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BRAUER ALGEBRAS OF SIMPLY LACED TYPE

BY

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ABSTRACT

The diagram algebra introduced by Brauer that describes the centralizer algebra of the n-fold tensor product of the natural representation of an orthogonal Lie group has a presentation by generators and relations that only depends on the path graph A_{n-1} on $n-1$ nodes. Here we describe an algebra depending on an arbitrary graph Q, called the Brauer algebra of type Q, and study its structure in the cases where Q is a Coxeter graph of simply laced spherical type (so its connected components are of type A_{n-1}, D_n, E_6, E_7, E_8). We find its irreducible representations and its dimension, and show that the algebra is cellular. The algebra is generically semisimple and contains the group algebra of the Coxeter group of type Q as a subalgebra. It is a ring homomorphic image of the Birman–Murakami–Wenzl algebra of type Q; this fact will be used in later work determining the structure of the Birman–Murakami–Wenzl algebras of simply laced spherical type.

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1. Introduction

Let Q be a graph. We define the Brauer monoid $\mathbf{BrM}(Q)$ to be the monoid generated by the symbols r_i and e_i for i a node of Q and δ, δ^{-1} subject to the relations of Table 1, where $\sim$ denotes adjacency between nodes of Q. The Brauer algebra $\mathbf{Br}(Q)$ of type Q is the monoid algebra $\mathbb{Z}[\mathbf{BrM}(Q)]$. As δ is in the center of $\mathbf{BrM}(Q)$, it is also an algebra over $\mathbb{Z}[\delta^{\pm 1}]$ and will often be regarded as such. For $Q = \mathrm{A}_{n-1}$, this algebra was introduced not by generators and relations but in terms of diagrams by Brauer [3]. It was related to the centralizer algebra of the n-th tensor power of the natural representation of a classical group where δ is the dimension of the representation. The BMW algebras, introduced by Birman & Wenzl [1] and Murakami [12], are deformations which play a similar role for quantum groups and are also a useful tool for introducing Kauffman polynomials, known from knot theory. In [4], we introduced BMW algebras of type Q for arbitrary Q. The results of the present paper will be of use in our determination of the structure of BMW algebras of type D_n [7, 6] in much the same way the Brauer algebra of type A_n was of use for Morton & Wasserman [11] in the structure determination of the BMW algebra of type A_n.

label	relation	label	relation
(δ)	δ is central	(δ^{-1})	$\delta\delta^{-1} = 1$
	for i		for i
(RSrr)	$r_i^2 = 1$	(RSer)	$e_i r_i = e_i$
(RSre)	$r_i e_i = e_i$	(HSee)	$e_i^2 = \delta e_i$
	for $i \not\sim j$		for $i \not\sim j$
(HCrr)	$r_i r_j = r_j r_i$	(HCer)	$e_i r_j = r_j e_i$
(HCee)	$e_i e_j = e_j e_i$		
	for $i \sim j$		for $i \sim j$
(HNrrr)	$r_i r_j r_i = r_j r_i r_j$	(HNrer)	$r_j e_i r_j = r_i e_j r_i$
(RNrre)	$r_j r_i e_j = e_i e_j$		

Table 1. Brauer relations

A look at (RSrr), (HCrr), and (HNrrr) makes it clear that products of the r_i belong to a subgroup of the monoid $\mathbf{BrM}(Q)$ isomorphic to a quotient of $W(Q)$, the Coxeter group of type Q. Modding out the ideal generated by all e_i, we see that the subgroup itself is in fact isomorphic to $W(Q)$ and that the r_i form a set of simple reflections. In particular, the rank of $\mathbf{Br}(Q)$ as a module over $\mathbb{Z}[\delta^{\pm 1}]$ is infinite if Q is not spherical in the sense of [2]. This means that, if $\mathbf{Br}(Q)$ is finite-dimensional, its connected components are isomorphic to Coxeter graphs of type A_n $(n \geq 1)$, D_n $(n \geq 4)$, or E_n $(n = 6, 7, 8)$, which we abbreviate to ADE. Also, if Q is disconnected, then $\mathbf{Br}(Q)$ is a direct sum of its connected components. This explains why we are interested in the cases where $Q \in \mathrm{ADE}$. Our main result reads as follows and is proved in Section 5.

THEOREM 1.1: *The Brauer algebra of type $Q \in \mathrm{ADE}$ over $\mathbb{Z}[\delta^{\pm 1}]$ is free of dimension as given in Table 2. When tensored with $\mathbb{Q}(\delta)$, the algebra is semisimple.*

Q	$\dim(\mathbf{Br}(Q))$
A_n	$(n+1)!!$
D_n	$(2^n + 1)n!! - (2^{n-1} + 1)n!$
E_6	$1,440,585$
E_7	$139,613,625$
E_8	$53,328,069,225$

Table 2. Brauer algebra dimensions

Here $k!! = 1 \cdot 3 \cdots (2k-1)$, the product of the first k odd natural numbers. As the submonoid $\langle \delta^{\pm 1} \rangle$ of $\mathbf{BrM}(Q)$ generated by δ and its inverse is a central subgroup of $\mathbf{BrM}(Q)$, the dimension given by the theorem is equal to the cardinality of the quotient monoid $\mathbf{BrM}(Q)/\langle \delta^{\pm 1} \rangle$.

These assertions, with the precise dimensions for the series A_n and D_n, were conjectured before, cf. [4]. The algebra presented by similar generators and relations for A_n was treated by Birman and Wenzl in [1]. The Brauer diagram algebra for A_n has the stated dimension by [3]. A similar approach for $Q = \mathrm{D}_n$ $(n \geq 4)$ using diagrams appears in [6].

We also prove that these algebras are cellular in the sense of [10]; cf. Section 6.

THEOREM 1.2: *Let $Q \in \mathrm{ADE}$ and let S be an integral domain that is a commutative algebra over $\mathbb{Z}[\delta^{\pm}]$ in which 2 is invertible if $Q \neq \mathrm{A}_n$, the number 3 is invertible if $Q \neq \mathrm{A}_n, \mathrm{D}_n$, and 5 is invertible if $Q = \mathrm{E}_8$. Then the Brauer algebra $\mathbf{BrM}(Q) \otimes_R S$ of type Q over S is cellular.*

This is an extension of the result in [10] that the ordinary Brauer algebras of type A_{n-1} are cellular. We need the additional assumptions on the coefficient ring S as we apply [9], where Geck proved that these conditions suffice to conclude that the Hecke algebras of the corresponding types over S are cellular. Cellularity is of importance in studying representations and semisimplicity.

In this paper, independent arguments are given that use rewrites of monomials to a certain standard form for upper bounding the dimensions and constructions of irreducible representations for lower bounding the dimensions.

We analyze the structure of $\mathbf{Br}(Q)$ in great detail. In order to describe the results, we recall some notions from [5]. Our standard reference for Coxeter groups and root systems is [2]. Corresponding to each root α (always normalized so that $(\alpha, \alpha) = 2$), there is a unique reflection $r_\alpha \in W$, and, conversely, each reflection r in W has a unique positive root β such that $r = r_\beta$. A set of mutually orthogonal positive roots corresponds bijectively to a set of commuting reflections in W. The group W acts on the sets of mutually orthogonal positive roots in a unique way corresponding to conjugation on the sets of reflections. We consider W-orbits under this action. A set B of mutually orthogonal positive roots of $W(Q)$ is called **admissible** if, whenever $\beta_1, \beta_2, \beta_3$ are distinct roots in B and there exists a root α for which $|(\alpha, \beta_i)| = 1$ for all i, the positive root of $\pm r_\alpha r_{\beta_1} r_{\beta_2} r_{\beta_3} \alpha$ is also in B; cf. Lemma 2.1 below. In [5], a partial ordering was defined on the W-orbit of an admissible set of mutually orthogonal positive roots. Each such W-orbit has a unique maximal element B_0 in this ordering, called the **highest element**; see [5, Corollary 3.6]. The set of nodes i of Q with α_i orthogonal to each element of B_0 (notation $\alpha_i \perp B_0$) is denoted $C_{\mathcal{B}}$. A basis for the Brauer algebras of type ADE will be found that is parametrized by triples consisting of an ordered pair of admissible sets of mutually orthogonal positive roots from the same W-orbit $\mathcal{B}$ and an element of $W(C_{\mathcal{B}})$; see Proposition 4.9 and Corollary 5.5 below. In this light, Theorem 1.1 can be clarified as follows.

LEMMA 1.3: *The dimensions of Table 2 are equal to* $\sum_{\mathcal{B}} |\mathcal{B}|^2 |W(C_{\mathcal{B}})|$, *where the summation is over all* W*-orbits* $\mathcal{B}$ *of admissible sets of mutually orthogonal positive roots. All orbits* $\mathcal{B}$ *of nonempty admissible sets are listed in Table 3.*

Proof. See [5] for the second statement (in [loc. cit.], the type of $C_{\mathcal{B}}$ for $Q = E_7$ and $|B_0| = 2$ is incorrect).

Q	$\|X\|$	$X^\perp \cap \Phi$	$C_{\mathcal{B}}$	$N_W(X)/C_W(X)$
A_n	t	A_{n-2t}	A_{n-2t}	Σ_t
D_n	t	$\mathrm{A}_1^t\mathrm{D}_{n-2t}$	$\mathrm{A}_1\mathrm{D}_{n-2t}$	Σ_t
D_n	$2t$	D_{n-2t}	A_{n-2t-1}	$W(\mathrm{B}_t)^*$
E_6	1	A_5	A_5	Σ_1
E_6	2	A_3	A_2	Σ_2
E_6	4	$\emptyset$	$\emptyset$	Σ_4
E_7	1	D_6	D_6	Σ_1
E_7	2	$\mathrm{A}_1\mathrm{D}_4$	$\mathrm{A}_1\mathrm{A}_3$	Σ_2
E_7	3	D_4	A_2	Σ_3
E_7	4	A_1^3	A_1	Σ_4
E_7	7	$\emptyset$	$\emptyset$	$\mathrm{L}(3,2)$
E_8	1	E_7	E_7	Σ_1
E_8	2	D_6	A_5	Σ_2
E_8	4	D_4	A_2	Σ_4
E_8	8	$\emptyset$	$\emptyset$	$2^3\mathrm{L}(3,2)$

Table 3. Nonempty admissible sets X of mutually orthogonal positive roots. Each line corresponds to the W-orbit of a single X for each possible choice of $|X|$ indicated in the second column except for the first line for D_n when $n = 2t$, in which case there are two W-orbits with $|X| = n/2$ conjugate by an outer automorphism. For D_4, these are conjugate by an outer automorphism to the entry in the second row for D_4 with $t = 1$. The values of t lie in $\mathbb{Z} \cap [1, n/2]$. The third column lists the Cartan type of the root system on the roots orthogonal to X. The centralizer $C_W(X)$ is the semi-direct product of the elementary abelian group of order $2^{|X|}$ generated by the reflections in W with roots in X and the subgroup $W(X^\perp \cap \Phi)$ of W generated by reflections with roots in $X^\perp \cap \Phi$. The fourth column lists $C_{\mathcal{B}}$ for $\mathcal{B} = WX$, the W-orbit of X, and the last column lists the structure of $N_W(X)/C_W(X)$. Here $W(\mathrm{B}_t)^*$ is understood to be $W(\mathrm{B}_t)$ if $t < n/2$ and $W(\mathrm{D}_t)$ if $t = n/2$. Here and in the third column, D_0 and D_1 are empty, $\mathrm{D}_2 = \mathrm{A}_1\mathrm{A}_1$, and $\mathrm{D}_3 = \mathrm{A}_3$, in the rows for $Q = \mathrm{D}_n$.

