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LIOUVILLE THEOREM, CONFORMALLY INVARIANT CONES AND UMBILICAL SURFACES FOR GRUSHIN-TYPE METRICS

BY

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ABSTRACT

We prove a classification theorem for conformal maps with respect to the control distance generated by a system of diagonal vector fields in $\mathbb{R}^n$. It turns out that in many cases all such maps can be obtained as compositions of suitable dilations, inversions and isometries. Our methods involve a study of the singular Riemannian metric associated with the vector fields. In particular, we identify some conformally invariant cones related to the Weyl tensor. The knowledge of such cones enables us to classify all umbilical hypersurfaces.

1. Introduction

The principal purpose of this paper is to classify maps which are conformal with respect to the control (Carnot–Carathéodory) distance d generated by a system of diagonal vector fields. Our principal result is that all such maps are compositions of a restricted class of elementary conformal maps: isometries, suitable dilations and inversions naturally associated with the distance d. The form of these elementary maps will be explicitly identified.

Consider in $\mathsf{M} := \mathbb{R}^p \times \mathbb{R}^q$ the diagonal vector fields

$$X_j = \frac{\partial}{\partial x_j}, \quad Y_\lambda = (\alpha+1)|x|^\alpha \frac{\partial}{\partial y_\lambda}, \quad j = 1, \dots, p, \quad \lambda = 1, \dots q. \tag{1.1}$$

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Here $\alpha > 0$ is fixed. Vector fields of the form (1.1) are usually referred to as Grushin vector fields and they are a subclass of the diagonal vector fields studied by Franchi and Lanconelli in [5]. Denote by $d : \mathsf{M} \times \mathsf{M} \to [0, +\infty[$ the control distance associated with the vector fields in (1.1) (see Subsection 2.2, or [5] for a complete account). We take here the following metric definition of conformal map. Let $\Omega, \Omega' \subset \mathsf{M}$ be open sets. A homeomorphism $f : \Omega \to \Omega'$ is conformal with respect to the metric d if there is a function $u : \Omega \to]0, +\infty[$ such that

$$\lim_{\zeta \to z} \frac{d(f(\zeta), f(z))}{d(\zeta, z)} = u(z)^{-1}, \tag{1.2}$$

for any $z = (x, y) \in \Omega$. We say that u is the **conformal factor** of f.

It is not difficult to check that the following maps are conformal:

$$(x, y) \mapsto \Gamma(x, y) = (Ax, By + b), \quad A \in O(p),\ B \in O(q),\ b \in \mathbb{R}^q; \tag{1.3}$$

$$(x, y) \mapsto \delta_t(x, y) := (tx, t^{\alpha+1} y), \qquad t > 0. \tag{1.4}$$

Maps of the form (1.3) are isometries. As the form of the vector fields Y_λ suggests, no translations in the variable x are admitted in (1.3). Note also that all the vector fields X_j, Y_λ are homogeneous of degree 1 with respect to the anisotropic dilations (1.4).

A less trivial example of conformal map, which makes the model studied here quite rich, is given by the following inversion. Define the "homogeneous norm" $\|z\| = \|(x, y)\| = \big(|x|^{2(\alpha+1)} + |y|^2\big)^{1/(2(\alpha+1))}$. Then, for any $z \in \mathsf{M} \setminus \{(0, 0)\}$, let

$$\Phi(z) = \delta_{\|z\|^{-2}} z. \tag{1.5}$$

The map (1.5) is a reflection in the homogeneous sphere of equation $\|z\| = 1$. It generalizes to the present setting the classical Möbius inversion $t \mapsto |t|^{-2} t$, where t belongs to a Euclidean space. See the discussion in Subsection 2.2. The conformality of the map Φ was already recognized in [18] by R. Monti and the author.

Compositions of the elementary maps described above provide easily more examples of conformal maps. Our main result states that, if $p \geq 3$, there are no further examples.

THEOREM 1.1: *Assume that $p \geq 3$ and $q \geq 1$. Let $\Omega, \Omega' \subset \mathsf{M}$ be connected open sets. Let $f : \Omega \to \Omega'$ be a conformal homeomorphism in the metric sense*

(1.2). *Then f has the form*

$$f(z) = \Gamma(\delta_{t\|(x,y-b)\|^s}(x, y - b)), \tag{1.6}$$

for all $z = (x, y) \in \Omega$. Here Γ is an isometry of the form (1.3), $t > 0$, $b \in \mathbb{R}^q$ *and $s = 0$ or -2.*

We immediately observe that the theorem is false for $p = 2$, $q \geq 1$. This is a consequence of the fact that the Riemannian metric $\widehat{g}$ (see (1.7) below) is conformally flat if $p = 2$. See Subsection 2.1 and Remark 3.3. Case $p = 1$ has been discussed by Payne [23].

The proof of Theorem 1.1 requires Riemannian arguments, because the control distance of the Grushin vector fields is Riemannian away from the (somehow small) set where x vanishes. Indeed, consider in $\mathsf{M}_0 := (\mathbb{R}^p \setminus \{0\}) \times \mathbb{R}^q$ the metric

$$\widehat{g} = |dx|^2 + \frac{|dy|^2}{(\alpha + 1)^2 |x|^{2\alpha}}. \tag{1.7}$$

The vector fields introduced in (1.1) form an orthonormal frame in $(\mathsf{M}_0, \widehat{g})$. It is easy to realize that their control distance agrees with the Riemannian distance $d_{\widehat{g}}$ associated with $\widehat{g}$ (lengths of curves are the same). Moreover, by a result of Ferrand [12], a conformal homeomorphism in a smooth Riemannian manifold must be smooth. Thus, given a homeomorphism $f : \Omega \to \Omega'$, where $\Omega, \Omega' \subset \mathsf{M}$, if f satisfies (1.2), then it is smooth in $\Omega \cap \mathsf{M}_0$ and it must satisfy the Cauchy–Riemann system

$$\widehat{g}(f_* U, f_* V) = u(z)^{-2} \widehat{g}(U, V), \tag{1.8}$$

for all vector fields U, V supported $\Omega \cap \mathsf{M}_0$ and for a suitable conformal factor u.

In view of the discussion above, it turns out that our main result Theorem 1.1 follows immediately from the following Liouville theorem for the manifold $(\mathsf{M}_0, \widehat{g})$.

THEOREM 1.2: *Let $p \geq 3$ and $q \geq 1$. If $\Omega, \Omega' \subset \mathsf{M}_0$ are open and connected and $f : \Omega \to \Omega'$ is a smooth diffeomorphism satisfying the Cauchy–Riemann system* (1.8), *then f has the form* (1.6).

In order to prove Theorem 1.2, we use the metric $g = (\alpha + 1)^2 |x|^{2\alpha} \widehat{g}$, which belongs to the same conformal class of $\widehat{g}$ and makes computations easier. A standard way to study conformal maps on Riemannian manifolds starts from the interpretation of the transformation formula for the Ricci tensor under conformal changes of metrics as a tool to obtain a system of partial differential equations for the conformal factor u. Indeed, given a metric g, letting $\widetilde{g} = u^{-2}g$, then we have the classical formula

$$\mathrm{Ric}_{\widetilde{g}} = \mathrm{Ric}_g + (n-2)u^{-1}\nabla^2 u - u^{-2}\big\{(n-1)|\nabla u|^2 - u\Delta u\big\}g. \tag{1.9}$$

Here $\nabla^2 u$ denotes the Hessian in the Levi Civita connection ∇ of the metric g, while $|\nabla u|$ is the length of the gradient and Δu the Laplacian. Next, if $f : \Omega \to \Omega'$ is a conformal diffeomorphism in (M_0, g), which means, by definition, $f^*g = u^{-2}g$ for some function u, then u satisfies

$$\begin{aligned}&\mathrm{Ric}(f_*U, f_*V)\\ &= \mathrm{Ric}(U,V) + (n-2)u^{-1}\nabla^2 u(U,V) - g(U,V)u^{-2}\big\{(n-1)|\nabla u|^2 - u\Delta u\big\},\end{aligned} \tag{1.10}$$

for any pair of vector fields U, V in Ω. Here $\mathrm{Ric} = \mathrm{Ric}_g$. We will show that, if $p \geq 3$, then all solutions u of (1.10) have the same form as the conformal factor of a suitable composition of maps of the form (1.3), (1.4) and (1.5). This will reduce the proof to a classification of local isometries of g, which is given in Section 2.

