 0\le n\le N-1\} =& \{u\in[0,1)^s : x \preceq u \prec z\}\\ =& \{u\in[0,1)^s : x\preceq u \prec a_1.0\dots 0x^1_{m_0}\dots\}\\ &\cup \{u\in[0,1)^s : a_1.0\dots 0x^1_{m_0}\dots \preceq u\prec z\}.\end{aligned}$$

Let

$$\tilde{P}(m_0,x,z) = \{u\in[0,1)^s : a_1.0\dots 0x^1_{m_0} \preceq u \prec z_0.z_1\dots z_{m_0}x_{m_0+1}\dots\}\,. \tag{4.31}$$

By (4.14) we obtain

$$\{T^n(x): 0\le n\le N-1\} = P(m_0,x)\cup \tilde{P}(m_0,x,z). \tag{4.32}$$

Similarly to (4.17), we have from (4.15), (4.29), and (4.30) that

$$\tilde{P}(m_0,x,z) = \bar{Q}(1,z) \bigcup_{2\le i\le m_0} \bigcup_{\substack{0\le j<q^s,\\ 0\preceq b_j\prec z_{i-1}}} Q(i,z,j) \bigcup_{\substack{0\le j<q^s,\\ x_{m_0}\prec b_j\prec z_{m_0}}} Q(m_0+1,z,j), \tag{4.33}$$

with

$$\bar{Q}(1,z) = \{u\in[0,1)^s : a_1.z_1\dots z_{m_0}x_{m_0+1}\dots \le u < z = z_0.z_1...z_{m_0}x_{m_0+1}\dots\}.$$

From (4.17), (4.32) and (4.33), we have

$$\begin{aligned}N =& 2r_2\sigma_1 + \sum_{2\le i\le m_0}\Bigg(\sum_{\substack{0\le j<q^s,\\ x_{i-1}\prec b_j}} \#Q(i,x,j) + \sum_{\substack{0\le j<q^s,\\ b_j\prec z_{i-1}}} \#Q(i,z,j)\Bigg)\\ &+ \sum_{\substack{0\le j<q^s,\\ x_{m_0}\prec b_j\prec z_{m_0}}} \#Q(m_0+1,z,j)\end{aligned} \tag{4.34}$$

with $0 \leq \sigma_1 \leq 1$. Let

(4.35)
$$m = m(x,N) = \max\{i \in [2,m_0] : x_{i-1} \prec b_{q^s-1}, \text{ or } 0 \prec z_i, \quad \text{or} \quad \exists\, j \in [0,q^s) \quad \text{with} \quad x^1_{m_0} \prec b_j \prec z_{m_0}\}.$$

Applying (4.9)–(4.12) and (4.28)–(4.30), we obtain

$$a_1.0\dots 0x^1_{m_0}\dots = a_1.0\dots 0x^1_m \dots \tag{4.36}$$

and

(4.37)
$$z = z_0.z_1\dots z_{m_0}x_{m_0+1}x_{m_0+2}\dots = \begin{cases} z_0.z_1\dots z_m x^1_{m+1}\dots & m < m_0, \\ z_0.z_1\dots z_m x_{m+1}x_{m+2}\dots & m = m_0. \end{cases}$$

Hence

$$z = z_0.z_1\dots z_m x^\nu_{m+1}\dots, \quad \text{with} \quad \nu = \nu(x,N) \in \{0,1\}. \tag{4.38}$$

We see that in (4.32) we can use m instead of m_0. Let $\mathcal{O} \subset [0,1)^s$ be an arbitrary polyhedron. By (4.17), and (4.32)–(4.38), we obtain

$$\begin{aligned} &|\#\{0 \leq n < N : T^n(x) \in \mathcal{O}\} - N\mathrm{vol}\mathcal{O}| \\ \leq& 4r_2 + \sum_{2 \leq i \leq m} \Bigg(\sum_{\substack{0\leq j<q^s,\\ x_{i-1}\prec b_j}} \#\tilde{Q}(i,x,j) + \sum_{\substack{0\leq j<q^s,\\ b_j\prec z_{i-1}}} \#\tilde{Q}(i,z,j)\Bigg) \\ &+ \sum_{\substack{0\leq j<q^s,\\ x_m\prec b_j\prec z_m}} \#\tilde{Q}(m+1,z,j), \end{aligned} \tag{4.39}$$

where

$$\tilde{Q}(i,x,j) = |\#\{Q(i,x,j)\cap\mathcal{O}\} - \#\{Q(i,x,j)\cap[0,1)^s\}\mathrm{vol}\mathcal{O}|.$$

Using (1.10), (4.6) and Lemma 4.1 we have

$$\#\{Q(i,x,j)\cap\mathcal{O}\} = (\det\Gamma)^{-1}\mathrm{vol}\mathcal{O}q^{(r_1+i-1)s} + \mathcal{R}(\mathcal{O},\Gamma_{i-1} + q^{-i+2}.b_j x_i x_{i+1}\dots),$$

and

$$\#Q(i,x,j) = (\det\Gamma)^{-1}q^{(r_1+i-1)s} + \mathcal{R}([0,1)^s,\Gamma_{i-1}+q^{-i+2}.b_j x_i x_{i+1}\dots)\,. \tag{4.40}$$

