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COUNTING CHARACTERS IN LINEAR GROUP ACTIONS

BY

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ABSTRACT

Let G be a finite group and V be a finite G-module. We present upper bounds for the cardinalities of certain subsets of $\mathrm{Irr}(GV)$, such as the set of those $\chi \in \mathrm{Irr}(GV)$ such that, for a fixed $v \in V$, the restriction of χ to $\langle v\rangle$ is not a multiple of the regular character of $\langle v\rangle$. These results might be useful in attacking the noncoprime $k(GV)$-problem.

1. Introduction

Let G be a finite group and V be a finite G-module of characteristic p. R. Knörr presented a beautiful argument, for $(|G|, |V|) = 1$, in [4, Theorem 2.2] showing how to obtain strong upper bounds for $k(GV)$ (the number of conjugacy classes of GV) by using only information on $C_G(v)$ for a fixed $v \in V$. Note that his result immediately implies the important special case that if G has a regular orbit on V (i.e., there is a $v \in V$ with $C_G(v) = 1$), then $k(GV) \leq |V|$, which was a crucial result in the solution of the $k(GV)$-problem. In this note we give a much shorter proof of this result (see 3.1 below).

The main objective of the paper, however, is to modify and generalize Knörr's argument in various directions to include noncoprime situations. This way we obtain a number of bounds on certain subsets of $\mathrm{Irr}(GV)$, such as the following:

Theorem A: *Let G be a finite group and let V be a finite G-module of characteristic p. Let $v \in V$ and $C = C_G(v)$ and suppose that $(|C|, |V|) = 1$. Then*

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the number of irreducible characters of GV whose restriction to $\langle v\rangle$ is not a multiple of the regular character of $\langle v\rangle$ is bounded above by

$$\sum_{i=1}^{k(C)} |C_V(c_i)|,$$

where the c_i are representatives of the conjugacy classes of C.

THEOREM B: *Let G be a finite group and V be a finite G-module. Let $g \in G$ be of prime order not dividing $|V|$. Then the number of irreducible characters of GV whose restriction to $A = \langle g\rangle$ is not a multiple of the regular character of $\langle g\rangle$ is bounded above by*

$$|C_G(g)|\ n(C_G(g), C_V(g)),$$

where $n(C_G(g), C_V(g))$ denotes the number of orbits of $C_G(g)$ on $C_V(g)$.

Stronger versions and refinements of these results are proved in the paper. It is hoped that these results will be useful in solving the noncoprime $k(GV)$-problem, as discussed, for instance, in [3] and [1]. Theorems A and B will be proved in Sections 3 and 4 below respectively. In Section 2, we will generalize a recent result of P. Schmid [6, Theorem 2(a)] stating that in the situation of the $k(GV)$-problem, if G has a regular orbit on V, then $k(GV) = |V|$ can only hold if G is abelian. We prove

THEOREM C: *Let G be a finite group and V a finite faithful G-module with $(|G|, |V|) = 1$. Suppose that G has a regular orbit on V. Then*

$$k(GV) \leq |V| - |G| + k(G).$$

Our proof is different from the approach taken in [6], and we actually prove a slightly stronger result including some non-coprime actions.

NOTATION: If the group A acts on the set B, we write $n(A, B)$ for the number of orbits of A on B. All other notation is standard or explained along the way.

2. $\mathbf{k(GV) = |V|}$ and regular orbits

In this paper we often work under the hypothesis of the $k(GV)$-problem which is the following.

2.1 Hypothesis: Let G be a finite group and let V be a finite faithful G-module such that $(|G|, |V|) = 1$. Write p for the characteristic of V.

In [6, Theorem 2(a)], P. Schmid proved that under 2.1, if G has a regular orbit on V, V is irreducible, and $k(GV) = |V|$, then G is abelian, and from this it follows easily that either $|G| = 1$ and $|V| = p$, or G is cyclic of order $|V| - 1$. The proof in [6] is somewhat technical.

The goal of this section is to give a short proof of a generalization of Schmid's result based on a beautiful argument of Knörr [4]. We word it in such a way that we do not even need the coprime hypothesis, so that the result may even be useful to study the noncoprime $k(GV)$-problem. To do this, for any group X and $x \in X$ we introduce the set

$$\mathrm{Irr}(X, x) = \{\chi \in \mathrm{Irr}(X) : \chi|_{\langle x\rangle} \text{is not an integer multiple of the regular character of } \langle x\rangle\}$$

and write

$$k(X, x) = |\mathrm{Irr}(X, x)|.$$

2.2 Theorem: *Let G be a finite group and let V be a finite G-module such that G possesses a regular orbit on V. Let $v \in V$ be a representative of such an orbit. Then*

$$k(GV, v) \leq |V| - |G| + k(G).$$

Proof. Let p be the characteristic if V. Let $A = \langle v\rangle$ and put $\eta = p1_A - \rho_A$, where ρ_A is the regular character of A. Then $\eta(1) = 0$ and $\eta(a) = p$ for $1 \neq a \in A$. So for $1 \neq a \in A$ we have

$$\eta^{GV}(a) = \frac{1}{p} \sum_{\substack{g \in G \\ u \in V}} \dot{\eta}(a^{gu}) = \frac{1}{p} \sum_{\substack{g \in N_G(A) \\ u \in V}} \dot{\eta}(a^g) = \frac{|V|}{p} \sum_{g \in N_G(A)} p = |V||N_G(A)|.$$

Now let x_i $(i = 1, \ldots, k(GV))$ be a set of representatives of the conjugacy classes of GV. As η^{GV} obviously vanishes on all conjuagacy classes of GV which intersect $A - 1$ trivially, and there are precisely $\frac{p-1}{|N_G(A)|}$ conjugacy classes of GV

which intersect $A-1$ nontrivially, we obtain

$$(1)\quad (p-1)|V| = \frac{p-1}{|N_G(A)|}|N_G(A)||V| = \sum_{i=1}^{k(GV)} \eta^{GV}(x_i) = \sum_{\tau\in\mathrm{Irr}(GV)} [\tau\eta^{GV},\tau]$$
$$= \sum_{\tau\in\mathrm{Irr}(GV)} [\tau_A\eta,\tau_A].$$