This strategy can be easily pursued in the Euclidean case and it reduces to a few lines thanks to the Ricci flatness of the Euclidean space. See [26, Chapter 6]. (See also [1] or Liouville's paper [13], for analytical proofs based on differentiation of the Cauchy–Riemann system.) Indeed, in the Euclidean case, Ricci flatness makes system (1.10) an easy overdetermined system in the only unknown u, whose solutions are particular quadratic polynomials. In our case, the metric g is not Ricci flat. Therefore it is not clear how to manage the Ricci terms in (1.10), especially the one on the left-hand side.

In order to make viable system (1.10), we introduce the following conformally invariant cones $\mathcal{U}_P \subset T_P\mathsf{M}_0$. Assume that the dimension of M_0 is greater than 3. Let R, Ric and Scal be the Riemann, Ricci and scalar curvature of g, respectively. Let

$$W = R + \frac{1}{n-2}(\mathrm{Ric} \odot g) - \frac{\mathrm{Scal}}{2(n-1)(n-2)}(g \odot g) \tag{1.11}$$

be the Weyl tensor. Here $(h\odot s)_{abcd} := h_{ad}s_{bc}+h_{bc}s_{ad}-h_{ac}s_{bd}-h_{bd}s_{ac}$ denotes the Kulkarni– Nomizu product of symmetric 2-tensors. Then define

$$\mathcal{U}_P = \{X \in T_P\mathsf{M}_0 : W(X,Y,U,V) = 0 \text{ for all } Y,U,V$$
$$\text{such that } X,Y,U,V \text{ are pairwise orthogonal with respect to } g\ \}.$$

The conformal invariance of the Weyl tensor (if $\widetilde{g} = u^{-2}g$, then $W_{\widetilde{g}} = u^{-2}W_g$) trivially implies that if $f : \Omega \to \Omega'$ is a conformal diffeomorphism between subsets of M_0, then

$$f_*(\mathcal{U}_P) = \mathcal{U}_{f(P)}, \quad \forall\ P \in \Omega.$$

In the case of the Grushin metric, the cones $\mathcal{U}_P$ will be determined explicitly at any $P \in \mathsf{M}_0$. They have a very clear structure in suitable cylindrical coordinates. Observe that the invariants $\mathcal{U}_P$ may also be used to study conformal maps in different Riemannian manifolds, provided the Weyl tensor has a nontrivial structure. See, for example, Remark 3.4. We also mention that different conformally invariant subsets (actually subspaces) of the tangent space constructed from the Weyl tensor were used by Listing [14].

With the explicit form of the cones $\mathcal{U}_P$ in hand, it becomes possible to deal with Ricci terms in system (1.10) and ultimately to solve it. Then Theorem 1.2 follows in a rather standard way. It turns out that all conformal maps preserve the Ricci tensor, in the sense that $\mathrm{Ric}_g(f_*U, f_*V) = \mathrm{Ric}_g(U,V)$, for all vector fields U,V. Therefore conformal maps are Liouville maps in the language of [10] and, in particular, Möbius maps in the terminology of [21]. Here the choice of the metric g in the conformal class $[\widehat{g}]$ is important.

Although our method of classification could be less efficient than other techniques, like the study of conformal Killing vector fields (see Payne [13], in the case $p = 1$), it should be emphasized that our approach provides more. Indeed, the study of the cones $\mathcal{U}_P$ is inspired by the fact that conformal maps must send umbilical hypersurfaces to umbilical hypersurfaces, see e.g. [11]. Indeed, given a manifold (M_0, g), of dimension at least 4, a standard obstruction to the existence of an umbilical surface Σ with given normal $N \in T_P\mathsf{M}_0$ at a point $P \in \Sigma$ is provided by Codazzi equations (see [20]), which for an umbilical hypersurface of curvature κ with respect to a unit normal N become

$$g(V,Z)(U\kappa) - g(U,Z)(V\kappa) = R(N,Z,U,V), \tag{1.12}$$

for all U,V,Z tangent to Σ. If U,V and Z are pairwise orthogonal, then $R(N,U,V,Z) = 0$. Moreover, $R(N,U,V,Z) = W(N,U,V,Z)$, by the form

(1.11) of the Weyl tensor. Therefore, equation (1.12) shows that a normal vector at P to an umbilical surface cannot belong to $\mathcal{U}_P$.

After the cones $\mathcal{U}_P$ are known, we are able to classify in Section 4 all umbilical hypersurfaces in (M_0, g) for $p \geq 3$. It turns out that they are rather rare, while for $p = 2$ the situation is different; see Remark 4.1. Here is our result.

THEOREM 1.3: *Let $p \geq 3$, $q \geq 1$. Then any connected umbilical hypersurface in (M_0, g) is contained in one among the following:*

(A1) *a homogeneous sphere of equation $|x|^{2(\alpha+1)} + |y - b|^2 = c^2, b \in \mathbb{R}^q, c > 0$;*
(A2) *a plane of equation $\langle a, y\rangle = c$, where $a \in \mathbb{R}^q$, $c \in \mathbb{R}$;*
(B) *a plane of equation $\langle a, x\rangle = 0$, where $a \in \mathbb{R}^p$.*

Observe that the choice of the metric g in the conformal class $[g]$ ensures that all umbilical surfaces have constant curvature; see Remark 4.1.

Grushin-type geometries have been studied rather recently. They pose interesting problems from the point of view of nonlinear analysis, sharp inequalities and search for symmetries related to the degenerate elliptic operator

$$\Delta_\alpha := \Delta_x + (\alpha + 1)^2 |x|^{2\alpha} \Delta_y. \tag{1.13}$$

See, for example, the papers [2, 17, 28, 6, 3, 15, 16], just to quote a few. The conformal inversion Φ in (1.5) is used in [18], in order to construct a Kelvin-type transform for a semilinear equation with critical nonlinearity of the form $-\Delta_\alpha u = u^r$, for a suitable $r > 1$. Our motivation for a better understanding of these conformal maps stems from the mentioned paper.

Concerning Liouville-type theorems in sub-Riemannian geometry, we mention the seminal papers by Korányi and Reimann [8, 9], where conformal maps in the Heisenberg group were classified (see also [4, 28]). More rigidity results are contained in [22, 25, 24]. The quoted paper are in the setting of Lie groups.

The plan of the paper is the following. In Section 2, we discuss some preliminary facts: the metric g, its isometries (Subsection 2.1), and the conformal inversion Φ (Subsection 2.2). In Section 3 we prove the Liouville theorem. We first study the cones $\mathcal{U}_P$, in Subsection 3.1; then, in Subsection 3.2, we solve system (1.10); finally we show, in Subsection 3.3, how the proof can be quickly concluded in view of the explicit knowledge of isometries. Section 4 is devoted to the classification of umbilical surfaces. Finally, we included a short appendix with some standard formulas on warped metrics in Riemannian manifolds.

Notation 1.4: Given a Riemannian metric g, we denote by ∇ the associated Levi Civita connection and by $R(X,Y)U = \nabla_X\nabla_Y U - \nabla_Y\nabla_X U - \nabla_{[X,Y]}U$ the curvature operator. We let $R(U,V,X,Y) = g(U,R(X,Y)V)$, so that $\mathrm{Ric}(X,Y) = \mathrm{trace}\{V \mapsto R(V,Y)X\}$. Moreover, $\langle\cdot,\cdot\rangle$ denotes Euclidean scalar product. Surfaces have codimension 1 and are orientable and connected. Unless otherwise stated, Latin indices i,j,k run from 1 to p, while λ,μ,σ go from 1 to q. For typographical reasons, we write ∂_j or ∂_{x_j} instead of $\partial/\partial x_j$ and we use the an analogous notation for $\partial/\partial y_\lambda$. Summation with respect to a repeated index (in the pertinent range) is sometimes omitted.

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2. Preliminary facts on Grushin geometry

2.1. The Grushin metric. In the conformal class of $\widehat{g}$ we choose the following metric g in M_0:

$$g = (\alpha+1)^2|x|^{2\alpha}|dx|^2 + |dy|^2. \tag{2.1}$$

It turns out that g is better than $\widehat{g}$ for our purposes. Observe that, if $p=2$, then the local holomorphic change of variable $x^{\alpha+1} = \xi$, or alternatively formula (1.9) show that, for any $\alpha \geq 0$, the metric $(\alpha+1)^2|x|^{2\alpha}|dx|^2$ is flat if $x \in \mathbb{R}^2 \setminus \{(0,0)\}$. Therefore, g is flat for $p=2$, $q \geq 1$.