As for the first statement, the size of the W-orbit $\mathcal{B}$ of an element X from Table 3 is equal to

$$\frac{|W(Q)|}{|N_W(X)|} = \frac{|W(Q)|}{2^{|X|}|W(X^\perp \cap \Phi)| \cdot |N_W(X)/C_W(X)|}.$$

All factors occurring in the last expression can be determined by means of Table 3 and the knowledge of orders of Coxeter groups of type ADE. The statement now follows from the following expressions of the relevant numbers for the individual types. For $Q = \mathrm{A}_{n-1}$ $(n \geq 2)$, the summation gives

$$\sum_{t=0}^{\lfloor n/2 \rfloor} \left(\frac{n!}{2^t t!(n-2t)!} \right)^2 (n-2t)!$$

which adds up to $n!!$. The equality between this summation and the expression of Table 2 can be proved directly as in [13, p. 113], or by counting Brauer diagrams in two different ways, as is clear from [3]. For $Q = \mathrm{D}_n$, the summation is

$$2^{n-1}n! + \sum_{t=1}^{\lfloor n/2 \rfloor} \left(\frac{n!}{t!(n-2t)!} \right)^2 2^{n-2t}(n-2t)! + \sum_{t=1}^{\lfloor n/2 \rfloor} \left(\frac{n!}{2^t t!(n-2t)!} \right)^2 (n-2t)!$$

which, by the formula for A_{n-1}, is easily seen to be

$$2^{n-1}n! + 2^n(n!! - n!) + (n!! - n!);$$

hence it coincides with the expression for D_n in Table 2. Here the expression in the first sum over t for $t = n/2$ is in fact a sum over the two orbits. It is $2\left(\frac{n!}{2(n/2)!}\right)^2 2$, where the leftmost 2 occurs because there are two orbits and the rightmost 2 accounts for the A_1 component in $C_{\mathcal{B}}$. Therefore, the summand becomes $\left(\frac{n!}{(n/2)!}\right)^2$, and so the expression $\left(\frac{n!}{t!(n-2t)!}\right)^2 2^{n-2t}(n-2t)!$ is valid for all $t \leq n/2$. In the second sum, the summand for $t = n/2$ also gives the expected answer by a deviation from the usual pattern: the group $C_W(X)$ has order $2^{n-1}(n-2t)!$ and $N_W(X)/C_W(X)$ has order $2^t t!$ for $t < n/2$, but the respective orders are $2^n(n-2t)!$ and $2^{t-1}t!$ if $n = 2t$ (as the type of the latter is $\mathrm{D}_{n/2}$ rather than $\mathrm{B}_{n/2}$), so $|N_W(X)| = 2^{n+t-1}(n-2t)!t!$ in all cases. An interpretation in terms of numbers of certain diagrams of type D_n will be given in [6]. For $Q = \mathrm{E}_6$, the summation is

$$|W(\mathrm{E}_6)| + 36^2|W(\mathrm{A}_5)| + 270^2|W(\mathrm{A}_2)| + 135^2,$$

for $Q = \mathrm{E}_7$,

$$|W(\mathrm{E}_7)| + 63^2|W(\mathrm{D}_6)| + 945^2|W(\mathrm{A}_1\mathrm{A}_3)| + 315^2|W(\mathrm{A}_2)| + 945^2|W(\mathrm{A}_1)| + 135^2,$$

and for $Q = \mathrm{E}_8$,

$$|W(\mathrm{E}_8)| + 120^2|W(\mathrm{E}_7)| + 3780^2|W(\mathrm{A}_5)| + 9450^2|W(\mathrm{A}_2)| + 2025^2. \qquad \blacksquare$$

After some preliminaries on admissibility in Section 2 and the construction of a presentation as maps and a linear representation of the monoid $\mathbf{BrM}(Q)$ in Section 3, we prove the upper bound of $\dim(\mathbf{Br}(Q))$ in Section 4 and the lower bound in Section 5. At the very end we describe how the Brauer diagrams for type A_{n-1} can be extended to a 'geometric' picture involving roots for other types in ADE. These are in terms of the triples described above Lemma 1.3. For those familiar with Brauer diagrams, the triples may be interpreted as knowledge of the horizontal lines on the top, the horizontal lines on the bottom, and the permutation of the remaining lines; see Remarks 4.10 and 5.7 below. As mentioned before, in Section 6 we show that the algebras are cellular.

Some of the work reported here grew out of the Master's thesis of one of us, [8]. The other two authors wish to acknowledge Caltech and Technische Universiteit Eindhoven for enabling mutual visits.

2. Admissibility

In this section we mention some basic properties of Brauer algebras related to admissible sets of mutually orthogonal positive roots that are useful for the proof of Theorem 1.1.

For $Q \in$ ADE, we need the notions of a root system Φ and a set of positive roots Φ^+. These can be found in [2], but for the convenience of the reader, we give their definitions here. The Coxeter group $W(Q)$ has a faithful linear representation on the real vector space $\bigoplus_i \mathbb{R}\alpha_i$ with formal basis α_i for i running over the nodes of Q. Let $(\cdot,\cdot)$ denote the symmetric bilinear form on this vector space determined by

$$(\alpha_i, \alpha_j) \quad = \quad \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \sim j, \\ 0 & \text{if } i \not\sim j, \end{cases}$$

for i, j nodes of Q. Then $(\cdot,\cdot)$ is positive definite and the faithful linear representation is determined by $r_i\alpha_j = \alpha_j - (\alpha_j, \alpha_i)\alpha_i$ for i and j nodes of Q. Now $\Phi = \bigcup_i W\alpha_i$ and $\Phi^+ = \Phi \cap (\bigoplus_i \mathbb{R}_{\geq 0}\alpha_i)$. It is well known that Φ is the disjoint union of Φ^+ and $-\Phi^+$.

For $\alpha, \beta \in \Phi$, we write $\alpha \sim \beta$ to denote $|(\alpha, \beta)| = 1$. Thus, for i and j nodes of Q, we have $\alpha_i \sim \alpha_j$ if and only if $i \sim j$.

Lemma 2.1: *Let $Q \in$ ADE and let β_1, β_2, β_3 be three mutually orthogonal roots of $W(Q)$. Then, up to sign, there is at most one $\langle r_{\beta_1}, r_{\beta_2}, r_{\beta_3}\rangle$-orbit of roots γ with $\gamma \sim \beta_i$ for $i = 1, 2, 3$. Moreover, there is a unique fourth positive root β_4 orthogonal to β_1, β_2, β_3 such that, for each such γ, we have $\beta_4 \sim \gamma$. This root satisfies $\beta_4 = \pm r_\gamma r_{\beta_1} r_{\beta_2} r_{\beta_3}\gamma$.*

Proof. Suppose that γ and γ' are roots with $\gamma \sim \beta_i \sim \gamma'$ for $i = 1, 2, 3$. After replacing each β_i by its negative if needed, we may assume $(\gamma, \beta_i) = -1$ for $i = 1, 2, 3$. Now $\beta_4 = r_\gamma r_{\beta_1} r_{\beta_2} r_{\beta_3}\gamma = 2\gamma + \beta_1 + \beta_2 + \beta_3 \in \Phi$ is a root orthogonal to β_1, β_2, β_3 with $(\beta_4, \gamma) = 1$. Also, $\varepsilon = r_\gamma \beta_4 = \gamma + \beta_1 + \beta_2 + \beta_3$ is a root. Replacing γ' by $r_{\beta_i}\gamma'$ if needed for successive values of i, we can arrange for $(\gamma', \beta_i) = -1$ if $i \in \{1, 2, 3\}$. If γ' does not coincide with γ, then $(\gamma', \gamma) \leq 1$, so

$$(\gamma', \varepsilon) = (\gamma', \gamma + \beta_1 + \beta_2 + \beta_3) = (\gamma', \gamma) - 3 \leq -2.$$

The only possibility of this being an integer with norm at most 2 occurs when $(\gamma', \varepsilon) = -2$, that is, $\gamma' = -\varepsilon = -r_{\beta_1} r_{\beta_2} r_{\beta_3}\gamma$, which, up to sign, belongs to the $\langle r_{\beta_1}, r_{\beta_2}, r_{\beta_3}\rangle$-orbit of γ.

As for uniqueness of β_4, observe that the linear span of β_1, β_2, β_3, and γ does not depend on the choice of γ and contains β_4. But in that 4-dimensional space, β_4 or $-\beta_4$ is the unique positive root orthogonal to β_1, β_2, and β_3. ∎

Let X be a set of mutually orthogonal positive roots. Then, by the lemma, for each triple of elements in X for which there exists a root γ non-orthogonal to each of the triple, there is a unique element of Φ^+ distinct from γ, non-orthogonal to γ, and orthogonal to each root from the triple. Therefore, the intersection of any collection of admissible sets of mutually orthogonal positive roots is again admissible. Consequently, the following notion is well defined as the intersection of all admissible sets containing X.

Definition 2.2: Given a set X of mutually orthogonal positive roots, the unique smallest admissible set containing X is called the **admissible closure** of X, and denoted X^{cl}.

In view of Lemma 2.1, the closure of X can be constructed by iteratively finding all $\beta_1, \beta_2, \beta_3 \in X$ for which there is a root γ with $\beta_i \sim \gamma$ for all i, and adjoining the positive root of $\pm r_\gamma r_{\beta_1} r_{\beta_2} r_{\beta_3}\gamma$ to X.

The significance of the admissible closure for Brauer algebras will become clear in Lemma 4.4. According to Lemma 1.3, a representative of each W-orbit of nonempty admissible sets is in Table 3.

3. Representations of the Brauer monoid

Throughout this section, we assume that Q is of type ADE. Set $W = W(Q)$. As mentioned before the statement of Theorem 1.1, W occurs as a subgroup of $\mathbf{BrM}(Q)$. The elements r_i of $\mathbf{BrM}(Q)$, for i nodes of Q, are a set of simple reflections of W, cf. [2]. It will be convenient to have more relations for $\mathbf{Br}(Q)$ at our disposal than those given in Table 1.

label	relation
	for $i \sim j$
(RNerr)	$e_i r_j r_i = e_i e_j$
(HNree)	$r_j e_i e_j = r_i e_j$
(RNere)	$e_i r_j e_i = e_i$
(HNeer)	$e_j e_i r_j = e_j r_i$
(HNeee)	$e_i e_j e_i = e_i$
	for $i \sim j \sim k$
(HTeere)	$e_j e_i r_k e_j = e_j r_i e_k e_j$
(RTerre)	$e_j r_i r_k e_j = e_j e_i e_k e_j$

Table 4. Additional relations

It may be worthy of mention that the labels of the relations are mnemonics as follows. The first capital is either H or R, depending on whether there is an equal number of r_l at both sides of the equality sign; if so, this is indicated by the letter H for homogeneous; otherwise, replacing the left hand side by the right hand side yields a smaller number of r_l and we use the letter R for reduction. The second letter is S, for self, C for commuting, or N for non-commuting, referring to the occurrence of a single node i, a pair of non-adjacent nodes i, j, and a pair of adjacent nodes i, j, in the respective cases. Finally, the small letters remind us of the pattern of the e_k and the r_l occurring at the left hand side.

Lemma 3.1: *The relations in Table 4 also hold in* $\mathbf{Br}(Q)$.

Proof. For (RNerr), we apply (HNrre), (RSrr), (HNrer), and (RSrr), respectively, to obtain $e_ie_j = r_jr_ie_j = r_jr_ie_jr_ir_i = r_jr_je_ir_jr_i = e_ir_jr_i$. For (HNeee) multiply $e_ir_jr_i = e_ie_j$, from (RNerr), by r_ir_j and use (RSrr) and (RNerr), respectively, to get $e_i = e_ie_jr_ir_j = e_ie_je_i$. For (HNree), use (RNrre) and (RSrr) to derive $r_je_ie_j = r_jr_jr_ie_j = r_ie_j$. For (RNere), use (RSrr), (RSer), (RNrre), and (HNeee) to compute $e_ir_je_i = e_ir_ir_ir_je_i = e_ie_je_i = e_i$. For (HNeer), use the reversed words of (HNree) and notice (RNerr) holds. For (HTeere), use (RNerr) and (RNrre) to find $e_je_ir_ke_j = e_jr_ir_jr_ke_j = e_jr_ie_ke_j$. Finally, for (RTerre), use (RSrr), (RNerr), and (RNrre) to compute $e_jr_ir_ke_j = e_jr_ir_jr_jr_ke_j = e_je_ie_ke_j$. ∎

By $\mathcal{A}$ we denote the collection of admissible sets of mutually orthogonal positive roots. Let $\mathcal{B}$ be a W-orbit in $\mathcal{A}$. Denote B_0 its highest element with respect to the partial order defined on $\mathcal{B}$; see [5] for this partial order and the proof of existence of B_0. The set of nodes i of Q for which $\alpha_i \perp B_0$ plays an important role in [5]; here it will be denoted $C_{\mathcal{B}}$ or, if no confusion is imminent, just C. It is well known, [2], that the subgroup $W(C)$ of W generated by the r_i for $i \in C$ is a Coxeter group whose type is the restriction of Q to C.

