ISRAEL JOURNAL OF MATHEMATICS **177** (2010), 335–348
DOI: 10.1007/s11856-010-0049-2

THE ZETA FUNCTIONS OF COMPLEXES FROM PGL(3): A REPRESENTATION-THEORETIC APPROACH

BY

MING-HSUAN KANG AND WEN-CHING WINNIE LI*

Department of Mathematics, The Pennsylvania State University
University Park, PA 16802, USA
e-mail: kang_m@math.psu.edu and wli@math.psu.edu

AND

CHIAN-JEN WANG**

Mathematics Division, National Center for Theoretical Sciences
National Tsing Hua University, Hsinchu, Taiwan, R.O.C.
e-mail: cjwang@math.cts.nthu.edu.tw

ABSTRACT

The zeta function attached to a finite complex X_Γ arising from the Bruhat–Tits building for $\mathrm{PGL}_3(F)$ was studied in [KL], where a closed form expression was obtained by a combinatorial argument. This identity can be rephrased using operators on vertices, edges, and directed chambers of X_Γ. In this paper we re-establish the zeta identity from a different aspect by analyzing the eigenvalues of these operators using representation theory. As a byproduct, we obtain equivalent criteria for a Ramanujan complex in terms of the eigenvalues of the operators on vertices, edges, and directed chambers, respectively.

---

* The research of the first and second authors are supported in part by the DARPA grant HR0011-06-1-0012. The second author is also supported in part by the NSF grants DMS-0457574 and DMS-0801096. She would like to thank Jiu-Kang Yu for inspiring conversations.

** Part of the work was done when the third author visited the Pennsylvania State University. She would like to thank the Penn State University for its hospitality, and National Center for Theoretical Sciences, Hsinchu, Taiwan, for its support.

Received September 7, 2008 and in revised form November 18, 2008

1. Introduction

Let F be a nonarchimedean local field with q elements in its residue field, and let π be a uniformizer of F. A finite quotient X_Γ of the Bruhat–Tits building of $G = \mathrm{PGL}_3(F)$ by a cocompact, discrete, and torsion-free subgroup Γ of G with $\mathrm{ord}_\pi(\det \Gamma) \subseteq 3\mathbb{Z}$ is a 2-dimensional complex. Each vertex of the complex has two kinds of neighboring vertices, of type one and type two. The edges from a vertex to its type i ($i = 1, 2$) neighbors are called type i edges.

The zeta function for the complex X_Γ was introduced and studied in [KL]. Similar to a graph zeta function, the complex zeta function counts tailless closed geodesics in X_Γ up to homotopy. More precisely, it is defined as

$$Z(X_\Gamma, u) = \prod_{[C]} \frac{1}{(1 - u^{l(C)})},$$

where the product runs through the equivalence classes of primitive tailless closed geodesics C in X_Γ consisting of solely type one edges or solely type two edges (up to based homotopy), and $l(C)$ denotes the length of C. Under the additional assumption that Γ is regular, the following closed form expression of $Z(X_\Gamma, u)$ is obtained in [KL]:

$$Z(X_\Gamma, u) = \frac{(1 - u^3)^{\chi(X_\Gamma)}}{\det(I - A_1 u + q A_2 u^2 - q^3 u^3 I) \det(I + L_B u)}, \tag{1.1}$$

in which $\chi(X_\Gamma)$ is the Euler characteristic of X_Γ, A_i ($i = 1, 2$) is the type i vertex adjacency matrix, and L_B is the directed chamber adjacency matrix (cf. [KL]).

As shown in [KL], the zeta function can easily be expressed in terms of the edge adjacency matrices, namely

$$Z(X_\Gamma, u) = \frac{1}{\det(I - L_E u) \det(I - (L_E)^t u^2)},$$

in which L_E is the type one edge adjacency matrix, and $(L_E)^t$, the transpose of L_E, is the type two edge adjacency matrix. The proof of the closed form expression (1.1) then boils down to establishing the following zeta function identity:

$$\frac{(1-u^3)^{\chi(X_\Gamma)}}{\det(I - A_1u + qA_2u^2 - q^3u^3I)} = \frac{\det(I + L_Bu)}{\det(I - L_Eu)\det(I - (L_E)^tu^2)}. \tag{1.2}$$

Each determinant in the identity (1.2) has a combinatorial interpretation. The proof given in [KL] follows the combinatorial meaning of each term; the identity is obtained by partitioning the closed geodesics into sets indexed by the conjugacy classes of Γ, and for each conjugacy class, counting and relating the numbers of closed geodesics of given length, with and without tails.

Since the vertices, type one edges, and directed chambers of the building attached to G are parametrized by cosets of G modulo the standard maximal compact subgroup K, a parahoric subgroup E, and an Iwahori subgroup B, respectively, A_1 and A_2 can be interpreted as Hecke operators supported on certain K-double cosets, L_E as a parahoric operator on an E-double coset, and L_B as an Iwahori–Hecke operator on a B-double coset. Their precise definitions, originally given in §2, §8 and §7 of [KL], are recalled in §2.1; their actions are described in detail in §2.3. Regarding A_1 and A_2 as operators on $L^2(\Gamma\backslash G/K)$, L_E as an operator on $L^2(\Gamma\backslash G/E)$, and L_B as an operator on $L^2(\Gamma\backslash G/B)$, in this paper we re-establish the zeta identity by computing the spectrum of each operator using representation theory. Our proof not only provides a totally different, new aspect of the identity (1.2), but also removes the regularity assumption on Γ.

