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# SCHMIDT'S GAME ON FRACTALS

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ABSTRACT

We construct $(\alpha, \beta)$ and $\alpha$-winning sets in the sense of Schmidt's game, played on the support of certain measures (absolutely friendly) and show how to compute the Hausdorff dimension for some.

In particular, we prove that if $K$ is the attractor of an irreducible finite family of contracting similarity maps of $\mathbb{R}^N$ satisfying the open set condition, (the Cantor's ternary set, Koch's curve and Sierpinski's gasket to name a few known examples), then for any countable collection of non-singular affine transformations, $\Lambda_i : \mathbb{R}^N \to \mathbb{R}^N$,

$$\dim K = \dim K \cap \left( \bigcap_{i=1}^{\infty} (\Lambda_i(\mathbf{BA})) \right)$$

where $\mathbf{BA}$ is the set of badly approximable vectors in $\mathbb{R}^N$.

## 0. Introduction

We shall be using Schmidt's game first introduced by W. M. Schmidt [S1] for estimating the Hausdorff dimension of certain sets. Let us first define the set of badly approximable vectors. A vector $\mathbf{x} \in \mathbb{R}^N$ is said to be badly approximable if there exists $\delta > 0$ such that for any $\mathbf{p} \in \mathbb{Z}^N$, $q \in \mathbb{N}^+$

$$d\left(\mathbf{x}, \frac{\mathbf{p}}{q}\right) \geq \delta q^{-\frac{N+1}{N}}, \tag{0.1}$$

where $d$ is the Euclidean distance function between points. We denote the set of all badly approximable vectors by $\mathbf{BA}$. The above mentioned game was used

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by Schmidt, among other things, to tackle the following questions concerning **BA**:

(1) If $\{\Lambda_i\}_{i=0}^{\infty}$ is a countable collection of non-singular affine transformations
$\Lambda_i : \mathbb{R}^N \to \mathbb{R}^N$, is $\bigcap_{i=1}^{\infty}(\Lambda_i(\mathbf{BA})) \neq \emptyset$?

(2) If $\bigcap_{i=1}^{\infty}(\Lambda_i(\mathbf{BA})) \neq \emptyset$, what is $\dim\bigcap_{i=1}^{\infty}(\Lambda_i(\mathbf{BA}))$?

Schmidt proved not only that the intersection is non-empty, but is, in fact, "large" dimensionwise, i.e., is of dimension $N$.

In recent years similar questions have been posed regarding the intersection of **BA** with certain subsets of $\mathbb{R}^N$. For example, let $K$ be any of the following sets: Cantor's ternary set, Koch's curve, Sierpinski's gasket, or in general, an attractor of an irreducible finite family of contracting similarity maps of $\mathbb{R}^N$ satisfying the open set condition. (This condition, due to J. E. Hutchinson [H], is discussed in Section 5). One may ask the following questions:

(1) Is $K \cap \mathbf{BA} \neq \emptyset$?

(2) If $K \cap \mathbf{BA} \neq \emptyset$, what is $\dim K \cap \mathbf{BA}$?

Answers to both of these questions have been independently given in [KW] and [KTV] proving that $\dim K \cap \mathbf{BA} = \dim K$ for a large family of sets including those mentioned above.

This paper's aim is to extend these results, utilizing Schmidt's game, by answering the following question: If $\{\Lambda_i\}_{i=0}^{\infty}$ is a countable collection of non-singular affine transformations $\Lambda_i : \mathbb{R}^N \to \mathbb{R}^N$, then what is

$$\dim K \cap \left(\bigcap_{i=1}^{\infty}(\Lambda_i(\mathbf{BA}))\right)?$$

It turns out that for a large family of sets the answer is analogous to Schmidt's result in $\mathbb{R}^N$, namely, we prove in Section 5

Corolarry 5.4: *Let $\{\phi_1, ..., \phi_k)$ be a finite irreducible family of contracting similarity maps of $\mathbb{R}^N$ satisfying the open set condition and let $K$ be its attractor. Then for any countable collection of non-singular affine transformations $\{\Lambda_i\}_{i=0}^{\infty}$, $\Lambda_i : \mathbb{R}^N \to \mathbb{R}^N$ the set*

$$\mathcal{S} = K \cap \left(\bigcap_{i=1}^{\infty}(\Lambda_i(\boldsymbol{BA}))\right)$$

*is a winning set on $K$. Furthermore, $\dim S = \dim K$.*

Our research follows closely in the footsteps of [KLW], [KW] and, consequently, [PV] and [KTV]. The definitions of measures given in the first and third of the above mentioned papers were not originally intended for creating a "friendly" environment for Schmidt's game on their support. It turns out however that in a sense to be made clearer later, these measures indeed provide an hospitable playground for this game.

Section 1 is devoted to establishing the link between the definitions given in [KLW], the stronger assumptions in [PV] and our work, exhibiting a geometric feature material for later discussion.

In Section 2, we follow the general setup introduced in [KTV] proving as a consequence of Corollary 2.1 and Theorem 2.2 that if a measure $\mu$ is absolutely friendly (see definition in Section 1) then under certain conditions

$$BA \cap \text{supp}(\mu) \text{ is an } (\alpha, \beta) - \text{winning set on } \text{supp}(\mu).$$

In Section 3, we formulate a sufficient condition for establishing a lower bound of a winning set's Hausdorff dimension, where the winning set is a subset of the support of an absolutely friendly measure.

In Section 4, we prove an analogue to the simplex lemma in [S1].

