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ON SOLUTIONS OF THE RICCI CURVATURE EQUATION AND THE EINSTEIN EQUATION

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ABSTRACT

We consider the pseudo-Euclidean space (R^n, g), with $n \geq 3$ and $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$, where at least one $\epsilon_i = 1$ and nondiagonal tensors of the form $T = \sum_{ij} f_{ij} dx_i dx_j$ such that, for $i \neq j, f_{ij}(x_i, x_j)$ depends on x_i and x_j. We provide necessary and sufficient conditions for such a tensor to admit a metric $\bar{g}$, conformal to g, that solves the Ricci tensor equation or the Einstein equation. Similar problems are considered for locally conformally flat manifolds. Examples are provided of complete metrics on R^n, on the n-dimensional torus T^n and on cylinders $T^k \times R^{n-k}$, that solve the Ricci equation or the Einstein equation.

1. Introduction

In the last two decades, different aspects of the following two problems have been considered by several authors.

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> Given a symmetric tensor T, of order two, defined on a manifold M^n, $n \geq 3$, does there exist a Riemannian metric g such that $Ric\, g = T$?
>
> Find necessary and sufficient conditions on a symmetric tensor T, so that one can find a metric g satisfying $Ric\, g - \frac{K}{2} g = T$, where K is the scalar curvature of g.

Both problems correspond to solving nonlinear second order differential equation. We call the first one the Ricci tensor equation. The second equation is called the Einstein field equation, when g is a Lorentzian metric on a four dimensional manifold.

When T is nonsingular, i.e. its determinant does not vanish, a local solution of the Ricci equation always exists, as it was shown by DeTurck [D1]. When T is singular, but still has constant rank and satisfies certain appropriate conditions, then the Ricci equation also admits local solutions [DG]. Rotationally symmetric nonsingular tensors were considered in [CD]. Other results can be found in [D2], [DK], [L], [H] and [DG].

As for the Einstein field equation, when $n = 4$, DeTurck [D3] considered the Cauchy problem for nonsingular tensors. Moreover, for tensors T that represent several physical situations, the equation has been studied by several authors (see [SKMHH] and its references).

In this paper, we consider a certain class of nondiagonal symmetric tensors T on a pseudo-Euclidean space (R^n, g), $n \geq 3$, and we determine all metrics, conformal to g, whose Ricci tensor is the given tensor T. A similar question is considered for the Einstein equation. The theory is also extended to locally conformally flat manifolds.

Our previous results with special classes of tensors T and conformal metrics can be found in [PT1–PT5] and [P], where all solutions to the problems were given explicitly. In this paper, we consider the pseudo-Euclidean space (R^n, g), with $n \geq 3$, coordinates $x = (x_1, \ldots, x_n)$ and $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$, where at least one ϵ_i is positive. We consider nondiagonal tensors of the form $T = \sum_i f_{ij} dx_i dx_j$, such that, for $i \neq j$, $f_{ij}(x_i, x_j)$ is a differentiable function of x_i and x_j. For such a tensor, we want to find metrics $\bar{g} = \frac{1}{\varphi^2} g$, that solve the Ricci equation or the Einstein equation. More precisely, we want to solve the following problems

$$(1)\qquad \begin{cases} \bar{g} = \frac{1}{\varphi^2} g \\ \text{Ric } \ \bar{g} = T. \end{cases}$$

$$(2)\qquad \begin{cases} \bar{g} = \frac{1}{\varphi^2} g \\ \text{Ric } \bar{g} - \frac{\bar{K}}{2}\bar{g} = T. \end{cases}$$

We will show that any such tensor, that solves (1) or (2), is of two types. Namely, up to a change of order of the independent variables, T is either of the form

$$T = \sum_{i,j=1}^{2} f_{ij}(x_1, x_2) dx_i dx_j + h(x_1, x_2) \sum_{i=3}^{n} dx_i^2$$

and $\varphi(x_1, x_2)$ is a solution of a hyperbolic equation, or T is determined by p, $3 \leq p \leq n$, nonconstant, differentiable functions $U_j(x_j)$. In the second case, φ and T are given explicitly in terms of U_j. This characterization is given in Theorem 1.1 for the Ricci tensor equation and in Theorem 1.2 for the Einstein equation. We also extend the results to locally conformally flat manifolds.