This paper is organized as follows. The eigenvalues of L_B, L_E, A_1, and A_2 are computed in §2 using representation theory. As shown in §3, the zeta identity (1.2) follows from comparing these eigenvalues. Recall from [Li] that a finite quotient X of the building of G is called Ramanujan if the nontrivial eigenvalues of A_1 and A_2 on X fall in the spectrum of those on the building. This definition, which involves only operators on the vertices of the complex, is a natural extension of Ramanujan graphs. As a byproduct of the spectral information of the operators, we show that a Ramanujan complex can be equivalently characterized in terms of the eigenvalues of the operator L_B on directed chambers or those of the operator L_E on edges (cf. Theorem 2).

2. Eigenvalues of L_B, L_E, A_1 and A_2

2.1. The relevant operators. We recall the definition of the operators L_B, L_E, A_1, A_2 from [KL]. Following the notation of [KL], denote by K the maximal compact subgroup of G, consisting of elements in G with entries in the ring of integers $\mathcal{O}_F$ of F and determinant in $\mathcal{O}_F^\times$. Let

$$\sigma = \begin{pmatrix} & 1 & \\ & & 1 \\ \pi & & \end{pmatrix}.$$

Define the Iwahori subgroup $B = K \cap \sigma K \sigma^{-1} \cap \sigma^{-1} K \sigma$ and the parahoric subgroup $E = K \cap \sigma K \sigma^{-1}$. Note that B is the set of elements in K congruent to the upper triangular matrices modulo π, while E consists of elements in K whose third row is congruent to $(0, 0, *)$ mod π. Clearly, $B \subset E \subset K$.

Write

$$t_2 = \begin{pmatrix} & & \pi^{-1} \\ & 1 & \\ \pi & & \end{pmatrix}.$$

The operator L_B acts on the space $L^2(\Gamma\backslash G/B)$ by sending a function f in $L^2(\Gamma\backslash G/B)$ to the function $L_B f$, where

$$L_B f(gB) = \sum_{w_i B \in B t_2 \sigma^2 B/B} f(g w_i B).$$

The operator L_E on $L^2(\Gamma\backslash G/E)$ sends a function f in $L^2(\Gamma\backslash G/E)$ to the function $L_E f$ given by

$$L_E f(gE) = \sum_{w'_j E \in E (t_2 \sigma^2)^2 E/E} f(g w'_j E).$$

The operators A_1, A_2 on $L^2(\Gamma\backslash G/K)$ are associated to the double cosets $K\mathrm{diag}(1, 1, \pi)K = \sqcup g_i K$ and $K\mathrm{diag}(1, \pi, \pi)K = \sqcup g'_j K$ respectively, and they are defined by

$$A_1 f(gK) = \sum_i f(g g_i K), \quad A_2 f(gK) = \sum_j f(g g'_j K),$$

for any f in the space $L^2(\Gamma\backslash G/K)$.

2.2. Representations containing B-invariant vectors. The group G acts on $L^2(\Gamma\backslash G)$ by right translations. Since Γ is a discrete cocompact subgroup of G, this representation decomposes into a direct sum of countably many irreducible unitary representations of G. The space $L^2(\Gamma\backslash G/B)$ is the direct sum of the functions in each irreducible subspace which are fixed by the Iwahori subgroup B, and similar decomposition holds for the spaces $L^2(\Gamma\backslash G/E)$ and $L^2(\Gamma\backslash G/K)$ as well. To understand the actions of L_B, L_E, and A_1,A_2, it then suffices to study the actions of these operators on each direct summand of the corresponding spaces. Recall from Casselman's paper [Ca] that an irreducible representation of G contains an Iwahori fixed vector if and only if it is an irreducible subquotient of an unramified principal series representation; by the work of Tadić [Ta], such a subquotient is unitary if it is equivalent to one of the following:

(a) The principal series representation $\mathrm{Ind}(\chi_1, \chi_2, \chi_3)$, where χ_1, χ_2, χ_3 are unramified unitary characters of $F^\times$ with $\chi_1\chi_2\chi_3 = id$; or the principal series representation $\mathrm{Ind}(\chi^{-2}, \chi|\ |^a, \chi|\ |^{-a})$, where χ is an unramified unitary character of $F^\times$, $0 < a < 1/2$, and $|\ |$ is the absolute value of F.

(b) The irreducible subrepresentation of $\mathrm{Ind}(\chi|\ |^{-1}, \chi, \chi|\ |)$, where χ is an unramified character of $F^\times$ with $\chi^3 = id$. This is a one-dimensional representation.

(c) The irreducible subrepresentation of $\mathrm{Ind}(\chi|\ |, \chi, \chi|\ |^{-1})$, where χ is an unramified character of $F^\times$ with $\chi^3 = id$. This is the Steinberg representation.