Hence

$$\begin{aligned}&\tilde{Q}(i,x,j)\\ &\leq |\mathcal{R}(\mathcal{O},\Gamma_{i-1}+q^{-i+2}.b_jx_ix_{i+1}\dots)|+|\mathcal{R}([0,1)^s,\Gamma_{i-1}+q^{-i+2}.b_jx_ix_{i+1}\dots)|,\end{aligned}$$

and

(4.41)

$$\begin{aligned}\sigma :=&|\#\{0\leq n<N : T^n(x)\in\mathcal{O}\}-N\mathrm{vol}\mathcal{O}|\\ \leq&4r_2+\sum_{i=2}^{m}\Bigg(\sum_{x_{i-1}\prec b_j}(|\mathcal{R}(q^{i-1}\mathcal{O}-.b_jx_ix_{i+1}\dots,\Gamma_1)|)\\ &\qquad+|\mathcal{R}(q^{i-1}[0,1)^s-.b_jx_ix_{i+1}\dots,\Gamma_1)|)\\ &+\sum_{b_j\prec z_{i-1}}(|\mathcal{R}(q^{i-1}\mathcal{O}-.b_jz_i\dots z_mx^{\nu}_{m+1}\dots,\Gamma_1)|\\ &\qquad+|\mathcal{R}(q^{i-1}[0,1)^s-.b_jz_i\dots z_mx^{\nu}_{m+1}\dots,\Gamma_1)|)\Bigg)\\ &+\sum_{x_m\prec b_j\prec z_m}(|\mathcal{R}(q^m\mathcal{O}-.b_jx^{\nu}_{m+1}\dots,\Gamma_1)|\\ &\qquad+|\mathcal{R}(q^m[0,1)^s-.b_jx^{\nu}_{m+1}\dots,\Gamma_1)|).\end{aligned}$$

Applying (1.11), we get

$$\sigma\leq 4r_2+2q^s\sum_{i=2}^{m+1}(R(q^{i-1}\mathcal{O},\Gamma_1)+R(q^{i-1}[0,1)^s,\Gamma_1)). \tag{4.42}$$

By (1.10), (1.19), (4.6) and Lemma 4.1 there exists a constant $c=c(\Gamma)$ such that

$$|\#Q(i,x,j)-(\det\Gamma)^{-1}q^{(r_1+i-1)s}|\leq c(\det\Gamma)^{-1}q^{(r_1+i-1)(s-1)}$$

for all $x\in[0,1)^s$, $j\in[0,q^s-1]$, and $i\geq 2$.

From (4.34), and (4.35), we derive

$$\begin{aligned}&(\det\Gamma)^{-1}q^{(m+r_1-1)s}(1-cq^{-m-r+1})\\ &\quad\leq N\\ &\quad\leq 6q^s(\det\Gamma)^{-1}q^{(m+r_1-1)s}(1+cq^{-m-r+1})+2r_2.\end{aligned}$$

Hence, there exists $N_1>0$ with

$$m_1-6\leq m=m(x,N)\leq m_1\quad\text{for } N\geq N_1, \tag{4.43}$$

where

$$m_1 = 3 - r_1 + \left[\frac{1}{s}\log_q(N\det\Gamma)\right]. \tag{4.44}$$

Let $\mathcal{O} = \prod_{i=1}^{s}[0, y_i)$. Using (1.1), (1.2), (1.19) and (4.42)–(4.44), we obtain (2.20). The ergodicity of the transformation $T = T_{2,\Gamma}$ follows from (1.2) and (2.20). Theorem 2.5 is proved. ∎

Similarly, using Propositions 2.1 and 2.2, we get Theorem 2.7 and Theorem 2.9 (case $\nu = 2$) from (4.42). Taking $N_k = \#P(k, 0)$, we obtain Theorem 2.6 from Theorem A and (4.39).

Proof of Theorem 2.8. We see that the parameter m defined in (4.35) depends on x. Hence, we cannot apply Minkowski's inequality to (4.39). Using (4.43) we can avoid this problem. Let

$$\mathcal{O}_1 = [0, y_1] \times \cdots \times [0, y_s] \quad \text{and} \quad \mathcal{O}_2 = [0, 1]^s. \tag{4.45}$$

We get from (1.1), (4.39), (4.41) and (3.23)

$$\begin{aligned} N\Big|\Delta\Big(\prod_{i=1}^{s}[0, y_i), (T^n(x))_{n=0}^{N-1}\Big)\Big| &\le 4r_2 + \sum_{i=2}^{m+1}\sum_{\mu=1}^{2}\sum_{j=0}^{q^s-1}(2s + |\mathcal{R}(q^{i-1}\mathcal{O}_\mu - .b_j x_i x_{i+1}\dots, \Gamma_1)| \\ &\qquad + |\mathcal{R}(q^{i-1}\mathcal{O}_\mu - .b_j z_i \dots z_m x_{m+1}^{\nu} \dots, \Gamma_1)|). \end{aligned}$$

Bearing in mind that $m = m(x, N) \in [m_1 - 6, m_1]$, and $\nu = \nu(x, N) \in \{0, 1\}$ (see (4.38) and (4.43)), we obtain

$$\begin{aligned} N\Big|\Delta\Big(\prod_{i=1}^{s}[0, y_i), (T^n(x))_{n=0}^{N-1}\Big)\Big| &\le 4r_2 + \sum_{m=m_1-6}^{m_1}\sum_{i=2}^{m+1}\sum_{\nu=0}^{1}\sum_{\mu=1}^{2}\sum_{j=0}^{q^s-1}(2s + |\mathcal{R}(q^{i-1}\mathcal{O}_\mu - .b_j x_i x_{i+1}\dots, \Gamma_1)| \\ &\qquad + |\mathcal{R}(q^{i-1}\mathcal{O}_\mu - .b_j z_i \dots z_m x_{m+1}^{\nu} \dots, \Gamma_1)|). \end{aligned} \tag{4.46}$$

Now we will prove that Theorem 2.8 easily follows from the following lemma:

Lemma 4.2: *With the above notation, we have*

$$(4.47)\quad \begin{aligned} I_1(i,m,\nu,\mu,j) &= \int_{[0,1)^s} |\mathcal{R}(q^{i-1}\mathcal{O}_\mu - .b_j z_i \dots z_m x^\nu_{m+1} \dots, \Gamma_1)|^p dx \\ &= O(i^{(s-1)p/2}), \end{aligned}$$

and

$$(4.48)\quad I_2(i,m,\mu,j) = \int_{[0,1)^s} |\mathcal{R}(q^{i-1}\mathcal{O}_\mu - .b_j x_i x_{i+1} \dots, \Gamma_1)|^p dx = O(i^{(s-1)p/2}),$$

where $z_i = z_i(x,N),\ 0 \le i \le m$ *(see (4.30)).*

Applying Minkowski's inequality (3.36), we obtain from (4.44) and (4.46)–(4.47)

$$\begin{aligned} &\left(\int_{[0,1)^s} \left| N\Delta\left(\prod_{i=1}^s [0,y_i), (T^n(x))_{n=0}^{N-1} \right) \right|^p dx \right)^{1/p} \\ \le& 4r_2 + \sum_{m=m_1-6}^{m_1} \sum_{i=2}^{m+1} \sum_{\nu=0}^{1} \sum_{\mu=1}^{2} \sum_{j=0}^{q^s-1} \left(2s + (I_2(i,m,\mu,j))^{1/p} + (I_1(i,m,\nu,\mu,j))^{1/p} \right) \\ =& O(m_1^{(s+1)/2}) = O((\ln N)^{(s+1)/2}). \end{aligned}$$

Now by (1.3), we get

$$\begin{aligned} &\int_{[0,1)^s} (ND_p(T^n(x))_{n=0}^{N-1})^p dx \\ &\quad = \int_{[0,1]^s} \int_{[0,1)^s} \left| N\Delta\left(\prod_{i=1}^s [0,y_i), (T^n(x))_{n=0}^{N-1} \right) \right|^p dxdy = O((\ln N)^{(s+1)p/2}). \end{aligned}$$

Theorem 2.8 is proved. ∎

Proof of Lemma 4.2. Let

$$f(u) = |\mathcal{R}(q^{i-1}\mathcal{O}_\mu - u, \Gamma_1)|^p.$$

We derive from (4.47) that

$$I_1 = I_1(i,m,\nu,\mu,j) = \int_{[0,1)^s} f(.b_j z_i \dots z_m x^\nu_{m+1} \dots) dx,$$

with $z_i = z_i(x,N) = z_i(\hat{x}_m, \bar{x}_{m+1}, N)$, $i = 1,2,\dots$ (see (4.30), (4.9) and (4.20)). Let

$$\delta(a=b) = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{otherwise.} \end{cases}$$

By (4.20)–(4.23), we get

$$(4.49)\quad \begin{aligned} I_1 &= \sum_{\hat{x}_m \in B_m} \int_{G_1(\hat{x}_m)} f(.b_j z_i \dots z_m x^{\nu}_{m+1} \dots) dx \\ &\leq \sum_{\hat{x}_m \in B_m} \int_{\mathbb{F}_{m+1}+x_0.x_1\dots x_m} f(.b_j z_i \dots z_m x^{\nu}_{m+1} \dots) dx \\ &= \sum_{\hat{x}_m \in B_m} \int_{\mathbb{F}_{m+1}} f(.b_j z_i \dots z_m x^{\nu}_{m+1} \dots) dx \\ &= \sum_{\hat{x}_m \in B_m} \int_{\mathbb{F}_{m+1}} f(.b_j u_i \dots u_m x^{\nu}_{m+1} \dots) \sum_{\hat{u}_m \in B_m} \delta(\hat{u}_m = \hat{z}_m) dx \\ &= \sum_{\hat{u}_m \in B_m} \int_{\mathbb{F}_{m+1}} f(.b_j u_i \dots u_m x^{\nu}_{m+1} \dots) \sum_{\hat{x}_m \in B_m} \delta(\hat{u}_m = \hat{z}_m(\hat{x}_m, \bar{x}_{m+1})) dx. \end{aligned}$$

Using (4.30) and (4.38), we obtain

$$x_0.x_1 \dots x_m x_{m+1} \dots = T^{-N}(z) = T^{-N}(z_0.z_1 \dots z_m x^{\nu}_{m+1} \dots).$$

Hence for fixed $\bar{x}_{m+1} = .x_{m+1} x_{m+2} \dots,\ \nu \in \{0,1\}$ and $\hat{u}_m \in B_m$, there exists at most one solution $\hat{x}_m \in B_m$ of the system of equations

$$z_i(\hat{x}_m, \bar{x}_{m+1}) = u_i \quad \text{for } i = 0, 1, \dots, m.$$

Therefore, the interior sum in (4.49) is no more than 1. Thus

$$I_1 \leq \sum_{\hat{u}_m \in B_m} \int_{\mathbb{F}_{m+1}} f(.b_j u_i \dots u_m x^{\nu}_{m+1} \dots) dx.$$

Now let $x \in \mathbb{F}_{m+1},\ v = q^m x = .v_1 v_2 \dots \in \mathbb{F}_1$. By (2.12), (4.2), and (4.6)–(4.10), we have

$$\begin{aligned} .b_j u_i \dots u_m x^{\nu}_{m+1} \dots &= .b_j u_i \dots u_m + \sum_{k=1}^{\infty} x^{\nu}_{m+1,k} / q^{r_1+m-i+k+2} \\ &= .b_j u_i \dots u_m + q^{-r_1-m+i-2} \sum_{k=1}^{\infty} v^{\nu}_k / q^k \\ &= .b_j u_i \dots u_m + q^{-m+i-2} T^{\nu}_{\mathbb{F}}(v). \end{aligned}$$

Therefore

$$I_1 \leq \sum_{\hat{u}_m \in B_m} q^{-ms} \int_{\mathbb{F}_1} f(.b_j u_i \dots u_m + q^{-m+i-2} T^{\nu}_{\mathbb{F}}(v)) dv.$$

Bearing in mind that $T_{\mathbb{F}}$ is an ergodic transformation of $\mathbb{F}_1$, we obtain

$$I_1 \leq q^{-ms} \sum_{\hat{u}_m \in B_m} \int_{\mathbb{F}_1} f(.b_j u_i \dots u_m + q^{-m+i-2} v) dv.$$