Now for any $\tau\in\mathrm{Irr}(GV)$ we have

$$(2)\quad [\tau_A\eta,\tau_A] = \frac{1}{|A|}\sum_{a\in A}\tau(a)(p-\rho_A(a))\overline{\tau(a)}$$
$$= \sum_{1\neq a\in A}|\tau(a)|^2 \begin{cases} = 0 & \text{if } \tau_A \text{ is an integer multiple of } \rho_A \\ \geq p-1 & \text{otherwise,} \end{cases}$$

where the last step follows from [2, Lemma (3.14)] (note that if τ_A is not a multiple of ρ_A, then $\tau(a)\neq 0$ for all $1\neq a\in A$, as all these $\tau(a)$ are Galois conjugate). Next, observe that if $\tau\in\mathrm{Irr}(GV)$ with $V\leq\ker\tau$, then $\tau\in\mathrm{Irr}(G)$ and clearly τ_A is not a multiple of ρ_A, then clearly

$$(3)\qquad [\tau_A\eta,\tau_A] = \sum_{1\neq a\in A}|\tau(a)|^2 = \sum_{1\neq a\in A}\tau(1)^2 = (p-1)\tau(1)^2.$$

Thus with (1), (2), and (3) we get

$$(p-1)|V| = \sum_{\tau\in\mathrm{Irr}(G)}[\tau_A\eta,\tau_A] + \sum_{\substack{\tau\in\mathrm{Irr}(GV),\\ V\not\leq\ker\tau}}[\tau_A\eta,\tau_A]$$
$$\geq \sum_{\tau\in\mathrm{Irr}(G)}(p-1)\tau(1)^2 + (k(GV,v)-k(G))(p-1)$$

which yields

$$|V| \geq \sum_{\tau\in\mathrm{Irr}(G)}\tau(1)^2 + k(GV,v) - k(G) = |G| + k(GV,v) - k(G).$$

This implies the assertion of the theorem, and we are done. ∎

The following consequence implies Schmid's result [6, Theorem 2(a)].

2.3 Corollary: *Assume 2.1 and that G has a regular orbit on V. Then*

$$k(GV) \leq |V| - |G| + k(G).$$

In particular, if $k(GV)=|V|$, then G is abelian.

Proof. By Ito's theorem and as $(|G|, |V|) = 1$, we know that $\chi(1)$ divides $|G|$ for every $\chi \in \mathrm{Irr}(GV)$, so in particular p does not divide $\chi(1)$. Thus, for any $v \in V^{\#}$, we see that $\chi|_{\langle v \rangle}$ cannot be an integer multiple of $\rho_{\langle v \rangle}$. Therefore, $k(GV, v) = k(GV)$. Now the assertion follows from 2.2. ∎

After seeing a preprint of this paper, P. Schmid informed me that he independently had obtained 2.3. This appeared, with a different proof, in his recent book on the $k(GV)$-problem (see Theorem 1.5d in [7]).

3. Bounds for $\mathbf{k(GV)}$

In this section we study more variations of Knörr's argument [4, Theorem 2.2] and generalize it to some noncoprime situations.

We begin, however, by looking at a classical application of it. An important and immediate consequence of Knörr's result is that if under 2.1 G has a regular orbit on V, then $k(GV) \leq |V|$. This important result can be obtained in the following shorter way.

3.1 Proposition: *Let G be a finite group and let V be a finite faithful G-module. Let $v \in V$. Then*

$$k(GV, v) \;\leq\; |C_G(v)||V|,$$

in particular, if $(|G|, |V|) = 1$ and G has a regular orbit on V, then $k(GV) \leq |V|$.

Proof. Put $A = \langle v \rangle$. If $\tau \in \mathrm{Irr}(GV, v)$, then by [2, Lemma (3.14)] we know that $\sum_{1 \neq a \in A} |\tau(a)|^2 \geq p - 1$. With this and well-known character theory we get

$$\begin{aligned}
(p-1)k(GV, v) \leq k(GV, v) \min_{\tau \in \mathrm{Irr}(GV,v)} \left(\sum_{1 \neq a \in A} |\tau(a)|^2 \right) &\leq \sum_{\tau \in \mathrm{Irr}(GV)} \sum_{1 \neq a \in A} |\tau(a)|^2 \\
&= \sum_{1 \neq a \in A} \sum_{\tau \in \mathrm{Irr}(GV)} \tau(a)\overline{\tau(a)} \\
&= \sum_{1 \neq a \in A} |C_{GV}(a)| \\
&= \sum_{1 \neq a \in A} |C_G(v)||V| \\
&= (p-1)|C_G(v)||V|.
\end{aligned}$$

This implies the first result. If $(|G|,|V|)=1$, then by Ito's result $\tau(1)$ divides $|G|$ for all $\tau\in\mathrm{Irr}(GV)$, so p cannot divide $\tau(1)$, and thus $k(GV,v)=k(GV)$, and the second result now follows by choosing v to be in a regular orbit of G on V. ∎

Now we turn to generalizing Knörr's argument. We discuss various ways to do so.

3.2 Remark: Let G be a finite group and let V be a finite faithful G-module of characteristic p. Let $v\in V$ and put $C=C_G(v)$ and $A=\langle v\rangle$. Let

$$\begin{aligned}\mathrm{Irr}(GV,&C,v)\\ &:=\mathrm{Irr}_0(GV)\\ &:=\mathrm{Irr}(GV)-\{\chi\in\mathrm{Irr}(GV):\chi|_{C\times\langle v\rangle}=\tau\times\rho_A \text{ for a character } \tau \text{ of } C\}\end{aligned}$$

and

$$\mathrm{Irr}_{p'}(GV)=\{\chi\in\mathrm{Irr}(GV): p \text{ does not divide } \chi(1)\},$$

so that clearly $\mathrm{Irr}_{p'}(GV)\subseteq\mathrm{Irr}_0(GV)$.
Note that if $(|G|,|V|)=1$, then by Ito $\mathrm{Irr}(GV)=\mathrm{Irr}_{p'}(GV)$.