(d) The irreducible subrepresentation of $\mathrm{Ind}(\chi|\ |^{-1/2}, \chi|\ |^{1/2}, \chi^{-2})$, where χ is an unramified unitary character of $F^\times$.

(e) The irreducible subrepresentation of $\mathrm{Ind}(\chi|\ |^{1/2}, \chi|\ |^{-1/2}, \chi^{-2})$, where χ is an unramified unitary character of $F^\times$.

The inductions are taken to be normalized and all the irreducible subrepresentations above are uniquely determined. We shall compute the eigenvalues of L_B, L_E, A_1 and A_2 for each type of representation listed above. Since the above representations can all be realized in induced spaces, the computation will be carried out using the standard model. Note that G acts by right translation on the induced space, so the Iwahori–Hecke operator L_B is defined by the same formula as in §2.1 in the standard model and so are the operators L_E, A_1 and A_2.

2.3. Explicit actions of the operators. We start with a principal series representation $V = \mathrm{Ind}(\chi_1, \chi_2, \chi_3)$ of G induced from three unramified characters χ_1, χ_2, χ_3 of $F^\times$ with $\chi_1\chi_2\chi_3 = id$. Write P for the standard Borel subgroup of G and W for the Weyl group. Then

$$W = \{id, \alpha_1, \alpha_2, \alpha_1\alpha_2, \alpha_2\alpha_1, \alpha_1\alpha_2\alpha_1\},$$

where

$$\alpha_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Denote by V^B the space of Iwahori fixed vectors of V, which has dimension six as one can observe from the decomposition $G = \bigsqcup_{\alpha \in W} P\alpha B$. Let $f_\alpha(x)$ be the function in V supported on the coset $P\alpha B$ with $f_\alpha(\alpha) = 1$. Then V^B has a basis consisting of $f_1 := f_{id}, f_2 := f_{\alpha_1}, f_3 := f_{\alpha_2}, f_4 := f_{\alpha_1\alpha_2\alpha_1}, f_5 := f_{\alpha_1\alpha_2}, f_6 := f_{\alpha_2\alpha_1}$. A straightforward computation using the coset decomposition

$$Bt_2\sigma^2 B = B \begin{pmatrix} 0 & 1 & 0 \\ \pi & 0 & 0 \\ 0 & 0 & \pi \end{pmatrix} B = \bigsqcup_{x \in \mathcal{O}_F/\pi\mathcal{O}_F} \begin{pmatrix} 0 & 1 & 0 \\ \pi & 0 & 0 \\ 0 & 0 & \pi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix} B$$

gives the following action of L_B:

$$\begin{aligned} &L_B(f_1) = \chi_1\chi_3(\pi) f_2, && L_B(f_2) = q\chi_2\chi_3(\pi) f_1, \\ &L_B(f_3) = (q-1)\chi_1\chi_3(\pi) f_2 + \chi_1\chi_2(\pi) f_6, \\ &L_B(f_4) = (q-1)q\chi_2\chi_3(\pi) f_3 + q\chi_1\chi_3(\pi) f_5, \\ &L_B(f_5) = (q-1)q\chi_2\chi_3(\pi) f_1 + \chi_1\chi_2(\pi) f_4, && L_B(f_6) = q\chi_2\chi_3(\pi) f_3. \end{aligned} \tag{2.1}$$

This shows that L_B on V^B has eigenvalues $\pm\sqrt{q\chi_1(\pi)}$, $\pm\sqrt{q\chi_2(\pi)}$ and $\pm\sqrt{q\chi_3(\pi)}$.

The operator L_E acts on the space of E-fixed vectors V^E. As $\alpha_1 \in E$, V^E is 3-dimensional, generated by $g_1 := f_1 + f_2$, $g_2 := f_3 + f_6$, and $g_3 := f_4 + f_5$. Applying the coset decomposition

$$E(t_2\sigma^2)^2 E = E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} E = \bigsqcup_{x,y \in \mathcal{O}_F/\pi\mathcal{O}_F} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x\pi & y\pi & \pi \end{pmatrix} E$$

to the definition of L_E, one obtains the following action of L_E:

$$\begin{aligned} L_E(g_1) &= q\chi_3(\pi) g_1 + (q-1)q\chi_3(\pi) g_2 + (q-1)q^2\chi_3(\pi) g_3, \\ L_E(g_2) &= q\chi_2(\pi) g_2 + (q-1)q\chi_2(\pi) g_3, \\ L_E(g_3) &= q\chi_1(\pi) g_3. \end{aligned} \tag{2.2}$$

Hence L_E on V^E has eigenvalues $q\chi_1(\pi)$, $q\chi_2(\pi)$ and $q\chi_3(\pi)$.