Now let $x = .b_j \dots u_i u_m + q^{-m+i-2} v$ with $v \in \mathbb{F}_1$. We have from(4.6), (4.7), (4.20) and (4.21) that $x \in \mathbb{F}_{m-i+3} + .b_j u_i \dots u_m$, and

$$\begin{aligned} I_1(i,m,\nu,\mu,j) \leq & q^{-ms} q^{(m-i+2)s} \sum_{\hat{u}_m \in B_m} \int_{\mathbb{F}_{m-i+3}+.b_j u_i \dots u_m} f(x)dx \\ = & q^{-(i-2)s} \sum_{\hat{u}_{i-1} \in B_{i-1}} \int_{\mathbb{F}_2 + .b_j} f(x)dx \\ = & r_2 q^s \int_{\mathbb{F}_2 + .b_j} f(x)d(x) \leq r_2 q^s \int_{\mathbb{F}_1} |\mathcal{R}(q^i \mathcal{O}_\mu - x, \Gamma_1)|^p dx. \end{aligned}$$

Applying Theorem A, we obtain (4.47). In a similar way, we get (4.48). Lemma 4.2 is proved. ∎

Appendix A. Proofs of the propositions.

In [Skr2], Skriganov found estimates for the error $R(tP, \Gamma)$ in the lattice point problem for a polyhedron tP, where tP is the dilatation of a given polyhedron P by a factor t $(t \to \infty)$. In this paper, we consider the set $\mathcal{V}(P_1)$ of all convex polyhedra $P \in [0,1)^n$, with each $(n-1)$-dimensional face parallel to some $(n-1)$-dimensional face of the polyhedron P_1. We will show that Skriganov's approach can be applied to estimate $\sup_{P \in \mathcal{V}(P_1)} R(tP, \Gamma)$. Mainly, we need to modify the values $t_\tau^- = t - \beta^{-1}\tau$ and $t_\tau^+ = t + \beta^{-1}\tau$ in the statement:

$$\text{(A.1)} \qquad \mathrm{vol}(t_\tau^+ P) - \mathrm{vol}(t_\tau^- P) \sim (2n\beta^{-1}\mathrm{vol}(P))\tau t^{n-1},$$

with $\beta = \beta(P)$ (see [Skr2, (12.4)]) . The left hand side of (A.1) is a part of $R(tP, \Gamma)$ (see [Skr2, (11.6)]). In our case, $\sup_{P \in \mathcal{V}(P_1)} \beta^{-1}(P) = \infty$. Hence, by (A.1), we cannot estimate $\sup_{P \in \mathcal{V}(P_1)} R(tP, \Gamma)$ with $t_\tau^\pm = t \pm \beta^{-1}\tau$. We need to apply several definitions and results from [Skr2] to ascertain the validity of the modification of $t_\tau^\pm$.

A.1. Auxiliary definitions and lemmas. Let $\Lambda \subset \mathbb{R}^n$ be an arbitrary lattice,

$$\begin{aligned} \text{(A.2)} \quad \Delta_r = \{ & \delta_m = \mathrm{diag}(2^{m_1}, \dots 2^{m_n}) \in SL(n, \mathbb{R}) : m = (m_1, \dots, m_n) \in \mathbb{Z}^n, \\ & m_1 + \dots + m_n = 0, \|m\| < r \}, \\ S(\Lambda, r) = & \sum_{\delta \in \Delta_r} \|\delta \Lambda\|^{-n}, \quad r \geq 0, \\ & \text{with} \quad \|\Lambda\| = \min\{\|\gamma\| : \gamma \in \Lambda \setminus \{0\}\}, \end{aligned}$$

and

$$\nu(\Lambda, \rho) = \min\{\mathrm{Nm}\gamma : \gamma \in \Lambda,\ 0 < \|\gamma\| < \rho\}, \quad \rho > 0. \tag{A.3}$$

LEMMA A.1 ([SkSt, Lemma 2]): *Let $\Lambda \in \mathbb{L}_n$ be an arbitrary lattice. Then for μ_n almost all orthogonal matrices $U \in SO(n)$*

$$S(U\Lambda, \rho) = O(r^{n-1+\varepsilon}), \quad r \to \infty,$$

with arbitrarily small $\varepsilon > 0$.

LEMMA A.2 ([Skr2, Lemma 4.3] and [SkSt, Lemma 3]): *Let $\Lambda \in \mathbb{L}_n$ be an arbitrary lattice. Then for μ_n almost all orthogonal matrices $U \in SO(n)$*

$$\nu(U\Lambda, \rho) > c_\varepsilon(U, \Lambda)(\log \rho)^{1-n-\varepsilon}, \quad \rho \to \infty, \tag{A.4}$$

with arbitrarily small $\varepsilon > 0$ and a constant $c_\varepsilon(U, \Lambda) > 0$.

A.2. THE MAIN RESULT. The proof of the propositions of Section 2 is based on the following modification of [Skr2, Theorem 6.1]:

THEOREM A.1: *Let an arbitrary polyhedron $P_1 \subset [0,1)^n$ and a lattice $\Gamma \in \mathbb{L}_n$ be given. Then the error $\bar{R}(tP_1, \Gamma)$ (see (1.10) and (2.29)) satisfies the bound*

$$\bar{R}(tP_1, \Gamma) = \sup_{P \in \mathcal{V}(P_1)} R(tP, \Gamma) < c(P_1, \Gamma, \theta)\Big(t^{n-1}\rho^{-\theta} + \sum_{\mathfrak{f}} S(\Gamma_{\mathfrak{f}}^{\perp}, r_{\mathfrak{f}})\Big), \tag{A.5}$$

with

$$r_{\mathfrak{f}} = \ln\Big(\frac{\rho^n}{\nu(\Gamma_{\mathfrak{f}}^{\perp}, \rho)}\Big). \tag{A.6}$$

Here $\rho > 1$ is an arbitrarily large parameter, $\theta \in (0,1)$ is arbitrarily fixed, the lattices $\Gamma_{\mathfrak{f}}^{\perp}$ are given in (2.28), the characteristics $S(\cdot,\cdot)$ and $\nu(\cdot,\cdot)$ are given in (A.2) and (A.3), respectively, and the summation in (A.5) is taken over all flags of faces $\mathfrak{f}$ of the polyhedron P_1.