As a consequence of Theorem 1.1, we show that for certain functions $\bar{K}$, depending on the functions of one variable $U_j(x_j)$, there exist metrics $\bar{g}$, conformal to the pseudo-Euclidean metric g, whose scalar curvature is $\bar{K}$. Equivalently, we find C^∞ solutions for the equation

$$(3)\qquad \frac{4(n-1)}{n-2} \Delta_g u + \bar{K} u^{\frac{n+2}{n-2}} = 0.$$

where Δ_g denotes the Laplacian in the pseudo-Euclidean metric g. This result is related to the prescribed scalar curvature problem: Given a differentiable function $\bar{K}$, on a Riemannian manifold (M, g), is there a metric $\bar{g}$, conformal to g, whose scalar curvature is $\bar{K}$? This problem has been studied by many authors. In particular, when $\bar{K}$ is constant, it is known as the Yamabe problem.

By applying the theory, we exhibit examples of complete metrics on R^n, on the n-dimensional torus T^n, or on cylinders $T^k \times R^{n-k}$, that solve the Ricci equation or the Einstein equation.

Main results

We will now state our main results. The proofs will be given in the following section. We will denote by $\varphi_{x_ix_j}$ and f_{ij,x_k} the second order derivative of φ with respect to x_ix_j and the derivative of f_{ij} with respect to x_k, respectively.

THEOREM 1.1: *Let (R^n, g), $n \geq 3$, be a pseudo-Euclidean space, with coordinates $x = (x_1, \ldots, x_n)$, $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Consider a nondiagonal symmetric tensor $T = \sum_{i,j=1}^{n} f_{ij} dx_i dx_j$. Assume that, for $i \neq j$, $f_{ij}(x_i, x_j)$ is a differentiable function of x_i and x_j. Then there exists a metric $\bar{g} = \frac{1}{\varphi^2} g$ such that Ric $\bar{g} = T$ if, and only if,*

$$f_{ii} = (n-2)\frac{\varphi_{x_ix_i}}{\varphi} + \epsilon_i \frac{\Delta_g \varphi}{\varphi} - \epsilon_i (n-1)\frac{|\nabla_g \varphi|^2}{\varphi^2} \quad \textit{for all } i \tag{4}$$

and up to a change of order of the independent variables, one of the following cases occur:

a) *$f_{12}(x_1, x_2)$ is any nonzero differentiable function, $f_{ij} \equiv 0$, for all $i \neq j$, such that $i \geq 3$ or $j \geq 3$ and $\varphi = \varphi(x_1, x_2)$ is a nonvanishing solution of the hyperbolic equation*

$$(n-2)\varphi_{x_1x_2} - f_{12}\varphi = 0. \tag{5}$$

b) *There exists an integer p, $3 \leq p \leq n$, such that $f_{ij} = 0$, if $i \neq j, i \geq p+1$ or $j \geq p+1$. Moreover, there exist nonconstant differentiable functions, $U_j(x_j)$, for $1 \leq j \leq p$, such that for all $i, j, 1 \leq i \neq j \leq p$,*

$$f_{ij} = (n-2)U_i' U_j', \quad \text{and} \quad \varphi = a e^{\sum_{j=1}^{p} U_j(x_j)} + b e^{-\sum_{j=1}^{p} U_j(x_j)}, \tag{6}$$

or

$$f_{ij} = -(n-2)U_i' U_j', \quad \text{and} \quad \varphi = a \cos\left(\sum_{j=1}^{p} U_j(x_j)\right) + b \sin\left(-\sum_{j=1}^{p} U_j(x_j)\right), \tag{7}$$

where a and b are real numbers such that $a^2 + b^2 \neq 0$. Moreover, in each case φ is defined on an open connected subset of R^n, where it does not vanish.

