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THE LOWER DIMENSIONAL BUSEMANN-PETTY PROBLEM FOR BODIES WITH THE GENERALIZED AXIAL SYMMETRY*

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ABSTRACT

The lower dimensional Busemann-Petty problem asks, whether n-dimensional centrally symmetric convex bodies with smaller i-dimensional central sections necessarily have smaller volumes. For $i = 1$, the affirmative answer is obvious. If $i > 3$, the answer is negative. For $i = 2$ or $i = 3$ $(n > 4)$, the problem is still open, however, when the body with smaller sections is a body of revolution, the answer is affirmative. The paper contains a solution to the problem in the more general situation, when the body with smaller sections is invariant under rotations, preserving mutually orthogonal subspaces of dimensions ℓ and $n - \ell$, respectively, so that $i+\ell \le n$. The answer essentially depends on ℓ. The argument relies on the notion of canonical angles between subspaces, spherical Radon transforms, properties of intersection bodies, and the generalized cosine transforms.

1. Introduction

Let $G_{n,i}$ be the Grassmann manifold of i-dimensional linear subspaces of $\mathbb{R}^n$, and let $\mathrm{vol}_i(\cdot)$ denote the i-dimensional volume function.

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QUESTION: Suppose that i is fixed, and let A and B be arbitrary origin-symmetric (o.s.) convex bodies in $\mathbb{R}^n$ satisfying

$$\text{vol}_i(A \cap \xi) \le \text{vol}_i(B \cap \xi) \quad \text{for every } \xi \in G_{n,i}. \tag{1.1}$$

Does it follow that

$$\text{vol}_n(A) \le \text{vol}_n(B)\ ? \tag{1.2}$$

This question generalizes the celebrated Busemann-Petty problem, corresponding to $i = n-1$ [BP]. The latter has a long history; see, e.g., [Ba, BFM, Ga1, Ga2, GKS, Gi, Ha], [K1]–[K4], [LR, Lu, Pa, R3, Z2]. The answer is really striking. It is "Yes" if and only if $n \le 4$; see [Ga3, GKS, K4, KY], and references therein. For $1 \le i \le n-2$, the problem is even more intriguing. We call it *the lower dimensional Busemann-Petty problem* (LDBP). If $i = 1$, the implication (1.1) → (1.2) is obvious for all o.s. star bodies without any convexity assumption. In the case $i = 2$, $n = 4$, an affirmative answer follows from the solution of the usual Busemann-Petty problem. For $3 < i \le n-1$, a negative answer was given by Bourgain and Zhang [BZ] and Koldobsky [K4]; see also [RZ]. In the cases $i = 2$ and $i = 3$ for $n > 4$, the answer is generally unknown, however, if the body with smaller sections is a body of revolution, the answer is affirmative; see [GZ], [Z1], [RZ]. It is also known [BZ] that when $i = 2$ and B is a Euclidean ball, the answer is affirmative provided that A is convex and sufficiently close to B. On the other hand [Mi2], for $i = 2$ or $i = 3$, there is a small perturbation A of a Euclidean ball, so that the implication (1.1) → (1.2) is true for arbitrary o.s. star body B. Modifications of the Busemann-Petty problem were studied in [CG, K4, KKZ, KYY, RZ, Y, Schu, Zv]; see also [Ga3], where one can find further references.

It is worth noting, that one of the ways to solve the original Busemann-Petty problem is to go through rotation invariant bodies. To the best of my knowledge, the first publication in this direction is due to Hadwiger [Ha] who considered a slightly more general problem for bodies of revolution in $\mathbb{R}^3$. An important breakthrough was made by Gardner [Ga1] who gave a positive answer to the Busemann-Petty problem for arbitrary origin-symmetric bodies in $\mathbb{R}^3$, and invoked bodies of revolution in his solution. First counter-examples for $\mathbb{R}^5$ were also based on rotation invariance [Ga2, Pa]. It is natural to expect that studying bodies which are invariant under more general groups of orthogonal transformations will lead to new results and bring new light to the lower dimensional Busemann-Petty problem. We investigate this conjecture in the present article.

Main results. We give a solution to the lower dimensional Busemann-Petty problem stated above, when the body with smaller sections is invariant under orthogonal transformations preserving mutually orthogonal subspaces, say, p and $p^\perp$, of dimensions ℓ and $n-\ell$ satisfying $i+\ell \le n,\ 1 \le \ell < n$ (the restriction $i+\ell \le n$ is discussed below in 2^0). Let us choose the coordinate system in $\mathbb{R}^n$ so that $p = \mathbb{R}^\ell = \bigoplus_{j=n-\ell+1}^{n} \mathbb{R}e_j$ and $p^\perp = \mathbb{R}^{n-\ell} = \bigoplus_{j=1}^{n-\ell} \mathbb{R}e_j$, where $e_1, e_2, \dots, e_n$ are the relevant coordinate unit vectors. Without loss of generality, we assume $\ell \le n-\ell$, i.e., $\ell \le n/2$ (otherwise, the coordinate subspaces can be renamed). The case $\ell = 1$ corresponds to bodies of revolution.

Consider the subgroup of orthogonal transformations

$$K_\ell = \left\{ \gamma \in O(n) : \gamma = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad \alpha \in O(n-\ell),\ \beta \in O(\ell) \right\}. \tag{1.3}$$

A star body A is K_ℓ-symmetric if $\gamma A = A$ for all $\gamma \in K_\ell$. Clearly, every K_ℓ -symmetric body A is origin-symmetric, that is $A = -A$. We set

$$x = (x', x'') \in \mathbb{R}^n, \quad x' \in \mathbb{R}^{n-\ell}, \quad x'' \in \mathbb{R}^\ell.$$

Every K_ℓ-symmetric body in $\mathbb{R}^n$ can be obtained, for instance, if we take a 2-dimensional body, which is symmetric with respect to coordinate axes in the plane (e_1, e_n), and rotate it about the subspaces $\mathbb{R}^\ell$ and $\mathbb{R}^{n-\ell}$. A typical example is the (q,ℓ)-ball

$$B^n_{q,\ell} = \{x : |x'|^q + |x''|^q \le 1\}, \quad q > 0, \tag{1.4}$$

where $|\cdot|$ denotes the Euclidean norm.

The basic idea of our approach is the following. We observe, that the relative position of two subspaces, say, $\xi \in G_{n,i}$ and $\eta \in G_{n,\ell}$, is determined by $m = \min(i,\ell)$ canonical angles $\omega_1, \dots, \omega_m$. This geometrical fact represents an important tool in diverse problems related to eigenvalues of matrices [C] and in statistics [J]. It was shown [OR] that every K_ℓ-invariant function f on $G_{n,i}$ is completely determined by a function f_0 of $m = \min(i,\ell)$ canonical angles, provided that $i + \ell \le n$.