The proof of Theorem A.1 will be given in §A.4. The proofs of Propositions 2.1 and 2.2 follow from Theorem A.1 in a way completely similar to the derivation of Theorem B from [Skr2, Theorem 6.1] (see [SkSt] and [Skr2]). For completeness we give the proof of Proposition 2.1 (see [Skr2, p. 32] and [SkSt, p. 1474]):

Proof of Proposition 2.1. Let $P \subset [0,1)^n$ be an arbitrary convex polyhedron, let $\Gamma \in \mathbb{L}_n$ be an arbitrary lattice, and let $\Gamma_U = U\Gamma, U \in SO(n)$. By (2.27) and

(2.28), we obtain the following relations

$$(\Gamma_U)_{\mathfrak{f}} = V_{\mathfrak{f}} U \Gamma, \quad (\Gamma_U)_{\mathfrak{f}}^{\perp} = V_{\mathfrak{f}} U \Gamma^{\perp}. \tag{A.7}$$

Since the set of flags $\mathfrak{f}$ of P and associated orthogonal matrices $V_{\mathfrak{f}}$ is finite, we derive from Lemmas A.1, A.2 and the relations (A.7) that for every flag $\mathfrak{f}$ and for μ_n almost all orthogonal matrices $U \in SO(n)$

$$\nu((\Gamma_U)_{\mathfrak{f}}^{\perp}, \rho) > c_\varepsilon (\ln \rho)^{1-n-\varepsilon}, \quad \varepsilon \to \infty, \tag{A.8}$$

$$S((\Gamma_U)_{\mathfrak{f}}^{\perp}, r) = O(r^{n-1+\varepsilon}), \quad r \to \infty, \tag{A.9}$$

with arbitrarily small $\varepsilon > 0$ and a constant $c_\varepsilon = c_\varepsilon(V_{\mathfrak{f}} U, \Gamma^{\perp}) > 0$.

From the bound (A.8) and the relation (A.6), we find that for every flag $\mathfrak{f}$ and for almost all orthogonal matrices $U \in SO(n)$, one has the bound

$$r_{\mathfrak{f}} = O(\ln \rho). \tag{A.10}$$

Substituting (A.9) and (A.10) in the main bound (A.5), we find that for almost all $U \in SO(n)$ one has the bound

$$R(P, t^{-1} U \Gamma) = R(tP, U\Gamma) = O(t^{n-1} \rho^{-\theta} + (\log \rho)^{n-1+\varepsilon}). \tag{A.11}$$

Choosing $\rho^\theta = t^{n-1}$ in (A.11), we obtain the bound (2.30). The proof of Proposition 2.1 is complete. ∎

The proof of Proposition 2.2 follows from Theorem A.1. This proof repeats derivations of [Skr2, Theorems 1.2 and 2.3] from [Skr2, Theorem 6.1].

A.3. The Fourier transform of the characteristic function of a polyhedron. The error $\mathcal{R}(\mathcal{O} + X, \Gamma)$ (see (1.10)) is a periodic function of $X \in \mathbb{R}^n$ with the period lattice $\Gamma \in \mathbb{L}_n$. The Voronoi-Hardy formula (see [Kr]) links the Fourier expansion of the error $\mathcal{R}(\mathcal{O} + X, \Gamma)$ and the Fourier transform of the characteristic function $\chi(\mathcal{O}, X)$, $X \in \mathbb{R}^n$, of the region $\mathcal{O} \in \mathbb{R}^n$:

$$\mathcal{R}(\mathcal{O} + X, \Gamma) = (\det \Gamma)^{-1} \sum_{\gamma \in \Gamma^{\perp} \setminus \{0\}} \hat{\chi}(\mathcal{O}, \gamma) e^{-2\pi i \langle \gamma, X \rangle} \tag{A.12}$$

and

$$\hat{\chi}(\mathcal{O}, Y) = \int_{\mathbb{R}^n} \chi(\mathcal{O}, X) e^{2\pi i \langle \gamma, X \rangle} dX = \int_{\mathcal{O}} e^{2\pi i \langle \gamma, X \rangle} dX. \tag{A.13}$$

However, the Fourier series (A.12) is not absolutely convergent. Thereby, a suitable “smoothing” method is required to handle the series (A.12) in the pointwise sense. Here we use a method given in [Skr2]. We recall the following:

Definition: (see [Skr2, Sec. 11]). Given a compact region $\mathcal{O} \subset \mathbb{R}^n$ and a number $\tau > 0$, a pair of compact regions $\mathcal{O}_\tau^-$ and $\mathcal{O}_\tau^+$ is a **τ-coapproximation** of $\mathcal{O}$ if $\mathcal{O}_\tau^- \subset \mathcal{O} \subset \mathcal{O}_\tau^+$, and the points of the boundaries $\partial\mathcal{O}_\tau^\pm$ are at a distance at least τ from the boundary $\partial\mathcal{O}$.

We fix a nonnegative function $\omega(X)$, $X \in \mathbb{R}^n$, of class C^∞, supported inside the unit ball $\|X\| \leq 1$, such that

$$\int_{\mathbb{R}^n} \omega(X) dX = 1. \tag{A.14}$$

Notice that the Fourier transform $\hat{\omega}(\centerdot)$ of the function $\omega(\centerdot)$ satisfies the bound

$$|\hat{\omega}(Y)| < c_A(1 + \|Y\|)^{-A} \tag{A.15}$$

with arbitrarily large $A > 0$.