Section 5 is our main example, an application to the Hutchinson construction.

As should be obvious from the discussion above, our conclusions strengthen results in [KW] and [KTV] regarding the Hausdorff dimension of the intersection of BA with certain sets. (See Corollary 1.2 in [KW] and conclusions from Theorem 1 in [KTV]). We should note however that in proving our theorems we are in fact using stronger assumptions on our measures in order to make sure that our target set—the set of badly approximable vectors, is indeed a winning set on the support of these measures.

Notation. $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the set of real, rational and natural numbers respectively.

$\mathbb{R}^+$ is the set of non-negative real numbers and $\mathbb{N}^+$ denotes the set of strictly positive integers.

Boldface lower case letters ($\mathbf{x}$, $\mathbf{y}$,... etc.) denote points in $\mathbb{R}^N$.

The function $d$ is the Euclidean distance function between points. If $A$ and $B$ are any two subsets of $\mathbb{R}^N$, $d(A,B) = \inf\{d(\mathbf{x},\mathbf{y}) : \mathbf{x} \in A, \mathbf{y} \in B\}$.

$\lambda_N$ denotes the Lebesgue measure in $\mathbb{R}^N$.

In the metric space $(\mathbb{R}^N, d)$, $B(x,r)$ denotes a closed ball of radius $r$ centered at $x$, i.e., $B(x,r) = \{z : d(x,z) \leq r\}$, $\partial B(x,r)$ the boundary of $B(x,r)$, i.e., $\{z : d(z,x) = r\}$ and $intB(x,r)$ denotes the interior of $B(x,r)$ i.e. $\{z : d(x,z) < r\}$.

An affine hyperplane of $\mathbb{R}^N$ will be denoted by $\mathcal{L}$ while $\mathcal{L}^{(\epsilon)}$ is defined to be the $\epsilon$ neighborhood of $\mathcal{L}$, i.e. $\mathcal{L}^{(\epsilon)} = \{\mathbf{x} \in \mathbb{R}^N : d(\mathbf{x}, \mathcal{L}) \leq \epsilon\}$ where $\epsilon$ is a non-negative, possibly zero, real number.

Unless otherwise stated, constants are real, strictly positive numbers.

Throughout the paper, $\mu$ will denote a Borel, locally finite measure on $\mathbb{R}^N$.

Whenever discussing a measure we denote its support by $\mathrm{supp}(\mu)$.

In order to avoid unnecessary repetitions, all affine transformations referred to in this paper are assumed to be non-singular.

Following conventional notation, for every $U \subset \mathbb{R}^N$ let

$$|U| = \sup\{d(\mathbf{x},\mathbf{y}) : x, y \in U\}.$$

If $F \subset \mathbb{R}^N$, $\delta > 0$ and $\{U_i\}$ is a countable or finite collection of sets we say that $\{U_i\}$ is a $\delta$-cover of $F$ if

$$F \subset \bigcup_{i=1}^{\infty} U_i \quad \text{and for every} \quad i\, 0 \leq |U_i| \leq \delta.$$

If $F \subset \mathbb{R}^N$ and $s \geq 0$, then for every $\delta > 0$ we define

$$H^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of F}\right\}$$

and

$$H^s(F) = \lim_{\delta \to 0} H^s_\delta(F)$$

is the $s$-Hausdorff measure.

The Hausdorff dimension of a set $F \subset \mathbb{R}^N$ is defined by

$$\dim F = \inf\{s : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}.$$

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## 1. Absolutely friendly measures

We first define absolutely friendly measures and show how it provides the right setting for our work and results. The class of friendly measures was first introduced in [KLW], followed by the more restrictive $\alpha$-absolutely friendly measures in [PV]. The definition of absolutely friendly coincides with that of $\alpha$-absolutely friendly, but as the constant $\alpha$ does not seem to have any special status in any of the formulas we use, we decided to use the term absolutely friendly instead.

*Definition 1:* Call a measure $\mu$ on $\mathbb{R}^N$ **absolutely friendly** if the following conditions are satisfied:

There exist constants $r_0$, $C$, $D$ and $a$ such that for every $0 < r \leq r_0$ and for every $\mathbf{x} \in \mathrm{supp}(\mu)$:

(i) for any $0 \leq \epsilon \leq r$, and any affine hyperplane $\mathcal{L}$,

$$\mu(B(\mathbf{x}, r) \cap \mathcal{L}^{(\epsilon)}) < C(\epsilon/r)^a \mu(B(\mathbf{x}, r)).$$

(ii) $\mu(B(\mathbf{x}, \frac{5}{6}r)) > D\mu(B(\mathbf{x}, r))$.

Two remarks are in order.

*Remark 1:* Notice that part (ii) of the above definition is equivalent (up to a change of the constant $D$) to the so called "Federer doubling property" with 1/2 replacing 5/6.

*Remark 2:* The reader should compare (i) with the following more general definition (Definition 2.5 in [KLW]), namely, given $C$, $a > 0$ and an open subset $U$ of $\mathbb{R}^N$ we say that $\mu$ is **absolutely $(C, a)$ -decaying on U** if for any nonempty open ball $B \subset U$ centered in $\mathrm{supp}(\mu)$, any affine hyperplane $\mathcal{L} \subset \mathbb{R}^N$ and any $\epsilon > 0$ one has

$$\mu(B \cap \mathcal{L}^{(\epsilon)}) \leq C\,(\epsilon/r)^a\,\mu(B) \tag{1.2}$$

where $r$ is the radius of $B$.