Let $\varrho = |x|$, $\vartheta = \frac{x}{|x|} \in \mathbb{S}^{p-1} \subset \mathbb{R}^p$. Then,

$$(\alpha+1)^2|x|^{2\alpha}|dx|^2 = (\alpha+1)^2\{\varrho^{2\alpha}d\varrho^2 + \varrho^{2(\alpha+1)}|d\vartheta|^2\},$$

where $|d\vartheta|^2$ is the standard metric on the sphere $\mathbb{S}^{p-1}$. Moreover, letting $\varrho^{\alpha+1} = r$, it follows quickly that

$$g = dr^2 + |dy|^2 + (\alpha+1)^2 r^2 |d\vartheta|^2. \tag{2.2}$$

Using the notation

$$\mathsf{H} = \{(r, y) \in \,]0, +\infty[\, \times \mathbb{R}^q\}, \quad g_{\mathsf{H}} = dr^2 + |dy|^2,$$
$$\mathsf{S} = \mathbb{S}^{p-1} = \{\vartheta \in \mathbb{R}^p : |\vartheta| = 1\}, \quad g_{\mathsf{S}} = |d\vartheta|^2,$$

we can write the manifold (M_0, g) as a warped product $\mathsf{H} \times_w \mathsf{S}$, with warping function $w(r, y) = (\alpha + 1)r$. Briefly, $g = g_{\mathsf{H}} + w^2 g_{\mathsf{S}}$. See the appendix for some standard facts about warped products. See [20, Chapter 7] for a complete introduction.

For any $P = (r, y, \vartheta) \in \mathsf{H} \times \mathsf{S}$, decompose any $U \in T_P(\mathsf{H} \times \mathsf{S})$ as

$$U = U_{\mathsf{H}} + U_{\mathsf{S}} \in T_P\mathsf{H} \oplus T_P\mathsf{S}, \tag{2.3}$$

where $T_P\mathsf{S}$ and $T_P\mathsf{H}$ denote the lifts at P of the tangent spaces $T_\vartheta\mathsf{S}$ and $T_{(r,y)}\mathsf{H}$, respectively.

Next we describe the connection in the warped model. Denote by ∇^{H}, ∇^{S} and ∇ the Levi Civita connections on H, S and $\mathsf{H} \times_w \mathsf{S}$, respectively. Since the factor H is Euclidean, by (A.1), covariant derivatives are Euclidean, namely (in the notation $\partial_r = \partial/\partial r$ and $\partial_\lambda = \partial/\partial y_\lambda$)

$$\nabla_{\partial_r}\partial_r = \nabla_{\partial_r}\partial_\lambda = \nabla_{\partial_\lambda}\partial_\mu = 0, \qquad \lambda, \mu = 1, \dots, q. \tag{2.4}$$

Then $\nabla^2 u(\partial_r, \partial_r) = \partial_r^2 u$, $\nabla^2 u(\partial_r, \partial_\lambda) = \partial_r\partial_\lambda u$ and $\nabla^2 u(\partial_\lambda, \partial_\mu) = \partial_\mu\partial_\lambda u$. Moreover, again by (A.1), given X, lifting of a vector field on the sphere, then

$$\nabla_{\partial_r} X = \frac{1}{r} X \quad \text{and} \quad \nabla_{\partial_\lambda} X = 0, \quad \lambda = 1, \dots, q. \tag{2.5}$$

Let $u = u(r, y, \vartheta)$ be a scalar function. Then

$$|\nabla u|^2 = (\partial_r u)^2 + |\nabla_y u|^2 + (\alpha + 1)^{-2} r^{-2} |\nabla^{\mathsf{S}} u|^2.$$

Moreover, given X, X' on the sphere, (A.2) provides

$$\nabla^2 u(X, X') = (\nabla^{\mathsf{S}})^2 u(X, X') + \frac{u_r}{r} g(X, X') \tag{2.6}$$

and

$$\Delta u = u_{rr} + \frac{p-1}{r} u_r + \Delta_y u + \frac{1}{(\alpha+1)^2 r^2} \Delta_{\mathsf{S}} u. \tag{2.7}$$

In order to compute the curvature, note that in our case $\nabla^2 w = 0$ (w is linear in r, y). Therefore, only the third line in (A.3) gives nonzero terms.

A short computation using the curvature of the standard sphere $\mathsf{S} = \mathbb{S}^{p-1}$, $R^{\mathsf{S}}(V_1, V_2, V_3, V_4) = g_{\mathsf{S}}(V_1, V_3) g_{\mathsf{S}}(V_2, V_4) - g_{\mathsf{S}}(V_1, V_4) g_{\mathsf{S}}(V_2, V_3)$, gives

(2.8)
$$\begin{aligned} R(U,V, X, Y) &\\ &= R(U_{\mathsf{S}}, V_{\mathsf{S}}, X_{\mathsf{S}}, Y_{\mathsf{S}}) \\ &= -\alpha(\alpha+2)(\alpha+1)^{-2} r^{-2} \Big\{ g(U_{\mathsf{S}}, X_{\mathsf{S}}) g(V_{\mathsf{S}}, Y_{\mathsf{S}}) - g(U_{\mathsf{S}}, Y_{\mathsf{S}}) g(V_{\mathsf{S}}, X_{\mathsf{S}}) \Big\}, \end{aligned}$$

where all the vectors U, V, X, Y are decomposed as in (2.3). We see again that the manifold is flat for $p = 2$ ($\mathbb{S}^{p-1} = \mathbb{S}^1$ and the curly bracket in (2.8) vanishes). Contracting,

(2.9)
$$\begin{aligned} \mathrm{Ric}(U, V) = \mathrm{Ric}(U_{\mathsf{S}}, V_{\mathsf{S}}) = -\alpha(\alpha+2)(p-2)(\alpha+1)^{-2} r^{-2} g(U_{\mathsf{S}}, V_{\mathsf{S}}) \quad \text{and} \\ \mathrm{Scal} = -\alpha(\alpha+2)(p-2)(p-1)(\alpha+1)^{-2} r^{-2}. \end{aligned}$$

Next we classify all local isometries of $\mathsf{H} \times_w \mathsf{S}$ for $p \geq 3$.

Proposition 2.1: *Let $p \geq 3$. Let $\Omega \subset \mathsf{M}_0$ be a connected open set. Let $f : \Omega \to \Omega' \subset \mathsf{M}_0$ be a local isometry in the metric g. Then f is a restriction of a map of the form*

$$(x, y) \mapsto (Ax, By + b),$$

where $A \in O(p)$, $B \in O(q)$ and $b \in \mathbb{R}^q$.

Proof. Write the map as $(x, y) \mapsto (\widetilde{x}(x, y), \widetilde{y}(x, y))$. Since isometries preserve scalar curvature, (2.9) gives $|\widetilde{x}(x, y)| = |x|$ for all $(x, y) \in \Omega$. Here the choice $p \geq 3$ is crucial.

Introduce the notation $\Sigma_\varrho = \{(x, y) : |x| = \varrho\}$. Next we claim that, for any $\varrho > 0$, the restriction of the map f to the set $\Omega \cap \Sigma_\varrho$ (provided the latter is nonempty) has the form

$$(x, y) \mapsto (A(\varrho)x, B(\varrho)y + b(\varrho)), \tag{2.10}$$

where $A(\varrho)$, $B(\varrho)$ are orthogonal and $b(\varrho) \in \mathbb{R}^q$. We assume without loss of generality that Ω is a product of the form $\Omega = \{\varrho \in (\varrho_0, \varrho_1), |y - y_0| < \varepsilon_0, |\vartheta - \vartheta_0| < \varepsilon_1\}$, so that $\Omega \cap \Sigma_\varrho$ is connected. Observe that the metric on Σ_ϱ has the form $(\alpha+1)^2 \varrho^{2(\alpha+1)} |d\vartheta|^2 + |dy|^2$, the product of a sphere (of dimension at least 2, because $p \geq 3$) with a Euclidean space. Therefore, the claim follows from the standard fact than a local isometry of a product of space forms of different curvature must be a product map of isometries of the factors (this fact can be easily proved by means of isometric invariance of sectional curvatures).

Finally, we prove that A, B, b are constant in ϱ. Take a point $z = (x, y)$. The normal vector $(\partial_\varrho)_z$ at the point $z = (x, y)$ to the surface Σ_ϱ is sent by f_* to the vector $\pm(\partial_\varrho)_{\widetilde{z}}$, normal to the same surface at the point $f(z) = \widetilde{z} = (\widetilde{x}, \widetilde{y})$. Since scalar curvature is increasing as ϱ increases, the sign must be $+$. Moreover, by the chain rule,

$$f_*((\partial_\varrho)_z) = \partial_\varrho\big(A_j^k x^j\big)(\partial_k)_{\widetilde{z}} + \partial_\varrho\big(B_\mu^\sigma y^\mu + b^\sigma\big)(\partial_\sigma)_{\widetilde{z}},$$

where we sum for $j, k = 1, \dots, p$ and for $\sigma, \mu = 1, \dots, q$. Therefore, the second term, the one with derivatives in y, must be zero. Thus, $(\partial_\varrho B)y + \partial_\varrho b = 0$. Differentiating in y, we get $\partial_\varrho B = 0$. Then $\partial_\varrho b = 0$. We have proved that B and b are constant.