We have a similar result for the Einstein equation. Observe that if (R^n, g) is a pseudo-Euclidean space and $\bar{g} = g/\varphi^2$ is a conformal metric, then the scalar curvature of $\bar{g}$ is given by

$$\bar{K} = (n-1)\left(2\varphi\Delta_g\varphi - n|\nabla_g\varphi|^2\right). \tag{8}$$

Theorem 1.2: *Let (R^n, g), $n \geq 3$, be a pseudo-Euclidean space, with coordinates $x = (x_1, \ldots, x_n)$, $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Consider a nondiagonal symmetric tensor $T = \sum_{i,j=1}^{n} f_{ij} dx_i dx_j$. Assume that, for $i \neq j$, $f_{ij}(x_i, x_j)$ is a differentiable function of x_i and x_j. Then there exists a metric $\bar{g} = \frac{1}{\varphi^2} g$ such that $Ric\, \bar{g} - \frac{\bar{K}}{2}\bar{g} = T$ if, and only if,*

$$f_{ii} = (n-2)\Big(\frac{\varphi_{x_i x_i}}{\varphi} - \epsilon_i \, \frac{\Delta_g \varphi}{\varphi} + \epsilon_i (n-1)\frac{|\nabla_g \varphi|^2}{2\varphi^2}\Big) \quad \text{for all } i \tag{9}$$

and up to a change of order of the independent variables, one of the following cases occur:

a) *$f_{12}(x_1, x_2)$ is any nonzero differentiable function, $f_{ij} \equiv 0$, for all $i \neq j$, such that $i \geq 3$ or $j \geq 3$ and $\varphi = \varphi(x_1, x_2)$ is a nonvanishing solution of the hyperbolic equation*

$$(n-2)\varphi_{x_1 x_2} - f_{12}\varphi = 0. \tag{10}$$

b) *There exists an integer p, $3 \leq p \leq n$, such that $f_{ij} = 0$, if $i \neq j$, $i \geq p+1$ or $j \geq p+1$. Moreover, there exist nonconstant differentiable functions, $U_j(x_j)$, for $1 \leq j \leq p$, such that for all i, j, $1 \leq i \neq j \leq p$,*

$$f_{ij} = (n-2)U_i' U_j' \quad \text{and} \quad \varphi = a\, e^{\sum_{j=1}^{p} U_j(x_j)} + b\, e^{-\sum_{j=1}^{p} U_j(x_j)}, \tag{11}$$

or

$$f_{ij} = -(n-2)U_i' U_j' \quad \text{and} \quad \varphi = a \cos\left(\sum_{j=1}^{p} U_j(x_j)\right) + b \sin\left(-\sum_{j=1}^{p} U_j(x_j)\right), \tag{12}$$

where a and b are real numbers such that $a^2 + b^2 \neq 0$. Moreover, in each case φ is defined on an open connected subset of R^n, where it does not vanish.

Corollary 1.3: *If (R^n, g) is the Euclidean space and $0 < |\varphi(x)| \leq C$ for some constant C, then the metrics given by Theorems 1.1 and 1.2 are complete on R^n.*

By considering $u = \varphi^{-(n-2)/2}$ and the expression of the scalar curvature obtained from the Ricci tensor T, one gets the following corollaries from Theorem 1.1. These corollaries are related to the prescribed scalar curvature problem, as one can see in Corollary 1.6.

Corollary 1.4: *Let (R^n, g) be a pseudo-Euclidean space, $n \geq 3$, with coordinates $x = (x_1, \ldots, x_n)$, $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Let $\bar{K} : R^n \to R$ be given by*

$$\bar{K} = (n-1)\Big\{2(a^2f^2 - b^2f^{-2})\sum_j \epsilon_j U_j'' + [2(n+2)ab - (n-2)(a^2f^2 + b^2f^{-2})]\sum_j \epsilon_j (U_j')^2\Big\} \tag{13}$$

where $U_j(x_j)$, $1 \leq j \leq p$, are arbitrary nonconstant differentiable functions, $3 \leq p \leq n$, $a^2 + b^2 \neq 0$ and $f = e^{\sum U_j}$. Then the differential equation

$$\frac{4(n-1)}{n-2}\Delta_g u + \bar{K}(x)u^{\frac{n+2}{n-2}} = 0 \tag{14}$$

where Δ_g denotes the Laplacian in the metric g, has a solution, globally defined on R^n, given by

$$u = (af + bf^{-1})^{-(n-2)/2}. \tag{15}$$

Corollary 1.5: *Let (R^n, g) be a pseudo-Euclidean space, $n \geq 3$, with coordinates $x = (x_1, \ldots, x_n)$, $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Let $\bar{K} : R^n \to R$ be given by*