As a consequence of Definition 1 we prove the following

LEMMA 1.1: *Suppose $\mu$ is absolutely friendly with constants as in Definition 1. Define $(D/C)^{1/a} = \alpha'$ and let $\mathcal{L}$ be any affine hyperplane. Then for every $0 < r \leq r_0$, if $0 < \alpha < \frac{1}{12}\alpha'$ and $0 \leq \epsilon_0 < \frac{1}{12}\alpha' r$, we have that for every $\mathbf{x} \in supp(\mu)$ there exists $\mathbf{x}_0 \in supp(\mu)$ such that*

1. $B(\mathbf{x}_0, \alpha r) \subset B(\mathbf{x}, r)$.
2. $d(B(\mathbf{x}_0, \alpha r), \mathcal{L}^{(\epsilon_0)}) > \alpha r$.
3. $d(B(\mathbf{x}_0, \alpha r), \partial B(\mathbf{x}, r)) > \alpha r$.

*Proof.* If $d(\mathbf{x}, \mathcal{L}^{(\epsilon_0)}) > 2\alpha r$ the first two conditions are evidently satisfied by choosing $\mathbf{x}_0 = \mathbf{x}$ while for the third notice that $r - \alpha r > \frac{11}{12} r > 2\alpha r$.

Otherwise let $d(\mathbf{x}, \mathcal{L}^{(\epsilon_0)}) \leq 2\alpha r$.

Let $\delta = 1 - \alpha$, $\epsilon = 5\alpha r + 2\epsilon_0$ and denote by $\mathcal{L}_{\mathbf{x}}$ an affine hyperplane parallel to $\mathcal{L}$ passing through $\mathbf{x}$. We observe that

$$\delta r - \epsilon = (1 - 6\alpha)r - 2\epsilon_0 > \left(1 - \frac{5}{6}\alpha'\right)r - \frac{1}{6}\alpha' r = (1 - \alpha')r \geq 0 \tag{1.3}$$

$$\mu(B(\mathbf{x}, \delta r)) = \mu(B(\mathbf{x}, (1 - \alpha)r)) \geq \mu\Big(B\Big(\mathbf{x}, \frac{5}{6}r\Big)\Big) \geq D\mu(B(\mathbf{x}, r)) \tag{1.4}$$

(1.5)
$$\mu(\mathcal{L}_{\mathbf{x}}^{(\epsilon)} \cap B(\mathbf{x}, r)) \leq C\left(\frac{\epsilon}{r}\right)^a \mu(B(\mathbf{x}, r)) = C\Big(5\alpha + \frac{2\epsilon_0}{r}\Big)^a \mu(B(\mathbf{x}, r))$$
$$< C(\frac{31}{36}\alpha')^a \mu(B(\mathbf{x}, r)) < C(\alpha')^a \mu(B(\mathbf{x}, r)) \leq D\mu(B(\mathbf{x}, r)).$$

Consequently, denoting by $\Xi = B(\mathbf{x}, \delta r) - \mathcal{L}_{\mathbf{x}}^{(\epsilon)}$, we have $\mu(\Xi \cap B(\mathbf{x}, r)) > 0$ and we may choose $\mathbf{x}_0$ to be any point in $\Xi \cap \mathrm{supp}(\mu)$.

The first condition is fulfilled by our choice of $\delta$. As for the second condition notice that for any $\mathbf{y} \in \Xi$ we have $d(\mathbf{y}, \mathcal{L}^{(\epsilon_0)}) \geq \epsilon - (2\alpha r + 2\epsilon_0) \geq 3\alpha r$. As $d(\Xi, \partial B(\mathbf{x}, r)) = \frac{1}{6} r > 2\alpha r$ the third condition is satisfied as well. ∎

## 2. Friendly Schmidt's game

Let $(X, d)$ be a complete metric space and let $\mathcal{S} \subset X$ be a given set (a target set). **Schmidt's game** [S1] is played by two players $A$ and $B$, each equipped with parameters $\alpha$ and $\beta$ respectively, $0 < \alpha, \beta < 1$. The game starts with player $B$ choosing $y_0 \in X$ and $r > 0$ hence specifying a closed ball $B_0 = B(y_0, r)$. Player $A$ may now choose any point $x_0 \in X$ provided that $A_0 = B(x_0, \alpha r) \subset B_0$. Next, player $B$ chooses a point $y_1 \in X$ such that $B_1 = B(y_1, (\alpha\beta)r) \subset A_0$. Continuing in the same manner we have a nested sequence of non-empty closed sets $B_0 \supset A_0 \supset B_1 \supset A_1 \supset \cdots \supset B_k \supset A_k \supset \cdots$ with diameters tending to zero as $k \to \infty$. As the game is played on a complete metric space, the intersection of these balls is a point $z \in X$. Call player $A$ the winner if $z \in \mathcal{S}$. Otherwise, player $B$ is declared winner. A strategy consists of specifications for a player's choices of centers for his balls as a consequence of his opponent's previous moves. If for certain $\alpha$ and $\beta$ player A has a winning strategy, i.e., a strategy for winning the game regardless of how well player B plays, we say that $\mathcal{S}$ is an **$(\boldsymbol{\alpha}, \boldsymbol{\beta})$-winning** set. If it so happens that $\alpha$ is such that $\mathcal{S}$ is an $(\alpha, \beta)$-winning set for all $0 < \beta < 1$, we say that $\mathcal{S}$ is an **$\boldsymbol{\alpha}$-winning** set. Call a set **winning** if such an $\alpha$ exists.