Finally, we look at the first term. Recall that $\varrho = |x|$, so that $\partial_\varrho x^j = x^j/\varrho$. Then,

$$\begin{aligned}(\partial_\varrho)_{\widetilde{z}} = f_*(\partial_\varrho)_z = \partial_\varrho\big(A_j^k x^j\big)(\partial_k)_{\widetilde{z}} &= \Big((\partial_\varrho A_j^k)x^j + A_j^k\frac{x^j}{\varrho}\Big)(\partial_k)_{\widetilde{z}} \\ &= \Big((\partial_\varrho A_j^k)x^j + \frac{1}{\varrho}\widetilde{x}^k\Big)(\partial_k)_{\widetilde{z}} = (\partial_\varrho A_j^k)x^j(\partial_k)_{\widetilde{z}} + (\partial_\varrho)_{\widetilde{z}}.\end{aligned}$$

Thus, $(\partial_\varrho A_j^k)x^j = 0$, which gives (differentiate in x) $\partial_\varrho A_j^k = 0$. The proof is concluded. ∎

2.2. Control distance and conformality of the inversion map. In this subsection we show that inversion is conformal. The same result has been proved in [18], but here we provide a shorter proof, using the warped model. Let $\Phi(z) = \delta_{\|z\|^{-2}} z$. Our aim is to check that, for any $z \neq (0,0)$,

$$\lim_{\zeta \to z} \frac{d(\Phi(\zeta), \Phi(z))}{d(\zeta, z)} = \|z\|^{-2}. \tag{2.11}$$

Before proving (2.11), we briefly recall the definition of control distance associated with the vector fields X_j, Y_λ, $j = 1, \dots, p, \lambda = 1, \dots, q$. See [5], see also [19]. An absolutely continuous path $\gamma : [0, T] \to \mathsf{M}$ is admissible if it satisfies almost everywhere $\dot\gamma = \sum a_j X_j(\gamma) + \sum b_\lambda Y_\lambda(\gamma)$ for suitable measurable functions $a_j, b_\lambda : [0, T] \to \mathbb{R}$. Define, for $z, z' \in \mathsf{M}$, $d(z, z') = \inf \int_0^T \sqrt{|a|^2 + |b|^2}$, where the infimum is taken among all the functions a_j, b_λ such that the corresponding path γ is admissible and connects z and z'.

We prove conformality by means of a suitable Cauchy–Riemann system. Indeed we prove that

$$\widehat{g}(\Phi_* U, \Phi_* U) = \|z\|^{-4}\widehat{g}(U,U), \tag{2.12}$$

for all $U \in \operatorname{span}\{X_j, Y_\lambda : j = 1,\dots,p, \lambda = 1,\dots,q\}$, $z = (x,y) \neq (0,0)$.

We first prove (2.12) in the set M_0, namely where $|x| > 0$. In the warped model of Subsection 2.1, the map Φ takes the form

$$\Phi(r,y,\vartheta) = \big(\varphi(r,y),\vartheta\big), \qquad \text{where } \varphi(r,y) = |(r,y)|^{-2}(r,y)$$

is a Euclidean Möbius map. The metric g at

$$P := (r,y,\vartheta)$$

is $dr^2 + |dy|^2 + (\alpha+1)^2 r^2 |d\vartheta|^2$, while in $\Phi(P) = (\varphi(r,y),\vartheta)$ it has the form $dr^2 + |dy|^2 + (\alpha+1)^2|(r,y)|^{-4}r^2|d\vartheta|^2$. Therefore, if we decompose, as in (2.3), $U = U_{\mathsf{H}} + U_{\mathsf{S}}$, we get $\Phi_*(U_{\mathsf{H}} + U_{\mathsf{S}}) = (\varphi_* U_{\mathsf{H}}) + U_{\mathsf{S}} \in T_{\Phi(P)}\mathsf{H} \oplus T_{\Phi(P)}\mathsf{S}$. Thus, since $T\mathsf{H}$ and $T\mathsf{S}$ are orthogonal,

$$g_{\Phi(P)}(\Phi_* U, \Phi_* U) = g_{\Phi(P)}(\varphi_* U_H, \varphi_* U_H) + g_{\Phi(P)}(U_{\mathsf{S}}, U_{\mathsf{S}}),$$

where, in order to be safe, we used the slightly cumbersome notation $g_{\Phi(P)}$ to indicate the metric g at the point $\Phi(P)$. Next look at the first term. By the properties of Euclidean Möbius maps, we have

$$g(\varphi_* U_{\mathsf{H}}, \varphi_* U_{\mathsf{H}}) = |(r,y)|^{-4} g(U_H, U_{\mathsf{H}}). \tag{2.13}$$

Moreover, looking at the second term, since the metric at the image point $(\varphi(r,y),\vartheta)$ is $dr^2 + |dy|^2 + (\alpha+1)^2|(r,y)|^{-4}r^2 g_{\mathsf{S}}$ (here $g_{\mathsf{S}} = |d\vartheta|^2$), we have

$$g_{\Phi(P)}(U_{\mathsf{S}}, U_{\mathsf{S}}) = (\alpha+1)^2|(r,y)|^{-4}r^2(g_{\mathsf{S}})_\vartheta(U_{\mathsf{S}}, U_{\mathsf{S}}) = |(r,y)|^{-4} g_P(U_{\mathsf{S}}, U_{\mathsf{S}}). \tag{2.14}$$

Putting together the three formulas above,

$$g(\Phi_* U, \Phi_* U) = |(r,y)|^{-4} g(U,U) = \|z\|^{-4(\alpha+1)} g(U,U), \tag{2.15}$$

which will be referred to in Section 3. Since

$$g = (\alpha+1)^2|x|^{2\alpha}\widehat{g} = (\alpha+1)^2 r^{2\alpha/(\alpha+1)}\widehat{g},$$

we also get

$$\widehat{g}(\Phi_* U, \Phi_* U) = \|z\|^{-4}\widehat{g}(U,U), \tag{2.16}$$

for every vector field U in M_0. Hence (2.12) is proved at any point of M_0.

Next, we prove (2.12) at points of the form $(0,y)$, $y \neq 0$. Here we may work in Cartesian coordinates. Observe that $\Phi(0,y) = (0, |y|^{-2}y)$ and $\partial_{x_j}(\|z\|)\big|_{(0,y)} = 0$. Therefore it is easy to see that $\Phi_*(\partial_{x_j})_{(0,y)} = |y|^{-2/(\alpha+1)}(\partial_{x_j})_{(0,|y|^{-2}y)}$. Thus, if $U = U^j(\partial_{x_j})_{(0,y)}$,

$$\widehat{g}(\Phi_* U, \Phi_* U) = |y|^{-4/(\alpha+1)}\widehat{g}(U,U). \tag{2.17}$$

Equations (2.16) and (2.17) together complete the proof of (2.12).

In order to prove conformality starting from (2.12), use the following routine argument. Take a point $z_0 \neq 0$. Let z be a close point and denote $\varepsilon = d(z, z_0)$. Take an arclength geodesic $\gamma : [0,\varepsilon] \to \mathsf{M}$, $\gamma(0) = z_0$, $\gamma(\varepsilon) = z$, with $\dot\gamma(t) = a_j(t)X_j(\gamma(t)) + b_\lambda(t)Y_\lambda(\gamma(t))$ and $|a(t)|^2 + |b(t)|^2 = 1$ at almost all t. We may assume that γ does not touch $(0,0)$, provided ε is small enough. Then

$$d\big(\Phi(z_0), \Phi(z)\big) \leq \int_0^\varepsilon \sqrt{\widehat{g}_{\Phi(\gamma)}(\Phi_*\dot\gamma, \Phi_*\dot\gamma)}dt = \int_0^\varepsilon \|\gamma\|^{-2}dt, \tag{2.18}$$

because $\widehat{g}(\dot\gamma,\dot\gamma) = 1$ almost everywhere. As $\varepsilon \to 0$ we get

$$\limsup_{\varepsilon\to 0}\frac{d\big(\Phi(z_0),\Phi(z)\big)}{d\big(z_0,z\big)} \leq \|z_0\|^{-2}.$$

The same argument applied to Φ^{-1} provides equality (2.11) at any point $z_0 \neq (0,0)$.