To work towards our next result, we again proceed somewhat similarly as in [4, Theorem 2.2], but we keep the presentation here self-contained. In the following, we work under the hypothesis that $(|C|,|V|)=1$. Let $N=N_G(A)$. Then $|N:C|$ divides $p-1$. Moreover, it is easy to see that for $c\in C$, $1\neq a\in A$, $g\in G$, $u\in V$ we know that

$$(ca)^{gu}\in C\times A \quad\text{if and only if}\quad g\in N \text{ and } u\in C_V(c^g).$$

With this, one can readily verify that if c_i $(i=1,\dots,k(C))$ with $c_1=1$ are representatives of the conjugacy classes of C and a_j $(j=1,\dots,\frac{p-1}{|N:C|})$ are representatives of the N–conjugacy classes of $A-1$ then, the c_ia_j are representatives of those conjugacy classes of GV which intersect $C\times(A-1)$ nontrivially.

Now define a character η on $C\times A$ by $\eta=1_C\times(p1_A-\rho_A)$. Then for $c\in C$, $a\in A$ we have

$$\eta(ca)=\begin{cases}p, & \text{if } a\neq 1\\ 0, & \text{if } a=1.\end{cases}$$

Therefore η^{GV} vanishes on all conjugacy classes of GV which intersect $C\times(A-1)$ trivially, whereas for $c\in C$, $1\neq a\in A$ we have that

$$\begin{aligned}\eta^{GV}(ca) &= \frac{1}{|C\times A|}\sum_{\substack{g\in G\\ u\in V}}\dot\eta((ca)^{gu}) = \frac{1}{p|C|}\sum_{g\in N}\sum_{u\in C_V(c^g)}\eta(c^g a^g)\\ &= \frac{1}{p|C|}\sum_{g\in N}|C_V(c^g)|p\\ &= \frac{1}{|C|}\sum_{g\in N}|C_V(c)|\\ &= |N:C|\ |C_V(c)|.\end{aligned}$$

Thus if x_i $(i=1,\dots,k(GV))$ are representatives of the conjugacy classes of GV, then we get

$$\begin{aligned}\sum_{i=1}^{k(GV)}\eta^{GV}(x_i) &= \sum_{i=1}^{k(C)}\sum_{j=1}^{\frac{p-1}{|N:C|}}\eta^{GV}(c_ia_j) = \sum_{i=1}^{k(C)}\sum_{j=1}^{\frac{p-1}{|N:C|}}|N:C|\ |C_V(c_i)|\\ &= \frac{p-1}{|N:C|}|N:C|\sum_{i=1}^{k(C)}|C_V(c_i)|\\ &= (p-1)\sum_{i=1}^{k(C)}|C_V(c_i)|,\end{aligned}$$

and thus

$$\begin{aligned}(p-1)\sum_{i=1}^{k(C)}|C_V(c_i)| = \sum_{i=1}^{k(GV)}\eta^{GV}(x_i) &= \sum_{\tau\in\mathrm{Irr}(GV)}[\tau\eta^{GV},\tau]\\ &= \sum_{\tau\in\mathrm{Irr}(GV)}[\tau_{C\times A}\eta,\tau_{C\times A}].\end{aligned}\tag{1}$$

Now if $\tau\in\mathrm{Irr}(GV)$, we can write

$$\tau|_{C\times A} = \sum_{\lambda\in\mathrm{Irr}(A)}\tau_\lambda\times\lambda\tag{2}$$

where τ_λ is a character of C or $\tau_\lambda = 0$. Then we see that

$$\begin{aligned}[]
[\tau_{C\times A}\eta, \tau_{C\times A}] &= \frac{1}{|C\times A|}\sum_{\substack{c\in C\\ a\in A}} \tau(ca)\eta(ca)\overline{\tau(ca)}\\
&= \frac{1}{|C|}\sum_{\substack{c\in C\\ 1\neq a\in A}} \tau(ca)\overline{\tau(ca)}\\
&= \frac{1}{|C|}\sum_{\substack{\lambda,\mu\in \mathrm{Irr}(A)\\ c\in C}} \tau_\lambda(c)\overline{\tau_\mu(c)}\sum_{1\neq a\in A}\lambda(a)\overline{\mu(a)}.
\end{aligned}$$

Since $\sum_{1\neq a\in A}\lambda(a)\overline{\mu(a)}$ equals $p-1$ if $\lambda=\mu$ and -1 if $\lambda\neq\mu$, we further conclude that

$$\begin{aligned}[]
[\tau_{C\times A}\eta, \tau_{C\times A}] &= p\sum_{\lambda\in\mathrm{Irr}(A)}[\tau_\lambda,\tau_\lambda] - \sum_{\lambda,\mu\in\mathrm{Irr}(A)}[\tau_\lambda,\tau_\mu]\\
&= \sum_{\lambda<\mu}[\tau_\lambda-\tau_\mu, \tau_\lambda-\tau_\mu], \qquad (3)
\end{aligned}$$

where "$\leq$" is some arbitrary ordering on $\mathrm{Irr}(A)$.