We present a useful representation of the Brauer monoid as a set of maps from $\mathcal{A}$ to itself. At the same time, for each W-orbit $\mathcal{B}$ within $\mathcal{A}$, we construct a linear representation of the Brauer algebra with basis indexed by $\mathcal{B}$ and with coefficients from the group ring of $W(C_{\mathcal{B}})$ over $\mathbb{Z}[\delta^{\pm 1}]$. We begin with the action on $\mathcal{A}$.

Definition 3.2: Let $\mathcal{A}$ be the disjoint union of all admissible W-orbits (so the empty set is a member of $\mathcal{A}$). The action of W on $\mathcal{A}$ is already given and corresponds to conjugation on sets of reflections. The action of δ is taken to be trivial, that is $\delta(X) = X$ for $X \in \mathcal{A}$. This action extends to an action of the generators e_i of the Brauer monoid in the following way, for $i \in Q$ and $B \in \mathcal{A}$:

$$e_iB = \begin{cases} B & \text{if } \alpha_i \in B, \\ (B \cup \{\alpha_i\})^{\mathrm{cl}} & \text{if } \alpha_i \perp B, \\ r_\beta r_i B & \text{if } \beta \in B \setminus \alpha_i^{\perp}. \end{cases} \tag{1}$$

Lemma 3.3: *For each admissible set B, set X of mutually orthogonal positive roots (not-necessarily admissibly closed), node i of Q, and positive root γ, the following properties hold.*

(i) $\alpha_i \in e_iB$.
(ii) *If* $\gamma \perp X$, *then* $\gamma \perp X^{\mathrm{cl}}$ *or* $\gamma \in X^{\mathrm{cl}}$.
(iii) *If* $(\alpha_i, \gamma) = 0$ *and* $\gamma \perp B$, *then* $\gamma \perp e_iB$ *or* $\gamma \in e_iB$.
(iv) *If* $w \in W$, *then* $wX^{\mathrm{cl}} = (wX)^{\mathrm{cl}}$.
(v) *The element* e_iB *is well defined.*

Proof. (i) is direct from the definition of the action of e_i. (Observe that $\alpha_i = \pm r_\beta r_i \beta$ if $\beta \sim \alpha_i$.)

(ii) Suppose that $\gamma_1, \gamma_2, \gamma_3$ are roots in X and $\alpha \in \Phi$ has inner products -1 with each of these. The admissible closure of X will then contain the positive root ζ of $\pm(\gamma_1 + \gamma_2 + \gamma_3 + 2\alpha)$; see Lemma 2.1. If γ is not orthogonal to ζ, then, by the assumption $\gamma \perp X$, we must have $0 \neq (\gamma, \zeta) = 2(\gamma, \alpha)$. Therefore, $(\gamma, \alpha) = \pm 1$ and $(\gamma, \zeta) = \pm 2$, which means $\gamma = \pm\zeta$. As both γ and ζ are positive, we find $\gamma = \zeta \in X^{\mathrm{cl}}$.

(iii) If $\alpha_i \in B$, then $e_iB = B$, and so the conclusion holds by the hypothesis $\gamma \perp B$. If there is $\beta \in B \setminus \alpha_i^\perp$, then $e_iB = r_\beta r_i B$, which consists fully of roots orthogonal to γ.

Finally, suppose $\alpha_i \perp B$. Then $e_iB = (B \cup \{\alpha_i\})^{\mathrm{cl}}$ and so the assertion follows from (ii).

(iv) If α, β, γ are mutually orthogonal roots joined to ζ, the same is true for the w images.

(v) Ambiguity arises if there are two choices, say β and γ, of roots in $B \setminus \alpha_i^\perp$. We need to show that then $r_\beta r_i B = r_\gamma r_i B$. Clearly, $r_\beta r_i(B \cap \alpha_i^\perp) = B \cap \alpha_i^\perp = r_\gamma r_i(B \cap \alpha_i^\perp)$. For simplicity choose β and γ so that the inner product with α_i is -1. Then $r_\beta r_i \gamma = r_\beta(\alpha_i + \gamma) = \alpha_i + \beta + \gamma = r_\gamma r_i \beta$. Now $r_\beta r_i\{\beta, \gamma\} = \{\alpha_i, \alpha_i + \beta + \gamma\} = r_\gamma r_i\{\beta, \gamma\}$.

Suppose that η is another root in $B \setminus \alpha_i^\perp$. Then, as B is admissibly closed, there will be a fourth root ζ in $B \setminus \alpha_i^\perp$. In fact, the fourth is $\zeta = \pm(\beta + \gamma + \eta + 2\alpha_i)$. Using this observation it is easily checked that both $r_\beta r_i$ and $r_\gamma r_i$ leave the set $\{\zeta, \eta\}$ invariant. ∎

We now define a linear representation of the Brauer algebra. In [5] simple reflections $h_{B,i}$ of $W(C_{\mathcal{B}})$ were defined for nodes i of Q and members B of $\mathcal{B}$ with $\alpha_i \perp B$. Extend this definition to all pairs (B, i) by $h_{B,i} = 1$ if $\alpha_i \not\perp B$. Let $V_{\mathcal{B}}$ be the free right $\mathbb{Z}[\delta^{\pm 1}][W(C_{\mathcal{B}})]$-module with basis ξ_B for $B \in \mathcal{B}$. For

$B \in \mathcal{B}$ and i a node of Q, set

$$r_i \xi_B = \xi_{r_i B} h_{B,i}. \tag{2}$$

LEMMA 3.4: *There is a unique linear representation $\rho_{\mathcal{B}} : W \to \mathrm{GL}(V_{\mathcal{B}})$ determined by* (2) *on the generators of W.*

Proof. It is shown in [5] that a similar map is a monoid representation. The value of m there can be taken to be 0 here, which simplifies some of the expressions. The only difference is that in [5] if $\alpha_i \in B$, the image under r_i on ξ_B in our set-up is 0. Here we have $r_i \xi_B = \xi_B$. Thus, we only treat the cases where this rule applies.

We first discuss the case where $\alpha_i \in B$. Here we have $r_i \xi_B = \xi_B$. It is immediate that in this case $r_i^2 \xi_B = \xi_B$ as needed. Suppose $i \not\sim j$ and so r_i and r_j commute. We must show $r_i r_j \xi_B = r_j r_i \xi_B$. Clearly $r_j r_i \xi_B = r_j \xi_B$. But this is $\xi_{r_j B} h_{B,j}$ by definition. As $\alpha_i \in B$, also $\alpha_i \in r_j B$, for $r_j \alpha_i = \alpha_i$ when $i \not\sim j$. This means $r_i r_j \xi_B = r_i \xi_{r_j B} h_{B,j} = \xi_{r_j B} h_{B,j} = r_j \xi_B = r_j r_i \xi_B$ and we are done. Suppose $i \sim j$. We need to show $r_i r_j r_i \xi_B = r_j r_i r_j \xi_B$. Now by definition $r_i r_j r_i \xi_B = r_i r_j \xi_B$. As $\alpha_j = r_i r_j \alpha_i \in r_i r_j B$, we also have $r_j r_i r_j \xi_B = r_i r_j \xi_B$ and we are done.

The only other possibility is that in acting by r_i in the case $i \not\sim j$ or by $r_i r_j$ in the case $i \sim j$ we would have $\alpha_j \in r_i B$ in the first case, or $\alpha_i \in r_j B$ or $\alpha_j \in r_i r_j B$ in the second case. If $i \not\sim j$ and $\alpha_i \in r_j B$, then $\alpha_i \in B$ and we are back in the previous case. Suppose therefore $i \sim j$. As $\alpha_j \in r_i r_j B$ implies $\alpha_i \in B$, it suffices to consider the case where $\alpha_i \in r_j B$. Now $\alpha_i + \alpha_j \in B$, so $\alpha_i \in r_j B$. Moreover, $r_j r_i r_j \xi_B = r_j r_i \xi_{r_j B} = r_j \xi_{r_j B} = \xi_B$. This is symmetric in i and j and we are done. ∎

The map $\rho_{\mathcal{B}}$ extends to a representation of $\mathbf{BrM}(Q)$. The action of δ is by homothety (so $\delta v = v\delta$ for $v \in V_{\mathcal{B}}$). Furthermore, the action of e_i is defined as follows:

$$e_i \xi_B = \begin{cases} \xi_B \delta & \text{if } \alpha_i \in B, \\ 0 & \text{if } \alpha_i \perp B, \\ r_\beta r_i \xi_B & \text{where } \beta \in B \text{ and } \beta \sim \alpha_i. \end{cases} \tag{3}$$

Before establishing that this is indeed a representation, we prove that the action of e_i on ξ_B is well defined. If $B \in \mathcal{B}$, we will write K_B for the subgroup $\{w \in W \mid w\xi_B = \xi_B\}$ of W. Clearly, $vK_Bv^{-1} = K_{vB}$ whenever $v \in W$.

Lemma 3.5: *If i is a node of Q and $B \in \mathcal{B}$ has elements β and γ with*

$$\beta \sim \alpha_i \sim \gamma,$$

then $r_i r_\beta r_\gamma r_i \in K_B$.

Proof. Take $w \in W$ with $w\alpha_k = \beta$ and $w\alpha_l = \gamma$ for nodes k and l of Q. Such a w always exists. Now, as r_i moves B, we have $r_\beta r_i \xi_B = r_\gamma r_i \xi_B$ if and only if $r_\beta \xi_{r_i B} = r_\gamma \xi_{r_i B}$, which holds if and only if $w r_k w^{-1} \xi_{r_i B} = w r_l w^{-1} \xi_{r_i B}$. This is in turn equivalent to $r_k w^{-1} \xi_{r_i B} = r_l w^{-1} \xi_{r_i B}$, and hence to $r_k \xi_{w^{-1} r_i B} c = r_l \xi_{w^{-1} r_i B} c$ for some $c \in W(C_{\mathcal{B}})$, which is obviously equivalent to $r_k \xi_{w^{-1} r_i B} = r_l \xi_{w^{-1} r_i B}$. Set $B' = w^{-1} r_i B$. Observe that $w^{-1} r_i \beta$, and $w^{-1} r_i \gamma$ belong to B' and are moved by r_k and r_l. Therefore, r_k and r_l move B', and so $r_k \xi_{B'} = \xi_{r_k B'}$ and $r_l \xi_{B'} = \xi_{r_l B'}$. But $r_k B' = r_k w^{-1} r_i B = w^{-1} r_\beta r_i B = w^{-1} r_\gamma r_i B = r_l w^{-1} r_i B = r_l B'$, whence $r_k \xi_{w^{-1} r_i B} = r_l \xi_{w^{-1} r_i B}$. Therefore $r_\beta r_i \xi_B = r_\gamma r_i \xi_B$, as required. ∎

Consequently, if $\alpha_i \sim \beta, \gamma \in B$, the two definitions $r_\beta r_i \xi_B$ and $r_\gamma r_i \xi_B$ of $e_i \xi_B$ coincide, so $e_i \xi_B$ is well defined.

For a set Y, we write $\mathcal{F}(Y)$ to denote the monoid of all maps from Y to itself.

Theorem 3.6: *For each $Q \in ADE$, corresponding Coxeter group $W = W(Q)$, and W-orbit $\mathcal{B}$ in $\mathcal{A}$, the following holds.*

- (i) *There is a unique homomorphim $\sigma : \mathbf{BrM}(Q) \to \mathcal{F}(\mathcal{A})$ of monoids determined by the usual action of the generators r_i and the e_i action of* (1) *on $\mathcal{A}$. If $Y, X \in \mathcal{A}$ and $a \in \mathbf{BrM}(Q)$ satisfy $Y \subseteq X$, then $aY \subseteq aX$.*
- (ii) *There is a unique linear representation, also denoted $\rho_{\mathcal{B}}$, of the Brauer algebra $\mathbf{Br}(Q)$ on $V_{\mathcal{B}}$ extending the map $\rho_{\mathcal{B}}$ of Lemma 3.4 with e_i acting according to* (3).

Proof. In order to show that σ and $\rho_{\mathcal{B}}$ are homomorphisms, we need to show that they respect the defining relations of $\mathbf{BrM}(Q)$. For σ, as the action by W is a group action, and for $\rho_{\mathcal{B}}$, as the restriction to W is a group representation by Lemma 3.4, we only need check the relations for $\mathbf{BrM}(Q)$ involving e_i's. We check both parts at the same time for each of these relations in Table 1.