We introduce the absolutely convergent Fourier series

$$\mathcal{R}_\tau^\pm(\mathcal{O}, X) = (\det\Gamma)^{-1} \sum_{\gamma \in \Gamma^\perp \setminus \{0\}} \hat{\chi}(\mathcal{O}_\tau^\pm, \gamma)\hat{\omega}(\tau\gamma)e^{-2\pi i\langle\gamma, X\rangle}, \tag{A.16}$$

where $\hat{\chi}(\mathcal{O}_\tau^\pm, Y), Y \in \mathbb{R}^n$, are the Fourier transforms (A.13) of the characteristic function $\chi(\mathcal{O}_\tau^\pm, Y), X \in \mathbb{R}^n$, of the regions $\mathcal{O}_\tau^\pm$.

The following assertion was given in [Skr2, Lemma 11.1] (with $\det\Gamma = 1$):

Lemma A.3: *Let a compact region $\mathcal{O} \subset \mathbb{R}^n$ and a lattice $\Gamma \in \mathbb{L}_n$ be given. Then, for any τ-coapproximation $\mathcal{O}_\tau^\pm$ of $\mathcal{O}$, one has the following bound for the error $R(\mathcal{O}, \Gamma)$ (see* (1.11)*):*

$$R(\mathcal{O}, \Gamma) \leq (\det\Gamma)^{-1}(\mathrm{vol}\mathcal{O}_\tau^+ - \mathrm{vol}\mathcal{O}_\tau^-) + \sup_{X \in \mathbb{R}^n} (|\mathcal{R}_\tau^+(\mathcal{O}, X)| + |\mathcal{R}_\tau^-(\mathcal{O}, X)|), \tag{A.17}$$

where $\mathcal{R}_\tau^\pm(\mathcal{O}, X)$ are the Fourier series (A.16).

Now we wish to estimate the Fourier transform of the characteristic function of a polyhedron $P \subset \mathbb{R}^n$:

$$\hat{\chi}(P, Y) = \int_P e^{2\pi i\langle Y, X\rangle} dX.$$

Lemma A.4 ([Skr2, Lemma 11.2]): *Let $P \subset \mathbb{R}^n$ be an arbitrary compact polyhedron. Then*

$$|\hat{\chi}(tP, Y)| \leq \frac{t^{n-1}}{2\pi\|Y\|} \mathrm{area}\, \partial P, \tag{A.18}$$

where area ∂P *denotes the* $(n-1)$*-dimensional Euclidean volume of the boundary* ∂P.

Let $(F_i)_{i=1}^{C(P)}$ be the set of $(n-1)$-dimensional faces of the polyhedron $P \subset [0,1)^n$. Bearing in mind that the $(n-1)$-dimensional volume of F_i is less than $n^{n/2}$, we obtain

$$\sup_{P \in \mathcal{V}(P_1)} \text{area } \partial P \leq 2n^{n/2} C(P_1). \tag{A.19}$$

Lemma A.5 ([Skr2, Lemma 11.3]): *Let* $P \subset \mathbb{R}^n$ *be an arbitrary compact polyhedron. Then*

$$\hat{\chi}(tP, Y) = \Big(\frac{1}{2\pi i}\Big)^n \sum_{\mathfrak{f}} \lambda(V_{\mathfrak{f}} Y) e^{2\pi i t \langle P^0, Y\rangle}. \tag{A.20}$$

The summation in (A.20) *is taken over all flags of faces of the polyhedron* P *(see* (A.2)*):*

$$\mathfrak{f} = \{P = P^n \supset P^{n-1} \supset \cdots \supset P^0,\ \dim P^j = j,\ n \geq j \geq 0\}, \tag{A.21}$$

and $V_{\mathfrak{f}} \in SO(n)$ *are the orthogonal matrices associated with the flags* $\mathfrak{f}$ *(see* (2.26)*). The function* $\lambda(\boldsymbol{.})$ *is given by*

$$\lambda(Y) = \frac{1}{\text{Nm} Y} \Phi(\hat{Y}),\ Y \in \mathbb{R}^n, \tag{A.22}$$

where $\Phi(\boldsymbol{.})$ *is the following function of class* C^∞*, depending only on the angular variable* $\hat{Y} = \|Y\|^{-1} Y$

$$\Phi(\hat{Y}) = \Phi\Big(\frac{y_1}{\|Y\|}, \ldots, \frac{y_n}{\|Y\|}\Big) = \prod_{j=1}^{n-1} \frac{y_j^2}{y_j^2 + \cdots + y_n^2}$$

A.4. Proof of Theorem A.1. Let $P_1 \subset [0,1)^n$ be an arbitrary polyhedron, $P \in \mathcal{V}(P_1)$. By definition (1.11) of the error $R(tP, \Gamma)$, we may consider the polyhedron P up to translations by vectors $X \in \mathbb{R}^n$. Thus, without loss of generality, we can assume that the origin $O \in P$. We set $a \prec^{(0)} b$ for $a < b$, and $a \prec^{(1)} b$ for $a \leq b$. Then the polyhedron P can be given by

$$P = \{X \in \mathbb{R}^n : \langle \mathbf{l}_j, X\rangle \prec^{(\epsilon_j)} \beta_j, \quad 1 \leq j \leq h\}, \tag{A.23}$$

where $\mathbf{l}_j,\ 1 \leq j \leq h$, are unit vectors of external normals to $(n-1)$-dimensional faces of P, $\epsilon_j \in \{0,1\}$, and $\beta_j > 0$ are some constants. With these notation the dilatation tP of P by a factor $t > 0$ can be given by

$$tP = \{X \in \mathbb{R}^n : \langle \mathbf{l}_j, X\rangle \prec^{(\epsilon_j)} \beta_j t, \quad 1 \leq j \leq h\}.$$