Now if $\tau_\lambda - \tau_\mu$ is a nonzero multiple of ρ_C, then

$$[\tau_\lambda-\tau_\mu, \tau_\lambda-\tau_\mu] \ \geq\ |C| \qquad (4)$$

and thus

$$[\tau_{C\times A}\eta, \tau_{C\times A}] \ \geq\ |C|.$$

Moreover, note that if $\tau\in\mathrm{Irr}_0(GV)$, then not all $\tau_\lambda-\tau_\mu$ can be equal to 0 since otherwise we see from (2) that $\tau_{C\times A}$ would be equal to $\tau_\lambda\times\rho_A$ for any λ. So we can partition the set $\mathrm{Irr}(A)$ into two disjoint nonempty subsets $\Lambda_1=\{\lambda\in\mathrm{Irr}(A):\tau_\lambda=\tau_1\}$ and $\Lambda_2=\{\lambda\in\mathrm{Irr}(A):\tau_\lambda\neq\tau_1\}$, and thus as $0\leq(|\Lambda_1|-1)(|\Lambda_2|-1)=|\Lambda_1||\Lambda_2|-|\Lambda_1|-|\Lambda_2|+1=|\Lambda_1||\Lambda_2|-(p-1)$, we see that $|\Lambda_1|\ |\Lambda_2|\geq p-1$, so there are at least $p-1$ pairs $\lambda,\mu\in\mathrm{Irr}(A)$ such that $\tau_\lambda-\tau_\mu\neq 0$. Thus

$$[\tau_{C\times A}\eta, \tau_{C\times A}]\geq p-1 \quad \text{for all } \tau\in\mathrm{Irr}_0(GV). \qquad (5)$$

Therefore, by (1) and (5) we get that

$$\begin{aligned}(p-1)\sum_{i=1}^{k(C)}|C_V(c_i)| &= \sum_{\tau\in\mathrm{Irr}(GV)}[\tau_{C\times A}\eta,\tau_{C\times A}] \\ &\geq \sum_{\tau\in\mathrm{Irr}_0(GV)}[\tau_{C\times A}\eta,\tau_{C\times A}] \geq (p-1)|\mathrm{Irr}_0(GV)|\end{aligned}$$

and thus

$$|\mathrm{Irr}_0(GV)| \leq \sum_{i=1}^{k(C)}|C_V(c_i)|. \tag{6}$$

From now on we assume that $C>1$.

Now we repeat the arguments of this proof, but replace η by

$$\eta_1 \;=\; (|C|1_C-\rho_C)\times(p1_A-\rho_A),$$

so for $c\in C$ and $a\in A$ we have

$$\eta_1(ca) \;=\; \begin{cases}|C|p, & \text{if } c\neq 1 \text{ and } a\neq 1\\ 0, & \text{if } c=1 \text{ or } a=1.\end{cases}$$

Now from the above we know that the c_ia_j ($i=2,\ldots,k(C)$, $j=1,\ldots,\frac{p-1}{|N:C|}$) are representatives of those conjugacy classes which intersect $(C-1)\times(A-1)$ nontrivially. Clearly η_1^{GV} vanishes on all conjugacy classes of GV which intersect $(C-1)\times(A-1)$ trivially, whereas for $1\neq c\in C$, $1\neq a\in A$, if $(|C|,|V|)=1$, we have that

$$\begin{aligned}\eta_1^{GV}(ca) &= \frac{1}{|C\times A|}\sum_{\substack{g\in G\\ u\in V}}\dot{\eta}_1((ca)^{gu}) = \frac{1}{p|C|}\sum_{g\in N}\sum_{u\in C_V(c^g)}\eta_1(c^ga^g)\\ &= |N|\;|C_V(c)|.\end{aligned}$$

Next we conclude that

$$\sum_{i=1}^{k(GV)}\eta_1^{GV}(x_i) = \sum_{i=2}^{k(C)}\sum_{j=1}^{\frac{p-1}{|N:C|}}\eta_1^{GV}(c_ia_j) = (p-1)|C|\sum_{i=2}^{k(C)}|C_V(c_i)|,$$

and so as in (1) we see that

$$(p-1)|C|\sum_{i=2}^{k(C)}|C_V(c_i)| = \sum_{\tau\in\mathrm{Irr}(GV)}[\tau_{C\times A}\eta_1,\tau_{C\times A}] \tag{7}$$

Now with (2), similarly as in our earlier calculation leading to (3), we see that

$$
\begin{aligned}
&= \frac{1}{|C\times A|}\sum_{\substack{c\in C\\ a\in A}} \tau(ca)\eta_1(ca)\overline{\tau(ca)} = \sum_{\substack{1\neq c\in C\\ 1\neq a\in A}} \tau(ca)\overline{\tau(ca)}\\
&= \sum_{\substack{1\neq c\in C\\ 1\neq a\in A}} \sum_{\lambda\in\mathrm{Irr}(A)} \tau_\lambda(c)\lambda(a) \sum_{\mu\in\mathrm{Irr}(A)} \overline{\tau_\mu(c)\mu(a)}\\
&= \sum_{\lambda,\mu\in\mathrm{Irr}(A)} \sum_{1\neq c\in C} \tau_\lambda(c)\overline{\tau_\mu(c)} \sum_{1\neq a\in A} \lambda(a)\overline{\mu(a)}\\
(8)\qquad &= (p-1)\sum_{\lambda\in\mathrm{Irr}(A)} \sum_{1\neq c\in C} \tau_\lambda(c)\overline{\tau_\lambda(c)} - \sum_{\substack{\lambda,\mu\in\mathrm{Irr}(A)\\ \lambda\neq\mu}} \sum_{1\neq c\in C} \tau_\lambda(c)\overline{\tau_\mu(c)}\\
&= \sum_{\lambda\in\mathrm{Irr}(A)} \sum_{1\neq c\in C} \tau_\lambda(c)\overline{\tau_\lambda(c)} - \sum_{\lambda,\mu\in\mathrm{Irr}(A)} \sum_{1\neq c\in C} \tau_\lambda(c)\overline{\tau_\mu(c)}\\
&= \sum_{\lambda<\mu} \sum_{1\neq c\in C} (\tau_\lambda(c)-\tau_\mu(c))(\overline{\tau_\lambda(c)}-\overline{\tau_\mu(c)})\\
&= sum_{\lambda<\mu} \sum_{1\neq c\in C} |\tau_\lambda(c)-\tau_\mu(c)|^2
\end{aligned}
$$

for some arbitrary ordering $\leq$ on $\mathrm{Irr}(A)$.