We abbreviate $C_{\mathcal{B}}$ to C. On several occasions, we will use the observation that, if $e_i \xi_B \neq 0$, then $e_i \xi_B \in \xi_{B'} W(C)$ for $B' \in \mathcal{B}$ with $\alpha_i \in B'$. We will then write $e_i \xi_B = \xi_{B'} h$ with $h \in W(C)$.

(RSer). For (i) we need to verify $e_ir_iB = e_iB$. If $\alpha_i \in B$ or $\alpha_i \perp B$, then $r_iB = B$, so we are done. Suppose, therefore, that there is $\beta \in B \setminus \alpha_i^\perp$. Then $r_i\beta \in r_i(B\setminus\alpha_i^\perp)$ and $\alpha_i \in e_iB$. This implies $r_ie_iB = e_iB$. Now $e_iB = r_i(e_iB) = r_ir_\beta r_iB = r_{r_i\beta}r_i(r_iB) = e_ir_iB$, as required.

For (ii), the representation, we must show $e_ir_i\xi_B = e_i\xi_B$. If $\alpha_i \in B$, then $r_i\xi_B = \xi_B$, so we are done. If $\alpha_i \perp B$ then $r_i\xi_B = \xi_Bh_{B,i}$. But $e_i\xi_B = 0$, so both sides are 0. Suppose, therefore, that there is $\beta \in B \setminus \alpha_i^\perp$. Then $e_i\xi_B = r_\beta r_i\xi_B = r_\beta\xi_{r_iB}$. Now r_iB contains $r_i\beta$ which is not perpendicular to α_i. Hence $e_ir_i\xi_B = e_i\xi_{r_iB} = r_{r_i\beta}r_i\xi_{r_iB} = r_ir_\beta\xi_{r_iB} = r_ir_\beta r_i\xi_B$. Notice that $\alpha_i \in r_\beta r_iB$, and so there is $h \in W(C)$ such that $e_ir_i\xi_B = r_ir_\beta r_i\xi_B = r_i\xi_{r_\beta r_iB}h = \xi_{r_\beta r_iB}h = r_\beta r_i\xi_B = e_i\xi_B$, as required.

(RSre). Here, for (i), we need to show $r_ie_iB = e_iB$. As $\alpha_i \in e_iB$, this is immediate.

For (ii) we need to show $r_ie_i\xi_B = e_i\xi_B$. If $\alpha_i \in B$, both sides are equal to ξ_B, and if $\alpha_i \perp B$, both sides are equal to 0. Suppose $\beta \in B$ is not perpendicular and not equal to α_i. Now $e_i\xi_B = r_\beta r_i\xi_B = \xi_{r_\beta r_iB}h$ for some $h \in W(C)$. As $\alpha_i \in r_\beta r_iB$, the reflection r_i fixes $\xi_{r_\beta r_iB}$ and so $r_ie_i\xi_B = r_i\xi_{r_\beta r_iB}h = \xi_{r_\beta r_iB}h = e_i\xi_B$, as required.

(HSee). For (i) we need to derive $e_ie_iB = e_iB$. As $\alpha_i \in e_iB$, this is immediate.

For (ii) we need $e_ie_i\xi_B = \delta e_i\xi_B$. If $\alpha_i \in B$ or $\alpha_i \perp B$, this is immediate. Otherwise $e_i\xi_B = \xi_{B'}h$ with $h \in W(C)$ and $B' \in \mathcal{B}$ containing α_i, and so the equality follows from $e_i\xi_{B'} = \delta\xi_{B'}$.

(HCer). Here $i \not\sim j$. For (i) we need to show $e_ir_jB = r_je_iB$. If $\alpha_i \in B^\perp$, then α_i is also in $(r_jB)^\perp$ and so the result in both cases is the closure of $r_jB \cup \{\alpha_i\}$. If $\alpha_i \in B$, then $r_j\alpha_i = \alpha_i$ and so $\alpha_i \in r_jB$. Now $e_ir_jB = r_jB$ and $r_je_iB = r_jB$. Now suppose there is $\beta \in B$ with $(\alpha_i, \beta) \neq 0$. Then $r_je_iB = r_jr_\beta r_iB$. Also $(\alpha_i, r_j\beta) \neq 0$ and so $e_ir_jB = r_{r_j\beta}r_ir_jB$. Now again $r_{r_j\beta} = r_jr_\beta r_j$ giving the last term $r_jr_\beta r_jr_ir_jB = r_jr_\beta r_iB$ as r_i and r_j commute and are of order two.

For (ii) we need to show $e_ir_j\xi_B = r_je_i\xi_B$. If $\alpha_i \in B^\perp$, then α_i is also in $(r_jB)^\perp$ and so the result is 0 in both cases. If $\alpha_i \in B$, then $r_j\alpha_i = \alpha_i$ and so $\alpha_i \in r_jB$. Now $e_ir_j\xi_B = e_i\xi_{r_jB}h_{B,j} = \delta\xi_{r_jB}h_{B,j} = r_j\xi_B\delta = r_je_i\xi_B$. Suppose there is $\beta \in B$ with $\alpha_i \sim \beta$. Then also $\alpha_i \sim r_j\beta$ and so $r_je_i\xi_B = r_jr_\beta r_i\xi_B = r_{r_j\beta}r_ir_j\xi_B = e_i\xi_{r_jB}h_{B,j} = e_ir_j\xi_B$.

(HCee). Here $i \not\sim j$. We need to show $e_ie_jB = e_je_iB$ and $e_ie_j\xi_B = e_je_i\xi_B$. Suppose $\alpha_i \in B$. Then $e_je_iB = e_jB$. Note $\alpha_i \in e_jB$ in all cases and so also

$e_ie_jB = e_jB$, so they are the same. For the linear representation, $e_je_i\xi_B = e_j\xi_B\delta$. If $e_j\xi_B = 0$ we are done as both sides of the required equality are 0. Otherwise, $e_j\xi_B = \xi_{B'}h$ with $h \in W(C)$ and $\alpha_i \in B' \in \mathcal{B}$ and so also $e_ie_j\xi_B = e_j\xi_B\delta$, as required. By symmetry of the argument in i and j, we may, and will, assume from now on that $\alpha_i, \alpha_j \notin B$.

Suppose next $\alpha_i \perp B$. Then $e_iB = (B \cup \{\alpha_i\})^{\mathrm{cl}}$. We will use the observation that, for $X \in \mathcal{A}$, we have $e_kX = (X \cup \{\alpha_k\})^{\mathrm{cl}}$ whenever $\alpha_k \in X \cup X^{\perp}$. Suppose first $\alpha_j \perp B$. Then, by Lemma 3.3(iii), $\alpha_j \in e_iB \cup (e_iB)^{\perp}$, so $e_je_iB = e_j(B \cup \{\alpha_i\})^{\mathrm{cl}} = ((B \cup \{\alpha_i\})^{\mathrm{cl}} \cup \{\alpha_j\})^{\mathrm{cl}} = (B \cup \{\alpha_i, \alpha_j\})^{\mathrm{cl}}$ is symmetric in i and j, so we are done.

If $\alpha_j \not\perp B$, then there is $\beta \in B$ with $\beta \sim \alpha_j$. As $\beta \perp \alpha_i \perp \alpha_j$, using Lemma 3.3 (iv), we find $e_je_iB = r_\beta r_j(B \cup \{\alpha_i\})^{\mathrm{cl}} = (r_\beta r_jB \cup \{\alpha_i\})^{\mathrm{cl}} = e_ir_\beta r_jB = e_ie_jB$, as required. For the representation, the arguments above for all cases where $\alpha_i \perp B$ give 0 here and there is nothing to prove. By symmetry, we can suppose, for the remainder of the proof of (HCee), that neither α_i nor α_j is in $B \cup B^{\perp}$.

This means there are β and β' in B with $\alpha_i \sim \beta$ and $\alpha_j \sim \beta'$. Suppose $\alpha_j \not\sim \beta$ and $\alpha_i \not\sim \beta'$ and $\beta \neq \beta'$. Then $e_ie_jB = e_ir_{\beta'}r_jB = r_\beta r_ir_{\beta'}r_jB$ as we may use β to give the action of r_i (here we use that $r_{\beta'}r_j\beta = \beta$). Similarly $e_je_iB = r_{\beta'}r_jr_\beta r_iB$. By orthogonality of the roots involved in commutation, $r_\beta r_ir_{\beta'}r_j = r_{\beta'}r_jr_\beta r_i$ and so $e_ie_jB = r_\beta r_ir_{\beta'}r_jB = r_{\beta'}r_jr_\beta r_iB = e_je_iB$, as required. For the representation replace each B by ξ_B and the result follows.

We are done if such a choice of β and β' is possible. Assume for the remainder of the proof of (HCee) that such a choice is not possible. During these arguments it will be useful to have a term for this. We say i and j **satisfy condition** $(*)$ if

> $i \not\sim j$, there is a $\beta \in B$ with $\alpha_i \sim \beta \sim \alpha_j$, and B has no pairs γ, γ' for which $\alpha_i \sim \gamma$, $\alpha_j \sim \gamma'$, $\alpha_i \not\sim \gamma'$, and $\alpha_j \not\sim \gamma$.

We suppose from now on in proving (HCee) that i and j satisfy condition $(*)$. Suppose β is the only element of B joined to α_i or α_j. Then $e_iB = \{\alpha_i\} \cup B \setminus \{\beta\}$ and $e_je_iB = (\{\alpha_j, \alpha_i\} \cup B \setminus \{\beta\})^{\mathrm{cl}}$. This is symmetric in i and j and we are done for the poset part. For the representation, as above, $e_i\xi_B = r_\beta r_i\xi_B = \xi_{\{\alpha_i\}\cup B\setminus\{\beta\}} = \xi_{e_iB}$. Now α_i is orthogonal to all elements in e_iB and so $e_je_i\xi_B = 0$. This is symmetric in i and j and we are done.

The most difficult condition is when there is a second root γ in B also joined to both α_i and α_j. We assume for the moment there is no such γ. This means up to interchanging i and j there are γ in B with $\gamma \sim \alpha_i$ and α_j is not joined to any of them. In fact now as i and j satisy $(*)$, α_j is joined only to β. Now $e_jB = \{\alpha_j\} \cup B \setminus \{\beta\}$ and $e_ie_jB = r_\gamma r_i(\{\alpha_j\} \cup B \setminus \{\beta\})$. Notice α_j is in this set, so $e_ie_jB = r_\gamma r_i r_\beta r_j B$. Now consider $e_iB = r_\gamma r_i B$. The only element of $r_\gamma r_i B$ not perpendicular to α_j is $r_\gamma r_i \beta$ and so $e_je_iB = (r_\gamma r_i r_\beta r_i r_\gamma) r_j r_\gamma r_i B$. But

$$(r_\gamma r_i r_\beta r_i r_\gamma) r_j r_\gamma r_i = r_\gamma r_i r_\beta r_i r_j r_i = r_\gamma r_i r_\beta r_j,$$

whence $e_ie_jB = e_je_iB$. The same computations work for $\rho_{\mathcal{B}}$.

Suppose now that B has roots β and γ, both joined to α_i and to α_j. There are two cases to be considered. Since the roots in elements of $\mathcal{A}$ are all supposed to be positive, we will take the liberty of indicating the positive root by its negative whenever convenient. Since confusion is minimal, we shall write $\{\alpha\}$ rather than $\{\pm\alpha\} \cap \Phi^+$. By changing positive roots to negatives we can assume that the inner products of α_j with β and γ are negative and that the inner product of α_i with γ is negative. There are now two choices ± 1 for (α_i, β).

If $(\alpha_i, \beta) = -1$, the Gram matrix has determinant 0 and an easy check shows $-\alpha_i = \alpha_j + \beta + \gamma$. In this case the roots involved generate a root system of type A_3; an example of the configuration occurs for $\alpha_i = \alpha_1$, $\gamma = \alpha_2$, $\alpha_j = \alpha_3$ and $\beta = -(\alpha_1 + \alpha_2 + \alpha_3)$ with $\alpha_1, \alpha_2, \alpha_3$ the simple roots of A_3.