We define the following (target) set. This definition is a modification of the one given in [KTV].

*Definition 2:* Suppose $\Omega \subset \mathbb{R}^N$ and let $\mathcal{U} = \{U_j \subset \mathbb{R}^N : j \in \mathbb{N}\}$ be a family of subsets of $\mathbb{R}^N$. If $I : \mathbb{N} \to \mathbb{R}^+$ is an increasing function tending to infinity as $j$ tends to infinity and $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $\rho(r) \to 0$ as $r \to \infty$ and decreasing for large enough $r$, let

$$\mathbf{Bad}^*(\mathcal{U}, I, \rho, \Omega) = \{\mathbf{x} \in \Omega : \exists \delta > 0 \text{ such that } d(\mathbf{x}, U_j) \geq \delta\rho(I(j))\ \forall j \in \mathbb{N}\} .$$

As an immediate consequence of the above definition we get:

Corollary 2.1: *For $\Omega \subset \mathbb{R}^N$, and $j \in \mathbb{N}^+$ defining $U_j = \left\{\mathbf{p}/j : \mathbf{p} \in \mathbb{Z}^N\right\}$, $I(j) = j$ and $\rho(I(j)) = j^{-(N+1)/N}$, we have*

$$\boldsymbol{BA} \cap \Omega = \boldsymbol{Bad}^*(\mathcal{U}, I, \rho, \Omega)$$

In the following theorem we shall show that under certain assumptions, $\mathbf{Bad}^*(\mathcal{U}, I, \rho, \Omega)$ is an $(\alpha, \beta)$-winning set.

Theorem 2.2: *Suppose $\mu$ is absolutely friendly (with constants as in Definition 1) and $(D/C)^{1/a} = \alpha'$. Let $\Omega = supp(\mu)$ and suppose $F : \mathbb{N} \to \mathbb{R}^+$ is an increasing function, with $F(k) \to \infty$ as $k \to \infty$. Define $F^0 = [0, F(0))$ and $F^k = [F(k-1), F(k))$ for any $k > 0$. Let $\mathcal{U} = \{U_j \subset \mathbb{R}^N : j \in \mathbb{N}\}$ be a family of subsets of $\mathbb{R}^N$.*

*Suppose $0 < \beta < 1$ and $0 < \alpha < \frac{1}{12}\alpha'$ satisfy:*

(1) *for every $k, l \in \mathbb{N}$, for every $\mathbf{x} \in supp(\mu)$ and for every $r \le r_0$, if $I(j_1), \ldots, I(j_l) \in F^k$ then $\left(\bigcup_{i=1}^{l} U_{j_i}\right) \bigcap B(\mathbf{x}, (\alpha\beta)^k r) \subset \mathcal{L}$ for some affine hyperplane $\mathcal{L}$,*
(2) *for every $k$, $(\alpha\beta)^k \ge \rho(F(k))$.*

*Then $\boldsymbol{Bad}^*(\mathcal{U}, I, \rho, \Omega)$ is an $(\alpha, \beta)$-winning set on $\Omega$.*

*Proof.* Player A's strategy is to play in an arbitrary manner until the the first ball of radius $r_I \le r_0$ is chosen by player B. Let $k_0 \in \mathbb{N}$ be such that $\beta^{k_0+1} r_0 < r_I \le \beta^{k_0} r_0$. Set $\delta = (\alpha\beta)^{k_0+1}\beta^{k_0} r_0$ and let $r' = (\alpha\beta)^{k_0} r_I$.

We "reset" our counter and specify player A's strategy from this point on. At his $k$-th move player $A$ has to choose a point $\mathbf{x} \in \mathrm{supp}(\mu)$ such that $A_k = B(\mathbf{x}, \alpha(\alpha\beta)^k r') \subset B_k = B(\mathbf{y}, (\alpha\beta)^k r')$ where $\mathbf{y} \in \mathrm{supp}(\mu)$ is player B's $k$-th choice. Let $\mathcal{U}_j = \bigcup_{i=1}^{l} U_{j_i}$ where $I(j_1), \ldots, I(j_l) \in F^k$.

(a) If $\mathcal{U}_j \cap B(\mathbf{y}, (\alpha\beta)^k r') = \emptyset$, player A may choose $\mathbf{x} = \mathbf{y}$.
By Lemma 1.1(3)

$$d\left(\mathcal{U}_j, A_k\right) > \alpha(\alpha\beta)^k r' \ge \delta(\alpha\beta)^k \ge \delta\rho(F(k)) > \delta\rho(I(j)).$$

(b) Otherwise, suppose $\mathcal{U}_j \cap B(\mathbf{y}, (\alpha\beta)^k r') \ne \emptyset$. by Lemma 1.1(2) player A can pick a point $\mathbf{x} = \mathbf{x}_k$ such that

$$d\left(\mathcal{U}_j \cap B(\mathbf{y}, (\alpha\beta)^k r'), A_k\right) > \alpha(\alpha\beta)^k r' > \delta\rho(I(i)).$$

Furthermore, if $\mathcal{U}_j - B(\mathbf{y}, (\alpha\beta)^k r') \ne \emptyset$ then by Lemma 1.1(3)

$$d\left(\mathcal{U}_j - B\left(\mathbf{y}, (\alpha\beta)^k r'\right), A_k\right) > \alpha(\alpha\beta)^k r' > \delta\rho(I(i)). \qquad \blacksquare$$

The following proposition, due to W. M. Schmidt [S1, Theorem 2], is material for later considerations.