We define

$$G^\ell_{n,i} = \{\xi \in G_{n,i} : \omega_1 = \cdots = \omega_m\} \tag{1.5}$$

to be the collection of all $\xi \in G_{n,i}$ such that all canonical angles between ξ and $\mathbb{R}^\ell$ are equal. The structure of the set $G^\ell_{n,i}$ can be understood as follows. Let $\lambda_1 = \cos^2\omega_1, \ldots, \lambda_m = \cos^2\omega_m$. These are eigenvalues of the positive semi-definite matrix

$$r = \begin{cases} \tau' \mathrm{P}_{\mathbb{R}^\ell} \tau & \text{if } i \le \ell \ , \\ \sigma' \mathrm{P}_\xi \sigma & \text{if } i > \ell, \end{cases} \tag{1.6}$$

where τ and σ denote arbitrarily fixed orthonormal frames which span ξ and $\mathbb{R}^\ell$, respectively; τ', σ', $\mathrm{P}_{\mathbb{R}^\ell}$, and P_ξ stand for the corresponding transposed matrices and orthogonal projections. We arrange $\lambda_1, \ldots, \lambda_m$ in nonincreasing order and regard $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ as a point of the simplex

$$\Lambda_m = \{\boldsymbol{\lambda} : 1 \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 0\}. \tag{1.7}$$

The edge $\{\lambda_1 = \cdots = \lambda_m\}$ of this simplex corresponds to $G^\ell_{n,i}$.

Our main results are the following.

Theorem 1.1: *Let $1 \le \ell \le n/2$, $i + \ell \le n$, and let A be a K_ℓ-symmetric star body in $\mathbb{R}^n$.*

(a) *If $1 \le i \le \ell$, then the implication*

$$\mathrm{vol}_i(A \cap \xi) \le \mathrm{vol}_i(B \cap \xi) \quad \textit{for every } \xi \in G^\ell_{n,i} \Longrightarrow \mathrm{vol}_n(A) \le \mathrm{vol}_n(B) \tag{1.8}$$

is true for every o.s. star body B.

(b) *If $i = \ell + 1$ or $i = \ell + 2$, then* (1.8) *holds for every o.s. star body B provided that A is convex.*

Theorem 1.2: *If $i > \ell + 2$, and $B = B^n_{4,\ell} = \{x : |x'|^4 + |x''|^4 \le 1\}$, then there is an infinitely smooth K_ℓ-symmetric convex body A, such that $\mathrm{vol}_i(A \cap \xi) \le \mathrm{vol}_i(B \cap \xi)$ for all $\xi \in G_{n,i}$, but $\mathrm{vol}_n(A) > \mathrm{vol}_n(B)$.*

Some comments are in order.

1^0. It might be surprising, that to make a positive conclusion in Theorem 1.1, we do not need *all* i-dimensional central sections, as suggested in the original problem. It suffices to consider only sections having equal canonical angles with respect to $\mathbb{R}^\ell$. More advantages of our *method of canonical angles* are described in Remark 2.6.

2^0. The condition $i + \ell \le n$ in Theorem 1.1 excludes the situation when $\dim(\xi \cap \mathbb{R}^\ell) \ge 1$ for all $\xi \in G_{n,i}$; see Remark 2.4. We actually assume

in (a): $i \le \min(\ell, n - \ell)$;

in (b): $\ell \le (n-1)/2$, if $i = \ell + 1$, and $\ell \le (n-2)/2$, if $i = \ell + 2$. Regarding (a), the situation, when inequalities $i + \ell > n$ and $i \le \ell$ hold simultaneously, is impossible, because in this case $\ell > n/2$, that contradicts our initial convention. Regarding (b), a simple examination shows that the following cases, which are admissible when $i + \ell > n$, are not presented in Theorem 1.1:

(i) $n = 2\ell$, when $i = \ell + 1$;
(ii) $n = 2\ell$ and $n = 2\ell + 1$, when $i = \ell + 2$;

The validity of the implication (1.1) → (1.2) in (i) and (ii) is an open problem. After several attempts to attack it, we have got an impression that the difficulties here have the same nature as those in the original LDBP for $i = 2$ or 3.

3^0. Another intriguing open problem is to check the following

Conjecture: *In the case (b) of Theorem 1.1, i.e., when* $i = \ell + 1$ *or* $i = \ell + 2$, *there exist a nonconvex* K_ℓ*-symmetric body* A *and an o.s. star body* B *so that*

$$\mathrm{vol}_i(A \cap \xi) \le \mathrm{vol}_i(B \cap \xi) \quad \textit{for all } \xi \in G_{n,i} \ \textit{(not only for } \xi \in G^\ell_{n,i}),$$

but $\mathrm{vol}_n(A) > \mathrm{vol}_n(B)$; *cf.* [Ga3, Theorem 8.2.4] *for* $n = 3$, $\ell = 1$.

The paper is organized as follows. In Section 2, we obtain new lower dimensional representations for the spherical Radon transform of K_ℓ-invariant functions; see Theorem 2.2 and Corollary 2.3. These results are used in Section 3 to prove Theorem 1.1. Theorem 1.2 is proved in Section 4, where we invoke some facts on intersection bodies and the generalized cosine transforms. The concept of intersection body was introduced by Lutwak [Lu] and extended by Zhang [Z1] and Koldobsky [K2] to lower dimensional sections. Useful information about these objects can be found in [K4], [Mi1], [R4].

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Notation: We use the standard notation $O(n)$ and $SO(n)$ for the orthogonal group and the special orthogonal group of $\mathbb{R}^n$ endowed with the invariant probability measure. For $1 \le i < n$, we denote by $G_{n,i}$ the Grassmann manifold of i-dimensional subspaces ξ of $\mathbb{R}^n$; $d\xi$ stands for the $O(n)$-invariant probability measure on $G_{n,i}$; S^{n-1} is the unit sphere in $\mathbb{R}^n$; $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of S^{n-1}; $e_1, e_2, \dots, e_n$ denote the coordinate unit vectors; $\mathfrak{M}_{n,i}$ is the space of real matrices having n rows and i columns. For $X \in \mathfrak{M}_{n,i}$, X' denotes the transpose of X, I_i is the identity $i \times i$ matrix;

$$V_{n,i} = \{\tau \in \mathfrak{M}_{n,i} : \tau'\tau = I_i\} = O(n)/O(n-i)$$

is the Stiefel manifold of orthonormal i-frames in $\mathbb{R}^n$. For $\tau \in V_{n,i}$, $\{\tau\}$ denotes the i-dimensional subspace spanned by τ. All vectors in $\mathbb{R}^n$ are interpreted as column-vectors.

2. The Spherical Radon Transform of K_ℓ-Invariant Functions

For functions $f(\theta)$ on S^{n-1} and $\varphi(\xi)$ on $G_{n,i}$, we define the spherical Radon transform $(R_i f)(\xi)$ and its dual $(R_i^* \varphi)(\theta)$ by

$$(2.1) \qquad (R_i f)(\xi) = \int_{S^{n-1}\cap\xi} f(\theta)\, d_\xi \theta, \qquad (R_i^*\varphi)(\theta) = \int_{\xi \ni \theta} \varphi(\xi)\, d_\theta \xi,$$

where measures $d_\xi\theta$ and $d_\theta\xi$ are normalized so that $R_i 1 = \sigma_{i-1}$ and $R_i^* 1 = 1$. The corresponding duality relation has the form

$$(2.2) \qquad \frac{1}{\sigma_{i-1}} \int_{G_{n,i}} (R_i f)(\xi)\varphi(\xi) d\xi = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f(\theta)(R_i^*\varphi)(\theta) d\theta$$

and is applicable whenever either side is finite for f and φ replaced by $|f|$ and $|\varphi|$, respectively; see [He], [R2].