3. Proof of the Liouville theorem

In this section we first study the cones $\mathcal{U}_P$ of the metric g. Then we use their form to find all admissible conformal factors u of a conformal map in (M_0, g), for $p \geq 3$. At the end of the section we give the easy argument which concludes the proof of Theorem 1.2 and hence of Theorem 1.1.

3.1. The cones $\mathcal{U}_P$ for the metric g. In the following proposition we identify the cones $\mathcal{U}_P$ defined in the introduction. We use the warped metric (2.2).

Proposition 3.1: *Let $p \geq 3$. Then, for any $P \in \mathsf{H}\times_w \mathsf{S}$, we have*

$$\mathcal{U}_P = \{X \in T_P(\mathsf{H}\times_w\mathsf{S}) : |X_\mathsf{H}|\,|X_\mathsf{S}| = 0\} = T_P\mathsf{S}\cup T_P\mathsf{H}.$$

Observe that, if $p = 2$ and $q \geq 2$, then we have $\mathcal{U}_P = T_P\mathsf{M}_0$, all the tangent space, because the metric is flat.

Proposition 3.1, together with a continuity argument, immediately gives corollary below, whose easy proof is omitted.

Corollary 3.2: *Let* $\Omega \subset \mathsf{H} \times \mathsf{S}$ *be a connected open set. Let* $f : \Omega \to f(\Omega) \subset \mathsf{H} \times \mathsf{S}$ *be a conformal diffeomorphism in the metric* $g = g_{\mathsf{H}} + w^2 g_{\mathsf{S}}$. *Assume that* $p \geq 3$. *Then, either*

$$\begin{cases} f_*(T_P\mathsf{S}) = T_{f(P)}\mathsf{S} \\ f_*(T_P\mathsf{H}) = T_{f(P)}\mathsf{H} \end{cases} \quad \forall P \in \Omega, \tag{3.1}$$

or

$$\begin{cases} f_*(T_P\mathsf{S}) = T_{f(P)}\mathsf{H} \\ f_*(T_P\mathsf{H}) = T_{f(P)}\mathsf{S} \end{cases} \quad \forall P \in \Omega. \tag{3.2}$$

Correspondingly, in cylindrical coordinates (r, y, ϑ), *the map is a product of one between the following types:*

$$(r, y, \vartheta) \mapsto (\widetilde{r}(r, y), \widetilde{y}(r, y), \widetilde{\vartheta}(\vartheta)), \tag{3.3}$$

or

$$(r, y, \vartheta) \mapsto (\widetilde{r}(\vartheta), \widetilde{y}(\vartheta), \widetilde{\vartheta}(r, y)). \tag{3.4}$$

Observe that, for dimensional reasons, (3.2) and the corresponding (3.4) may happen only if S and H have the same dimension, namely when $p - 1 = q + 1$.

Remark 3.3: Since in case $p = 2$ the metric g is flat, it is easy to realize that in this situation there are conformal maps which satisfy neither (3.1), nor (3.2). More precisely, given any point P and any $X, Y \in T_P\mathsf{M}_0$, there is a local isometry f around P such that $f(P) = P$ and $f_*X = Y$.

Proof of Proposition 3.1. Observe first that $W(X, U, V, Z) = R(X, U, V, Z)$, provided X, U, V, Z form an orthogonal family. This follows from (1.11).

The proof will be accomplished in two steps.

Step 1. If $X = X_{\mathsf{H}} + X_{\mathsf{S}} \in T_P\mathsf{H} \oplus T_P\mathsf{S}$ with $|X_{\mathsf{H}}| \neq 0$ and $|X_{\mathsf{S}}| \neq 0$, then $X \notin \mathcal{U}_P$.

Step 2. If $X = X_{\mathsf{H}} \in T_P\mathsf{H}$ or $X = X_{\mathsf{S}} \in T_P\mathsf{S}$, then $X \in \mathcal{U}_P$.

Proof of Step 1. Write $X = X_{\mathsf{H}} + X_{\mathsf{S}} \in T_P\mathsf{H} + T_P\mathsf{S}$. Recall that both X_{H} and X_{S} are nonzero. Take two nonzero vectors $X_{\mathsf{S}}^{\perp} \in T_P\mathsf{S}$ with $g(X_{\mathsf{S}}, X_{\mathsf{S}}^{\perp}) = 0$ and $X_{\mathsf{H}}^{\perp} \in T_P\mathsf{H}$ with $g(X_{\mathsf{H}}, X_{\mathsf{H}}^{\perp}) = 0$. This choice is possible, because $\dim T_P\mathsf{S} \geq 2$ ($p \geq 3$) and $\dim T_P\mathsf{H} = q + 1 \geq 2$. Then take $V = X_{\mathsf{S}} - c_1 X_{\mathsf{H}}$, where c_1 is such that $g(X,V)=0$, and $U = X_{\mathsf{S}}^{\perp} + X_{\mathsf{H}}^{\perp}$, $Z = X_{\mathsf{S}}^{\perp} - c_2 X_{\mathsf{H}}^{\perp}$, where c_2 is such that $g(U,Z)=0$. Then X,U,V,Z form an orthogonal family and moreover, by (2.8), $R(X,U,V,Z) = R(X_{\mathsf{S}}, X_{\mathsf{S}}^{\perp}, X_{\mathsf{S}}, X_{\mathsf{S}}^{\perp}) \neq 0$. Step 1 is proved.

Proof of Step 2. If $X = X_{\mathsf{H}} \in T_P\mathsf{H}$ and X,U,V,Z form an orthogonal family, then $W(X,U,V,Z) = R(X,U,V,Z) = 0$, by (2.8).

If $X = X_{\mathsf{S}} \in T_P\mathsf{S}$, take U,V,Z an orthogonal triple, where all the vectors U,V and Z are orthogonal to X. There are two cases.

First case: all the vectors U,V,Z have nonzero projection along $T_P\mathsf{S}$, $U = U_{\mathsf{H}} + U_{\mathsf{S}}$, $V = V_{\mathsf{H}} + V_{\mathsf{S}}$ and $Z = Z_{\mathsf{H}} + Z_{\mathsf{S}}$, with $|U_{\mathsf{S}}|\,|V_{\mathsf{S}}|\,|Z_{\mathsf{S}}| \neq 0$. But then, since U,V,Z,X are orthogonal and $X_{\mathsf{H}} = 0$, all $U_{\mathsf{S}}, V_{\mathsf{S}}$ and Z_{S} must be orthogonal to X_{S}. Hence, by (2.8),

$$\begin{aligned} W(X,U,V,Z) &= R(X,U,V,Z) = R(X_{\mathsf{S}}, U_{\mathsf{S}}, V_{\mathsf{S}}, Z_{\mathsf{S}}) \\ &= -\alpha(\alpha+2)(\alpha+1)^2 r^2 R^{\mathsf{S}}(X_{\mathsf{S}}, U_{\mathsf{S}}, V_{\mathsf{S}}, Z_{\mathsf{S}}) = 0, \end{aligned}$$

by elementary properties of the curvature R^{S} of the sphere.

Second case: at least one among the vectors U,V,W has zero projection along $T_P\mathsf{S}$. Then $W(X,U,V,W) = R(X,U,V,W) = 0$, by (2.8) again. ∎

Remark 3.4: The argument of the proof above can be used to show a similar result on the cones $\mathcal{U}_P$ for a map f conformal in the product of a standard sphere $\mathbb{S}^k$ with a Euclidean space $\mathbb{R}^m$, $k \geq 2$, $m+k \geq 4$. It turns out that $\mathcal{U}_P = T_P\mathbb{S}^k \cup T_P\mathbb{R}^m$. Moreover, all arguments of the following Subsection 3.2 reduce to a few lines and it is easy to see that a conformal map on a connected open set $\Omega \subset \mathbb{S}^k \times \mathbb{R}^m$ must be the restriction of a local isometry.

3.2. The conformal factor u. Here we find all functions u which can be conformal factors of some conformal maps. We begin by proving in the following easy lemma that the function u must be a product. Write $h = (r,y)$ and denote by (h,s) points in $\mathsf{H} \times \mathsf{S}$.