The Hecke operators A_1 and A_2 act on the 1-dimensional space of K-fixed vectors V^K, which is generated by $h = \sum_{i=1}^6 f_i$. It follows from the coset decomposition

$$K \begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix} K = \bigsqcup_{a,b \in \mathcal{O}_F/\pi\mathcal{O}_F} \begin{pmatrix} \pi & a & b \\ & 1 & \\ & & 1 \end{pmatrix} K \bigsqcup_{c \in \mathcal{O}_F/\pi\mathcal{O}_F} \begin{pmatrix} 1 & & \\ & \pi & c \\ & & 1 \end{pmatrix} K \bigsqcup \begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix} K$$

that the operator A_1 acts on V^K via the scalar multiplication by

$$\lambda := q(\chi_1(\pi) + \chi_2(\pi) + \chi_3(\pi)). \tag{2.3}$$

Similarly the action of A_2 is obtained from the coset decomposition of $K\mathrm{diag}(1, \pi, \pi)K$, as scalar multiplication by

$$q(\chi_1\chi_2(\pi) + \chi_2\chi_3(\pi) + \chi_3\chi_1(\pi)) = q(\chi_1(\pi)^{-1} + \chi_2(\pi)^{-1} + \chi_3(\pi)^{-1}). \tag{2.4}$$

We remark that the operator A_2 is the transpose of A_1 (cf. [Li]), so (2.4) should be the complex conjugation of (2.3). This is indeed the case by checking the representations of types (a)–(e).

2.4. Computation of eigenvalues. Let (ρ, V_ρ) be an irreducible representation in $L^2(\Gamma\backslash G)$ with a nontrivial Iwahori-fixed vector, so ρ is isomorphic to one of the representations described in §2.2, and we identify V_ρ with its image in the induced space under this isomorphism. We are now ready to determine, type by type, the eigenvalues of L_B on V_ρ^B, L_E on V_ρ^E, and A_1 and A_2 on V_ρ^K, whenever the underlying space is nontrivial. We will also record the factors in $\det(I - A_1u + qA_2u^2 - q^3u^3I)$, $\det(I - L_Eu)$, and $\det(I + L_Bu)$ arising from ρ.

First recall a result of Borel [Bo] and Casselman [Ca] on the dimension of V_ρ^B, which helps to determine the space V_ρ^B. Let N be the upper triangular maximal unipotent subgroup of G and let $V_{\rho,N} = V_\rho/V_\rho(N)$ be the Jacquet module of ρ. Here $V_\rho(N)$ is the subspace of V_ρ generated by vectors of the form $\rho(n)v - v$ for all $n \in N$, $v \in V_\rho$. Write $M_0 = T \cap K$, where T is the diagonal subgroup of G. Then the canonical projection $V_\rho \to V_{\rho,N}$ induces a linear isomorphism between the B-fixed space V_ρ^B and the M_0-fixed space $V_{\rho,N}^{M_0}$. In particular, $\dim V_\rho^B = \dim V_{\rho,N}^{M_0}$.

Case (a) ρ is a principal series representation with $(\rho, V_\rho) = \mathrm{Ind}(\chi_1, \chi_2, \chi_3)$, where χ_1, χ_2, χ_3 are unramified characters of $F^\times$ and $\chi_1\chi_2\chi_3 = id$. It follows immediately from §2.3 that the Iwahori–Hecke operator L_B acting on V_ρ^B has eigenvalues $\pm\sqrt{q\chi_1(\pi)}$, $\pm\sqrt{q\chi_2(\pi)}$ and $\pm\sqrt{q\chi_3(\pi)}$, which yield the factor $(1 - q\chi_1(\pi)u^2)(1 - q\chi_2(\pi)u^2)(1 - q\chi_3(\pi)u^2)$ of $\det(I + L_Bu)$. The operator L_E on V_ρ^E has eigenvalues $q\chi_1(\pi)$, $q\chi_2(\pi)$, and $q\chi_3(\pi)$ which give rise to the factor $(1 - q\chi_1(\pi)u)(1 - q\chi_2(\pi)u)(1 - q\chi_3(\pi)u)$ of $\det(I - L_Eu)$. The operators A_1 and A_2 on V_ρ^K have eigenvalues $q(\chi_1(\pi) + \chi_2(\pi) + \chi_3(\pi))$ and $q(\chi_1\chi_2(\pi) + \chi_2\chi_3(\pi) + \chi_3\chi_1(\pi))$ respectively, and they yield the factor $(1 - q\chi_1(\pi)u)(1 - q\chi_2(\pi)u)(1 - q\chi_3(\pi)u)$ of $\det(I - A_1u + qA_2u^2 - q^3u^3I)$.

Case (b) ρ is the 1-dimensional subrepresentation occurring in $V = \mathrm{Ind}(\chi|\ |^{-1}, \chi, \chi|\ |)$ for an unramified character χ with $\chi^3 = id$. Then $\rho(g) = \chi(\det g)$ for all $g \in G$ by [Ze]. Since χ is unramified, V_ρ is generated by the K-invariant function $h = \sum_{i=1}^6 f_i$ in V. Therefore $V_\rho = V_\rho^K = V_\rho^E = V_\rho^B = \mathbb{C} \cdot h$, and we have $L_B(h) = q\chi^2(\pi)h$ by appealing to (2.1) with $\chi_1 = \chi|\ |^{-1}, \chi_2 = \chi$ and $\chi_3 = \chi|\ |$. This gives rise to the factor $1 + q\chi^2(\pi)u$ of $\det(I + L_B u)$ arising from ρ. Similarly, using the action of L_E given in (2.2), we find that $L_E(h) = q^2\chi(\pi)h$, which gives rise to the factor $1 - q^2\chi(\pi)u$ of $\det(I - L_E u)$. Applying (2.3) and (2.4) with $\chi_1 = \chi|\ |^{-1}, \chi_2 = \chi$ and $\chi_3 = \chi|\ |$, we see that the actions of A_1 and A_2 on V_ρ are given by multiplication by $\chi(\pi) + q\chi(\pi) + q^2\chi(\pi)$ and $\bar{\chi}(\pi) + q\bar{\chi}(\pi) + q^2\bar{\chi}(\pi)$, respectively, and hence they yield the factor $(1 - \chi(\pi)u)(1 - q\chi(\pi)u)(1 - q^2\chi(\pi)u)$ of $\det(I - A_1 u + qA_2 u^2 - q^3 u^3)$.