Now recall that if $\tau\in\mathrm{Irr}_0(GV)$, then not all of the $\tau_\lambda-\tau_\mu$ are 0. So choose $\lambda,\mu\in\mathrm{Irr}(C)$ such that $\tau_\lambda-\tau_\mu\neq 0$. If all the τ_μ ($\mu\in\mathrm{Irr}(A)$) are integer multiples of ρ_C then put $\Lambda_1=\{\phi\in\mathrm{Irr}(A):\tau_\phi=\tau_\lambda\}$ and $\Lambda_2=\{\phi\in\mathrm{Irr}(A):\tau_\phi\neq\tau_\lambda\}$, so $\Lambda_1\neq\emptyset$ and $\Lambda_2\neq\emptyset$ and from $0\leq(|\Lambda_1|-1)(|\Lambda_2|-1)$ we clearly deduce that $|\Lambda_1||\Lambda_2|\geq p-1$, so there are at least $p-1$ pairs $(\phi_1,\phi_2)\in\mathrm{Irr}(A)\times\mathrm{Irr}(A)$ such that $\tau_{\phi_1}-\tau_{\phi_2}$ is a nonzero multiple of ρ_C.

So next we assume that τ_λ is not a multiple of ρ_C. Put

$$\Gamma_1=\{\phi\in\mathrm{Irr}(A):\tau_\lambda-\tau_\phi \text{ is a multiple of } \rho_C\}$$

and

$$\Gamma_2=\{\phi\in\mathrm{Irr}(A):\tau_\lambda-\tau_\phi \text{ is not a multiple of } \rho_C\}.$$

Clearly $\lambda\in\Gamma_1$, so $\Gamma_1\neq\emptyset$. If $\Gamma_2=\emptyset$, then $\mathrm{Irr}(A)=\Gamma_1$, and if we define Λ_1, Λ_2 as in the previous argument, we see that there are at least $(p-1)$ pairs $(\phi_1,\phi_2)\in\mathrm{Irr}(A)\times\mathrm{Irr}(A)$ such that $\tau_{\phi_1}-\tau_{\phi_2}$ is a nonzero multiple of ρ_C.

So now suppose $\Gamma_2\neq\emptyset$. Then $|\Gamma_1|+|\Gamma_2|=p$, and if $\phi_1\in\Gamma_1$ and $\phi_2\in\Gamma_2$, then $\tau_{\phi_1}-\tau_{\phi_2}=(\tau_{\phi_1}-\tau_\lambda)+(\tau_\lambda-\tau_{\phi_2})$ clearly is not a multiple of ρ_C, and by the same argument as used before we see that $|\Gamma_1||\Gamma_2|\geq p-1$, so there are at least $(p-1)$ pairs $(\phi_1, \phi_2) \in \mathrm{Irr}(A) \times \mathrm{Irr}(A)$ such that $\tau_{\phi_1} - \tau_{\phi_2}$ is not a multiple of ρ_C.

Altogether we have shown that for any $\tau \in \mathrm{Irr}_0(GV)$ one of the following holds:

(A) There are at least $(p-1)$ pairs $(\phi_1, \phi_2) \in \mathrm{Irr}(A) \times \mathrm{Irr}(A)$ such that

$$\tau_{\phi_1} - \tau_{\phi_2} \text{ is a nonzero multiple of } \rho_C, \text{ or}$$

(B) there are at least $(p-1)$ pairs $(\phi_1, \phi_2) \in \mathrm{Irr}(A) \times \mathrm{Irr}(A)$ such that

$$\tau_{\phi_1} - \tau_{\phi_2} \text{ is not a multiple of } \rho_C.$$

Now it remains to consider two cases:

Case 1: At least half of the $\tau \in \mathrm{Irr}_0(GV)$ satisfy (A). Then for any of these τ by (3) and (4) we have

$$[\tau_{C\times A}\eta, \tau_{C\times A}] = \sum_{\lambda<\mu}[\tau_\lambda - \tau_\mu, \tau_\lambda - \tau_\mu] \geq (p-1)|C|$$

and so by (1) we see that

$$(p-1)\sum_{i=1}^{k(C)} |C_V(c_i)| \geq \sum_{\tau\in\mathrm{Irr}_0(GV)} [\tau_{C\times A}\eta, \tau_{C\times A}] \geq \frac{1}{2}|\mathrm{Irr}_0(GV)|(p-1)|C|$$

which implies

$$|\mathrm{Irr}_0(GV)| \leq \frac{2}{|C|}\sum_{\imath=1}^{k(C)} |C_V(c_i)|. \tag{9}$$

Case 2: At least half of the $\tau \in \mathrm{Irr}_0(GV)$ satisfy (B).
Then for any of these τ by (8) and [5, Corollary 4] we have

$$[\tau_{C\times A}\eta_1, \tau_{C\times A}] \geq (p-1)(k(C)-1).$$

Thus by (7) we have that

$$\begin{aligned}(p-1)|C|\sum_{i=2}^{k(C)} |C_V(c_i)| &\geq \sum_{\tau\in\mathrm{Irr}_0(GV)} [\tau_{C\times A}\eta_1, \tau_{C\times A}] \\ &\geq \frac{1}{2}|\mathrm{Irr}_0(GV)|(p-1)\cdot(k(C)-1)\end{aligned}$$

whence

$$\text{(10)} \qquad |\mathrm{Irr}_0(GV)| \le \frac{2|C|}{k(C)-1} \sum_{i=2}^{k(C)} |C_V(c_i)|.$$

Now we drop the assumption $(|C|,|V|)=1$ and work towards a general bound for $|\mathrm{Irr}_0(GV)|$.