Proposition 2.3: *The intersection of countably many $\alpha$-winning sets is $\alpha$-winning.*

## 3. Full Hausdorff dimension

We are now in position to formulate a sufficient condition for establishing a lower bound of a winning set's Hausdorff dimension, where the winning set is a subset of the support of an absolutely friendly measure.

The main ideas in this section are due to W. M. Schmidt [S1]. We have decided nonetheless to include the definitions, results and proofs for the sake of a clearer understanding of the connection to the previous definitions and results.

*Definition 3:* For a metric space $(X,d)$, given $x \in X$, and real numbers $r > 0$, $0 < \beta < 1$, denote by $N_X(\beta, x, r)$ the maximum number of disjoint balls of radius $\beta r$ contained in $B(x,r)$.

THEOREM 3.1: *Let $\mu$ be absolutely friendly and denote $X = supp(\mu)$. Suppose the following condition is satisfied:*

*There exists constants $r_1 \leq 1$, $M$ and $\delta$ such that for every $0 < r \leq r_1$, $0 < \beta < 1$ and $\mathbf{x} \in X$,*

$$N_X(\beta, \mathbf{x}, r) \geq M\beta^{-\delta}. \tag{3.6}$$

*Then if $\mathcal{S}$ is a winning set on $(X,d)$ then $dim\mathcal{S} \geq \delta$.*

In the course of the proof we shall use the following auxiliary lemma. (Lemma 20 in [S1]).

PROPOSITION 3.2: *Let $\mathcal{H}$ be a Hilbert space and let $w_0 = 2\sqrt{3} - 1$. For any $r \in \mathbb{R}^+$ let $\mathcal{M}$ be any collection of balls $\{B(x_i, r) : i \in \mathbb{N},\ x_i \in \mathcal{H}\}$ such that*

$$intB(x_i, r) \cap intB(x_j, r) = \emptyset \quad \text{for every } i \neq j.$$

*Then for any $r_0 < w_0 r$ and $x \in \mathcal{H}$ the ball $B(x, r_0)$ has a non empty intersection with at most two balls from $\mathcal{M}$.*

*Proof of Theorem 3.1.* Let $\mu$ be an absolutely friendly measure satisfying condition (3.6) and $\beta \leq (M/2)^{1/\delta}$. Thus $N_X(\beta, \mathbf{x}, r) \geq 2$ for every $\mathbf{x} \in \mathrm{supp}(\mu)$. In order to estimate the Hausdorff dimension of a winning set $\mathcal{S}$ assume player $A$ is playing to win the game using some strategy. This means that given choices of balls $B_0 \supset A_0 \supset \cdots A_{k-1} \supset B_k$, played by the two players prior to player $A$'s k-th turn, the strategy of player $A$ applies his strategy and chooses a ball $A_k \subset B_k$. Since the strategy is winning, $\bigcap A_k = \bigcap B_k$ will be in $\mathcal{S}$ regardless of player $B$'s choices. Here we will describe many possible strategies for player $B$, resulting in many points in $\mathcal{S}$.

We consider the game from the loser's point of view, player $B$. Fix $\beta$ such that

$$2 \le N(\beta) = \min \{ N_X(\beta, \mathbf{x}, r) : \mathbf{x} \in X,\ 0 < r \le r_1 \}.$$

At each stage of the game player $B$ may direct the game to $N(\beta)$ disjoint balls and we restrict his moves to these $N(\beta)$ choices. Thus, for each sequence of choices made by player $B$ with the restriction above, we obtain a parametrization of the sequence of balls chosen by him. Let $B_0$ be his initially chosen ball, and for $k \in \mathbb{N}^+$, corresponding to his k-th move, let $B_k = B_k(j_1, \ldots, j_k)$, with $j_i \in \{0, \ldots, N(\beta) - 1\}$ $i = 1, 2, \ldots, k$. Notice also that given a sequence of positive integers $i_1$,$i_2$,... there is a *unique* point $x = x(i_1, i_2, \ldots)$ contained in *all* balls $B_k = B_k(j_1, \ldots, j_k)$. By considering the $N(\beta)$ ways in which player $B$ may direct the game we consider the function

$$f : \{0, ..., N(\beta) - 1\}^{\mathbb{N}} \to \mathcal{S}, (t_k)_{k \in \mathbb{N}} \mapsto \bigcap_{k \in \mathbb{N}} B_k(t_1, \ldots, t_k) = \{x(t)\}.$$

As every number in the closed unit interval has at least one expansion in base $N(\beta)$ we map the image of $f$, $\mathcal{S}^* \subset \mathcal{S}$ onto $[0, 1]$ by

$$g : \mathcal{S}^* \to [0, 1], x(t) \mapsto 0.t_1 t_2 \ldots.$$

In view of Proposition 3.2, for $0 < w < w_0$ and $0 < \alpha < 1$ any ball of radius $w(\alpha\beta)^k$ intersects at most two of the balls $B_k(j_1, \ldots, j_k)$. Let $\mathcal{C} = \{C_l\}_{l \in \mathbb{N}}$ be a cover of $\mathcal{S} \cap K$ of balls with radius $\rho(C_l) = \rho_l$. As $\mathcal{C}$ covers $\mathcal{S}^*$ we have that $g(\mathcal{C})$ covers $[0, 1]$. Let $\overline{\lambda}$ denote the outer Lebesgue measure. We have

$$\sum_{l=1}^{\infty} \overline{\lambda}(g(C_l)) \ge \overline{\lambda}\left( \bigcup_{l=1}^{\infty} g(C_l) \right) \ge 1. \tag{3.7}$$