Case (c) ρ is the Steinberg representation, that is, the subrepresentation in the induced space $V = \mathrm{Ind}(\chi|\ |, \chi, \chi|\ |^{-1})$ for an unramified character χ with $\chi^3 = id$. It follows from Proposition 2.10 of [Ze] that the Jacquet module of ρ is a one-dimensional unramified character on the diagonal subgroup T in G, hence $\dim V_{\rho,N}^{M_0} = \dim V_{\rho,N} = \dim V_\rho^B = 1$, and V_ρ^B is generated by a single function ϕ.

To determine ϕ, consider the intertwining maps

$$T_{\alpha_1} : \mathrm{Ind}(\chi|\ |, \chi, \chi|\ |^{-1}) \to \mathrm{Ind}(\chi, \chi|\ |, \chi|\ |^{-1})$$

and

$$T_{\alpha_2} : \mathrm{Ind}(\chi|\ |, \chi, \chi|\ |^{-1}) \to \mathrm{Ind}(\chi|\ |, \chi|\ |^{-1}, \chi)$$

defined as in §3 of [Ca]. Applying Theorem 3.4 in [Ca], we see that

$$T_{\alpha_1}(f_1) = \frac{1}{q}(f_1' + f_2'), \quad T_{\alpha_1}(f_2) = f_1' + f_2', \quad T_{\alpha_1}(f_3) = \frac{1}{q}(f_3' + f_5'),$$
$$T_{\alpha_1}(f_4) = f_4' + f_6', \quad T_{\alpha_1}(f_5) = f_3' + f_5', \quad T_{\alpha_1}(f_6) = \frac{1}{q}(f_4' + f_6');$$

and

$$T_{\alpha_2}(f_1) = \frac{1}{q}(f_1'' + f_3''), \quad T_{\alpha_2}(f_2) = \frac{1}{q}(f_2'' + f_6''), \quad T_{\alpha_2}(f_3) = f_1'' + f_3'',$$
$$T_{\alpha_2}(f_4) = f_4'' + f_5'', \quad T_{\alpha_2}(f_5) = \frac{1}{q}(f_4'' + f_5''), \quad T_{\alpha_2}(f_6) = f_2'' + f_6''.$$

Here $\{f_1, f_2, \ldots, f_6\}$ (resp. $\{f'_1, f'_2, \ldots, f'_6\}$ and $\{f''_1, f''_2, \ldots, f''_6\}$) is a basis of the space of the Iwahori-fixed vectors in $\mathrm{Ind}(\chi|\,|, \chi, \chi|\,|^{-1})$ (resp. $\mathrm{Ind}(\chi, \chi|\,|, \chi|\,|^{-1})$ and $\mathrm{Ind}(\chi|\,|, \chi|\,|^{-1}, \chi)$) as defined in §2.3. Since the Steinberg representation ρ is irreducible, and it does not appear as a subrepresentation in $\mathrm{Ind}(\chi, \chi|\,|, \chi|\,|^{-1})$, T_{α_1} must be trivial on V_ρ (cf. [Ze]). The same reason shows that T_{α_2} is also trivial on V_ρ. In particular, we have $T_{\alpha_1}(\phi) = T_{\alpha_2}(\phi) = 0$, and this condition characterizes ϕ. Indeed, writing ϕ as a linear combination of $f_1, \ldots, f_6$ and solving $T_{\alpha_1}(\phi) = 0$ and $T_{\alpha_2}(\phi) = 0$ simultaneously, we find that ϕ is a (nonzero) constant multiple of the function $f_1 - \frac{1}{q}f_2 - \frac{1}{q}f_3 - \frac{1}{q^3}f_4 + \frac{1}{q^2}f_5 + \frac{1}{q^2}f_6$. The action of L_B as described by (2.1) with $\chi_1 = \chi|\,|, \chi_2 = \chi, \chi_3 = \chi|\,|^{-1}$ gives $L_B(\phi) = -\chi^2(\pi)\phi$, which in turn yields the factor $1 - \chi^2(\pi)u$ of $\det(I + L_B u)$ arising from the Steinberg representation ρ. Note that $V_\rho^E = V_\rho^K = \{0\}$ since ϕ is not fixed by E.