In this section we obtain explicit "lower dimensional" expressions for $R_i f$ when f is K_ℓ-invariant. We remind that

$$(2.3) \qquad \mathbb{R}^n = \mathbb{R}^{n-\ell} \oplus \mathbb{R}^\ell, \quad \mathbb{R}^{n-\ell} = \bigoplus_{j=1}^{n-\ell} \mathbb{R}e_j, \quad \mathbb{R}^\ell = \bigoplus_{j=n-\ell+1}^{n} \mathbb{R}e_j,$$

$1 \le \ell \le n-1$, and set

$$(2.4) \qquad \sigma = [e_{n-\ell+1}, \dots, e_n] = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix}.$$

Every $\theta \in S^{n-1}$ is represented in bi-spherical coordinates as

$$(2.5) \qquad \theta = \begin{bmatrix} u \sin\omega \\ v\cos\omega \end{bmatrix}, \quad u \in S^{n-\ell-1}, \quad v \in S^{\ell-1}, \quad 0 \le \omega \le \frac{\pi}{2},$$

so that $d\theta = \sin^{n-\ell-1}\omega \cos^{\ell-1}\omega \, du dv d\omega$; see, e.g., [VK]. Clearly, $\cos^2\omega = \theta'\sigma\sigma'\theta = \theta' \mathrm{P}_{\mathbb{R}^\ell}\theta$, where $\mathrm{P}_{\mathbb{R}^\ell}$ denotes the orthogonal projection onto $\mathbb{R}^\ell$. The following statement is an immediate consequence of (2.5).

LEMMA 2.1: *A function f on S^{n-1} is K_ℓ-invariant if and only if there is a function f_0 on $[0,1]$ such that $f(\theta) = f_0(t)$, where $t^{1/2} = (\theta' \mathrm{P}_{\mathbb{R}^\ell}\theta)^{1/2}$ is the cosine of the angle between the unit vector θ and the coordinate subspace $\mathbb{R}^\ell$. Moreover,*

$$\int_{S^{n-1}} f(\theta)\, d\theta = c \int_0^{\pi/2} \sin^{n-\ell-1}\omega \cos^{\ell-1}\omega \, f_0(\cos^2\omega)\, d\omega$$

$$= \frac{c}{2} \int_0^1 t^{\ell/2-1}(1-t)^{(n-\ell)/2-1} f_0(t)\, dt, \quad c = \sigma_{\ell-1}\sigma_{n-\ell-1}. \tag{2.6}$$

THEOREM 2.2: *Let $1 \le i, \ell \le n-1$; $m = \min(i,\ell)$. Let $\omega_1, \dots, \omega_m$ be canonical angles between the subspace $\xi \in G_{n,i}$ and the coordinate plane $\mathbb{R}^\ell$,*

$$\boldsymbol{\lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_m), \quad \lambda_1 = \cos^2\omega_1, \dots, \lambda_m = \cos^2\omega_m. \tag{2.7}$$

Suppose that f is a K_ℓ-invariant function on S^{n-1}, so that $f(\theta) = f_0(t)$, $t = \cos^2\omega$, where ω is the angle between θ and $\mathbb{R}^\ell$. Then the Radon transform $R_i f$ has the form $(R_i f)(\xi) = F(\boldsymbol{\lambda})$, where

$$F(\boldsymbol{\lambda}) = \frac{\sigma_{i-\ell-1}}{2} \int_{S^{\ell-1}} \frac{dv}{(v'\boldsymbol{\lambda}v)^{i/2-1}} \int_0^{v'\boldsymbol{\lambda}v} t^{\ell/2-1}(v'\boldsymbol{\lambda}v - t)^{(i-\ell)/2-1} f_0(t)\, dt \tag{2.8}$$

if $i > \ell$, and

$$F(\boldsymbol{\lambda}) = \int_{S^{i-1}} f_0(v'\boldsymbol{\lambda}v)\, dv \tag{2.9}$$

if $i \le \ell$.

Proof. We set

$$p_i = \begin{bmatrix} I_i \\ 0 \end{bmatrix} \in V_{n,i}, \quad \{p_i\} = \bigoplus_{j=1}^{i} \mathbb{R}e_j, \quad \sigma = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} \in V_{n,\ell},$$

and let $\rho_\xi \in SO(n)$ be a rotation that takes the subspace $\{p_i\}$ to $\xi \in G_{n,i}$. Then (set $\theta = \rho_\xi \eta$)

$$(R_i f)(\xi) = \int_{S^{n-1}\cap\xi} f_0(\theta'\sigma\sigma'\theta)\, d_\xi\theta = \int_{S^{i-1}} f_0(\eta'\rho'_\xi\sigma\sigma'\rho_\xi\eta)\, d\eta,$$

S^{i-1} being the unit sphere in $\{p_i\}$. Let

$$u = \rho'_\xi \sigma = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in V_{n,\ell}, \quad u_1 = p'_i u = p'_i \rho'_\xi \sigma \in \mathfrak{M}_{i,\ell}, \quad u_2 \in \mathfrak{M}_{n-i,\ell}. \tag{2.10}$$

Then $\eta' u = \eta' u_1$, and we have

$$(R_i f)(\xi) = \int_{S^{i-1}} f_0(\eta' u u' \eta)\, d\eta = \int_{S^{i-1}} f_0(\eta' u_1 u'_1 \eta)\, d\eta. \tag{2.11}$$

Consider the case $\ell < i$ and write u_1 in the form (cf. [Mu, p. 589])

$$u_1 = \gamma p_\ell\, r^{1/2}, \quad \gamma \in SO(i), \quad p_\ell = \begin{bmatrix} I_\ell \\ 0 \end{bmatrix} \in V_{i,\ell},$$

where r is a positive semi-definite $\ell \times \ell$ matrix defined by

$$r = u'_1 u_1 = u' p_i p'_i u = \sigma' \rho_\xi p_i p'_i \rho'_\xi \sigma = \sigma' \mathrm{P}_\xi \sigma. \tag{2.12}$$

Hence,

$$(R_i f)(\xi) = \int_{S^{i-1}} f_0(\eta' \gamma p_\ell\, r p'_\ell\, \gamma' \eta)\, d\eta = \int_{S^{i-1}} f_0(\zeta' p_\ell\, r p'_\ell\, \zeta)\, d\zeta.$$