Define integers

$$k_l = [k_l^*] \quad \text{where } k_l^* = log_{\alpha\beta}(2w^{-1}\rho_l).$$

Notice that, $(2w^{-1}\rho_l)^{\frac{log N(\beta)}{|log(\alpha\beta)|}} = N(\beta)^{-k_l^*}$, and since $k_l^* < k_l + 1$ we get

$$N(\beta)^{-k_l} < N(\beta) N(\beta)^{-k_l^*} = N(\beta)(2w^{-1}\rho_l)^{\frac{log N(\beta)}{|log(\alpha\beta)|}}. \tag{3.8}$$

Assuming, without loss of generality, that for every $l$, $\rho_l \le w/2$, there exists $n_0 \in \mathbb{N}$ such that $\frac{w}{2}(\alpha\beta)^{n_0+1} < \rho_l \le \frac{w}{2}(\alpha\beta)^{n_0}$. It follows that $k_l = n_0$ and so

$$\rho_l < w(\alpha\beta)^{k_l}. \tag{3.9}$$

This implies that the ball $C_l$ intersects at most two of the balls $B_l(j_1,\dots,j_{k_l})$. As the length of the interval $g(B_l(j_1,\dots,j_l))$ is $N(\beta)^{-k_l}$ we have $\overline{\lambda}(g(C_l)) \leq 2N(\beta)^{-k_l}$. Combining with 3.7,

$$1 \leq \sum_{l=1}^{\infty} \overline{\lambda}(g(C_l)) \leq \sum_{l=1}^{\infty} 2N(\beta)^{-k_l} < 2N(\beta)(2w^{-1})^{\frac{log(N(\beta))}{|log(\alpha\beta)|}} \sum_{l=1}^{\infty} \rho_l^{\frac{log(N(\beta))}{|log(\alpha\beta)|}}.$$

By definition, $\dim\mathcal{S} \geq \frac{log(N(\beta))}{|log(\alpha\beta)|} \geq \frac{\delta|logC_0\beta|}{|log\alpha|+|log\beta|} \to \delta$ as $\beta \to 0$. ∎

*Remark 3:* If it so happens that $\delta$=dim(supp($\mu$)) then obviously

$$\dim\mathcal{S} = \delta.$$

## 4. Simplex lemma

Before giving our main example in the following section, we prove a version of the simplex lemma following ideas credited by W. M. Schmidt in [S1] to Davenport.

THEOREM 4.1: *Let $\Lambda : \mathbb{R}^N \to \mathbb{R}^N$ be an affine map and denote by $\mathcal{A}$ the $N \times N$ matrix associated with the linear part of $\Lambda$. For every $\theta \in (0,1)$ let $R = \theta^{\frac{-N}{N+1}}$ and for every $k \in \mathbb{N}^+$ let*

$$U_k = \left\{ \Lambda(\frac{\mathbf{p}}{q}) : q \in \mathbb{N}^+, \mathbf{p} \in \mathbb{Z}^N \text{ and } R^{k-1} \leq q < R^k \right\}.$$

*Denote by $V_N$ the volume of the $N$-dimensional unit ball. Then for every $r > 0$ such that $r^N < |det\mathcal{A}|\,(N!)^{-1}V_N^{-1}\theta^N$ and for every $\mathbf{x}$ there exists an affine hyperplane $L$ such that*

$$U_k \cap B(\mathbf{x}, \theta^{k-1}r) \subset \mathcal{L}.$$

*Proof.* Assume the contrary and let $\{V_i\}_{i=0}^N$, $V_i = (v_i^1, \dots, v_i^N)$ be $N+1$ independent points in $U_k \cap B(\mathbf{x}, \theta^{k-1}r)$, i.e., not belonging to any single affine hyperplane. Denote by $\Delta$ the $N$-dimensional simplex subtended by them. By a well-known result from calculus we have

$$\lambda_N(\Delta) = (N!)^{-1}\left|\det L'\right| > 0, \text{ where } L' = \begin{pmatrix} v_1^1 - v_0^1 & . & . & . & v_1^N - v_0^N \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ v_N^1 - v_0^1 & . & . & . & v_N^N - v_0^N \end{pmatrix}.$$

As $\lambda_N(\Delta) > 0$ we have $\det L' \neq 0$.
Consider now the $(N+1 \times N+1)$ matrix

$$L = \begin{pmatrix} 1 & v_0^1 & . & . & v_0^N \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & v_N^1 & . & . & v_N^N \end{pmatrix}.$$