For this, fix $g_0 \in C$ such that g_0 is of prime order q and put $C_0 = \langle g_0 \rangle$ and $N_0 = N_G(C_0)$. Trivially there are at most $|C_0|(p-1) = q(p-1)$ conjugacy classes of GV that intersect $C_0 \times (A-1)$ nontrivially, and given $1 \ne c \in C_0$, $1 \ne a \in A$, we see that for $g \in G$, $u \in V$

$$(ca)^{gu} = c^g[c^g,u]a^g \; \in \; C_0 \times A \text{ first implies } c^g \in C_0, \text{ i.e., } g \in N_0$$

and each fixed $g \in N_0$, the equation $[c^g,u]a^g \in A$ implies $[c^g,u] \in Aa^{-g}$ which has at most $|C_V(c^g)|\;|Ag^{-1}| = p|C_V(g_0)|$ solutions u. Moreover, if $c=1$, then

$$(ca)^{gu} = a^{gu} = a^g \text{ implies } g \in N_G(A) = N \text{ and } u \in V.$$

Now we define the character η_2 on $C_0 \times A$ by $\eta_2 = 1_{C_0} \times (p1_A - \rho_A)$. Thus η_2^{GV} vanishes on all conjugacy classes of GV which intersect $C_0 \times (A-1)$ trivially, whereas for $1 \ne c \in C_0$, $1 \ne a \in A$ we get

$$\eta_2^{GV}(ca) = \frac{1}{|C_0 \times A|} \sum_{\substack{g \in G \\ u \in V}} \dot{\eta}\left((ca)^{gu}\right) \le \frac{1}{qp} \sum_{g \in N_0} p|C_V(g_0)|p = \frac{p}{q}|N_0||C_V(g_0)|,$$

and for $c=1$, $1 \ne a \in A$ we get

$$\eta_2^{GV}(ca) = \eta_2^{GV}(a) = \frac{1}{qp} \sum_{q \in N} |V|p = \frac{1}{q}|N||V|.$$

Thus if x_i $(i = 1, \ldots, k(GV))$ are representatives of the conjugacy classes of GV, then

$$\sum_{i=1}^{k(GV)} \eta_2^{GV}(x_i) \le (p-1)\frac{1}{q}|N||V| + (q-1)(p-1)\frac{p}{q}|N_0||C_V(g_0)|$$

and as in (1) we see that

$$\sum_{i=1}^{k(GV)} \eta_2^{GV}(x_i) = \sum_{\tau \in \mathrm{Irr}(GV)} [\tau_{C_0 \times A}\eta_2, \tau_{C_0 \times A}].$$

Now arguing as in (2), (3), (5) and (6) above yields

$$|\mathrm{Irr}_{p'}(GV)| \leq k(GV, v) \leq |\mathrm{Irr}(GV, C_0, v)| \leq \frac{1}{q}(|N||V| + (q-1)p|N_0||C_V(g_0)|),$$

where $\mathrm{Irr}(GV, C_0, v)$ is as defined at the beginning of 3.2. Putting the main results together, we have proved the following:

3.3 Theorem: *Let G be a finite group and let V be a finite faithful G-module of characteristic p. Let $v \in V$ and put $C = C_G(v)$. If c_i, $i = 1, \ldots, k(C)$, are representatives of the conjugacy classes of C, then the following hold:*

(a) *If $(|C|, |V|) = 1$, then*

$$|\mathrm{Irr}_0(GV)| \leq \sum_{i=1}^{k(C)} |C_V(c_i)|$$

and if $C > 1$, then

$$|\mathrm{Irr}_0(GV)| \leq \max\left\{\frac{2}{|C|}\sum_{i=1}^{k(C)} |C_V(c_i)|, \frac{2|C|}{k(C)-1}\sum_{i=2}^{k(C)} |C_V(c_i)|\right\}.$$

(b) *If $(|G|, |V|) = 1$, then*

$$\mathrm{Irr}_0(GV) = \mathrm{Irr}(G), \text{ so } k(GV) = |\mathrm{Irr}_0(GV)|$$

and the bounds in (a) hold true for $k(GV)$ instead of $|\mathrm{Irr}_0(GV)|$.

(c) *In general, if $g \in C$ such that $o(g) = q$ is a prime, then*

$$|\mathrm{Irr}_{p'}(GV)| \leq k(GV, v) \leq \frac{1}{q}\Big(|N_G(\langle v\rangle)||V| + (q-1)p|N_G(\langle g\rangle)||C_V(g)|\Big).$$

4. The dual approach

In the previous section, we always fixed $v \in V$ and obtained bounds on the size of suitable subsets of $\mathrm{Irr}(GV)$ in terms of properties of the action of $C_G(v)$ on V. In this section we consider a "dual" approach:

We fix $g \in G$ and find bounds in terms of the action of $C_G(g)$ on $C_V(g)$. For this, put

$$\begin{aligned}\mathrm{Irr}_g(GV) = \{\chi \in \mathrm{Irr}(G) : &\chi|_{\langle g\rangle \times C_V(g)} \text{ cannot be written as} \\ &\rho_{\langle g\rangle} \times \psi \text{ for a character } \psi \text{ of } C_V(g)\}.\end{aligned}$$

In particular, $\mathrm{Irr}(GV, g) \subseteq \mathrm{Irr}_g(GV)$.

4.1 Theorem: *Let G be a finite group and V be a finite G–module. Let $g \in G$ such that $(o(g), |V|) = 1$. Write $A = \langle g \rangle$, $N = N_G(A)$ and $C = C_V(g)$. Then*

(a) $|\mathrm{Irr}_g(GV)| \le \frac{(n(N,A)-1)n(C_G(A),C)}{(|A|-1)|C|} \max_{1 \ne a \in A}(|N_G(\langle a \rangle)||C_V(a)|)$;

(b) *if g is of prime order, then*

$$|\mathrm{Irr}_g(GV)| \le |C_G(A)| n(C_G(A), C);$$

(c) *there are $X, Y \subseteq \mathrm{Irr}_g(GV)$ such that $\mathrm{Irr}_g(GV)$ is a disjoint union of X and Y, and*

$$|X| \le \frac{(n(N,A)-1)n(C_G(A),C)}{(|A|-1)|C|^2} \max_{1 \ne a \in A}(|N_G(\langle a \rangle)||C_V(a)|) \text{ and}$$

$$|Y| \le \frac{(n(N,A)-1)(n(C_G(A),C)-1)}{(|A|-1)|C|} \max_{1 \ne a \in A}(|N_G(\langle a \rangle)||C_V(a)|);$$

(d) *if g is of prime order and X, Y are as in (c), then*

$$|X| \le \frac{|C_G(A)| n(C_G(A), C)}{|C|} \text{ and } |Y| \le |C_G(A)|(n(C_G(A), C) - 1).$$

Proof. If $a_1, a_2 \in A$ and $c_1, c_2 \in C - \{1\}$, then it is straightforward to see that $(a_1, c_1)^{GV} = (a_2, c_2)^{GV}$ implies that $a_1^G = a_2^G$. Hence, if T is a set of representatives of the orbits of N on $A - \{1\}$, then every conjugacy class of GV that intersects nontrivially with $(A - \{1\}) \times C$ has a representative ac for some $a \in T$ and $c \in C$. Moreover, for each $a \in T$ we have that if c_3, $c_4 \in C$ are $C_G(A)$-conjugate, then ac_3 and ac_4 are $C_G(A)$-conjugate and thus $(ac_3)^G = (ac_4)^G$.