$$\bar{K} = -(n-1)(a^2+b^2) \times \sum_j \epsilon_j \Big\{\sin 2\Big(\sum U_k + \theta\Big)U_j'' + \Big[(n-2)\sin^2\Big(\sum U_k + \theta\Big) + 2\Big](U_j')^2\Big\}, \tag{16}$$

where $U_j(x_j)$, $1 \leq j \leq p$, are arbitrary nonconstant differentiable functions, $3 \leq p \leq n$, $a^2 + b^2 \neq 0$ and θ is defined by $\cos\theta = a/\sqrt{a^2+b^2}$ and $\sin\theta = -b/\sqrt{a^2+b^2}$. Then the differential equation

$$\frac{4(n-1)}{n-2}\Delta_g u + \bar{K}(x)u^{\frac{n+2}{n-2}} = 0$$

where Δ_g denotes the Laplacian in the metric g, has a solution, globally defined on R^n, given by

$$u = \left(\sqrt{a^2+b^2}\cos\left(\sum_j U_j + \theta\right)\right)^{-\frac{n-2}{2}}. \tag{17}$$

Observe that considering $a = 1$ and $b = 0$ in (13), we get a particular case of Corollary 1.4. Let

$$\bar{K}(x) = (n-1)e^{2\sum_j U_j(x_j)} \sum_{j=1}^{p} \epsilon_j \left[2U_j'' - (n-2)(U_j')^2\right], \tag{18}$$

where $U_j(x_j)$, $1 \leq j \leq p$, are nonconstant differentiable functions and $3 \leq p \leq n$. Then the differential equation (14) has a solution, globally defined on R^n, given by

$$u = \left(e^{-\sum_j U_j}\right)^{-\frac{n-2}{2}}. \tag{19}$$

The geometric interpretation of the above results is the following:

Corollary 1.6: *Let (R^n, g) be a pseudo-Euclidean space, $n \geq 3$ and $\bar{K}$ a function given by* (13) *(resp.,* (16)*). Then there exists a metric $\bar{g} = u^{4/(n-2)}g$, where u is given by* (15) *(resp.,* (17)*), whose scalar curvature is $\bar{K}$. In particular, if (R^n, g) is the Euclidian space and u is a bounded function then $\bar{g}$ is a complete metric.*

Examples 1.7: As a direct consequence of Theorems 1.1, 1.2 and Corollary 1.3 we get the following examples, where we are considering (R^n, g), $n \geq 3$, the pseudo-Euclidean space with coordinates $(x_1, \ldots, x_n)$ such that $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$.

a) Consider for each $j = 1, \ldots, n$, the function $U_j = -x_j^{2m_j}$, where m_j is a positive integer and the tensor T determined as in Theorem 1.1, with $a = 1$, $b = 0$. We observe that although this tensor may have singular points (depending on the integers m_j), there exists $\bar{g} = \frac{1}{\varphi^2}g$ such that $Ric\,\bar{g} = T$, globally defined on R^n with $\varphi = \exp(-\sum_j x_j^{2m_j})$. Moreover, it follows from Corollary 1.3, that in the Euclidean case, the metric $\bar{g}$, is a complete metric on R^n, with negative Ricci curvature.

b) Consider any periodic nonconstant function $U_j(x_j)$ for each $j = 1, \ldots, n$. Then the symmetric tensor $T = \sum f_{ij}dx_i dx_j$, defined as in Theorem 1.1, where we choose positive constants a and b, admits a metric $\bar{g}$, on an n-dimensional torus, T^n, conformal to the pseudo-Euclidean metric, whose Ricci tensor is T. Observe that in the Euclidean case ($\epsilon_k = 1$, for all k), $\bar{g}$ is a complete metric on T^n. If we consider k periodic functions U_j, $3 \leq k < n$, we get metrics defined on $T^k \times R^{n-k}$, conformal to the pseudo-Euclidean metric. In the Euclidean case, if, moreover, φ is a bounded function, then $\bar{g}$ is a complete metric on $T^k \times R^{n-k}$.

c) As a consequence of Theorem 1.2, we observe that periodic functions $U_j(x_j)$, for each $j = 1, \dots, n$, determine a tensor T which admits a solution $\bar{g}$, conformal to g, for the Einstein equation, defined on T^n. If we consider k periodic functions U_j, $3 \leq k < n$, we get solutions for the Einstein equation on $T^k \times R^{n-k}$. In the Euclidean case, if, moreover, φ is a bounded function, $\bar{g}$ is a complete metric.