By repeatedly subtracting the first row from all others we get $\det L = \det L''$ where

$$L'' = \begin{pmatrix} 1 & v_0^1 & . & . & v_0^N \\ 0 & v_1^1 - v_0^1 & . & . & v_1^N - v_0^N \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & v_N^1 - v_0^1 & . & . & v_N^N - v_0^N \end{pmatrix}$$

and so $\det L = \det L'$.
Hence, $\lambda_N(\Delta) = |\det A| \, (N!)^{-1} \, |\det L|$ where

$$L = \begin{pmatrix} 1 & \frac{p_0^1}{q_0} & . & . & \frac{p_N^1}{q_0} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & \frac{p_0^N}{q_N} & . & . & \frac{p_N^N}{q_N} \end{pmatrix}$$

and $\det L \neq 0$ by our assumption.
Notice also that

$$q_0 \cdot q_1 \cdot \cdots \cdot q_N \cdot L = \begin{pmatrix} q_0 & p_0^1 & . & . & p_N^1 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ q_N & p_0^N & . & . & p_N^N \end{pmatrix},$$

and as all entries in $q_0 \cdot q_1 \cdot \cdots \cdot q_N \cdot L$ are integers it follows that

$$q_0 q_1 \cdot \cdots \cdot q_N \cdot |\det L| \geq 1.$$

And so,

(4.10)

$$\lambda_N(\Delta) = (N!)^{-1}\,|\det A|\,|\det L| \geq (N!)^{-1}\frac{|\det\mathcal{A}|}{q_0\cdot\cdots\cdot q_N} > (N!)^{-1}\,|\det\mathcal{A}|\,R^{-k(N+1)}.$$

But,

$$\text{(4.11)}\quad \lambda_N(B(\mathbf{x},\theta^{k-1}r)) = (\theta^{k-1}r)^N V_N = \theta^{(k-1)N}r^N V_N < |\det\mathcal{A}|\,\theta^{kN}(N!)^{-1},$$

$$\text{(4.12)}\qquad \theta^{kN} = (\theta^{\frac{-N}{N+1}})^{-k(N+1)} = R^{-k(N+1)},$$

and so

$$\text{(4.13)}\qquad \lambda_N(B(\mathbf{x},\theta^{k-1}r)) \leq |\det\mathcal{A}|\,(N!)^{-1}R^{-k(N+1)}.$$

by our assumption on $U_k$.

As $\Delta \subset B(\mathbf{x},\theta^{k-1}r)$, (4.10) contradicts (4.13). ∎

## 5. Application to Hutchinson's construction

Before turning our attention to our main example we state and prove the following theorem which is material for what follows.

*Definition 4:* Say that $\mu$ satisfies the power law if there exist real numbers $a, b, \delta > 0$ such for every $\mathbf{x} \in \mathrm{supp}(\mu)$, $0 < r \leq 1$

$$ar^\delta \leq \mu(B(\mathbf{x},r)) \leq br^\delta.$$

Theorem 5.1: *Let $\mu$ satisfy the power law. Then $\mu$ satisfies condition* (3.6).

*Proof.* Let $r \leq 1$, $0 < \beta < 1$ and consider a ball $B(\mathbf{x},r)$ with $\mathbf{x} \in K$. Denote by $\{\mathbf{x}_i\}$, $i \in \{0,\dots,N_X(\beta,\mathbf{x},r)\}$ the centers of the $N_X(\beta,\mathbf{x},r)$ balls under consideration. Then, for every $i$, $\mathbf{x}_i \in B(\mathbf{x},(1-\beta)r)\cap K$.

By a simple geometric argument we see that the collection of balls $B(\mathbf{x}_i, 3\beta r)$ cover $B(\mathbf{x},(1-\beta)r)$. For, otherwise, there exists $\mathbf{y} \in B(\mathbf{x},(1-\beta)r)$ such that $d(\mathbf{y},\mathbf{x}_i) \geq 3\beta r$ for every $i$. It follows that $B(\mathbf{y},\beta r)$ could be added to the original collection of balls, which is a contradiction to the maximality assumption on $N_X(\beta,\mathbf{x},r)$. We may assume that $\beta \leq 1/2$ with no loss of generality, as for $1/2 < \beta < 1$ we may choose $M \leq 2^{-\delta} \Rightarrow M\beta^{-\delta} \leq 1$. Notice also that $\delta \leq N$.

And so,

$$a(1-\beta)^\delta r^\delta \le \mu(B(\mathbf{x},(1-\beta)r) \le N_X(\beta,\mathbf{x},r)\mu(B(\mathbf{x}_i,3\beta r)) \le N_X(\beta,\mathbf{x},r)b3^\delta\beta^\delta r^\delta.$$

$$N_X(\beta,\mathbf{x},r) \ge ab^{-1}3^{-1}(1-\beta)^\delta\beta^{-\delta} \ge ab^{-1}3^{-1}2^{-N}\beta^{-\delta}. \tag{5.14}$$

Thus condition (3.6) is satisfied with $r_1 = 1$ and $M = ab^{-1}3^{-1}2^{-N}$. ∎

A map $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is a **similarity** if it can be written as

$$\phi(\mathbf{x}) = \rho\Theta(\mathbf{x}) + \mathbf{y},$$

where $\rho \in \mathbb{R}^+$, $\Theta \in O(N,\mathbb{R})$ and $\mathbf{y} \in \mathbb{R}^N$. It is said to be **contracting** if $\rho < 1$. It is known (see [Hu] for a more general statement) that for any finite family $\phi_1,\ldots,\phi_m$ of contracting similarities there exists a unique non-empty compact set $K$, called the **attractor** or **limit set** of the family, such that

$$K = \bigcup_{i=1}^m \phi_i(K).$$

Say that $\phi_1,\ldots,\phi_m$ as above satisfy the **open set condition** if there exists an open subset $U \subset \mathbb{R}^N$ such that

$$\phi_i(U) \subset U \quad \text{for all} \;\; i = 1,\ldots,m\,,$$

and

$$i \ne j \Longrightarrow \phi_i(U) \cap \phi_j(U) = \varnothing\,.$$

The family $\{\phi_i\}$ is called **irreducible** if there is no finite collection of proper affine subspaces which is invariant under each $\phi_i$. Well-known self-similar sets, like Cantor's ternary set, Koch's curve or Sierpinski's gasket, are all examples of attractors of irreducible families of contracting similarities satisfying the open set condition.