Lemma 3.5: *Let $\Omega \subset \mathsf{H} \times \mathsf{S}$ be a connected open set. Let $f : \Omega \to f(\Omega) \subset \mathsf{H} \times \mathsf{S}$ be a conformal diffeomorphism with respect to the warped metric $g = g_{\mathsf{H}} + w^2 g_{\mathsf{S}}$. Assume that f is a product map of the form either*

$$(h,s) \mapsto (\widetilde{h}(h), \widetilde{s}(s)), \tag{3.5}$$

or

$$(h,s) \mapsto (\widetilde{h}(s), \widetilde{s}(h)), \tag{3.6}$$

for all $(h,s) \in \Omega$. *Then the conformal factor* u *is a product:* $u(h,s) = A(h)B(s)$.

Proof. The Cauchy–Riemann system $g(f_*X, f_*X) = u^{-2}g(X,X)$ for every vector field X holds. In case (3.5), fix a (lifted) horizontal vector field X. Observe that $g(X,X) = g_{\mathsf{H}}(X,X)$ depends on h only. Moreover, by (3.5), we have $f_*(X) = \widetilde{h}_*(X)$. Therefore $g(f_*(X), f_*(X)) = g_{\mathsf{H}}(\widetilde{h}_*(X), \widetilde{h}_*(X))$ is a function of h only. Therefore, by the Cauchy–Riemann system, u depends on h only.

In case (3.6), which may happen only if H and S have the same dimension, given the same X as above, we have $f_*(X) = \widetilde{s}_*(X)$, a vertical vector field. Therefore, by the warped form of g,

$$g(f_*X, f_*X) = g(\widetilde{s}_*(X), \widetilde{s}_*(X)) = w(\widetilde{h}(s))^2 g_{\mathsf{S}}(\widetilde{s}_*(X), \widetilde{s}_*(X)) = \varphi(s)\psi(h),$$

a product of suitable functions φ and ψ of s and h, respectively. Therefore the Cauchy–Riemann system gives $u(h,s) = A(h)B(s)$. ∎

Before discussing the system (1.10), we prove the following proposition.

PROPOSITION 3.6: *Let* $f : \Omega \to f(\Omega) \subset \mathsf{H} \times \mathsf{S}$ *be a conformal diffeomorphism in the metric* $g = g_{\mathsf{H}} + w^2 g_{\mathsf{S}}$, $p \geq 3$. *Assume that* (3.1) *holds. Then* f *preserves the Ricci tensor.*

Proof. We need to prove that $\mathrm{Ric}(f_*U, f_*V) = \mathrm{Ric}(U,V)$ for all vectors U, V. Assumption (3.1) and the form (2.9) of the Ricci tensor show that it suffices to assume $U, V \in TS$. In this case we have

$$\begin{aligned}
&\mathrm{Ric}(f_*U, f_*V) - \mathrm{Ric}(U,V) \\
&\qquad = -\alpha(\alpha+2)(p-2)(\alpha+1)^{-2}\big(\widetilde{r}^{-2} g(f_*U, f_*V) - r^{-2} g(U,V)\big) \\
&\qquad = -\alpha(\alpha+2)(p-2)(\alpha+1)^{-2}\big(\widetilde{r}^{-2} u^{-2} - r^{-2}\big) g(U,V).
\end{aligned}$$

To prove the proposition it suffices to show that $u^{-2}\widetilde{r}^{-2} - r^{-2} = 0$. We use the Weyl tensor. Let $X, Y \in T\mathsf{S}$ be orthogonal vectors. Then, by (1.11), (2.8) and (2.9) it is easy to see that

$$W(X,Y,X,Y) = C_0 r^{-2} g(X,X) g(Y,Y), \tag{3.7}$$

where

$$C_0 = -\frac{q^2+q}{(n-1)(n-2)} \frac{\alpha(\alpha+2)}{(\alpha+1)^2} < 0, \quad \text{if } \alpha > 0.$$

Conformal invariance of W gives

$$W(f_*X, f_*Y, f_*X, f_*Y) = u^{-2} W(X,Y,X,Y).$$

Using (3.7) in both sides together with the CR system

$$g(f_*Z, f_*Z) = u^{-2}g(Z,Z), \quad Z = X, Y,$$

we conclude that $u^{-2}\widetilde{r}^{-2} = r^{-2}$. Thus, the proposition is proved. ∎

Now we are ready to solve system (1.10).

THEOREM 3.7: *Let $p \geq 3$ and let $\Omega \subset \mathsf{M}_0$ be a connected open set. Let $f : \Omega \to f(\Omega) \subset \mathsf{M}_0$ be a smooth diffeomorphism, conformal in the metric g. Then, either its conformal factor u is constant, or it has the form*

$$u = a\left(r^2 + |y-b|^2\right) = a\left(|x|^{2(\alpha+1)} + |y-b|^2\right) = a\|(x, y-b)\|^{2(\alpha+1)}, \tag{3.8}$$

for suitable $a > 0$, $b \in \mathbb{R}^q$, $(x,y) \in \Omega$.

Proof. Write the map in the form $(r, y, \vartheta) \mapsto (\widetilde{r}, \widetilde{y}, \widetilde{\vartheta})$. We first write system (1.10) in both cases (3.1) and (3.2). Assume that (3.1) holds. Then by Proposition 3.6, system (1.10) becomes

$$(n-2)u^{-1}\nabla^2 u(U,V) - g(U,V)u^{-2}\big\{(n-1)|\nabla u|^2 - u\Delta u\big\} = 0, \tag{3.9}$$

for any $U, V \in T\Omega$. Here $\widetilde{r} = \widetilde{r}(r,y)$. In case (3.2) it turns out that, given $U = U_{\mathsf{H}} + U_{\mathsf{S}}$, we have $(f_*U)_{\mathsf{S}} = f_*(U_{\mathsf{H}})$. Then,

$$\begin{aligned}\alpha(\alpha+2)(p-2)(\alpha+1)^{-2}&\Big\{r^{-2}g(U_{\mathsf{S}}, V_{\mathsf{S}}) - \widetilde{r}^{-2}u^{-2}g\big(U_{\mathsf{H}}, V_{\mathsf{H}}\big)\Big\} \\ &= (n-2)u^{-1}\nabla^2 u(U,V) - g(U,V)u^{-2}\big\{(n-1)|\nabla u|^2 - u\Delta u\big\},\end{aligned} \tag{3.10}$$

with $\widetilde{r} = \widetilde{r}(\vartheta)$.

Next we start to analyze the systems just obtained. The first part of the discussion is the same for case (3.9) and (3.10). Indeed, since the connection is Euclidean in variables r, y_λ, in both cases we have $\partial_r\partial_\lambda u = \nabla^2 u(\partial_r, \partial_\lambda) = 0$, $\lambda = 1, \ldots, q$. We also have $\partial_\lambda\partial_\mu u = \nabla^2 u(\partial_\lambda, \partial_\mu) = 0$, for all $\lambda \neq \mu$. Then $u(\vartheta, r, y) = F(\vartheta, r) + \sum_\lambda G^{(\lambda)}(\vartheta, y_\lambda)$, for suitable functions $F, G^{(\lambda)}$. Moreover, since $\nabla_{\partial_r}\partial_r = 0$, $\nabla_{\partial_\lambda}\partial_\lambda = 0$, both (3.9) and (3.10) give

$$u_{rr} = \nabla^2 u(\partial_r, \partial_r) = \nabla^2 u(\partial_\lambda, \partial_\lambda) = \frac{\partial^2 u}{\partial y_\lambda^2}, \quad \lambda = 1, \ldots, q. \tag{3.11}$$

Recall also that, by Lemma 3.5, u must be a product. Thus its form is

$$u(r, \vartheta, y) = H(\vartheta)\Big\{\frac{1}{2}(r^2 + |y|^2) + lr + \langle m, y\rangle + n\Big\}, \tag{3.12}$$

$l, n \in \mathbb{R}$, $m \in \mathbb{R}^p$. Here we used the fact that Ω is connected.

Next we use condition $\nabla^2 u(\partial_r, X) = 0$, for any X on the sphere, which holds in both cases (3.9) and (3.10). Let X be (the lifting of) a vector field on the sphere. By (2.5) we get $\partial_r X u = \frac{1}{r} X u$, which gives $Xu(\vartheta, r, y) = K(\vartheta, y) r$, where K is a function depending on the vector field X. Applying X to (3.12) and equating homogeneous powers of r, we deduce $XH = 0$. Thus H is constant, u is constant on the sphere and has the form

$$u(r, \vartheta, y) = \frac{1}{2} H(r^2 + |y|^2) + Lr + \langle M, y\rangle + N, \tag{3.13}$$

for some $M \in \mathbb{R}^q$, $L, N \in \mathbb{R}$.