d) Consider the Euclidean space (R^n, g) and a tensor T as in Theorem 1.1, with $a = 1$, $b = 0$, determined by

$$f_{ij} = (n-2)U_i'U_j', \quad 1 \leq i \neq j \leq p, \quad f_{ij} = 0 \text{ for } i \neq j, i \geq p+1, \text{ or } j \geq p+1,$$

$$f_{ii} = (n-2)U_i'' + \sum_j U_j'' - (n-2)\sum_{j\neq i}(U_j')^2,$$

where $U_j(x_j)$ are arbitrary differentiable functions such that $U_j'' < 0$ for all j, $1 \leq j \leq p$ and $p \geq 3$. Then the metric $\bar{g}$ has negative Ricci curvature. If, moreover, φ is bounded then $\bar{g}$ is a complete metric on R^n.

We now consider a Riemannian manifold locally conformally flat (M^n, g), then one can consider problems (1) and (2) for any neighborhood $V \subset M$ such that there are local coordinates $(x_1, \dots, x_n)$ with $g_{ij} = \delta_{ij}/F^2$, where F is a nonvanishing differentiable function on V. It is easy to see that the following results hold.

Theorem 1.8: *Let (M^n, g), $n \geq 3$ be Riemannian manifold, locally conformally flat. Let V be an open subset of M with coordinates $(x_1, \dots, x_n)$ such that $g_{ij} = \delta_{ij}/F^2$. Consider a nondiagonal symmetric tensor $T = \sum_{i,j=1}^n f_{ij} dx_i dx_j$. Assume that, for $i \neq j$, $f_{ij}(x_i, x_j)$ depends on x_i and x_j. Then there exists $\bar{g} = \frac{1}{\psi^2} g$ such that Ric $\bar{g} = T$ if, and only if, $\psi = \varphi/F$ and, up to a change of order of the independent variables, φ satisfies a) or b) of Theorem 1.1, with $\epsilon_i = 1$, for all i.*

The following result provides the analogue theorem for the Einstein equation.

Theorem 1.9: *Let (M^n, g), $n \geq 3$, be Riemannian manifold, locally conformally flat. Let V be an open subset of M with coordinates $(x_1, \dots, x_n)$ such that $g_{ij} = \delta_{ij}/F^2$. Consider a nondiagonal symmetric tensor $T = \sum_{i,j=1}^n f_{ij} dx_i dx_j$.*

Assume that, for $i \neq j$, $f_{ij}(x_i, x_j)$ depends on x_i and x_j. Then there exists a metric $\bar{g} = \frac{1}{\psi^2} g$ such that Ric $\bar{g} - \frac{\bar{K}}{2}\bar{g} = T$ if, and only if, $\psi = \varphi/F$ and, up to a change of order of the independent variables, φ satisfies a) or b) of Theorem 1.2, with $\epsilon_i = 1$, for all i.

We observe that there are similar results for manifolds that are locally conformal to the pseudo-Euclidean space.

Proof of the main results

In order to prove our main results we will need the following lemmas.

LEMMA 1.10: *Assume $\varphi(x_1, \ldots, x_p)$, $p \geq 3$, is a nonvanishing differentiable function that satisfies a system of equations*

$$\varphi_{x_i x_j} - f_{ij}(x_i, x_j)\varphi = 0, \quad \textit{for all } i \neq j, \tag{20}$$

where $f_{ij} = f_{ji}$ is a differentiable function of x_i and x_j. Assume there is an open subset $U \subset R^p$, where all f_{ij} do not vanish. Then there is an open dense subset of U where $\prod_i \varphi_{x_i}$ does not vanish. On each connected component of this subset, there exist differentiable functions $V_i(x_i) \neq 0$, $i = 1, \ldots p$, such that

$$f_{ij} = \epsilon V_i(x_i) V_j(x_j), \quad \epsilon = 1 \textit{ or } \epsilon = -1 \quad \textit{for all } 1 \leq i \neq j \leq p. \tag{21}$$

Proof. Since φ is a nonvanishing solution of (20) and all f_{ij} do not vanish on U, it follows that for each i the set $S_i = \{x \in U \subset R^p;\ \varphi_{x_i}(x) = 0\}$ has measure zero. Therefore, there is an open dense subset of U where all φ_{x_i} do not vanish. For the rest of the proof we restrict ourselves to a connected component of this subset.