Case (d) (ρ, V_ρ) is the subrepresentation of $V = \mathrm{Ind}(\chi|\,|^{-1/2}, \chi|\,|^{1/2}, \chi^{-2})$ for an unramified unitary character χ. Then the dimension of the Jacquet module $V_{\rho,N}$ is three (cf. [Ze]), and so is the dimension of V_ρ^B. The intertwining map $T_{\alpha_1} : \mathrm{Ind}(\chi|\,|^{-1/2}, \chi|\,|^{1/2}, \chi^{-2}) \to \mathrm{Ind}(\chi|\,|^{1/2}, \chi|\,|^{-1/2}, \chi^{-2})$ when restricted on V^B is given by

$$T_{\alpha_1}(f_1) = -f'_1 + \frac{1}{q}f'_2, \qquad T_{\alpha_1}(f_2) = f'_1 - \frac{1}{q}f'_2, \qquad T_{\alpha_1}(f_3) = -f'_3 + \frac{1}{q}f'_5,$$
$$T_{\alpha_1}(f_4) = f'_6 - \frac{1}{q}f'_4, \qquad T_{\alpha_1}(f_5) = f'_3 - \frac{1}{q}f'_5, \qquad T_{\alpha_1}(f_6) = -f'_6 + \frac{1}{q}f'_4,$$

where $\{f_1, f_2, \ldots, f_6\}$ is a basis for the space of B-fixed vectors in $\mathrm{Ind}(\chi|\,|^{-1/2}, \chi|\,|^{1/2}, \chi^{-2})$ and $\{f'_1, f'_2, \ldots, f'_6\}$ that in $\mathrm{Ind}(\chi|\,|^{1/2}, \chi|\,|^{-1/2}, \chi^{-2})$, defined as in §2.3. The representation ρ does not occur as a subrepresentation of $\mathrm{Ind}(\chi|\,|^{1/2}, \chi|\,|^{-1/2}, \chi^{-2})$ (cf. [Ze]), which implies that $T_{\alpha_1}(f) = 0$ for all f in V_ρ^B. (Note that in this case, T_{α_2} gives an isomorphism between $\mathrm{Ind}(\chi|\,|^{-1/2}, \chi|\,|^{1/2}, \chi^{-2})$ and $\mathrm{Ind}(\chi|\,|^{-1/2}, \chi^{-2}, \chi|\,|^{1/2})$, so it adds no condition in determining V_ρ^B.) Now it is easy to see that V_ρ^B is a 3-dimensional space generated by $f_1 + f_2$, $f_3 + f_5$, and $f_4 + f_6$; V_ρ^E is a 2-dimensional space generated by $g_1 = f_1 + f_2$ and $g_2 + g_3 = \sum_{3 \le i \le 6} f_i$; and V_ρ^K is a 1-dimensional space generated by $h = \sum_{1 \le i \le 6} f_i$.

Then a routine calculation using (2.1), (2.2), (2.3), and (2.4) shows that the operator L_B on V_ρ^B has eigenvalues $q^{1/2}\chi^{-1}(\pi)$, $-q^{3/4}\chi^{1/2}(\pi)$ and $q^{3/4}\chi^{1/2}(\pi)$;

Type	Zeros of $\det(I+L_Bu)$	Zeros of $\det(I-L_Eu)$	Zeros of $\det(I-A_1u+qA_2u^2-q^3u^3)$
(a)	$\pm q^{-1/2}\chi_i^{-1/2}(\pi)$, $i=1,2,3$	$q^{-1}\chi_i^{-1}(\pi)$, $i=1,2,3$	$q^{-1}\chi_i^{-1}(\pi)$, $i=1,2,3$
(b)	$-q^{-1}\chi^{-2}(\pi)$	$q^{-2}\chi^{-1}(\pi)$	$\chi^{-1}(\pi)$, $q^{-1}\chi^{-1}(\pi)$, $q^{-2}\chi^{-1}(\pi)$
(c)	$\chi^{-2}(\pi)$	none	none
(d)	$-q^{-1/2}\chi(\pi)$, $\pm q^{-3/4}\chi^{-1/2}(\pi)$	$q^{-1}\chi^2(\pi)$, $q^{-3/2}\chi^{-1}(\pi)$	$q^{-1}\chi^2(\pi)$, $q^{-1/2}\chi^{-1}(\pi)$, $q^{-3/2}\chi^{-1}(\pi)$
(e)	$q^{-1/2}\chi(\pi)$, $\pm q^{-1/4}\chi^{-1/2}(\pi)$	$q^{-1/2}\chi^{-1}(\pi)$	none

Table 1

the operator L_E on V_ρ^E has eigenvalues $q\chi^{-2}(\pi)$ and $q^{3/2}\chi(\pi)$; the Hecke operators A_1 and A_2 on V_ρ^K have eigenvalues $q^{3/2}\chi(\pi)+q^{1/2}\chi(\pi)+q\chi^{-2}(\pi)$ and $q^{3/2}\chi^{-1}(\pi)+q^{1/2}\chi^{-1}(\pi)+q\chi^2(\pi)$, respectively. Thus the representation ρ gives rise to the factor $(1+q^{1/2}\chi^{-1}(\pi)u)(1-q^{3/2}\chi(\pi)u^2)$ in $\det(I+L_Bu)$, the factor $(1-q\chi^{-2}(\pi)u)(1-q^{3/2}\chi(\pi)u)$ in $\det(I-L_Eu)$, and $(1-q^{3/2}\chi(\pi)u)(1-q^{1/2}\chi(\pi)u)(1-q\chi^{-2}(\pi)u)$ in $\det(I-A_1u+qA_2u^2-q^3u^3)$.