Next we are ready to rule out case (3.10). Indeed, letting $U = V = \partial_r$ in (3.10), we get

$$-\alpha(\alpha+2)(p-2)(\alpha+1)^{-2}\widetilde{r}^{-2}u^{-2} = (n-2)u^{-1}H - u^{-2}\{(n-1)|\nabla u|^2 - u\Delta u\}.$$

Multiplying by u^2 and using the fact that $\widetilde{r} = \widetilde{r}(\vartheta)$ (compare (3.4)), we get an equation of the form $\widetilde{r}(\vartheta)^{-2} = \varphi(r, y)$, where φ is a suitable function. Therefore it must be $\widetilde{r} =$ constant. But this is impossible, because in this case the map f would become singular.

We are left with the study of (3.9). Given any unit vector $X \in T\mathsf{S}$ we have

$$\nabla^2 u(X, X) = \nabla^2 u(\partial_r, \partial_r).$$

Taking the form (3.13) of u and (2.6) into account, we get $\nabla^2 u(X, X) = (H + L/r) g(X, X) = H + L/r$. Moreover, since $\nabla^2 u(\partial_r, \partial_r) = H$, we conclude that $L = 0$. Thus

$$u(r, \vartheta, y) = \frac{1}{2} H(r^2 + |y|^2) + \langle M, y\rangle + N, \tag{3.14}$$

$M \in \mathbb{R}^q$, $L, N \in \mathbb{R}$.

Taking the trace of (3.9), we obtain

$$2u\Delta u - n|\nabla u|^2 = 0. \tag{3.15}$$

Some computations based on (3.14) and (2.7) give

$$\begin{aligned} |\nabla u|^2 &= |\partial_r u|^2 + |\nabla_y u|^2 \\ &= H^2 r^2 + |Hy + M|^2 = 2Hu - 2NH + |M|^2, \\ \Delta u &= u_{rr} + \frac{p-1}{r} u_r + \Delta_y u = (p-1)H + (q+1)H = nH. \end{aligned}$$

Inserting this information into (3.15), we easily see that $|M|^2 = 2NH$. Ultimately, if $H = 0$, then $M = 0$ and $u = N > 0$. If instead $H > 0$, then we can write $u = \frac{1}{2}H(r^2 + |y + M/H|^2)$, as desired. ∎

3.3. Conclusion of the argument. Let $\Omega \subset \mathsf{H} \times_w \mathsf{S}$ be a connected open set. Let $f : \Omega \to f(\Omega) \subset \mathsf{H} \times_w \mathsf{S}$ be a conformal diffeomorphism with respect to g. Then, either its conformal factor is constant or it has the form given in (3.8). Recall that the map $\Phi(z) = \delta_{t\|(x,y-b)\|^{-2}}(x, y - b)$ has conformal factor $u_\Phi(z) = t^{-\alpha+1}\|(x, y-b)\|^{2(\alpha+1)}$ (see Subsection 2.2, especially equation (2.15)). Write $f(z) = F(\Phi(z))$ and note that $u_{F\circ\Phi}(z) = u_F(\Phi(z))u_\Phi(z)$. Then, letting $t^{-(\alpha+1)} = a$, the map F turns out to be a local isometry. The proof is easily concluded, because local isometries are classified in Proposition 2.1.

4. Umbilical surfaces

In this section we prove Theorem 1.3. Let $\Sigma \subset (\mathsf{M}_0, g)$ be a smooth orientable connected hypersurface. Fix a unit normal vector field N. Recall that Σ is umbilical if at any point $P \in \Sigma$ there is $\kappa(P) \in \mathbb{R}$ such that the shape operator L satisfies $L(X) := -\nabla_X N = \kappa(P)X$, for all $X \in T_P\Sigma$.

Let $p \geq 3$. As discussed in the introduction, the identification of the cones $\mathcal{U}_P$ and Codazzi equations give the following obstruction. If Σ is an umbilical surface, $P \in \Sigma$ and N is a normal vector to Σ at P, then it must be $N \in \mathcal{U}_P$, which means

$$|N_{\mathsf{H}}|\,|N_{\mathsf{S}}| = 0. \tag{4.1}$$

Hence, if (4.1) is not satisfied for a given $N \in T_P\mathsf{M}_0$, then there is no umbilical surface containing P and with normal N at P.

Before proving Theorem 1.3 observe the following facts:

Remark 4.1: (1) Since for $p = 2$, $q \geq 1$ the manifold (M_0, g) is flat, then for any point P and $N \in T_P\mathsf{M}_0$ there is Σ umbilical and with normal N at P.

(2) The notion of umbilical surface is conformally invariant, while curvature depends on the metric. The choice of the particular metric g makes all umbilical surfaces to have constant curvature. More precisely, spheres A1 (as defined in the statement of Theorem 1.3) have curvature $1/c$, while planes A2 and B are geodesic surfaces.

(3) Surfaces A1 can be conformally mapped in surfaces of type A2, while surfaces of type B cannot.

(4) The homogeneous spheres A1 have the same level sets of the function $\Gamma(z) = \|z\|^{-Q+2}$, $Q = p + (\alpha+1)q$, which is a singular solution of the equation $\Delta_\alpha \Gamma = 0$ (see (1.13)) and plays an important role in analysis and potential theory (see [18]).

Proof of Theorem 1.3. The proof will be accomplished in three steps:

Step 1. Surfaces A1, A2 and B are umbilical.

Step 2. If Σ is umbilical and has normal $\bar{N} \in T_{\bar{P}}\mathsf{H}$ at some $\bar{P} \in \Sigma$, then Σ is contained in a surface of type A1 or A2.

Step 3. If Σ is umbilical and has normal $\bar{N} \in T_{\bar{P}}\mathsf{S}$ at some $\bar{P} \in \Sigma$, then Σ is contained in a plane of type B.

Proof of Step 1. We start from type A1. Without loss of generality we take $c = 1$ and $b = 0$, so that our surface Σ has equation $|x|^{2(\alpha+1)} + |y|^2 = 1$. In the warped model with metric (2.2), a unit normal vector field has the form $N = -(r\partial_r + y^\lambda \partial_\lambda)$. Let $P \in \Sigma$ and $U \in T_P\Sigma$. By linearity of the shape operator, it suffices to consider separately the cases $U \in T_P\mathsf{S}$ and $U = a\partial_r + c_\lambda \partial_\lambda$, where $ar + \sum c_\lambda y^\lambda = 0$. In the first case, $L(U) = -\nabla_U N = \nabla_U(r\partial_r + y^\lambda\partial_\lambda) = U$, in view of (2.5). In the second case, if $U = a\partial_r + c_\lambda\partial_\lambda$, then $L(U) = -\nabla_U N = \nabla_{a\partial_r + c_\lambda\partial_\lambda}(r\partial_r + y^\mu\partial_\mu) = U$, because in these variables the connection is Euclidean. The proof for planes A2 is analogous and we omit it.

Next we pass to Type B. Here Σ has equation $\sum_k a_k x_k = 0$. Assume that $\sum a_k^2 = 1$. We use Cartesian coordinates (x^j, y^λ) and the metric g. A unit normal vector field to Σ is $N = (\alpha+1)^{-1}|x|^{-\alpha} a_k \partial_{x_k}$. Again by linearity of L, it suffices to consider separately vectors of the form $U = \partial_\lambda$, with $\lambda = 1, \dots, q$, and $U = U^j\partial_j$, which are tangent provided $U^k a_k = 0$. In first case, $L(\partial_\lambda) = -(\alpha+1)^{-1}\nabla_{\partial_\lambda}|x|^{-\alpha}a_k\partial_k = 0$, because $\nabla_{\partial_\lambda}\partial_k = 0$. In the second case, since the Christoffel symbols of the metric $(\alpha+1)^2|x|^{2\alpha}|dx|^2$ in $\mathbb{R}^p \setminus \{0\}$ are $\Gamma_{ij}^k = \alpha|x|^{-2}\big\{\delta_{ik}x_j + \delta_{jk}x_i - \delta_{ij}x_k\big\}$, we get

$$\begin{aligned} L(U^j\partial_j) &= -U^j\nabla_{\partial_j}(\alpha+1)^{-1}|x|^{-\alpha}a_k\partial_k \\ &= -(\alpha+1)^{-1}U^j a_k\big\{-\alpha|x|^{-\alpha-2}x_j\partial_k + |x|^{-\alpha}\Gamma_{jk}^i\partial_i\big\} \\ &= \alpha(\alpha+1)^{-1}|x|^{-\alpha-2}\{-(a_kx_k)U + U^k a_k x_i\partial_i\} \\ &= 0, \end{aligned}$$

because $\langle a, x\rangle = 0$ and U is tangent to the plane. Thus $L(U) = 0$. We have proved that Σ is a geodesic surface. ∎

Proof of Step 2. Let $\bar{P} \in \mathsf{M}_0$ and let $\bar{N} = \bar{N}_{\mathsf{H}} \in T_{\bar{P}}\mathsf{H}$. Let Σ be an umbilical surface with normal $\bar{N}$ at $\bar{P}$ with respect to g. Examples of surfaces A1 and A2 show that there is at least one surface with these properties. We want to show that Σ is contained in a surface of type A1 or A2. Denote by N the unit normal vector field to Σ which agrees with $\bar{N}$ at $\bar{P}$. By continuity and by (4.1) it must be that $N = N_{\mathsf{H}} \in T_P\mathsf{H}$ at any point $P \in \Sigma$.