Since $\ell < i$, then $\{p_\ell\} \subset \{p_i\}$, and we can write ζ in bi-spherical coordinates

$$\zeta = \begin{bmatrix} v \cos\psi \\ w \sin\psi \end{bmatrix}, \quad v \in S^{\ell-1}, \quad w \in S^{i-\ell-1}, \quad 0 \le \psi \le \frac{\pi}{2},$$

so that $d\zeta = \cos^{\ell-1}\psi \, \sin^{i-\ell-1}\psi\, dv dw d\psi$. This gives $p'_\ell\, \zeta = v\cos\psi$, and therefore,

$$\begin{aligned}(R_i f)(\xi) &= \sigma_{i-\ell-1} \int_{S^{\ell-1}} dv \int_0^{\pi/2} f_0(v' r v \cos^2\psi) \cos^{\ell-1}\psi\, \sin^{i-\ell-1}\psi\, d\psi \\ &= \frac{\sigma_{i-\ell-1}}{2} \int_{S^{\ell-1}} \frac{dv}{(v' r v)^{i/2-1}} \int_0^{v' r v} t^{\ell/2-1} (v' r v - t)^{(i-\ell)/2-1} f_0(t)\, dt. \end{aligned} \tag{2.13}$$

Finally, we diagonalize $r = \sigma' \mathrm{P}_\xi \sigma$ by setting $r = \gamma' \boldsymbol{\lambda} \gamma$, where $\gamma \in O(\ell)$ and $\boldsymbol{\lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_\ell)$. Changing variables, we obtain (2.8).

Consider the case $\ell \ge i$. We replace u_1 in (2.11) by $p'_i \rho'_\xi \sigma$ from (2.10) and let $\tau \in V_{n,i}$ be an arbitrary orthonormal i-frame in ξ. We can always choose ρ_ξ so that $\rho_\xi p_i = \tau$. Then $u_1 u'_1 = p'_i \rho'_\xi \sigma \sigma' \rho_\xi p_i = \tau' \sigma \sigma' \tau$. The $i \times i$ matrix $s = \tau' \sigma \sigma' \tau$ is positive semi-definite and can be diagonalized as above. Hence, (2.11) yields

$$(R_i f)(\xi) = \int_{S^{i-1}} f_0(\eta' s \eta)\, d\eta = \int_{S^{i-1}} f_0(\eta' \boldsymbol{\lambda} \eta)\, d\eta,$$

as desired. ∎

Corollary 2.3: *If all canonical angles in Theorem 2.2 are equal, that is,* $\lambda_1 = \cdots = \lambda_m = \lambda$*, then* $(R_i f)(\xi) = F(\lambda)$*, where*

$$F(\lambda)=\frac{\sigma_{i-\ell-1}\,\sigma_{\ell-1}}{2\lambda^{i/2-1}}\int_0^\lambda t^{\ell/2-1}(\lambda-t)^{(i-\ell)/2-1}f_0(t)\,dt \tag{2.14}$$

if $i > \ell$*, and*

$$F(\lambda) = \sigma_{i-1}\, f_0(\lambda) \tag{2.15}$$

if $i \le \ell$*.*

Remark 2.4: If $i + \ell > n$, then every $\xi \in G_{n,i}$ has at least one-dimensional intersection with $\mathbb{R}^\ell$. It means that some canonical angles between ξ and $\mathbb{R}^\ell$ are necessarily zero and therefore, some of the eigenvalues $\lambda_1, \dots, \lambda_m$ equal 1. It follows that for $i + \ell > n$, equalities (2.14) and (2.15) are available only for $\lambda = 1$. This situation is not favorable for our purposes, because we will need (2.14) and (2.15) to be available *for all* $\lambda \in (0, 1)$. The latter is guaranteed if $i \le n - \ell$, when we have "sufficiently many" i-dimensional subspaces with the property $\dim(\xi \cap \mathbb{R}^\ell) = 0$.

Corollary 2.3 motivates the following

Definition 2.5: We denote by $G^\ell_{n,i}$ the submanifold of all i-dimensional subspaces ξ with the property that all canonical angles between ξ and $\mathbb{R}^\ell$ are equal.

Remark 2.6: It is known that the Radon transform is overdetermined if the dimension of the target space is greater than the dimension of the source space. If f is K_ℓ-invariant and $i \le n - \ell$, then, by Corollary 2.3 and Remark 2.4, the overdeterminicity can be eliminated if we restrict $(R_i f)(\xi)$ to $\xi \in G^\ell_{n,i}$. Here one should mention the general method of the kappa-operator, which allows us to reduce overdeterminicity by invoking the relevant permissible complexes of subspaces; see, e.g., [GGR] and references therein. The advantage of our *method of canonical angles*, which is applicable to the particular case of K_ℓ -invariant functions, is the following. If $i > \ell$, then to recover f from $R_i f$, it suffices to invert a simple Abel integral (2.14). If $i \le \ell$, then f expresses through $R_i f$ without any integro-differential operations.

3. K_ℓ -Symmetric Bodies and Comparison of Volumes

3.1. Preliminaries. An origin-symmetric (o.s.) star body B in $\mathbb{R}^n$, $n \geq 2$, is a compact set with nonempty interior such that $tB \subset B \; \forall t \in [0,1]$, $B = -B$, and the **radial function** $\rho_B(\theta) = \sup\{\lambda \geq 0 : \lambda\theta \in B\}$ is continuous on S^{n-1}. The **Minkowski functional** of B is defined by $\|x\|_B = \min\{a \geq 0 \, : \, x \in aB\}$, so that $\|\theta\|_B = \rho_B^{-1}(\theta)$. An o.s. star body B is called infinitely smooth if $\rho_B(\theta) \in C^\infty_{even}(S^{n-1})$.

If $\xi \in G_{n,i}, 1 < i < n$, then

$$\text{vol}_i(B \cap \xi) = i^{-1} \int_{S^{n-1}\cap\xi} \rho_B^i(\theta)\, d_\xi\theta = i^{-1}(R_i\rho_B^i)(\xi). \tag{3.1}$$

Similarly, $\text{vol}_n(B) = n^{-1} \int_{S^{n-1}} \rho_B^n(\theta)\, d\theta$.

Problem: *Let i be a fixed integer, $1 \leq i \leq n-1$. We wonder, for which o.s. star bodies A and B in $\mathbb{R}^n$ the inequality*

$$\text{vol}_i(A \cap \xi) \leq \text{vol}_i(B \cap \xi) \quad \textit{for every } \xi \in G_{n,i} \tag{3.2}$$

implies

$$\text{vol}_n(A) \leq \text{vol}_n(B). \tag{3.3}$$

For $i = 1$ the affirmative answer is obvious. Unlike the question in the Introduction, here we do not assume a priori that A and B are convex. The reason is that the implication (3.2) $\to$ (3.3) may be valid without any convexity assumption (see Theorem 1.1 (a)) and we want to understand how the convexity comes into play in this context.

It is worth noting that connection of the original Busemann-Petty problem ($i = n-1$) reformulated as above, with convexity was studied a lot, starting from Gardner [Ga1] and Gardner, Koldobsky and Schlumprecht [GKS]. The introduction of the so-called slicing function in these publications is an essential step. The breakthrough advantage of the method in [GKS] (and in other Koldobsky's works) is that the slicing function makes clear connection of the Busemann-Petty problem to Brunn-Minkowski theory (via Brunn's theorem). However, our attempts to extend "the method of slicing function" to the lower dimensional problem $i < n-1$ did not lead to the solution (see [RZ, p. 492] for details), and new ideas are needed.