This shows that for each $a \in T$ there are at most $n(C_G(A), C)$ conjugacy classes of GV intersecting nontrivially with $\{a\} \times C$. Hence, altogether we see that there are at most

$$|T| n(C_G(A), C) = (n(N, A) - 1) n(C_G(A), C) \tag{1}$$

conjugacy classes of GV which intersect $(A - \{1\}) \times C$ nontrivially.

Moreover, observe that for $1 \ne a \in A$, $c \in C$, $h \in G$ and $u \in V$ we have

$$(ac)^{hu} \in A \times C \text{ if and only if } h \in N_G(\langle a \rangle),\ c^h \in C \text{ and } u \in C_V(a)$$

because the condition $(ac)^{hu} = a^h[a^h, u]c^h \in A \times C$ first forces $a^h \in A$ which implies (as A is cyclic) $a^h \in \langle a \rangle$, so $h \in N_G(\langle a \rangle)$, and then as $c \in C \le C_V(\langle a \rangle)$, it follows that $c^h \in C_V(\langle a \rangle)$ and $[a^h, u] \in [\langle a \rangle, V]$. Now, by our hypothesis, we have $V = C_V(\langle a \rangle) \times [\langle a \rangle, V]$, we see that $(ac)^{hu} \in A \times C$ now forces $[a^h, u] = 1$ and $c^h \in C$. Hence $u \in C_V(a^h) = C_V(a)$.

Note that the direct product $A \times C$ is a subgroup of GV. We now define a generalized character η on $A \times C$ by

$$\eta = (|A| \cdot 1_A - \rho_A) \times 1_C,$$

where ρ_A is the regular character of A. So for $a \in A$, $c \in C$ we have

$$\eta(ac) = \begin{cases} 0, & a = 1 \\ |A|, & a \neq 1. \end{cases}$$

Therefore, η^{GV} vanishes on all conjugacy classes of GV which intersect $(A - \{1\}) \times C$ trivially, whereas for $c \in C$ and $1 \neq a \in A$ we have

$$\begin{aligned} \eta^{GV}(ac) &= \frac{1}{|A \times C|} \sum_{\substack{h \in G \\ u \in V}} \dot{\eta}((ac)^{hu}) = \frac{1}{|A||C|} \sum_{\substack{h \in N_G(\langle a \rangle) \\ \text{with } c^h \in C}} \sum_{u \in C_V(a)} \eta((ac)^{hu}) \\ &= \frac{1}{|A||C|} \sum_{\substack{h \in N_G(\langle a \rangle) \\ \text{with } c^h \in C}} \sum_{u \in C_V(a)} \eta(a^h c^h) \\ &= \frac{|C_V(a)|}{|A||C|} \sum_{\substack{lh \in N_G(\langle a \rangle) \\ \text{with } c^h \in C}} |A| \\ &\leq (|N_G(\langle a \rangle)||C_V(a)|)/|C|. \end{aligned} \tag{2}$$

Thus if $\{x_i \mid i = 1, \ldots, k(GV)\}$ is a set of representatives for the conjugacy classes of GV, then by (1) and (2) we see that

$$\begin{aligned} (n(N, A) - 1)n(C_G(A), C) \cdot \frac{1}{|C|} \max_{1 \neq a \in A} (|N_G(\langle a \rangle)||C_V(a)|) \\ \geq \sum_{i=1}^{k(GV)} \eta^{GV}(x_i) \\ = \sum_{\tau \in \mathrm{Irr}(GV)} [\tau \eta^{GV}, \tau] \\ = \sum_{\tau \in \mathrm{Irr}(GV)} [\tau_{A \times C} \eta, \tau_{A \times C}]. \end{aligned} \tag{3}$$

Observe that in case that A is of prime order, then

$$n(N, A) - 1 = \frac{|A| - 1}{|N : C_G(A)|} = \frac{(|A| - 1)|C_G(A)|}{|N|}$$

and $\max_{1\neq a\in A}(|N_G(\langle a\rangle)||C_V(a)|)=|N||C|$, so that (3) becomes

$$|C_G(A)|(|A|-1)n(C_G(A),C)\geq\sum_{\tau\in\mathrm{Irr}(GV)}[\tau_{A\times C}\eta,\tau_{A\times C}]. \tag{3a}$$

Since $A\times C$ is a direct product, we can write

$$\tau_{A\times C}=\sum_{\lambda\in\mathrm{Irr}(C)}(\tau_\lambda\times\lambda),$$

where τ_λ is a character of A or $\tau_\lambda=0$. Then

$$\begin{aligned}
[\tau_{A\times C}\eta,\tau_{A\times C}]&=\frac{1}{|A\times C|}\sum_{\substack{a\in A\\ c\in C}}\tau(ac)\eta(ac)\overline{\tau(ac)}=\frac{1}{|A||C|}\sum_{\substack{1\neq a\in A\\ c\in C}}\tau(ac)|A|\overline{\tau(ac)}\\
&=\frac{1}{|C|}\sum_{\substack{1\neq a\in A\\ c\in C}}\sum_{\lambda\in\mathrm{Irr}(C)}\tau_\lambda(a)\lambda(c)\sum_{\mu\in\mathrm{Irr}(C)}\overline{\tau_\mu(a)\mu(c)}\\
&=\sum_{1\neq a\in A}\sum_{\lambda,\mu\in\mathrm{Irr}(C)}\tau_\lambda(a)\overline{\tau_\mu(a)}\frac{1}{|C|}\sum_{c\in C}\lambda(c)\overline{\mu(c)}\\
&=\sum_{1\neq a\in A}\sum_{\lambda,\mu\in\mathrm{Irr}(C)}\tau_\lambda(a)\overline{\tau_\mu(a)}[\lambda,\mu].
\end{aligned}$$