If $(\alpha_i, \beta) = 1$, the roots involved generate a root system of type D_4. An example of the configuration occurs for $\alpha_i = \alpha_1$, $\gamma = \alpha_2$, $\alpha_j = \alpha_3$ and $\beta = \alpha_2 + \alpha_3 + \alpha_4$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the simple roots of D_4 with 2 the triple node.

We suppose first that all roots β' of B other than β and γ are orthogonal to α_i and α_j. In the A_3 case we have $\gamma + \alpha_j + \beta = -\alpha_i$ and so

$$e_jB = r_\beta(\{\beta + \alpha_j, \gamma + \alpha_j\} \cup B \setminus \{\beta, \gamma\}) = \{\alpha_j, \alpha_i\} \cup B \setminus \{\beta, \gamma\},$$

whence $e_ie_jB = \{\alpha_j, \alpha_i\} \cup (B \setminus \{\beta, \gamma\})$. The other order gives the same result and so $e_ie_jB = e_je_iB$. For the representation, $e_j\xi_B = r_\beta r_j \xi_B$. We have just seen $e_jB = \{\alpha_j, \alpha_i\} \cup (B \setminus \{\beta, \gamma\})$. This means $e_j\xi_B = \xi_{r_\beta r_j B}$ and so $e_ie_j\xi_B = r_\beta r_j \xi_B \delta$. Similarly, $e_je_iB = r_\beta r_i \xi_B \delta$. For these to be equal we would need $r_i\xi_B = r_j\xi_B$. From the definition this is $\xi_{r_iB} = \xi_{r_jB}$ and so is equivalent to $r_iB = r_jB$. As, up to the signs of roots, $r_i\{\beta, \gamma\} = \{\beta + \alpha_i, \gamma + \alpha_i\} = \{\beta + \alpha_j, \gamma + \alpha_j\} = r_j\{\beta, \gamma\}$, this is indeed the case.

There is one other case in which all roots of B other than β and γ are orthogonal to α_i and to α_j, viz., $(\alpha_i, \beta) = 1$ and $\alpha_i, \alpha_j, \beta, \gamma$ generate a root system of type D_4. Now $e_jB = r_\beta r_j B = \{\alpha_j, \gamma + \alpha_j + \beta\} \cup (B \setminus \{\beta, \gamma\})$. Notice $(\alpha_i, \gamma + \alpha_j + \beta) = 0$ and so

$$e_i(\{\alpha_j, \gamma + \alpha_j + \beta\} \cup B \setminus \{\beta, \gamma\}) = (\{\alpha_i, \alpha_j, \gamma + \alpha_j + \beta\} \cup B \setminus \{\beta, \gamma\})^{\mathrm{cl}}.$$

Also

$$e_je_iB = e_j(\{\alpha_i, \beta - \alpha_i - \gamma\} \cup (B \setminus \{\beta, \gamma\})) = (\{\alpha_j, \alpha_i, \beta - \alpha_i - \gamma\} \cup (B \setminus \{\beta, \gamma\}))^{\mathrm{cl}}.$$

Now $(\gamma, \beta - \alpha_i - \gamma) = -1$ and so in the closure of

$$\{\alpha_j, \alpha_i, \beta - \alpha_i - \gamma\} \cup (B \setminus \{\beta, \gamma\})$$

there is $\alpha_i + \alpha_j + \beta - \gamma - \alpha_i + 2\gamma = \alpha_j + \beta + \gamma$. This means

$$(\{\alpha_j, \alpha_i, \beta - \alpha_i - \gamma\} \cup (B \setminus \{\beta, \gamma\}))^{\mathrm{cl}} = (\{\alpha_i, \alpha_j, \gamma + \alpha_j + \beta, \beta - \gamma - \alpha_i\} \cup (B \setminus \{\beta, \gamma\}))^{\mathrm{cl}}$$

and so $e_je_iB = e_ie_jB$. For the representation, the actions are all the 0 action and so the required equality is trivially satisfied.

This concludes the cases where α_i and α_j are joined only to β and γ. In the remaining cases, we may assume α_j is not orthogonal to at least three roots in B and so, because B is admissible, α_j is orthogonal to four roots of B. This means there is $\varepsilon \in B$ with $(\alpha_j, \varepsilon) = -1$ and $\eta = \beta + \gamma + \varepsilon + 2\alpha_j$ is also in B. If α_i were not joined to all the roots $\{\beta, \gamma, \epsilon, \eta\}$ but joined to another we would contradict condition $(*)$. If it were joined to three it would be joined to four by the admissibility.

If α_i were joined to all four of them consider the 4-dimensional linear subspace of $\mathbb{R}^n$ spanned by the roots $\beta, \gamma, \epsilon, \eta$. Both α_i and α_j lie in this space and so the six roots generate a root system of type D_4. An easy check shows that r_i and r_j act the same on β, γ, ϵ, η, and fix the remaining roots of B. This means that the actions of r_i and r_j on B are the same and so the actions of e_i and e_j on B are the same. In particular $e_ie_jB = e_i^2B = e_j^2B = e_je_iB$. Also, for the representation the actions of r_i and r_j must be the same and $e_ie_j\xi_B = e_i^2\xi_B = e_je_i\xi_B$.

The only remaining case occurs when α_i is joined to just β and γ as discussed. Now α_i, α_j, β, γ, ϵ, and η generate a root system of type D_5. In computing e_je_iB we can use β first and then η to get $e_je_iB = r_\eta r_j r_\beta r_i B$. In the other order we can use η first and then $\beta + \alpha_j + \eta$ to compute $e_i e_j B = r_{\beta+\alpha_j+\eta} r_i r_\eta r_j B$. Now we use $r_{\beta+\alpha_j+\eta} = r_\beta r_\eta r_j r_\beta r_\eta$ and derive

$$r_{\beta+\alpha_j+\eta} r_i r_\eta r_j = r_\beta r_\eta r_j r_\beta r_\eta r_i r_\eta r_j = r_\eta r_\beta r_j r_\beta r_i r_j = r_\eta r_\beta r_j r_\beta r_j r_i = r_\eta r_j r_\beta r_i.$$

Thus, $e_j e_i B = r_\eta r_j r_\beta r_i B = r_{\beta+\alpha_j+\eta} r_i r_\eta r_j B = e_i e_j B$, as required. The same computations work for $\rho_{\mathcal{B}}$, which finishes the proof of (HCee).

(HNrer). Here $i \sim j$. We need to show $r_i e_j r_i B = r_j e_i r_j B$ and $r_i e_j r_i \xi_B = r_j e_i r_j \xi_B$. Suppose first that there is $\beta \in B$ with $r_i \beta \not\perp \alpha_j$. Then $r_i e_j r_i B = r_i r_{r_i\beta} r_j r_i B = r_\beta r_j r_i r_j B$ and $r_i e_j r_i \xi_B = r_i r_{r_i\beta} r_j r_i \xi_B = r_\beta r_j r_i r_j \xi_B$. On the other hand, also $(\alpha_i, r_j\beta) = (r_j\alpha_i, \beta) = (r_i\alpha_j, \beta) = (\alpha_j, r_i\beta) \neq 0$, so $\beta \not\perp \alpha_i$ and so $r_j e_i r_j B = r_j r_{j\beta} r_i r_j B = r_\beta r_i r_j r_i B$ and $r_j e_i r_j \xi_B = r_j r_{j\beta} r_i r_j \xi_B = r_\beta r_i r_j r_i \xi_B$. In view of the braid relation $r_j r_i r_j = r_i r_j r_i$, both sides are equal.

Next suppose that $\alpha_j \perp r_i B$. Then $r_i e_j r_i B = r_i(r_i B \cup \{\alpha_j\})^{\mathrm{cl}} = (B \cup r_i\{\alpha_j\})^{\mathrm{cl}}$ and $e_j r_i \xi_B = e_j \xi_{r_i B}$ or 0 if $\alpha_i \perp B$. Also, $\alpha_i = r_j r_i \alpha_j \perp r_j r_i r_i B = r_j B$ and so $r_j e_i r_j B = (B \cup r_j\{\alpha_i\})^{\mathrm{cl}}$ and $e_i r_j \xi_B = 0$. As $r_i \alpha_j = r_j \alpha_i$, the two sides are equal.

Finally, suppose that $\alpha_j \in r_i B$. This means $\alpha_i + \alpha_j \in B$. Now $r_i e_j r_i B = r_i r_i B = B$. Moreover, $\alpha_i = r_j r_i \alpha_j \in r_j B$ and so $r_j e_i r_j B = r_j r_j B = B$, as required. For the representation, this means $r_i e_j r_i \xi_B = r_j r_i r_{r_i\alpha_j} r_i \xi_B$. However $r_j r_i r_j r_i r_j r_i = 1$ and so both sides are the same.

(HNrre). Here $i \sim j$. For (i), we need to show $r_j r_i e_j B = e_i e_j B$. As α_j is in $e_j(B \setminus \alpha_i^\perp)$, we have $e_i e_j B = r_j r_i e_j B$, and we are done.

For (ii) we need to show $r_j r_i e_j \xi_B = e_i e_j \xi_B$. We may assume that $e_j \xi_B$ is not 0. If $\alpha_j \in B$, then $e_i e_j \xi_B = \delta e_i \xi_B = \delta r_j r_i \xi_B$. As $e_j \xi_B = \delta \xi_B$ we are done. If there is $\beta \in B$ not perpendicular to α_j, then $e_j \xi_B = r_\beta r_j \xi_B = \xi_{r_\beta r_j B} h$. Notice $r_\beta r_j B$ contains α_j. Now $e_i e_j \xi_B = r_j r_i e_j \xi_B$, and we are done.

There is one more property we need to show: if $Y \subseteq X$, then $aY \subseteq aX$. The action by r_i is just the group action which preserves inclusion, so we need only check the actions by e_i. Let $Y \subseteq X$.

Suppose $\alpha_i \in X$. Then $e_i X = X$. If $\alpha_i \in Y$ then $e_i Y = Y$ and we are done. If $\alpha_i \notin Y$ then $\alpha_i \perp Y$ as $\alpha_i \in X$ and elements in X are mutually orthogonal. Consequently, $e_i Y = (\{\alpha_i\} \cup Y)^{\mathrm{cl}} \subseteq X = e_i X$, as required.

For the remainder of the proof, we may assume $\alpha_i \notin X$. If $\alpha_i \perp X$ then $\alpha_i \perp Y$. This means $e_i Y = (\{\alpha_i\} \cup Y)^{\mathrm{cl}}$ and $e_i X = (\{\alpha_i\} \cup X)^{\mathrm{cl}}$ so $Y \cup \{\alpha_i\} \subseteq X \cup \{\alpha_i\}$, and hence $e_i Y \subseteq e_i X$, as required.

Suppose that there is $\beta \in Y$ with $\alpha_i \sim \beta$. Then $e_iY = r_\beta r_i Y$ and $e_iX = r_\beta r_i X$ while $r_\beta r_i Y \subseteq r_\beta r_i X$. The only case left is $\alpha_i \perp Y$ but there is β in X with $(\alpha_i, \beta) \neq 0$. Clearly $\beta \notin Y$. Now $e_iY = (\{\alpha_i\} \cup Y)^{\mathrm{cl}}$ and $e_iX = r_\beta r_i X$. By Lemma 3.3(ii), $\alpha_i \in e_iX$. As $r_\beta r_i Y = Y$, we find

$$e_iY = (\{\alpha_i\} \cup Y)^{\mathrm{cl}} = (\{\alpha_i\} \cup r_\beta r_i Y)^{\mathrm{cl}} \subseteq (e_iX)^{\mathrm{cl}} = e_iX.$$

so the assertion holds. ∎

Corollary 3.7: *For X the highest element of $\mathcal{B}$, the permutation stabilizer $N_W(X)$ of X in W is the semi-direct product of K_X and $W(C_\mathcal{B})$.*

Proof. From (ii) of the theorem we see $N_W(X) = \{w \in W \mid w\xi_X \in \xi_X W(C_\mathcal{B})\}$. As $h_{X,i} = r_i$ for i a node of $C_\mathcal{B}$, the subgroup $W(C_\mathcal{B})$ of $N_W(X)$ satisfies $W(C_\mathcal{B})\xi_X = \xi_X W(C_\mathcal{B})$ and so is a complement to K_X in $N_W(X)$. ∎

4. Rewriting elements and upper bounding the dimension

The main goal of this section is to prove that every element of $\mathbf{BrM}(Q)$ can be written in a certain standard form, which corresponds to the well-known Brauer diagrams if $Q = \mathrm{A}_{n-1}$. This will lead to the following upper bound of the dimension of $\mathbf{Br}(Q)$. Recall that $C_\mathcal{B}$ is the set of nodes of Q whose corresponding roots are orthogonal to the highest element of $\mathcal{B}$.