Below we study the problem stated above in the particular case, when the body A with smaller sections is symmetric with respect to some mutually orthogonal subspaces, say, p and $p^\perp$, of dimensions ℓ and $n-\ell$, respectively. We fix the coordinate system so that $p = \mathbb{R}^\ell = \bigoplus_{j=n-\ell+1}^n \mathbb{R}e_j$ and $p^\perp = \mathbb{R}^{n-\ell} = \bigoplus_{j=1}^{n-\ell} \mathbb{R}e_j$. Then $K_\ell A = A$, where K_ℓ is the group (1.3). An o.s. star body with this property is said to be K_ℓ -symmetric.

By Lemma 2.1, the radial function $\rho_A(\theta)$ of a K_ℓ-symmetric body A is completely determined by the angle ω between the unit vector θ and the subspace $\mathbb{R}^\ell$. Hence, we set

$$\rho_A(\theta) = \tilde{\rho}_A(t), \quad t = \cos^2\omega = \theta' \mathrm{P}_{\mathbb{R}^\ell} \theta. \tag{3.4}$$

By Theorem 2.2, the Radon transform $(R_i f)(\xi)$, $\xi \in G_{n,i}$, of every K_ℓ-invariant function f is actually a function of the canonical angles between $\xi \in G_{n,i}$ and $\mathbb{R}^\ell$. Restricting $(R_i f)(\xi)$ to $\xi \in G^\ell_{n,i}$ (see Definition 2.5), we can remove overdeterminicity of $R_i f$. As we shall see below, the corresponding lower dimensional Busemann-Petty problem is also overdetermined. We can remove this overdeterminicity by considering sections by subspaces $\xi \in G^\ell_{n,i}$.

We will need the following auxiliary lemmas.

Lemma 3.1: *The group K_ℓ preserves canonical angles between $\xi \in G_{n,i}$ and $\mathbb{R}^\ell$.*

Proof. The proof relies on (1.6). Let first $\ell < i$, $\xi = \{\tau\}$, $\tau \in V_{n,i}$. It suffices to check that for every $\gamma \in K_\ell$, matrices $r = \sigma'\tau\tau'\sigma$ and $r_\gamma = \sigma'\gamma\tau\tau'\gamma'\sigma$ have the same eigenvalues. Let $\gamma = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, where $\alpha \in O(n-\ell)$, $\beta \in O(\ell)$. Multiplying matrices, we have $\gamma'\sigma = \sigma\beta'$. Hence, $r_\gamma = \beta\sigma'\tau\tau'\sigma\beta' = \beta r \beta'$. Since $\beta r \beta'$ and r have the same eigenvalues, we are done.

If $\ell \geq i$, we compare eigenvalues of matrices $s = \tau'\sigma\sigma'\tau$ and $s_\gamma = \tau'\gamma'\sigma\sigma'\gamma\tau$. These matrices coincide, because, as we have already seen, $\gamma'\sigma = \sigma\beta'$, and therefore, $s_\gamma = \tau'\sigma\beta'\beta\sigma'\tau = \tau'\sigma\sigma'\tau = s$. ∎

Definition 3.2 (K_ℓ-symmetrization): Given an o.s. star body B in $\mathbb{R}^n$, we define the associated K_ℓ-symmetric body B_0 by

$$\rho_{B_0}(\theta) = \left(\int_{K_\ell} \rho_B^i(\gamma\theta)\, d\gamma \right)^{1/i}. \tag{3.5}$$

Lemma 3.3: $\mathrm{vol}_n(B_0) \leq \mathrm{vol}_n(B)$.

Proof. By the generalized Minkowski inequality,

$$\mathrm{vol}_n^{i/n}(B_0) = \left(\frac{1}{n}\int_{S^{n-1}}\Big[\int_{K_\ell}\rho_B^i(\gamma\theta)\,d\gamma\Big]^{n/i}d\theta\right)^{i/n}$$
$$\leq \int_{K_\ell}\left[\frac{1}{n}\int_{S^{n-1}}\rho_B^n(\gamma\theta)\,d\theta\right]^{i/n}d\gamma = \mathrm{vol}_n^{i/n}(B),$$

and the result follows. ∎

Lemma 3.4: *Let A and B be o.s. star bodies in $\mathbb{R}^n$, $1 \le \ell \le n-1$. If A is K_ℓ-symmetric, and*

$$\mathrm{vol}_i(A\cap\xi) \le \mathrm{vol}_i(B\cap\xi) \quad \textit{for every } \xi \in G_{n,i}^\ell, \tag{3.6}$$

then $\mathrm{vol}_i(A\cap\xi) \le \mathrm{vol}_i(B_0\cap\xi)$ *for all* $\xi \in G_{n,i}^\ell$.

Proof. Fix $\xi \in G_{n,i}^\ell$. By Lemma 3.1, $\gamma\xi \in G_{n,i}^\ell$ for every $\gamma \in K_\ell$. Owing to (3.6), $\mathrm{vol}_i(A\cap\gamma\xi) \le \mathrm{vol}_i(B\cap\gamma\xi)$ or $(R_i\rho_A^i)(\gamma\xi) \le (R_i\rho_B^i)(\gamma\xi)$ for all $\gamma\in K_\ell$. Integrating this inequality in γ and taking into account that R_i commutes with orthogonal transformations, we obtain

$$(R_i\rho_A^i)(\xi) \le R_i\left[\int_{K_\ell}\rho_B^i(\gamma\theta)\,d\gamma\right](\xi) = (R_i\rho_{B_0}^i)(\xi). \tag{3.7}$$

This implies $\mathrm{vol}_i(A\cap\xi) \le \mathrm{vol}_i(B_0\cap\xi)$. ∎

3.2. The case $i \le \min(\ell, n-\ell)$. The following proposition represents part (a) of Theorem 1.1.

Proposition 3.5: *Let* $1 \le i,\ell \le n-1$; $i \le \min(\ell, n-\ell)$. *If A is a K_ℓ-symmetric body in $\mathbb{R}^n$, then the implication*

$$\mathrm{vol}_i(A\cap\xi) \le \mathrm{vol}_i(B\cap\xi) \textit{ for every } \xi\in G_{n,i}^\ell \Longrightarrow \mathrm{vol}_n(A) \le \mathrm{vol}_n(B) \tag{3.8}$$

is true for every o.s. star body B.