We consider the following two polyhedra :

$$P^{\pm}_{t,\tau} = \{X \in \mathbb{R}^n : \langle \mathbf{l}_j, X\rangle \prec^{(\epsilon_j)} \beta_j t \pm \tau, \quad 1 \leq j \leq h\}. \tag{A.24}$$

with $0 < \tau < 1$. If $P^{-}_{t,\tau} = \emptyset$, then we put $\mathrm{vol} P^{-}_{t,0} = 0$, $\hat{\chi}(P^{-}_{t,\tau}, \gamma) = 0$ for all $\gamma \in \Gamma^{\perp}$ and $\mathcal{R}^{-}_{\tau}(P^{-}_{t,\tau}, X) = 0$

Comparing definitions (A.23) and (A.24), we conclude that the polyhedra $P^{-}_{t,\tau}$ and $P^{+}_{t,\tau}$ form τ-coapproximations of the polyhedron tP. It is easy to see that all flags of the faces of $P^{-}_{t,\tau}$ and $P^{+}_{t,\tau}$ belong to the set of flags of the polyhedron P_1. Therefore, the set of orthogonal matrices $V_{\mathfrak{f}} \in SO(n)$ associated with flags of $P^{-}_{t,\tau}$ and $P^{+}_{t,\tau}$ belongs to the set of orthogonal matrices $V_{\mathfrak{f}} \in SO(n)$ of the polyhedron P_1. We wish to apply Lemma A.3 with $\mathcal{O} = tP$ and $\mathcal{O}^{\pm}_{\tau} = P^{\pm}_{t,\tau}$ to estimate the error $R(tP, \Gamma)$. First we have the bound

$$0 < \mathrm{vol} P^{+}_{t,\tau} - \mathrm{vol} P^{-}_{t,\tau} \leq 2\tau \text{ area } \partial(tP) \leq 2n^{n/2}\tau t^{n-1} C(P) \leq 4n^{n/2} C(P_1)\tau t^{n-1}, \tag{A.25}$$

where $C(P)$ is the number of $(n-1)$-dimensional faces of P (see (A.19)).

Now we consider the Fourier series (A.16) with $\mathcal{O} = tP$, $\mathcal{O}^{\pm}_{\tau} = P^{\pm}_{t,\tau}$, and $\omega(X) = \omega_1(X)$. Here $\omega_1(X)$, $X \in \mathbb{R}^n$ is a C^{∞}-function supported inside the ball $\|X\| \leq 1/4$ and satisfying the following conditions

$$\begin{cases} 0 \leq \omega_1(X) \leq 1 & \text{if } X \in \mathbb{R}^n, \\ \omega_1(X) = 1 & \text{if } \|X\| \leq 1/8. \end{cases} \tag{A.26}$$

Let $\omega_2(X) = \hat{\omega}_1(X)$, $X \in \mathbb{R}^n$, be the Fourier transform of the function $\omega_1(\cdot)$. Obviously, the function $\omega_2(\cdot)$ satisfies (A.15). Moreover, we assume that both of the functions $\omega_1(\cdot)$ and $\omega_2(\cdot)$ are spherically symmetric:

$$\omega_j(VX) = \omega_j(X), \ V \in SO(n), \quad j = 1, 2. \tag{A.27}$$

The Fourier series (A.16) can be written as

$$\mathcal{R}^{\pm}_{\tau}(tP, X) = \mathcal{A}^{\pm}_{\tau,\rho}(tP, X) + \mathcal{B}^{\pm}_{\tau,\rho}(tP, X), \tag{A.28}$$

where $\rho \geq 1$ is an arbitrary large parameter

$$\mathcal{A}^{\pm}_{\tau,\rho}(tP, X) = (\det \Gamma)^{-1} \sum_{\gamma \in \Gamma^{\perp} \setminus \{0\}} \hat{\chi}(P^{\pm}_{\tau,\rho}, \gamma)\omega_2(\tau\gamma)\omega_1(\rho^{-1}\gamma) e^{-2\pi i \langle \gamma, X\rangle} \tag{A.29}$$

are sums with finitely many terms and

$$\mathcal{B}^{\pm}_{\tau,\rho}(tP, X) = (\det \Gamma)^{-1} \sum_{\gamma \in \Gamma^{\perp} \setminus \{0\}} \hat{\chi}(P^{\pm}_{\tau,\rho}, \gamma)\omega_2(\tau\gamma)(1 - \omega_1(\rho^{-1}\gamma)) e^{-2\pi i \langle \gamma, X\rangle} \tag{A.30}$$

are absolutely convergent series over the lattice points $\gamma \in \Gamma^\perp$ with $\|\gamma\| > \frac{1}{8}\rho$ (see(A.26)).

Substituting the formula (A.20) for the Fourier transform of the characteristic functions of the polyhedra $P^{\pm}_{\tau,\tau}$ in (A.29), we obtain

$$\text{(A.31)}\quad \mathcal{A}^{\pm}_{\tau,\rho}(tP,X) = \Big(\frac{1}{2\pi i}\Big)^n (\det\Gamma)^{-1} \sum_{\gamma\in\Gamma^\perp\setminus\{0\}} \sum_{\mathfrak{f}} \lambda(V_{\mathfrak{f}}\gamma)\omega_2(\tau\gamma)\omega_1(\rho^{-1}\gamma)e^{2\pi i\langle\gamma, P^{\pm 0}_{t,\tau}-X\rangle}.$$

In the second sum in (A.31), the summation is taken over all flags of faces $\mathfrak{f}$ of the polyhedron P. The function $\lambda(\cdot)$ is given by (A.22). In (A.31), the points $P^{\pm 0}_{t,\tau}$ are zero-dimensional faces (vertices) of the polyhedron $P^{\pm}_{t,\tau}$ belonging to the corresponding flags $\mathfrak{f}$ (see (A.21)).