Proposition 4.1: *The dimension of the Brauer algebra of type Q is at most*

$$\sum_{\mathcal{B}} |\mathcal{B}|^2 |W(C_\mathcal{B})|.$$

This will be proved in a series of lemmas and propositions and completed at the end of this section.

Lemma 4.2: *Let i and j be nodes of Q. If $w \in W$ satisfies $w\alpha_i = \alpha_j$, then $we_iw^{-1} = e_j$.*

Proof. By [4, Proposition 3.2], there is a unique element w_{ij} of minimal length such that $w_{ij}\alpha_i = \alpha_j$. This can be proved exactly as in [4, Lemma 3.1(iv)], by use of (HNree) and (HNeer). It remains to verify that $C_W(\alpha_i)$ centralizes e_i. This is proved as in [4, Lemma 3.9], where it was shown $s_is_\beta = s_\beta s_i$ for any root β of W orthogonal or equal to α_i, where s_β is the product in the Artin group of the simple generators corresponding to a minimal length word for $r_\beta \in W$.

Here we replace s_i by e_i and use (HNree) and (HNeer) appropriately. Since $C_W(\alpha_i)$ is generated by such reflections r_β, this establishes the lemma. ∎

Consider a positive root β and a node i of Q. There exists $w \in W$ such that $\beta = w\alpha_i$. Define the element e_β of $\mathbf{Br}(Q)$ by

$$e_\beta = we_iw^{-1}. \tag{4}$$

Lemma 4.2 implies that e_β is well defined. The relations in $\mathbf{Br}(Q)$ involving the elements e_β extend the relations already described for fundamental elements e_i.

Lemma 4.3: *Let β and γ be positive roots of W.*

(i) $e_\beta r_\beta = r_\beta e_\beta = e_\beta$ *and* $e_\beta^2 = \delta e_\beta$.
(ii) *If* $(\beta,\gamma) = \pm 1$ *then*
  (a) $e_\beta r_\gamma e_\beta = e_\beta$,
  (b) $r_\beta r_\gamma e_\beta = e_\gamma r_\beta r_\gamma = e_\gamma e_\beta$,
  (c) $e_\beta e_\gamma e_\beta = e_\beta$.
(iii) *If* $(\beta,\gamma) = 0$, *then* $e_\beta r_\gamma = r_\gamma e_\beta$ *and* $e_\beta e_\gamma = e_\gamma e_\beta$.

Proof. If β and γ are simple roots, this is direct from the defining relations of $\mathbf{BrM}(Q)$. Otherwise, there are $w \in W$ and nodes i, j of Q such that $w\alpha_i = \beta$ and $w\alpha_j = \gamma$, and the result follows from (4) by conjugation. ∎

We next extend the definition of e_β to arbitrary sets of mutually orthogonal positive roots. For such a set B, we define the element e_B of $\mathbf{Br}(Q)$ by

$$e_B = \prod_{\beta \in B} e_\beta. \tag{5}$$

This definition is unambiguous as e_β and e_γ commute whenever β and γ are orthogonal (cf. Lemma 4.3(iii)). Clearly, e_B behaves well under conjugation by W in the sense that $ue_Bu^{-1} = e_{uB}$.

An important difference between $\mathbf{Br}(\mathrm{A}_n)$ and the Brauer algebras of other types is the fact that the orbit of B under the action of W need not correspond bijectively with the orbit of e_B under W by conjugation. For example, when $Q = \mathrm{D}_4$, with the labeling of the nodes as in [2], the set $B = \{\alpha_1, \alpha_3, \alpha_4\}$ is distinct from wB, where $w = r_2r_1r_3r_2$, but $we_Bw^{-1} = e_B$. For this reason, we need compare the action of W on e_B with the conjugation action on its admissible closure $B^{\rm cl}$ rather than B. The necessary transition from B to $B^{\rm cl}$ is expressed in the next lemma.

Lemma 4.4: *If X is a set of mutually orthogonal positive roots of W, then*

$$e_{X^{\rm cl}} = e_X \delta^{|X^{\rm cl}|-|X|}.$$

Proof. Suppose that $\alpha \in \Phi^+ \setminus X$ is non-orthogonal to the pair β_2, β_3 in X. By Lemma 4.3,

$$\begin{aligned} r_\alpha r_{\beta_2} r_{\beta_3} r_\alpha e_{\beta_2} e_{\beta_3} &= r_\alpha r_{\beta_2} r_{\beta_3} r_\alpha e_{\beta_3} e_{\beta_2} = r_\alpha r_{\beta_2} e_\alpha e_{\beta_3} e_{\beta_2} = e_{\beta_2} e_\alpha e_{\beta_3} e_{\beta_2} \\ &= e_{\beta_2} e_\alpha e_{\beta_2} e_{\beta_3} \\ &= e_{\beta_2} e_{\beta_3}, \end{aligned}$$

whence $r_\alpha r_{\beta_2} r_{\beta_3} r_\alpha e_X = e_X$.

Suppose now that β_1 is a third root of X that is not orthogonal to α. Let γ be the unique positive root in X non-orthogonal to α and orthogonal to β_1, β_2, and β_3; cf. Lemma 2.1. Then $\gamma = r_\alpha r_{\beta_2} r_{\beta_3} r_\alpha \beta_1$. As $r_\alpha r_{\beta_2} r_{\beta_3} r_\alpha r_{\beta_1} = r_\gamma r_\alpha r_{\beta_2} r_{\beta_3} r_\alpha$, using Lemma 4.2 we find

$$r_\alpha r_{\beta_2} r_{\beta_3} r_\alpha e_X \delta = r_\alpha r_{\beta_2} r_{\beta_3} r_\alpha e_{\beta_1} e_X = e_\gamma r_\alpha r_{\beta_2} r_{\beta_3} r_\alpha e_X = e_\gamma e_X = e_{\{\gamma\}\cup X}.$$

This procedure can be repeated until we have reached $X^{\rm cl}$. The lemma follows. ∎

Proposition 4.5: *Let X be an admissible set and let Y be a set of mutually orthogonal positive roots (not necessarily admissible). Then*

$$e_Y \xi_X \in \xi_Z W(C) \delta^k \cup \{0\},$$

for some $k \in \mathbb{N}$ with $k \le |Y|$ and $Z \in WX$ with $Y \subseteq Z$. Moreover, $e_Y \xi_X \neq 0$ with $k = |Y|$ if and only if $Y \subseteq X$, in which case $e_Y \xi_X = \xi_X \delta^{|Y|}$.

Proof. The proof of the first assertion is by induction on the size of Y.

Suppose that $|Y| = 1$. There exists a positive root α such that $Y = \{\alpha\}$ and $e_Y = e_\alpha$. If α is not a simple root, choose $w \in W$ for which $w\alpha_i = \alpha$ where α_i is simple. Then $e_\alpha = w e_i w^{-1}$ and $w \xi_X = \xi_{wX} h$ with $h \in W(C)$. The conditions on subsets are preserved. Therefore, we may, and shall, assume that α is simple. There are three cases to consider.

$\alpha \in X$. Then $e_Y \xi_X = e_\alpha \xi_X = \xi_X \delta$. Now, for $k = 1$ and $Z = X$ we have $Y \subseteq Z$ and $k = |Y|$, as required.

$\alpha \perp X$. Then $e_Y \xi_X = e_\alpha \xi_X = 0$ and the assertions hold.

$\alpha \sim \beta \in X$. Then $e_Y\xi_X = e_\alpha\xi_X = r_\beta r_\alpha \xi_X \in \xi_{r_\beta r_\alpha X}W(C)$. Moreover, $\alpha = r_\beta r_\alpha \beta \in r_\beta r_\alpha X$, so the assertions hold with $Z = r_\beta r_\alpha X$ and $k = 0$.

Next, assume $|Y| > 1$. Take $\alpha \in Y$ and set $Y_0 = Y \backslash \{\alpha\}$. We compute $e_Y\xi_X = e_\alpha e_{Y_0}\xi_X$. If $e_{Y_0}\xi_X = 0$, then clearly $e_Y\xi_X = 0$. Assume therefore $e_{Y_0}\xi_X \neq 0$. By the induction hypothesis, $e_{Y_0}\xi_X = \xi_Z v\delta^k$ with $Y_0 \subseteq Z$, $v \in W(C)$, and $k \leq |Y_0|$. Now

$$e_Y\xi_X = e_\alpha \xi_Z v\delta^k. \tag{6}$$

Put $Z_0 = Z \setminus Y_0$. We have $\alpha \in Y_0^\perp$. Moreover, every element of Z_0 commutes with every element of Y_0. Again, there are three cases to consider,

$\alpha \in Z_0$. Then $e_Y\xi_X = e_\alpha\xi_Z v\delta^k = \xi_Z v\delta^{k+1}$ and $Y = Y_0 \cup \{\alpha\} \subseteq Z$. Furthermore, $k \leq |Y_0| = |Y| - 1$, so $k + 1 \leq |Y|$. This proves the proposition in this case.

$\alpha \perp Z$. Then $e_\alpha\xi_Z = 0$, so $e_Y\xi_X = e_\alpha\xi_Z v\delta^k = 0$ by (6).

$\alpha \sim \beta \in Z$. Then

$$e_Y\xi_X = e_\alpha\xi_Z v\delta^k = r_\beta r_\alpha \xi_Z v\delta^k.$$

Now $r_\beta r_\alpha Z = r_\beta r_\alpha Z_0 \cup r_\beta r_\alpha Y_0$. As $\alpha, \beta \perp Y_0$, we have $r_\beta r_\alpha Y_0 = Y_0$. Hence $r_\beta r_\alpha Z = r_\beta r_\alpha Z_0 \cup Y_0$. As before, $\alpha \in r_\beta r_\alpha Z_0$. Hence $Y = Y_0 \cup \{\alpha\} \subseteq r_\beta r_\alpha Z$. Furthermore, $k \leq |Y_0| < |Y|$, as required for the proof of the first assertion.

In order to settle the second assertion, suppose that $k = |Y|$ and $e_Y\xi_X \neq 0$. If $Y = \emptyset$ the assertions $Y \subseteq X$ and $e_Y\xi_X = \xi_X\delta^{|Y|}$ hold trivially. Let $k > 0$ and proceed by induction on k. Take $\beta \in Y$ and set $Y' = Y \setminus \{\beta\}$. Clearly $e_{Y'}\xi_X \neq 0$ and $k - 1 = |Y'|$, so, by the induction hypothesis, $Y' \subseteq X$ and $e_{Y'}\xi_X = \xi_X\delta^{k-1}$, whence $e_Y\xi_X = e_\beta\xi_X\delta^{k-1}$. If $\beta \perp X$, then $e_Y\xi_X = 0$, a contradiction. If $\beta \sim \gamma \in X$, then $e_Y\xi_X = r_\gamma r_\beta \xi_X\delta^{k-1} \in \xi_X W(C)\delta^{k-1}$ contradicting the assumption $e_Y\xi_X \in \xi_X W(C)\delta^k$, so we must have $\beta \in X$. It follows that $Y = Y' \cup \{\beta\} \subseteq X$ and $e_Y\xi_X = \xi_X\delta^k$ as required for the only if part. For the converse use the case $|Y| = 1$ above repeatedly. This establishes the second assertion. ∎

Corollary 4.6: *Let $\mathcal{B}$ be an admissible W-orbit and $X, Y \in \mathcal{B}$. Then*

$$e_Y\xi_X \in \xi_Y W(C)\delta^k \cup \{0\},$$

where $k \leq |Y|$. Moreover, if $k = |Y|$ and $e_Y\xi_X \neq 0$, then $Y = X$ and $e_X\xi_X = \xi_X\delta^k$.

Proof. Suppose that $e_Y\xi_X \neq 0$. By Proposition 4.5 there are $Z \in W X$, $w \in W(C)$, and $k \in \mathbb{N}$ such that $e_Y\xi_X = w\xi_Z\delta^k$. Moreover, $Y \subseteq Z$ and $k \leq |Y|$. Since $Z \in \mathcal{B}$ we know that $|Y| = |X| = |Z|$. Thus $Y = Z$.