Proof. By Lemma 3.4, for all $\xi \in G_{n,i}^\ell$ we have

$$\mathrm{vol}_i(A\cap\xi) \le \mathrm{vol}_i(B_0\cap\xi) \quad \text{or} \quad (R_i\rho_A^i)(\xi) \le (R_i\rho_{B_0}^i)(\xi).$$

Hence, by (2.15) and (3.4), $\tilde\rho_A^i(\lambda) \le \tilde\rho_{B_0}^i(\lambda)$ for all $\lambda\in(0,1)$ (see Remark 2.4), and therefore, $\rho_A(\theta) \le \rho_{B_0}(\theta)$ for all $\theta \in S^{n-1}$. By Lemma 3.3, it follows that $\mathrm{vol}_n(A) \le \mathrm{vol}_n(B_0) \le \mathrm{vol}_n(B)$. ∎

3.3. THE CASE $\ell < i \le n-\ell$. We will need some sort of duality which is a one-dimensional analogue of (2.2) and serves as a substitute for the Lutwak's connection [Lu] between the Busemann-Petty problem and intersection bodies. According to (2.14), the Radon transform $(R_i\rho_A^i)(\xi)$, restricted to $\xi \in G_{n,i}^\ell$, is represented by the Abel type integral

$$(3.9) \qquad (R_i\rho_A^i)(\xi) = \frac{c_1}{\lambda^{i/2-1}} \int_0^\lambda t^{\ell/2-1}(\lambda-t)^{(i-\ell)/2-1}\tilde\rho_A^i(t)\,dt,$$

$$c_1 = \sigma_{i-\ell-1}\,\sigma_{\ell-1}/2,$$

where $\lambda^{1/2} \in (0,1)$ is the cosine of the canonical angles between ξ and $\mathbb{R}^\ell$ (we remind that these angles are equal when $\xi \in G_{n,i}^\ell$ and (3.9) is available for all $\lambda \in (0,1)$; see Remark 2.4). Denote the right-hand side of (3.9) by $(I_+\tilde\rho_A^i)(\lambda)$ and define the dual integral operator

$$(3.10) \qquad (I_-\psi)(t) = c_1\, t^{\ell/2-1} \int_t^1 (\lambda-t)^{(i-\ell)/2-1}\psi(\lambda)\frac{d\lambda}{\lambda^{i/2-1}},$$

so that

$$(3.11) \qquad \int_0^1 (I_+\tilde\rho_A^i)(\lambda)\,\psi(\lambda)\,d\lambda = \int_0^1 \tilde\rho_A^i(t)\,(I_-\psi)(t)\,dt.$$

Expression (3.10) resembles the classical Riemann-Liouville integral

$$(3.12) \qquad (I_-^\alpha g)(t) = \frac{1}{\Gamma(\alpha)} \int_t^1 g(\lambda)(\lambda-t)^{\alpha-1}\,d\lambda, \quad \alpha > 0.$$

LEMMA 3.6: *Let* $1 \le \ell < i \le n-\ell$ *and suppose that* A *is a* K_ℓ*-symmetric body in* $\mathbb{R}^n$. *If there is a nonnegative function* g *on* $(0,1)$, *which is integrable on every interval* $(\delta,1)$, $0<\delta<1$, *and such that*

$$(3.13) \qquad (1-t)^{(n-\ell)/2-1}\tilde\rho_A^{n-i}(t) = (I_-^\alpha g)(t), \quad \alpha = (i-\ell)/2, \quad t \in (0,1),$$

then the implication

$$(3.14) \qquad \mathrm{vol}_i(A\cap\xi) \le \mathrm{vol}_i(B\cap\xi) \text{ for every } \xi \in G_{n,i}^\ell \Longrightarrow \mathrm{vol}_n(A) \le \mathrm{vol}_n(B)$$

holds for every o.s. star body B.

Proof. By (2.6),

$$\mathrm{vol}_n(A) = \frac{1}{n}\int_{S^{n-1}} \rho_A^n(\theta)\,d\theta = c_2 \int_0^1 \tilde\rho_A^n(t) t^{\ell/2-1}(1-t)^{(n-\ell)/2-1}\,dt,$$

$$c_2 = \sigma_{\ell-1}\sigma_{n-\ell-1}/2n.$$

Hence, owing to (3.10), (3.11) and (3.13),

$$\mathrm{vol}_n(A) = c_2 \int_0^1 \tilde\rho_A^i(t) t^{\ell/2-1} (I_-^\alpha g)(t)\, dt = \frac{c_2}{c_1} \int_0^1 \tilde\rho_A^i(t)(I_-\psi)(t)\, dt$$

$$= \frac{c_2}{c_1} \int_0^1 (I_+\tilde\rho_A^i)(\lambda)\,\psi(\lambda)\, d\lambda, \quad \psi(\lambda) = \frac{\lambda^{i/2-1} g(\lambda)}{\Gamma(\alpha)} \geq 0. \tag{3.15}$$

If $\mathrm{vol}_i(A \cap \xi) \leq \mathrm{vol}_i(B \cap \xi)$ for every $\xi \in G_{n,i}^\ell$, then, by (3.9) and (3.7),

$$(I_+\tilde\rho_A^i)(\lambda) = (R_i\rho_A^i)(\xi) \leq (R_i\rho_{B_0}^i)(\xi) = (I_+\tilde\rho_{B_0}^i)(\lambda),$$

and therefore,

$$\begin{aligned}\mathrm{vol}_n(A) &\leq \frac{c_2}{c_1} \int_0^1 (I_+\tilde\rho_{B_0}^i)(\lambda)\,\psi(\lambda)\, d\lambda \\ &= c_2 \int_0^1 \tilde\rho_{B_0}^i(t)\tilde\rho_A^{n-i}(t) t^{\ell/2-1}(1-t)^{(n-\ell)/2-1}\, dt \\ &= \frac{1}{n} \int_{S^{n-1}} \rho_{B_0}^i(\theta)\rho_A^{n-i}(\theta)\, d\theta.\end{aligned}$$

Now Hölder's inequality yields $\mathrm{vol}_n(A) \leq \mathrm{vol}_n(B_0)$, and the result follows by Lemma 3.3. ∎

Up to now, the K_ℓ-symmetric body A with smaller sections was arbitrary. To handle the case $i > \ell$, we additionally assume that A is convex. The following lemma enables us to reduce consideration to smooth bodies.

LEMMA 3.7: *Let A and B be o.s. star bodies in $\mathbb{R}^n$. If the implication*

$$\mathrm{vol}_i(A \cap \xi) \leq \mathrm{vol}_i(B \cap \xi) \textit{ for every } \xi \in G_{n,i}^\ell \Longrightarrow \mathrm{vol}_n(A) \leq \mathrm{vol}_n(B) \tag{3.16}$$

is true for every infinitely smooth K_ℓ-symmetric convex body A, then it is true when A is an arbitrary K_ℓ-symmetric convex body.