Using definition (2.27) of the lattices $\Gamma_{\mathfrak{f}} = V_{\mathfrak{f}}\Gamma$, (2.28), relation (A.27), and formula (A.31), we find that

$$\text{(A.32)}\qquad \mathcal{A}^{\pm}_{\tau,\rho}(tP,X) = \Big(\frac{1}{2\pi i}\Big)^n (\det\Gamma)^{-1} \sum_{\mathfrak{f}} W_{\tau,\rho}(\Gamma_{\mathfrak{f}}, X^{\pm}_{\tau,\mathfrak{f}})$$

with

$$\text{(A.33)}\qquad W_{\tau,\rho}(\Gamma_{\mathfrak{f}}, X^{\pm}_{\tau,\mathfrak{f}}) = \sum_{\gamma\in\Gamma^\perp\setminus\{0\}} \lambda(\gamma)\omega_2(\tau\gamma)\omega_1(\rho^{-1}\gamma)e^{2\pi i\langle\gamma, X^{\pm}_{\tau,\mathfrak{f}}\rangle},$$

where we used the notation $X^{\pm}_{\tau,\mathfrak{f}} = V_{\mathfrak{f}}(P^{\pm 0}_{t,\tau} - X)$.

The following assertion was given in [Skr2, Lemma 10.1].

Lemma A.6: *Let $\Gamma \in \mathbb{L}_n$ be an arbitrary lattice. Assume that the dual lattice $\Gamma^\perp$ is weakly admissible. Then the special Fourier series* (A.33) *satisfies the bound*

$$\text{(A.34)}\qquad \max_{X\in\mathbb{R}^n} |W_{\tau,\rho}(X)| < cS(\Gamma^\perp, r)$$

with

$$\text{(A.35)}\qquad r = 2\tau_n + \Big|\log\frac{\rho^n}{\nu(\Gamma^\perp,\rho)}\Big|.$$

In the bound (A.34), *the constant $c = c(\Phi,\omega_1,\omega_2)$ is independent of the parameters $0<\tau<1$ and $\rho>0$.*

From Lemma A.6 and the relation (A.32), we obtain the bound

$$\text{(A.36)}\qquad \det\Gamma \sup_{P\in\mathcal{V}(P_1)} \max_{X\in\mathbb{R}^n} |\mathcal{A}^{\pm}_{\tau,\rho}(tP,X)| < c\sum_{\mathfrak{f}} S(\Gamma^\perp_{\mathfrak{f}}, r_{\mathfrak{f}})$$

with

$$r_{\mathfrak{f}} = 2\kappa_n + \left| \log \frac{\rho^n}{\nu(\Gamma^\perp, \rho)} \right|. \tag{A.37}$$

In the bound (A.36), the constant $c = c(\omega_1, \omega_2)$ is independent of the lattice Γ and the parameters $t > 0,\ \rho > 0$ and $0 < \tau < 1$.

Now we consider the series (A.30). Substituting bounds (A.18)–(A.19) for the Fourier transform of the characteristic functions of the polyhedra $P^{\pm}_{t,\tau}$, and the bound (A.15) for the function $\omega_2(\cdot) = \hat{\omega}_1(\cdot)$ with $A > n$, we obtain as $t \to \infty$

$$\begin{aligned} &\det \Gamma \sup_{P \in \mathcal{V}(P_1)} \max_{X \in \mathbb{R}^n} |\mathcal{B}^{\pm}_{\tau,\rho}(tP, X)| \\ &\leq \sup_{P \in \mathcal{V}(P_1)} (\text{area } \partial P) t^{n-1} \sum_{\gamma \in \Gamma^\perp} \omega_2(\tau\gamma)(1 - \omega_1(\rho^{-1}\gamma))(2\pi \|\gamma\|)^{-1} \\ &\leq n^{n/2} c_A C(P_1) t^{n-1} \tau^{-A} \sum_{\gamma \in \Gamma^\perp : \|\gamma\| > \frac{1}{8}\rho} \|\gamma\|^{-A-1}. \end{aligned} \tag{A.38}$$

Notice that for every lattice $\Gamma \in \mathbb{R}^n$, one has the bound (see, e.g., [Kr])

$$\sum_{\gamma \in \Gamma^\perp : \|\gamma\| > \frac{1}{8}\rho} \|\gamma\|^{-A-1} < C_A(\Gamma) \rho^{n-A-1} \ (A > n) \tag{A.39}$$

as $\rho \to \infty$. The bounds (A.38)–(A.39) imply

$$\sup_{P \in \mathcal{V}(P_1)} \max_{X \in \mathbb{R}^n} |\mathcal{B}^{\pm}_{\tau,\rho}(tP, X)| < C(A, P_1, \Gamma) t^{n-1} \tau^{-A} \rho^{n-A-1}. \tag{A.40}$$

Substituting the bounds (A.25), (A.36)–(A.37) and (A.40) in (A.28) and (A.17), we obtain

(A.41)

$$\sup_{P \in \mathcal{V}(P_1)} R(tP, \Gamma) < C(A, P_1, \Gamma) \left(t^{n-1}\tau + t^{n-1}\tau^{-A}\rho^{n-A-1} + \sum_{\mathfrak{f}} S(\Gamma^\perp_{\mathfrak{f}}, r_{\mathfrak{f}}) \right).$$

Further we assume that

$$\tau = \rho^\theta \quad \text{with} \quad \theta = \frac{A + 1 - n}{A + 1} = 1 - \frac{n}{A + 1}, \tag{A.42}$$

and, moreover, the exponent $\theta \in (0, 1)$ may be chosen arbitrarily close to 1, since $A > n$ is arbitrary large. From (A.42) we obtain

$$t^{n-1}\tau = t^{n-1}\tau^{-A}\rho^{n-A-1} = t^{n-1}\rho^{-\theta}. \tag{A.43}$$

Now the bound (A.5)–(A.6) follows from (A.41) and (A.43). The proof of Theorem A.1 is complete. ∎

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