Suppose that $k = |Y|$. Then $Y \subseteq X$ by Proposition 4.5. Since $|Y| = |X|$ we conclude $Y = X$. ∎

For X a set of mutually orthogonal positive roots, define the **annihilator** of e_X, denoted A_X, to be

$$A_X = \{w \in W \mid we_X = e_X\}. \tag{7}$$

and the **centralizer** of e_X, denoted N_X, to be

$$N_X = \{w \in W \mid e_X w = we_X\}. \tag{8}$$

In view of Lemmas 3.3(iv) and 4.4, $N_W(X) \leq N_W(X^{\rm cl}) \leq N_X$. Also, by Proposition 4.5, $A_X \leq A_{X^{\rm cl}} \lhd N_X$. Some further properties of these subgroups are listed in the next proposition, the second item of which we could only prove by means of a case by case verification.

Before Lemma 3.5 we introduced the notation K_X for the kernel of the restriction of $\rho_{\mathcal{B}}$ to N_X on $\xi_X\mathbb{Z}[W(C), \delta^{\pm 1}]$.

Proposition 4.7: *Let X be the highest element in its W-orbit and put $C = C_{WX}$.*

(i) $N_X = N_W(X)$.

(ii) *The normal subgroup A_X of N_X coincides with K_X. It is generated by*

$$\{r_\beta, r_\alpha r_\beta r_\gamma r_\alpha \mid \alpha \in \Phi^+, \beta, \gamma \in X, \beta \sim \alpha \sim \gamma\}.$$

(iii) *N_X is the semi-direct product of A_X and $W(C)$.*

Proof. (i) Above, we observed that $N_W(X) \leq N_X$. By Proposition 4.5, $we_X = e_X w$ for $w \in W$ implies $\xi_{wX} \in \xi_X W(C)\delta^{\mathbb{Z}}$. Therefore, N_X leaves invariant the 1-dimensional subspace $\xi_X\mathbb{Z}[W(C), \delta^{\pm 1}]$ of $V_{\mathcal{B}}$. This proves $N_X \leq N_W(X)$.

(ii) If $w \in W$ satisfies $we_X = e_X$, then there is $h \in W(C)$ such that $\xi_{wX}h\delta^{\mathbb{Z}} = w\xi_X\delta^{\mathbb{Z}} = we_X\xi_X\delta^{\mathbb{Z}} = e_X\xi_X\delta^{\mathbb{Z}} = \xi_X\delta^{\mathbb{Z}}$. But then $wX = X$, so $w \in N_W(X) = N_X$ by (i), and $h = 1$. This proves that A_X is contained in K_X.

Let L_X be the subgroup of W with the generators specified in the assertion. If $\beta \in X$ then, by Lemma 4.3, $r_\beta e_X = r_\beta e_\beta e_X \delta^{-1} = e_\beta e_X \delta^{-1} = e_X$, so $r_b \in A_X$. Let $\alpha \in \Phi^+$ and assume β and γ in X are as stated. Then $r_\alpha r_\beta r_\gamma r_\alpha \in A_X$ by the first paragraph of the proof of Lemma 4.4. Hence L_X is contained in

A_X. Now L_X is a normal subgroup of $N_W(X)$ contained in K_X, so the product $L_X W(C)$ is a subgroup of N_X. A case by case analysis shows that the action of L_X induced on X coincides with the action of N_X. Also, by inspection of cases, for every root of $\beta \in \Phi$ orthogonal to X there is an element $u \in L_X$ with $ur_\beta \in W(C)$. This implies that L_X coincides with K_X and hence with A_X.

(iii) By (i) and (ii), this is a restatement of Corollary 3.7. ∎

LEMMA 4.8: *Let X be a set of mutually orthogonal positive roots, $w \in W$, and $\beta \in \Phi^+$.*

(i) *If $X \in \mathcal{A}$ and $w \in W$ is of minimal length in its coset wN_X, then $w\xi_X = \xi_{wX}$.*
(ii) *The product $e_\beta e_X$ can be expanded as follows:*

$$e_\beta e_X = \begin{cases} e_X\delta & \text{if } \beta \in X^{\mathrm{cl}}, \\ e_{X\cup\{\beta\}} & \text{if } \beta \perp X, \\ r_\gamma r_\beta e_X & \text{where } \gamma \in X, \text{ if } \beta \sim \gamma \in X. \end{cases}$$

Proof. (i) In a minimal expression $s_1 \cdots s_q$ of w as a product of simple reflections, each s_i will move $s_{i+1}\cdots s_q X$. Then $s_i \xi_{s_{i+1}\cdots s_q X} = \xi_{s_i\cdots s_q X}$.

(ii) If $\beta \in X^{\mathrm{cl}}$, the result follows from Lemma 4.4. If $\beta \perp X$, the assertion is immediate from the definition of $e_{\{\beta\}\cup X}$. Finally, suppose that there is some $\gamma \in X$ with $\beta \sim \gamma$. Then the assertion follows from Lemma 4.3(ii)(b). ∎

Let $\mathcal{A}_0$ be the set of highest elements from the W-orbits in $\mathcal{A}$. For $X \in \mathcal{A}_0$, let D_X be a set of right coset representatives for $N_X = N_W(X)$ in W. By convention, if $X = \emptyset$ we take $e_\emptyset$ to be the identity, $N_{W\emptyset}$ to be W, and $C_{W\emptyset}$ also to be W.

PROPOSITION 4.9: *Each element of the Brauer monoid* $\mathbf{BrM}(Q)$ *can be written in the form $ue_X zv\delta^k$, where $X \in \mathcal{A}_0$, $u, v^{-1} \in D_X$, $z \in W(C_{WX})$, and $k \in \mathbb{Z}$.*

Proof. By Lemma 4.8(ii), any expression of the form $e_\beta w e_{X'}$ with $\beta \in \Phi^+$, $w \in W$ and X' a set of mutually orthogonal positive roots, can be rewritten in the form $ve_Y\delta^k$ with $v \in W$, $Y \in \mathcal{A}$ and $k \in \mathbb{Z}$. Consequently, up to a power of δ, every element of $\mathbf{BrM}(Q)$ is equal to $w_1 e_X w_2^{-1}$ for some $X \in \mathrm{A}_0$ and $w_1, w_2 \in W$. Now, using Proposition 4.7(iii), write $w_1 = uy_1z_1$ and $w_2 = vy_2z_2$ with $u, v \in D_X$, $y_1, y_2 \in A_X$, and $z_1, z_2 \in W(C)$. Then $w_1 e_X w_2^{-1} =$

$uy_1e_Xz_1z_2^{-1}y_2^{-1}v^{-1} = ue_Xz_1z_2^{-1}y_2^{-1}v^{-1} = uz_1z_2^{-1}e_Xy_2^{-1}v^{-1} = uz_1z_2^{-1}e_Xv^{-1} = ue_Xz_1z_2^{-1}v^{-1}$. Taking $z = z_1z_2^{-1}$, we find an expression as required. ∎

Proof of Proposition 4.1. The dimension of $\mathbf{Br}(Q)$ is equal to the size of the quotient monoid $\mathbf{BrM}(Q)\langle\delta^{\pm 1}\rangle$, which, by Proposition 4.9, is at most

$$\sum_{X\in\mathcal{A}_0} |D_X|^2 \cdot |W(C_{WX})|.$$

The proposition follows as $|D_X| = |WX|$. ∎

Remark 4.10: To finish this section, we describe the usual Brauer diagram on n strands corresponding to $ue_Xzv\delta^k$ for $k \in \mathbb{N}$ when $Q = \mathrm{A}_{n-1}$. It contains k circles. The horizontal strands at the top are determined by uX in the following way: each root in uX is of the form $\varepsilon_i - \varepsilon_j$ in the standard representation of Φ^+, where each ε_t denotes the t-th standard basis vector of $\mathbb{R}^n$; in the diagram there is a corresponding horizontal strand from i to j. The bottom of the diagram is obtained by the same interpretation of $v^{-1}X$. Finally, the element z determines the vertical strands in terms of a permutation on the remaining nodes up to a translation from the highest root to X. See Remark 5.7 below on how to obtain it.

5. Irreducibility of representations and lower bounding the dimension

Corollary 4.6 allows us to find irreducible representations of the Brauer algebra $\mathbf{Br}(Q)$: in fact, one for each pair of a W-orbit $\mathcal{B}$ inside $\mathcal{A}$ and an irreducible representation of $W(C_\mathcal{B})$. This will enable us to find a lower bound for the dimension of $\mathbf{Br}(Q)$, which together with Proposition 4.1 gives the exact dimension. Fix a W-orbit $\mathcal{B}$ inside $\mathcal{A}$ and recall the notation $\rho_\mathcal{B}$ from Theorem 3.6(ii). We shall often abbreviate $V_\mathcal{B}$ and $C_\mathcal{B}$ to V and C, respectively, where $V_\mathcal{B}$ was defined just above Lemma 3.4.

Proposition 5.1: *Suppose that $v = \sum \xi_B w\lambda_{B,w}$ is a nonzero element of V where the sum is over all $w \in W(C)$ and over all $B \in \mathcal{B}$. Then there is some $Y \in \mathcal{B}$ for which $e_Yv \neq 0$.*

Proof. Suppose that $e_Y v = 0$ for all $Y \in \mathcal{B}$. By Proposition 4.5 there are coefficients $T_{Y,u;B,w} \in \{0\} \cup \delta^{\mathbb{Z}}$, where $Y, B \in \mathcal{B}$ and $u, w \in W(C)$, such that

$$e_Y \xi_B w = \sum_{u \in W(C)} \xi_Y u T_{Y,u;B,w}.$$

After an ordering of $\mathcal{B} \times W(C)$, the coefficients $T_{Y,u;B,w}$ can be considered entries of a square matrix, T, over $\mathbb{Q}(\delta)$ whose rows and columns are both indexed by the pairs in $\mathcal{B} \times W(C)$.

Let λ be the column vector with entries $\lambda_{B,w}$ indexed in the same order as used for T. Now

$$\begin{aligned} 0 = e_Y v = \sum e_Y \xi_B w \lambda_{B,w} &= \sum_{u \in W(C)} \sum_{B,w} \xi_Y u T_{Y,u;B,w} \lambda_{B,w} \\ &= \sum_{u \in W(C)} \xi_Y u (T\lambda)_{Y,u}. \end{aligned}$$

As this equality holds for all $(Y, u) \in \mathcal{B} \times W$, we find $T\lambda = 0$. By Corollary 4.6, the exponent of δ in an entry $T_{Y,u;B,w}$ of T is $|B|$ on the diagonal as $e_B \xi_B w = \delta^{|B|} w$, whereas, at nonzero off-diagonal entries, only lower powers of δ occur. Consequently, $\det(T)$ is a nonzero element of $\mathbb{Q}[\delta^{\pm 1}]$. This means that T is nonsingular over the field $\mathbb{Q}(\delta)$, and so $T\lambda = 0$ implies $\lambda = 0$, that is, $v = 0$, a contradiction. Hence the proposition. ∎

PROPOSITION 5.2: *Suppose that U is the regular representation space of $W(C)$ over $\mathbb{Q}(\delta)$ and U_1 is an invariant subspace of U for $W(C)$. Then $\sum_{B \in \mathcal{B}} \xi_B U_1$ is an invariant subspace of $V \otimes_{\mathbb{Z}[\delta^{\pm 1}]} \mathbb{Q}(\delta)$ for $\mathbf{Br}(Q)$.*

Proof. This follows from the actions of r_i and e_i on $\xi_B u$ for $u \in U$. In each case the result is of the form 0 or $\xi_Y w u$ with $w \in W(C_{\mathcal{B}})$, and so if $u \in U_1$ then so is wu_1. ∎

PROPOSITION 5.3: *If U_1 is an irreducible invariant subspace of the regular $W(C)$-representation space U over $\mathbb{Q}(\delta)$, then the representation over $\mathbb{Q}(\delta)$ of $\mathbf{Br}(Q)$ on $V_1 = \sum_{B \in \mathcal{B}} \xi_B U_1$ of Proposition 5.2 is irreducible. Moreover, if U_1 is absolutely irreducible, then so is V_1.*

Proof. Let v be a nonzero vector in $V_1 = \sum_{B \in \mathcal{B}} \xi_B U_1$. We know from Proposition 5.1 that there is a $B \in \mathcal{B}$ for which $e_B v \neq 0$. Suppose that v is a nonzero element of an invariant subspace of V_1. Then this subspace also contains $e_B v$, which, by Corollary 4.6, is equal to $\xi_B u_1$ for some nonzero $u_1 \in U_1$. If the representation is irreducible, then, by letting $W(C_{\mathcal{B}})$ act, we can obtain all of $\xi_B U_1$ in U_1. As W is transitive on ξ_Y for $Y \in \mathcal{B}$, the invariant subspace U_1 contains V_1 and so coincides with V_1. Therefore, the representation is irreducible.