Proof. Given a K_ℓ-symmetric convex body A, let $A^* = \{x : |x \cdot y| \leq 1\ \forall y \in A\}$ be the polar body of A with the support function $h_{A^*}(x) = \max\{x \cdot y : y \in A^*\}$. Since $h_{A^*}(\cdot)$ coincides with Minkowski's functional $\|\cdot\|_A$, then $h_{A^*}(\cdot)$ is K_ℓ-invariant, and therefore, A^* is K_ℓ-symmetric. It is known [Schn, pp. 158–161], that any o.s. convex body in $\mathbb{R}^n$ can be approximated by infinitely smooth convex bodies with positive curvature and the approximating operator commutes with rigid motions. Hence, there is a sequence $\{A_j^*\}$ of infinitely smooth K_ℓ-symmetric convex bodies with positive curvature such that $h_{A_j^*}(\theta)$ converges to

$h_{A^*}(\theta)$ uniformly on S^{n-1}. It follows that for the relevant sequence of infinitely smooth K_ℓ-symmetric convex bodies $A_j = (A_j^*)^*$,

$$\lim_{j\to\infty} \max_{\theta\in S^{n-1}} |\, \|\theta\|_{A_j} - \|\theta\|_A| = 0.$$

This implies convergence in the radial metric, i.e.,

$$\lim_{j\to\infty} \max_{\theta\in S^{n-1}} |\rho_{A_j}(\theta) - \rho_A(\theta)| = 0. \tag{3.17}$$

Let us show that the sequence $\{A_j\}$ in (3.17) can be modified so that $A_j \subset A$. The idea of this argument was borrowed from [RZ]. Without loss of generality, assume that $\rho_A(\theta) \geq 1$. Choose A_j so that

$$|\rho_{A_j}(\theta) - \rho_A(\theta)| < \frac{1}{j+1} \quad \text{for all } \theta \in S^{n-1}$$

and set $A'_j = \frac{j}{j+1} A_j$. Then, obviously, $\rho_{A'_j}(\theta) \to \rho_A(\theta)$ uniformly on S^{n-1} as $j \to \infty$, and

$$\rho_{A'_j} = \frac{j}{j+1}\rho_{A_j} < \frac{j}{j+1}\Big(\rho_A + \frac{1}{j+1}\Big) \leq \rho_A.$$

Hence, $A'_j \subset A$.

Now suppose that $\mathrm{vol}_i(A\cap\xi) \leq \mathrm{vol}_i(B\cap\xi)$ for every $\xi \in G^\ell_{n,i}$. Then this is true when A is replaced by A'_j, and, by the assumption of the lemma, $\mathrm{vol}_n(A'_j) \leq \mathrm{vol}_n(B)$. Passing to the limit as $j \to \infty$, we obtain $\mathrm{vol}_n(A) \leq \mathrm{vol}_n(B)$. ∎

The next proposition gives part (b) of Theorem 1.1.

Proposition 3.8: *Let A be a K_ℓ-symmetric convex body in $\mathbb{R}^n$, and let $2 \leq i \leq n-\ell$. If*

$$i = \ell + 1 \quad (\textit{in this case } \ell \leq (n-1)/2)$$

or

$$i = \ell + 2 \quad (\textit{in this case } \ell \leq (n-2)/2),$$

then the implication

$$\mathrm{vol}_i(A\cap\xi) \leq \mathrm{vol}_i(B\cap\xi) \textit{ for every } \xi \in G^\ell_{n,i} \Longrightarrow \mathrm{vol}_n(A) \leq \mathrm{vol}_n(B) \tag{3.18}$$

holds for every o.s. star body B.

Proof. By Lemma 3.7, we can assume $\rho_A \in C^\infty(S^{n-1})$. If $i = \ell+2$, then (3.13) becomes $(1-t)^{(n-i)/2}\tilde{\rho}_A^{n-i}(t) = \int_t^1 g(\lambda)\,d\lambda$, which implies

$$g(t) = -\frac{d}{dt}[(1-t)^{(n-i)/2}\tilde{\rho}_A^{n-i}(t)] \in L^1(0,1).$$

To check that g is nonnegative, we set $t = 1-s$, $r(s) = s^{1/2}\tilde\rho_A(1-s)$, $s = \sin^2\omega$, and get

$$g(1-s) = \frac{d}{ds}[r^{n-i}(s)] = (n-i)r^{n-i-1}(s)r'(s).$$

If $\theta = u\sin\omega + v\cos\omega \in S^{n-1}$, $u \in S^{n-\ell-1} \subset \mathbb{R}^{n-\ell}$, $v \in S^{\ell-1} \subset \mathbb{R}^\ell$, and $P_{u,v}$ is a 2-plane spanned by u and v, then $A \cap P_{u,v}$ is a convex domain, which is symmetric with respect to the u and v axes. Since $s = \sin^2\omega$, then $r(s) = s^{1/2}\tilde\rho_A(1-s)$ is non-decreasing, and therefore, $r'(s) \geq 0$. This gives $g(1-s) \geq 0$, $s \in (0,1)$, or, equivalently, $g(t) \geq 0$ for all $0 < t < 1$. Now the result follows by Lemma 3.6.

Let $i = \ell + 1$. We set $\varkappa_A(t) = (1-t)^{(n-i-1)/2}\tilde\rho_A^{n-i}(t)$ and reconstruct $g(t)$ from (3.13) using fractional differentiation as follows:

$$\begin{aligned} g(t) &= -\frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_t^1 (s-t)^{-1/2}\varkappa_A(s)\,ds \quad (\text{set } p = 1-t,\ q = 1-s) \\ &= \frac{1}{\sqrt{\pi}}\frac{d}{dp}\int_0^p (p-q)^{-1/2}\varkappa_A(1-q)\,dq \\ &= \frac{1}{\sqrt{\pi}}\frac{d}{dp}\left[p^{1/2}\int_0^1 (1-\eta)^{-1/2}\varkappa_A(1-p\eta)\,d\eta\right]. \end{aligned}$$

This gives

$$\begin{aligned} g(t) &= \frac{1}{\sqrt{\pi}}\frac{d}{dp}\int_0^1 [(p\eta)^{1/2}\tilde\rho_A(1-p\eta)]^{n-i}\frac{d\eta}{\sqrt{\eta(1-\eta)}}\,d\eta \\ &= \frac{1}{\sqrt{\pi}}\frac{d}{dp}\int_0^1 \frac{r^{n-i}(p\eta)}{\sqrt{\eta(1-\eta)}}\,d\eta, \quad r(s) = s^{1/2}\tilde\rho_A(1-s). \end{aligned}$$

The last integral is a nondecreasing function of p, and therefore, the derivative of it is nonnegative. Hence, $g(t) \geq 0$ for all $0 < t < 1$ and, by Lemma 3.6, we are done. ∎

4. The negative result

The proof of the negative result in Theorem 1.2 relies on Koldobsky's generalizations of the Lutwak's concept of intersection body (see [K4], [Lu]) and properties of the generalized cosine transforms [R4].

We recall some basic facts. The generalized cosine transform of a function f on S^{n-1} is defined by

$$(M^\alpha f)(u) = \gamma_n(\alpha) \int_{S^{n-1}} f(\theta)|\theta \cdot u|^{\alpha-1}\, d\theta, \tag{4.1}$$

$$\gamma_n(\alpha)=\frac{\sigma_{n-1}\,\Gamma\big((1-\alpha)/2\big)}{2\pi^{(n-1)/2}\Gamma(\alpha/2)}, \quad \text{Re}\,\alpha>0, \quad \alpha\neq 1,3,5,\dots. \tag{4.2}$$

The integral (4.1) is absolutely convergent for any $f \in L^1(S^{n-1})$. If f is infinitely differentiable, then $M^\alpha f$ extends as meromorphic function of α with the poles $\alpha = 1, 3, 5, \dots$. The following statement is a consequence of the relevant spherical harmonic decomposition.