Suppose $\{\phi_i\}_{i=1}^m$ is a family of contracting similarities of $\mathbb{R}^N$ satisfying the open set condition, let $K$ be its attractor, $\delta$ the Hausdorff dimension of $K$, and $\mu$ the restriction of the $\delta$-dimensional Hausdorff measure to $K$.

J. Hutchinson [H] gave a simple formula for calculating $\delta$ and proved that $\mu(K)$ is positive and finite. Furthermore,

PROPOSITION 5.2: *$\mu$ satisfies the power law with $\delta$=dimK.*

As a consequence of Proposition 5.2 and Theorem 5.1 we prove the following.

Corollary 5.3: *Let* $\{\phi_1, \dots, \phi_k)$ *be a finite irreducible family of contracting similarity maps of* $\mathbb{R}^N$ *satisfying the open set condition. Let* $K$ *be its attractor. Let* $\mu$ *be the restriction of* $H^\delta$ *to* $K$*. Then* $\mu$ *is absolutely friendly satisfying condition* (3.6) *with* $\dim K = \delta$.

*Proof.* By Theorem 5.1, Condition (3.6) is satisfied.

Set $r_0 = 1$. It is easily seen that the power law implies that condition (ii) of Definition 1 is satisfied with $D = a/b(5/6)^\delta$.

Following [KLW](Theorem 2.3, Lemmas 8.2 and 8.3), there exist $C$ and $a$ such that $\mu$ is absolutely $(C, a)$-decaying (see Remark 2) on any ball of radius $r = 1$ centered in supp$(\mu)$.

Using the notation of Definition 1, $\mu$ is absolutely friendly with $r_0 = 1$. ∎

We are now ready to prove our main example.

Corollary 5.4: *Let* $\{\phi_1, \dots, \phi_k)$ *be a finite irreducible family of contracting similarity maps of* $\mathbb{R}^N$ *satisfying the open set condition. Let* $K$ *be its attractor and* $\alpha'$ *as in Lemma 1.1. Then for any countable collection of affine transformations* $\{\Lambda_i\}_{i=0}^\infty$, *with* $\Lambda_i : \mathbb{R}^N \to \mathbb{R}^N$ *the set*

$$\mathcal{S} = K \cap (\textstyle\bigcap_{i=1}^\infty (\Lambda_i(\boldsymbol{BA})))$$

*is an* $\alpha$*-winning set on* $K$ *for any* $0 < \alpha < \frac{1}{12}\alpha'$*. Furthermore, dimS=dimK.*

*Proof.* In view of Proposition 2.3 it suffices to prove that for each $i$, $K \cap \Lambda_i(\mathbf{BA})$ is $\alpha$-winning. Given an affine transformation $\Lambda$ and following Corollary 2.1 we prove that $\mathbf{Bad}^*(\mathcal{U}, I, \rho, \Omega)$ is an $\alpha$ winning set on $\Omega = K$ where for every $q \in \mathbb{N}^+$

$$U_q = \left\{\Lambda\left(\frac{\mathbf{p}}{q}\right) : \mathbf{p} \in \mathbb{Z}^N\right\}, \tag{5.15}$$

$I(q) = q$ and $\rho(I(q)) = \rho(q) = q^{\frac{-N+1}{N}}$. Following the notation of Theorem 2.2 and Theorem 4.1 let $\theta = \alpha\beta$ and for every $k \in \mathbb{N}^+$ let $F(k) = R^k = (\alpha\beta)^{\frac{-Nk}{N+1}}$. Define

$$U_k = \left\{\Lambda\left(\frac{\mathbf{p}}{q}\right) : q \in \mathbb{N}^+, \mathbf{p} \in \mathbb{Z}^N \text{ and } R^{k-1} \le q < R^k\right\}. \tag{5.16}$$

By Theorem 4.1 we get that the first condition of Theorem 2.2 is satisfied by any $\beta$. As by our definition $\rho(F(k)) = (\alpha\beta)^k$, the second condition is satisfied as well. Thus $K \cap T_i(\mathbf{BA})$ is an $(\alpha, \beta)$-winning set for every $\beta$, rendering it an $\alpha$-winning set.

Furthermore, as $\mu$ is absolutely friendly satisfying condition (3.6) with the exponent of the condition being $\delta$=dim$K$, by Theorem 3.1, followed by Remark 3 we are done. ∎

## References

[H] J. E. Hutchinson, *Fractals and self-similarity*, Indiana University Mathematics Journal **30** (1981), 713–747.

[KLW] D. Kleinbock, E. Lindenstrauss and B. Weiss, *On fractal measures and diophantine approximation*, Selecta Mathematica New series **10** (2004), 479–523.

[KW] D. Kleinbock and B. Weiss, *Badly approximable vectors on fractals*, Israel Journal of Mathematics **149** (2005), 137–170.

[KTV] S. Kristensen, R. Thorn and S. L. Velani, *Diophantine approximation and badly approximable sets*, Advances in Mathematics **203** (2006), 132–169.

[PV] A. D. Pollington and S. L. Velani, *Metric Diophantine approximation and 'absolutely friendly' measures*, Selecta Mathematica **11** (2005), 297–307.

[S1] W. M. Schmidt, *On badly approximable numbers and certain games*, Transactions of the American Mathematical Society **123** (1966), 27–50.