Since the above argument works for any extension field of $\mathbb{Q}(\delta)$, the second assertion of the theorem also follows. ∎

PROPOSITION 5.4: *The irreducible representations obtained in Proposition 5.3 are not equivalent.*

Proof. Suppose that U_1 and U_2 are inequivalent irreducibles of $W(C_{\mathcal{B}})$. These occur as subspaces of the regular representation space U of Proposition 5.3. Consider now the irreducible representations of $\mathbf{Br}(Q)$ obtained in Proposition 5.3 for U_1 and U_2, respectively. When restricted to $W(C_{\mathcal{B}})$, these representations are $|\mathcal{B}|U_1$ and $|\mathcal{B}|U_2$, which are inequivalent. Therefore, they cannot be equivalent. ∎

Proof of Theorem 1.1. The above shows that, for each irreducible representation τ of $W(C_{\mathcal{B}})$, there is an irreducible representation $\rho_{\mathcal{B}} \otimes \tau$ of

$$\mathbf{Br}(Q) \otimes_{\mathbb{Z}[\delta^{\pm 1}]} \mathbb{Q}(\delta)[W(C_{\mathcal{B}})].$$

In particular, the algebra $\mathbf{Br}(Q) \otimes_{\mathbb{Z}[\delta^{\pm 1}]} \mathbb{Q}(\delta)$ maps homomorphically onto a direct sum of matrix algebras of dimensions $|\mathcal{B}|\tau(1)$ over $\mathbb{Q}(\delta)$ for $\mathcal{B}$ running over the admissible W-orbits in $\mathcal{A}$ and τ over the irreducible representations of $W(C_{\mathcal{B}})$. Therefore, $\dim(\mathbf{Br}(Q)) \geq \sum_{\mathcal{B},\tau} |\mathcal{B}|^2\tau(1)^2 = \sum_{\mathcal{B}} |\mathcal{B}|^2|W(C_{\mathcal{B}})|$. In Proposition 4.1, this number was proved to be an upper bound for $\dim(\mathbf{Br}(Q))$, so, in view of Lemma 1.3, the homomorphism onto a direct sum of matrix algebras is an isomorphism and $\mathbf{Br}(Q) \otimes_{\mathbb{Z}[\delta^{\pm 1}]} \mathbb{Q}(\delta)$ is split semisimple, so Theorem 1.1 is proved. ∎

With the notation of Proposition 4.9 and as an immediate consequence of this proposition and the theorem, we have the following two corollaries.

COROLLARY 5.5: *For $Q \in$ ADE, the Brauer algebra $\mathbf{Br}(Q)$ over $\mathbb{Z}[\delta^{\pm 1}]$ has a basis of the form $ue_X zv$ for $X \in \mathcal{A}_0$, $u, v^{-1} \in D_X$, and $z \in W(C_{WX})$.*

COROLLARY 5.6: *For $Q \in$ ADE, the Brauer algebra $\mathbf{Br}(Q) \otimes_{\mathbb{Z}[\delta^{\pm 1}]} \mathbb{Q}(\delta)$ over $\mathbb{Q}(\delta)$ is a direct sum of matrix algebras of size $|\mathcal{B}| \cdot \tau(1)$ for $(\mathcal{B}, \tau)$ running over* *all pairs of a W-orbit $\mathcal{B}$ inside $\mathcal{A}$ and an irreducible representation τ of $W(C_{\mathcal{B}})$. The irreducibles are indexed by the irreducibles of $W(C_{\mathcal{B}})$ over all $\mathcal{B}$.*

Remark 5.7: We finish this section by describing how to compute from a Brauer monomial $a \in \mathbf{BrM}(Q)$ the triple (L, R, z) consisting of two elements L, R of the same W-orbit $\mathcal{B} = WX$ inside $\mathcal{A}$, where $X \in \mathcal{A}_0$, and of the element $z \in W(C_{\mathcal{B}})$ for which $a = ue_X zv\delta^k$ as in Proposition 4.9 with $L = uX$ and $Y = vX$. First compute $L = a(\emptyset)$ and $R = a^{\mathrm{op}}(\emptyset)$, where a^{op} is the element of $\mathbf{BrM}(Q)$ obtained by reading backwards an expression of a as a word in the generators (this element is well defined as the operation $\cdot^{\mathrm{op}}$ is an anti-involution; see [4] or note that the set of relations shown in Tables 1 and 4 is invariant under opposition). As a consequence of Proposition 4.9, L and R belong to the same W-orbit inside $\mathcal{A}$. Let $X \in \mathcal{A}_0$ be the highest element of this orbit. Pick $u, v^{-1} \in D_X$ such that $L = uX$ and $R = v^{-1}X$. Now compute $u^{-1}av^{-1}\xi_X$. The result will be an element of the form $\xi_X z\delta^s$ for some $s \in \mathbb{Z}$ and $z \in W(C_{WX})$. Then $a = ue_X zv\delta^k$ with $k = s - |X|$, as required. As discussed in Remark 4.10, for $Q = \mathrm{A}_{n-1}$, the sets L and R determine the horizontal strands at the top and bottom, respectively, of the corresponding Brauer diagram, whereas z determines the permutation corresponding to the vertical strands of the diagram. In view of Corollary 5.5, these triples may be thought of as the abstract Brauer diagrams for any $Q \in \mathrm{ADE}$. For $Q = \mathrm{D}_n$, there is a diagrammatic description of $\mathbf{BrM}(\mathrm{D}_n)$ in [6].

6. Cellularity

In this section we prove Theorem 1.2, which states that $\mathbf{Br}(Q) \otimes_{\mathbb{Z}[\delta^{\pm 1}]} S$ is cellular in the sense of Graham–Lehrer in [10, Definition 1.1] provided the coefficient ring S is as specified in the theorem. Recall from [10] that an associative algebra A over a commutative ring S is cellular if there is a quadruple $(\Lambda, T, C, *)$ satisfying the following three conditions.

(C1) Λ is a finite partially ordered set. Associated to each $\lambda \in \Lambda$, there is a finite set $T(\lambda)$. Also, C is a map from $T(\lambda) \times T(\lambda)$ to A. It satisfies

$$C : \coprod_{\lambda \in \Lambda} T(\lambda) \times T(\lambda) \to A$$

is an injective map whose image is an S-basis of A.

(C2) If $\lambda \in \Lambda$ and $s', t' \in T(\lambda)$, write $C(s',t') = C^\lambda_{s',t'} \in A$. Then $* : A \to A$ is an S-linear anti-involution such that $(C^\lambda_{s',t'})^* = C^\lambda_{t',s'}$.

(C3) If $\lambda \in \Lambda$ and $s', t' \in T(\lambda)$, then, for any element $a \in A$, we have

$$aC^\lambda_{s',t'} \equiv \sum_{u' \in T(\lambda)} r_a(u',s') C^\lambda_{u',t'} \mod A(< \lambda),$$

where $r_a(u', s') \in S$ is independent of t' and where $A(< \lambda)$ is the S-submodule of A spanned by $\{C^\mu_{s'',t''} \mid \mu < \lambda \ ; \ s'', t'' \in T(\mu)\}$.

Such a quadruple will be called a cell datum for A. Now let Q be a fixed diagram of type ADE and consider $A = \mathbf{Br}(Q) \otimes_{\mathbb{Z}[\delta^{\pm 1}]} S$. We introduce a quadruple $(\Lambda, T, C, *)$ and prove that it is a cell datum for A. The map $*$ on A will be the opposition map $\cdot^{\mathrm{op}}$ on A that linearly extends the opposition map of Remark 5.7. As discussed in Section 5, it is an anti-automorphism of A as it preserves the defining relations.

By Corollary 5.5, the Brauer algebra A over S has a basis of the form $ue_X zv$ for $X \in \mathcal{A}_0$, $u, v^{-1} \in D_X$, and $z \in W(C_{WX})$. Recall $\mathcal{A}_0$ is the set of highest elements from the W-orbits in the admissible sets, $\mathcal{A}$. Also, D_X is a set of right coset representatives for $N_X = N_W(X)$ in W.

The groups $W(C_{WX})$ are all Weyl groups of type ADE or direct products. For $X = \emptyset$ this is the Weyl group of type Q. For the others they are the Weyl groups of types $C_{\mathcal{B}}$ appearing in the fouth column of Table 3. As the coefficient ring S satisfies the conditions of [9, Theorem 1.1], this implies by [9, Corollary 3.2] that the group rings $S[W(C_{WX})]$ are all cellular. Each is a subalgebra of A.

Let $(\Lambda_X, T_X, C_X, *_X)$ be a cell datum for $S[W(C_{WX})]$. Note that by [9, Section 3], $*_X$ is the map $\cdot^{\mathrm{op}}$ on $S[W(C_{WX})]$ and so $*_X$ is the restriction of $* = \cdot^{\mathrm{op}}$ to $S[W(C_{WX})]$.

We now define a cell datum, $(\Lambda, T, C, *)$, for A. The underlying set of the poset Λ will be the disjoint union of all Λ_X over all admissible sets $X \in \mathcal{A}_0$. We make Λ into a poset as follows. For a fixed X, we keep the partial order within Λ_X. If X, Y are two admissible sets, we say $X > Y$ if and only if some element in WX is properly contained in some element of WY. If $X > Y$ we order all elements of Λ_X greater than all elements of Λ_Y. In particular, the elements of $\Lambda_\emptyset$ are greater than the elements of Λ_X for any $X \neq \emptyset$. No further pairs of Λ are ordered.

The set $T(X)$ is the set of pairs (u,s) for $u \in D_X$ and $s \in T_X$. The map C is given by $C((u,s),(v,t)) = uC_X(s,t)v^{-1}$. The union of these over all $u \in D_X$ and all $s \in T_X$ is a basis by Corollary 5.5 and the fact that the set $C(s,t)$ over all s,t in T_X is a basis for $S[W(C_{WX})]$ by (C1) of the cellularity of $S[W(C_{WX})]$. This is (C1) for $(\Lambda, T, C, *)$.

For (C2) notice $\left(uC_X(s,t)v^{-1}\right)^{\mathrm{op}} = v(C_X(s,t))^{\mathrm{op}}u^{-1}$. Now $(C_X(s,t))^{\mathrm{op}} = C_X(t,s)$ by the cellularity condition (C2) for $S[W(C_{WX})]$ and so (C2) holds for the cell datum $(\Lambda, T, C, *)$.

We have now only to check condition (C3) for $(\Lambda, T, C, *)$. For this we need to consider $r_i u e_X C_X(s,t)v^{-1}$ and $e_i u e_X C_X(s,t)v^{-1}$, where $u, v \in D_X$ and $s,t \in T_X$. It follows from Section 4 that $r_i u e_X = u' e_X z$ for some $z \in W(C_{WX})$ and $u' \in D_X$ and that $e_i u e_X = u' e_X \delta^k z$ or $u' e_{X'} \delta^k z v'$ for some $k \in \mathbb{N}$, $z \in W(C_{WX})$, $X' < X$, and $u', v' \in D_{X'}$. In the latter case the expression does not depend on the pair (v,t) and is equal to 0 modulo lower terms in

$$\{C^{X'}_{s'',t''} \mid X' < X \ ; \ s'', t'' \in T(X')\}.$$

We need then just check the (C3) condition for $u' e_X z$. But by the (C3) condition for the cellularity of $S[W(C_{WX})]$ we get

$$zC^X_{s,t} \equiv \sum_{u' \in T(X)} r_z(u',s) C^X_{u',t} \mod A(< X),$$

where $r_z(u',s) \in S$ is independent of t. Hence the condition (C3) holds for $(\Lambda, T, C, *)$.

This establishes that $(\Lambda, T, C, *)$ is a cell datum for A and so completes the proof of Theorem 1.2.

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