Case (e) (ρ,V_ρ) is a subrepresentation of $V=\mathrm{Ind}(\chi|\ |^{1/2},\chi|\ |^{-1/2},\chi^{-2})$ with an unramified unitary character χ. Arguing as before, we see that V_ρ^B is a 3- dimensional space generated by f_2-qf_1, f_5-qf_3, f_4-qf_6; V_ρ^E is a 1-dimensional space generated by $f_4+f_5-qf_3-qf_6$; while there are no nontrivial K-fixed vectors in V_ρ.

The eigenvalues of L_B on V_ρ^B are $-q^{1/2}\chi^{-1}(\pi)$, $-q^{1/4}\chi^{1/2}(\pi)$, $q^{1/4}\chi^{1/2}(\pi)$; and the eigenvalue of L_E on V_ρ^E is $q^{1/2}\chi(\pi)$. In this case, ρ gives rise to the factor $(1-q^{1/2}\chi^{-1}(\pi)u)(1-q^{1/2}\chi(\pi)u^2)$ in $\det(I+L_Bu)$ and the factor $1-q^{1/2}\chi(\pi)u$ in $\det(I-L_Eu)$.

3. Main results

Table 1 summarizes the computations in §2 of the zeros of $\det(I+L_Bu)$, $\det(I-L_Eu)$, and $\det(I-A_1u+qA_2u^2-q^3u^3)$ arising from each type of the representations in $L^2(\Gamma\backslash G)$. A representation will have no contribution if it does not contain a nontrivial B-fixed vector.

Type	dim V^B	dim V^E	dimV^K	Number of representations
(a)	6	3	1	$N_0 - D - 3$
(b)	1	1	1	3
(c)	1	0	0	$3N_0 - 3N_1 + 3N_2 - 3$
(d)	3	2	1	D
(e)	3	1	0	$N_1 - 3N_0 + D + 6$
total	$3N_2$	N_1	N_0	

Table 2

Recall that the characters χ_1, χ_2, χ_3 in case (a) are all unramified with the product equal to identity; the character χ in cases (b) and (c) is unramified with order dividing 3; and the character χ in cases (d) and (e) is unramified and unitary.

Denote by N_0, N_1, and N_2 the number of vertices, edges, and chambers in X_Γ, respectively, so that $\dim L^2(\Gamma\backslash G/K) = N_0$, $\dim L^2(\Gamma\backslash G/E) = N_1$, and $\dim L^2(\Gamma\backslash G/B) = 3N_2$. Denote by D the number of type (d) representations in $L^2(\Gamma\backslash G)$. There are three 1-dimensional representations (type (b)), so the number of representations of types (a) and (d) together is $N_0 - 3$. Comparing the total dimensions of V^K and V^E, we conclude that the number of type (e) representations minus the number of type (d) representations is $N_1 - 3N_0 + 6$. Combined with the total dimension of V^B, we get that the total number of Steinberg representations (type (c)), counting multiplicities, is equal to $3\chi(X_\Gamma) - 3$, where $\chi(X_\Gamma) = N_0 - N_1 + N_2$ is the Euler characteristic of X_Γ. The result is summarized in Table 2.

As there are only three Steinberg representations, which differ by cubic unramified twists, each occurs with multiplicity $\chi(X_\Gamma) - 1$. We have shown

Proposition 1: (1) *Each Steinberg representation of type (c) occurs in $L^2(\Gamma\backslash G)$ with multiplicity equal to $\chi(X_\Gamma) - 1$.*

(2) *The number of representations of type (e) occurring in $L^2(\Gamma\backslash G)$ is at least $N_1 - 3N_0 + 6$, where N_0 and N_1 denote the number of vertices and edges in X_Γ.*

The first statement is in agreement with Garland's result in [Ga], which is for a general p-adic group. The corresponding assertion for graphs is discussed in Hashimoto's paper [Ha].

The zeros arising from 1-dimensional representations (case (b)) are called **trivial** zeros of the respective determinant, as they correspond to the trivial eigenvalues of the operators L_B, L_E, A_1 and A_2. The remaining zeros are **nontrivial**.

Recall also that X_Γ is a Ramanujan complex if and only if the nontrivial zeros of $\det(I - A_1u + qA_2u^2 - q^3u^3I)$ have absolute value q^{-1} (cf. [Li], [LSV]). It is natural to ask whether this property is also reflected on other operators. The theorem below answers this question.

Theorem 2: *The following statements are equivalent.*

(1) X_Γ *is Ramanujan.*

(2) *The nontrivial zeros of* $\det(I - A_1u + qA_2u^2 - q^3u^3)$ *have absolute value* q^{-1}.

(3) *The nontrivial zeros of* $\det(I - L_Eu)$ *have absolute values* q^{-1} *and* $q^{-1/2}$.

(4) *The nontrivial zeros of* $\det(I + L_Bu)$ *have absolute values 1,* $q^{-1/2}$*, and* $q^{-1/4}$.