Lemma 4.1 ([R1]): *Let $\alpha, \beta \in \mathbb{C}$; $\alpha, \beta \neq 1, 3, 5, \dots$. If $\alpha + \beta = 2 - n$ and $f \in C^\infty_{even}(S^{n-1})$, then*

$$M^\alpha M^\beta f = f. \tag{4.3}$$

If $\alpha, 2-n-\alpha \neq 1, 3, 5, \dots$, then M^α is an automorphism of the space $C^\infty_{even}(S^{n-1})$ endowed with the standard topology.

We will also need the following statement, which is a particular case of Lemma 3.5 from [R4].

Lemma 4.2: *Let $f \in C^\infty_{even}(S^{n-1})$. Then*

$$(R_i M^{1-i} f)(\xi) = c\,(R_{n-i} f)(\xi^\perp), \quad \text{for } \xi \in G_{n,i}, \tag{4.4}$$

where $c = c(n, i)$ is a positive constant.

Definition 4.3 ([K4]): An o.s. star body K in $\mathbb{R}^n$ is a k-intersection body if there is a nonnegative finite Borel measure μ on S^{n-1}, so that for every Schwartz ϕ,

$$\int_{\mathbb{R}^n} \|x\|_K^{-k} \phi(x)\, dx = \int_{S^{n-1}} \left[\int_0^\infty t^{k-1} \hat{\phi}(t\theta)\, dt \right] d\mu(\theta),$$

where $\hat{\phi}$ is the Fourier transform of ϕ. We denote by $\mathcal{I}^n_k$ the class of all k-intersection bodies in $\mathbb{R}^n$.

In some applications of k-intersection bodies, alternative definitions are more convenient than the original Definition 4.3; see, e.g., in [R4, Remark 8.2]. The following equivalent definition is a particular case of the more general [R4, Definition 5.4].

Definition 4.4: An o.s. star body K in $\mathbb{R}^n$ is a k-intersection body if there is a nonnegative finite Borel measure μ on S^{n-1}, so that $\rho_K^k = M^{1-k}\mu$, i.e., $(\rho_K^k, \varphi) = (\mu, M^{1-k}\varphi)$ for any $\varphi \in C_{even}^\infty(S^{n-1})$.

For more details about equivalent definitions and generalizations of the concept of k-intersection body the reader is addressed to [R4].

The next proposition plays a key role in the proof of the negative result in this section.

LEMMA 4.5: *Let B be an infinitely smooth K_ℓ-symmetric convex body with positive curvature. If $B \notin \mathcal{I}_{n-i}^n$, then there is an infinitely smooth K_ℓ-symmetric convex body A such that $\mathrm{vol}_i(A \cap \xi) \le \mathrm{vol}_i(B \cap \xi)$, for every $\xi \in G_{n,i}$, but $\mathrm{vol}_n(A) > \mathrm{vol}_n(B)$.*

Proof. By Definition 4.4 with $k = n-i$, there is a function φ in $C_{even}^\infty(S^{n-1})$, which is negative on some open set $\Omega \subset S^{n-1}$ and such that $\rho_B^{n-i} = M^{1+i-n}\varphi$. Since B is K_ℓ-symmetric and operators M^α commute with orthogonal transformations, then φ is K_ℓ-invariant and $\varphi < 0$ on the whole orbit $\Omega_\ell = K_\ell \Omega$. Choose a function $h \in C_{even}^\infty(S^{n-1})$ so that $h \not\equiv 0$, $h(\theta) \ge 0$ if $\theta \in \Omega_\ell$ and $h(\theta) \equiv 0$ otherwise. Without loss of generality, we can assume h to be K_ℓ-invariant (otherwise, we can replace it by $\tilde{h}(\theta) = \int_{K_\ell} h(\gamma\theta)d\gamma$). Define an origin-symmetric smooth body A by $\rho_A^i = \rho_B^i - \varepsilon M^{1-i}h$, $\varepsilon > 0$. Clearly, A is K_ℓ -symmetric. If ε is small enough, then A is convex. This conclusion is a consequence of Oliker's formula [Ol], according to which the Gaussian curvature of an o.s. star body expresses through the first and second derivatives of the radial function. Since by (4.4), $(R_i M^{1-i}h)(\xi) = c\,(R_{n-i}h)(\xi^\perp) \ge 0$, then $R_i\rho_A^i \le R_i\rho_B^i$. This gives $\mathrm{vol}_i(A \cap \xi) \le \mathrm{vol}_i(B \cap \xi)$, for every $\xi \in G_{n,i}$. On the other hand, by (4.3),

$$(\rho_B^{n-i}, \rho_B^i - \rho_A^i) = \varepsilon(M^{1+i-n}\varphi, M^{1-i}h) = \varepsilon(\varphi, h) < 0,$$

or $(\rho_B^{n-i}, \rho_B^i) < (\rho_B^{n-i}, \rho_A^i)$. By Hölder's inequality, this implies $\mathrm{vol}_n(B) < \mathrm{vol}_n(A)$. ∎

Consider the (q,ℓ)-ball $B_{q,\ell}^n = \{x = (x', x'') : |x'|^q + |x''|^q \le 1\}$, where $x' \in \mathbb{R}^{n-\ell} = \bigoplus_{j=1}^{n-\ell} \mathbb{R}e_j$, $x'' \in \mathbb{R}^\ell = \bigoplus_{j=n-\ell+1}^{n} \mathbb{R}e_j$.

LEMMA 4.6: (*cf.* [K3, Theorem 4.21]) *Let*

$$B_{q,1}^{m+1} = \{(x', y) : |x'|^q + |y|^q \le 1,\ x' \in \mathbb{R}^m, y \in \mathbb{R}\}. \tag{4.5}$$

If $q > 2$ and $m \ge k+3$, then $B_{q,1}^{m+1} \notin \mathcal{I}_k^{m+1}$.

LEMMA 4.7: *If* $q > 2$ *and* $\ell + 2 < i \leq n-1$, *then* $B^n_{q,\ell} \notin \mathcal{I}^n_{n-i}$.

Proof. Suppose the contrary and consider the section of $B^n_{q,\ell}$ by the $(n-\ell+1)$-dimensional plane $\eta = \mathbb{R}e_n \oplus \mathbb{R}^{n-\ell}$. By Proposition 3.17 from [Mi2] (see also more general Theorem 5.12 in [R4]) $B^n_{q,\ell} \cap \eta \in \mathcal{I}^{n-\ell+1}_{n-i}$ in η, but this contradicts Lemma 4.6, according to which (set $m = n - \ell$) $B^n_{q,\ell} \cap \eta$ is not an $(n-i)$-intersection body when $i > \ell + 2$. ∎

For $q = 4$, Lemmas 4.5 and 4.7 imply the following negative result.

PROPOSITION 4.8: *If* $\ell + 2 < i \leq n - 1$, *then there is an infinitely smooth* K_ℓ*-symmetric convex body* A *such that*

$$\mathrm{vol}_i(A \cap \xi) \leq \mathrm{vol}_i(B^n_{4,\ell} \cap \xi) \quad \forall \xi \in G_{n,i}, \qquad \textit{but} \quad \mathrm{vol}_n(A) > \mathrm{vol}_n(B^n_{4,\ell}).$$

This is Theorem 1.2.

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