We prove first that Σ has constant curvature κ. It suffices to prove that $U\kappa = 0$ for any vector U tangent to Σ. Since $N = N_{\mathsf{H}}$, (2.8) gives $R(N, U, V, W) = 0$, for any U, V, W orthogonal to N. Thus, Codazzi equations (1.12) show that κ must be constant.

In the warped model (r, y, ϑ), the vector field N has the form $N = a\partial_r + N^\lambda\partial_\lambda$, where a, N^λ are suitable functions on Σ. Since Σ is umbilical, given any $V = V_{\mathsf{H}} + V_{\mathsf{S}} \in T_P\Sigma$, it must be that

(4.2)
$$\begin{aligned}-\kappa V_H - \kappa V_{\mathsf{S}} = -\kappa V = \nabla_V N &= (Va)\partial_r + a\nabla_V\partial_r + (VN^\lambda)\partial_\lambda + N^\lambda\nabla_V\partial_\lambda\\ &= (Va)\partial_r + \frac{a}{r}V_{\mathsf{S}} + (VN^\lambda)\partial_\lambda,\end{aligned}$$

where we used (2.4) and (2.5). Comparing like terms, we get

$$-\kappa = a/r, \quad \Rightarrow N = -r\kappa\partial_r + N^\lambda\partial_\lambda. \tag{4.3}$$

Since $|N| = 1$, it must be that

$$r^2\kappa^2 + \sum_\lambda (N^\lambda)^2 = 1. \tag{4.4}$$

Write in (4.2) $V_{\mathsf{H}} = V^\lambda\partial_\lambda + V^r\partial_r$ and take components along ∂_λ. Thus

$$-\kappa V^\lambda = VN^\lambda, \quad \forall\, V \in T\Sigma. \tag{4.5}$$

There are two cases: if $\kappa = 0$, then (4.5) implies that N^λ is constant. Therefore Σ is contained in the plane of equation $N^\lambda y^\lambda$ =constant. If instead $\kappa \neq 0$, (4.5) gives $V(y^\lambda + \kappa^{-1}N^\lambda) = 0$. Therefore $y^\lambda + \kappa^{-1}N^\lambda = b_\lambda$, where b_λ is a constant. Thus (4.4) becomes $1 = r^2\kappa^2 + \kappa^2\sum_\lambda(b_\lambda - y_\lambda)^2$, as desired. ∎

Proof of Step 3. Let $P \in \mathsf{M}_0$ and let $\bar{N} = \bar{N}_{\mathsf{S}} \in T_{\bar{P}}\mathsf{S}$ be a unit vector. Consider an umbilical surface Σ with normal $\bar{N}$ at $\bar{P}$. Surfaces of type B show that there is at least one surface with this property. Our aim is to show that Σ is contained in a plane of type B.

Let N be the unit normal to Σ which agrees with $\bar{N}$ at $\bar{P} = (\bar{r}, \bar{y}, \bar{\vartheta})$. Since Σ is umbilical, by (4.1) and by continuity, it must be that $N_{\mathsf{H}} = 0$ for all $P = (r, y, \vartheta) \in \Sigma$. Therefore, given a local frame X_j, $j = 1, \dots, p-1$, on the sphere $\mathbb{S}^{p-1}$ around $\bar{\vartheta}$, N has the form $N = \sum_{j=1}^{p-1} b_j(r, y, \vartheta) X_j$.

Next take the tangent vector $\partial_{y_1} \in T_P\Sigma$, for any P close to $\bar{P}$. Let κ be the curvature of Σ. Then

$$\kappa\partial_{y_1} = L(\partial_{y_1}) = -\sum_{j\le p-1} \left\{(\partial_{y_1} b_j)X_j + b_j \nabla_{\partial_{y_1}} X_j\right\} = -\sum_{j\le p-1} (\partial_{y_1} b_j)X_j,$$

by (2.5). Therefore it must be that $\kappa = 0$. Hence Σ is a geodesic surface. Thus it must be contained in the plane of equation $\sum_k \bar{N}^k x_k = 0$, which is, by the previous Step 1, a geodesic surface too. ∎

Appendix

We collect here some standard formulas on warped products. See [20, Chapter 7] for a complete discussion. Let $(\mathsf{H}, g_{\mathsf{H}})$ and $(\mathsf{S}, g_{\mathsf{S}})$ be Riemannian manifolds. Given $w : \mathsf{H} \to]0, +\infty[$, the warped product $\mathsf{H} \times_w \mathsf{S}$ is the manifold $\mathsf{H} \times \mathsf{S}$ equipped with the metric $g = g_{\mathsf{H}} + w^2 g_S$. Given any $P = (h, s)$, decompose as usual $T_P M$ as the orthogonal sum of $T_P\mathsf{H}$ and $T_P\mathsf{S}$, the lifts at P of $T_h\mathsf{H}$ and $T_s\mathsf{S}$, respectively. We use the same notation for a vector and its lifting. Lifting of vector fields on H and on S are usually denoted by $\mathcal{L}(\mathsf{H})$ and $\mathcal{L}(\mathsf{S})$. They are often called **lifted horizontal** or **lifted vertical** vector fields. Vector fields and their liftings are denoted by the same symbol. Observe that for a function φ depending on h only, the gradient gradφ of φ in the metric g is nothing but the obvious lifting of $\mathrm{grad}_{g_{\mathsf{H}}}\varphi$.

Next, let ∇^{H}, ∇^{S} and ∇ be the Levi Civita connections on H, S and $\mathsf{H} \times_w \mathsf{S}$, respectively. Then, the following formulas hold (below $A, B, C, D \in \mathcal{L}(\mathsf{H})$ and $X, Y, Z, V \in \mathcal{L}(\mathsf{S})$):

$$\begin{aligned} &\nabla_A B = \nabla^{\mathsf{H}}_A B, \\ \text{(A.1)} \qquad &\nabla_A X = \nabla_X A = w^{-1}(Aw)X, \\ &\nabla_X Y = \nabla^{\mathsf{S}}_X Y - g(X, Y) w^{-1} \mathrm{grad} w. \end{aligned}$$

Therefore, given $u : \mathsf{H} \times \mathsf{S} \to \mathbb{R}$, with slight abuse of notation,

$$\begin{aligned}\nabla^2 u(A,B) &= (\nabla^{\mathsf{H}})^2 u(A,B),\\ \nabla^2 u(X,Y) &= (\nabla^{\mathsf{S}})^2 u(X,Y) + w^{-1} g(X,Y)(\mathrm{grad} w)u,\\ \Delta u &= \Delta_{\mathsf{H}} u + w^{-2}\Delta_{\mathsf{S}} u + \dim(\mathsf{S}) w^{-1}(\mathrm{grad} w)u.\end{aligned}\tag{A.2}$$

A computation using (A.1) provides also

$$\begin{aligned}&R(A,B)C = R_{\mathsf{H}}(A,B)C,\\ &R(A,X)B = w^{-1}\nabla^2 w(A,B)X,\\ &R(X,Y)Z = R_{\mathsf{S}}(X,Y)Z + w^{-2}|\nabla w|^2\big\{g(X,Z)Y - g(Y,Z)X\big\}.\end{aligned}$$

Therefore,

$$\begin{aligned}R(D,C,A,B) :=& g(D, R(A,B)C) = R_{\mathsf{H}}(D,C,A,B),\\ R(Y,B,A,X) =& w^{-1}\nabla^2 w(A,B) g(X,Y),\\ R(V,Z,X,Y) =& w^2 R_{\mathsf{S}}(V,Z,X,Y)\\ &+ w^{-2}|\nabla w|^2\big\{g(X,Z)g(Y,V) - g(Y,Z)g(X,V)\big\}.\end{aligned}\tag{A.3}$$

The remaining nonzero components of R can be obtained by the standard symmetries $R_{abcd} = -R_{bacd} = R_{cdab}$ of the curvature tensor R.

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