Proof. In view of Table 1, X_Γ is Ramanujan if and only if the representations of type (d) do not occur and the principal series representations of type (a) are induced from unitary unramified characters. These conditions reflected on $\det(I - A_1u + qA_2u^2 - q^3u^3)$, $\det(I - L_Eu)$, and $\det(I + L_Bu)$ are as stated since representations of types (c) and (e) do occur by Proposition 1. ∎

Remark: If the group Γ arises from an inner form H of GL_3 as described in §4 of [KL], then, by the strong approximation theorem, the functions on vertices of X_Γ can be viewed as automorphic forms on H. The underlying automorphic representations then correspond to cuspidal automorphic representations of PGL_3 under the Jacquet–Langlands correspondence established in [JPSS]. As such, its local components are generic. In view of Theorem 9.7 of [Ze], representations of type (d) cannot occur.

In particular, when the local field F has positive characteristic, it was shown in [LSV] that X_Γ is always Ramanujan for Γ arising from an inner form.

We are now ready to prove the main result of this paper.

Theorem 3: *The following identity holds:*

$$(1-u^3)^{\chi(X_\Gamma)} = \frac{\det(I - A_1u + qA_2u^2 - q^3u^3I)\det(I + L_Bu)}{\det(I - L_Eu)\det(I - (L_E)^tu^2)}.$$

Proof. Denote by $R(u)$ the right hand side of the identity. As a consequence of Table 1, we have the following list describing the contribution to $R(u)$ arising from each type of representation:

- Type (a): 1;
- Type (d): $\frac{(1-\chi(\pi)u)(1-q\chi(\pi)u)}{1-q\chi^2(\pi)u}$;
- Type (c): $1-\chi^2(\pi)u$;
- Type (d): $\frac{1-q^{1/2}\chi(\pi)u}{1-q^{1/2}\chi^{-1}(\pi)u}$;
- Type (e): $\frac{1-q^{1/2}\chi^{-1}(\pi)u}{1-q^{1/2}\chi(\pi)u}$.

Note that the character χ occurring in representations of types (b) and (e) are unitary. Since each determinant is a polynomial with coefficients in $\mathbb{Z}$, if χ is not real-valued, then there is a representation of the same type with χ replaced by χ^{-1}, and the two representations occur with the same multiplicity. Consequently, the contributions from representations of types (a), (d) and (e) all cancel out.

The characters χ occurring in representations of types (d) and (c) satisfy $\chi^3 = id$. Taking the product over all such χ of the contributions in $R(u)$ from type (b) representations yields $1-u^3$, while the product of those of type (c) is $(1-u^3)^{\chi(X_\Gamma)-1}$ by Proposition 1. This shows that the total contribution from all representations in $R(u)$ is equal to $(1-u^3)^{\chi(X_\Gamma)}$, which is equal to the left hand side of the identity. ∎

As defined in §4.1 of [KL], an element γ in Γ is rank-one split if its eigenvalues generate a quadratic extension of the base field F. Table 1 shows that the eigenvalues of L_B and L_E are never zero, hence the operators L_B and L_E have full rank. Since the rank of L_B is greater than twice the rank of L_E, the left hand side of Theorem 26 in §8.3 of [KL] is nonzero, from which we conclude

Corollary 4: *The group* Γ *contains rank-one split elements.*

This is different from the case of graphs. More precisely, the non-identity elements in a regular torsion-free discrete cocompact subgroup of $\mathrm{PGL}_2(F)$ are hyperbolic (called split in §4.1 of [KL]), while a regular torsion-free discrete cocompact subgroup of $\mathrm{PGL}_3(F)$ contains, in addition to split elements, also rank-one split elements.

References

[Bo] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Inventiones Mathematicae **35** (1976), 233–259.

[Ca] W. Casselman, *The unramified principal series of p-adic groups. I. The spherical function*, Compositio Mathematica **40** (1980), 387–406.

[Ga] H. Garland, *p-adic curvature and the cohomology of discrete subgroups of p-adic groups*, Annals of Mathematics **97** (1973), 375–423.

[Ha] K. Hashimoto, *Zeta functions of finite graphs and representations of p-adic groups*, in *Automorphic Forms and Geometry of Arithmetic Varieties,* in *Advanced Studies in Pure Mathematics* **15**, Academic Press, Boston, MA, 1989, pp. 211–280.

[JPSS] H. Jacquet, I. I. Piatetski-Shapiro and J. Shalika, *Automorphic forms on* GL(3), *I & II*, Annals of Mathematics **109** (1979), 169–258.

[KL] M.-H. Kang and W.-C. W. Li, *Zeta functions of complexes arising from* PGL(3), (2008), submitted.

[Li] W.-C. W. Li, *Ramanujan hypergraphs*, Geometric and Functional Analysis **14** (2004), 380–399.

[LSV] A. Lubotzky, B. Samuels and U. Vishne, *Ramanujan complexes of type* $\widetilde{A}_d$, Israel Journal of Mathematics **149** (2005), 267–299.

[Ta] M. Tadić, *Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case)*, Annales Scientifiques de l'École Normale Supérieure. Quatriéme Série **19** (1986), 335–382.

[Ze] A. V. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of* GL(n), Annales Scientifiques de l'École Normale Supérieure. Quatriéme Série **13** (1980), 165–210.