As $[\lambda,\mu]=\begin{cases}1, & \lambda=\mu\\ 0, & \lambda\neq\mu\end{cases}$, we further obtain

$$\begin{aligned}
[\tau_{A\times C}\eta,\tau_{A\times C}]&=\sum_{1\neq a\in A}\sum_{\lambda\in\mathrm{Irr}(C)}\tau_\lambda(a)\overline{\tau_\lambda(a)}\\
&=\sum_{\lambda\in\mathrm{Irr}(C)}\sum_{1\neq a\in A}|\tau_\lambda(a)|^2.
\end{aligned} \tag{4}$$

Now observe that $\tau(1)=\sum_{\lambda\in\mathrm{Irr}(C)}\tau_\lambda(1)$.
If all the τ_λ are multiples of ρ_A, then clearly $\tau_1\notin\mathrm{Irr}_g(GV)$, and so if $\tau\in\mathrm{Irr}_g(GV)$, then by [5, Corollary 4] with (4) we see that

$$[\tau_{A\times C}\eta,\tau_{A\times C}]\geq|A|-1. \tag{5}$$

So (3) and (5) yield

$$|\mathrm{Irr}_g(GV)|\leq\frac{(n(N,A)-1)n(C_G(A),C)}{(|A|-1)|C|}\max_{1\neq a\in A}(|N_G(\langle a\rangle)||C_V(a)|), \tag{6}$$

and if g is of prime order, then (3a) and (5) yield

$$|\mathrm{Irr}_g(GV)|\leq|C_G(A)|n(C_G(A),C). \tag{6a}$$

Now as in Section 3, we repeat the same arguments, but use

$$\eta_1 = (|A|1_A - \rho_A) \times (|C|1_C - \rho_C)$$

instead of η.
One can then easily check that

$$\text{(3b)} \quad (n(N,A)-1)(n(C_G(A),C)-1) \cdot \frac{1}{|C|} \max_{1 \neq a \in A} (|N_G(\langle a \rangle)||C_V(a)|) \geq \sum_{\tau \in \mathrm{Irr}(GV)} [\tau_{A\times C}\eta_1, \tau_{A\times C}]$$

and if g is of prime order, then

$$\text{(3c)} \qquad |C_G(A)|(|A|-1)(n(C_G(A),C)-1) \geq \sum_{\tau \in \mathrm{Irr}(GV)} [\tau_{A\times C}\eta_1, \tau_{A\times C}].$$

Moreover it is easily seen that

$$\begin{aligned}[]
[\tau_{A\times C}\eta_1, \tau_{A\times C}] &= \sum_{\substack{1\neq a\in A \\ 1\neq c\in C}} \tau(ac)\overline{\tau(ac)} \\
&= \sum_{1\neq a\in A} \sum_{\lambda,\mu\in \mathrm{Irr}(C)} \tau_\lambda(a)\overline{\tau_\mu(a)} \sum_{1\neq c\in C} \lambda(c)\overline{\mu(c)},
\end{aligned}$$

and as $\sum_{1\neq c\in C} \lambda(c)\overline{\mu(c)} = \begin{cases} -1, & \text{if } \lambda \neq \mu \\ |C|-1, & \text{if } \lambda = \mu \end{cases}$, it follows that

$$\text{(7)} \qquad [\tau_{A\times C}\eta_1, \tau_{A\times C}] = \sum_{\lambda<\mu} \sum_{1\neq a\in A} |\tau_\lambda(a) - \tau_\mu(a)|^2,$$

where "$\leq$" is an arbitrary ordering on $\mathrm{Irr}(C)$.

Next, suppose that there are exactly a characters $\tau \in \mathrm{Irr}_g(GV)$ such that there is a character ψ of A (depending on τ) and there are $a_\lambda \in \mathbb{Z}$ ($\lambda \in \mathrm{Irr}(C)$) such that $\tau_\lambda = \psi + a_\lambda \rho_A$ for all $\lambda \in \mathrm{Irr}(C)$ and ψ is not a multiple of ρ_A. Then by (4) and [5, Corollary 4] we know that

$$[\tau_{A\times C}\eta, \tau_{A\times C}] = \sum_{\lambda\in\mathrm{Irr}(C)} \sum_{1\neq a\in A} |\psi(a)|^2 \geq |C|(|A|-1)$$

and hence by (3) we get

$$\text{(8)} \qquad a \leq \frac{(n(N,A)-1)n(C_G(A),C)}{(|A|-1)|C|^2} \max_{1\neq a\in A}(|N_G(\langle a\rangle)||C_V(a)|),$$

and if g is of prime order, then by (3a) even

$$a \leq \frac{|C_G(A)|n(C_G(A),C)}{|C|}. \tag{8a}$$

Now let b be the number of $\tau \in \mathrm{Irr}_g(GV)$ such that there is no such ψ. Then there exist $\lambda, \mu \in \mathrm{Irr}(C)$ with

$$\sum_{1 \neq a \in A} |\tau_\lambda(a) - \tau_\mu(a)|^2 \neq 0,$$

and thus by [5, Corollary 4] we have

$$[\tau_{A\times C}\eta_1, \tau_{A\times C}] \geq |A| - 1. \tag{9}$$

So (3b) and (9) yield

$$b \leq \frac{(n(N,A)-1)(n(C_G(A),C)-1)}{|C|(|A|-1)} \max_{1\neq a \in A}(|N_G(\langle a\rangle)||C_V(a)|) \tag{10}$$

and, if g is of prime order, then by (3c)

$$b \leq |C_G(A)|(n(C_G(A),C)-1), \tag{10b}$$

and clearly $a + b = |\mathrm{Irr}_g(GV)|$, and all the assertions follow and we are done. ∎

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