If φ is a solution of (20), then for each triple (i, j, k), of distinct indices

$$\varphi_{x_i x_j x_k} = f_{ij}\varphi_{x_k}.$$

Hence,

$$f_{ij}\varphi_{x_k} = f_{ik}\varphi_{x_j} = f_{jk}\varphi_{x_i}, \quad \text{for all } i, j, k, \text{ distinct.} \tag{22}$$

In particular,

$$f_{1j}\varphi_{x_k} = f_{1k}\varphi_{x_j}, \quad \text{for all } j \neq k \geq 2.$$

Hence, for all $j \geq 2$, all the quotients φ_{x_j}/f_{1j} are equal and, therefore, there exists a nonvanishing function $\beta_1(x_1, \ldots, x_p)$ such that

$$\varphi_{x_j} = \beta_1 f_{1j}, \quad \text{for all } j \geq 2. \tag{23}$$

Consider the derivative of this equation with respect to x_1 and substitute (20). It follows that for $j \neq k$, we have

$$\beta_{1,x_1} f_{1j} + \beta_1 f_{1j,x_1} = f_{1j}\varphi,$$
$$\beta_{1,x_1} f_{1k} + \beta_1 f_{1k,x_1} = f_{1k}\varphi.$$

Therefore,

$$(f_{1k} f_{1j,x_1} - f_{1j} f_{1k,x_1})\beta_1 = 0, \quad \text{for all } j \neq k \geq 2.$$

Moreover, f_{1j} depends only on (x_1, x_j). Therefore, there exists a function $V_1(x_1)$, such that

$$\frac{f_{1j,x_1}}{f_{1j}} = \frac{V_1'(x_1)}{V_1}, \quad \text{for all } j \geq 2.$$

We conclude that there exists $V_j(x_j)$ such that

$$f_{1j} = V_1(x_1)V_j(x_j), \quad \text{for all } j \geq 2. \tag{24}$$

We will now show by induction that for any $l \geq 2$, $f_{lk} = cV_lV_k$, for all $k \geq 2, k \neq l$, where $c \neq 0$ is a constant.

We start proving for $l = 2$. From (22) we get

$$\varphi_{x_k} = \beta_2(x_1, \ldots, x_p) f_{2k}, \quad \text{for all } k \neq 2.$$

Consider this equation for $k = 1$ and $k \geq 3$ and take the derivative of each equation with respect to x_2, we get that

$$f_{21,x_2} f_{2k} - f_{2k,x_2} f_{21} = 0 \quad \text{for all } k \geq 3.$$

Using (24) we have

$$f_{2k} = V_2 \tilde{V}_k(x_k), \quad \text{for all } k \geq 3. \tag{25}$$

We will now relate $\tilde{V}_k$ with V_k. From (22) we have

$$f_{1k}\varphi_{x_2} = f_{2k}\varphi_{x_1}, \quad \text{for all } k \geq 3. \tag{26}$$

It follows from this equation and its derivative with respect to x_k, after using (20), (24) and (25),

$$\tilde{V}_k = c_{2k} V_k, \quad \text{for all } k \geq 3, \tag{27}$$

where $c_{2k} \neq 0$ is a constant. If $p \geq 4$, we will show that all the constants c_{2k} are equal. In fact, taking the derivative of (26) with respect to x_l, $l \neq k$, $l \geq 3$, using (20), (24), (25) and (27), it follows that $c_{2k} = c_{2l}$. We denote this constant by c. We conclude that we have shown that

(28) $$f_{1j} = V_1 V_j, \quad \text{for all } j \geq 2 \quad \text{and} \quad f_{2k} = cV_2 V_k, \quad \text{for all } k \geq 3.$$

Claim: Assume that for a fixed l, $2 \leq l < p-1$ we have that

(29) $$f_{ik} = cV_i V_k, \quad \text{for all } i,\ 2 \leq i \leq l-1, \text{ for all } k \geq 2,\ k \neq i,$$

then $f_{lk} = cV_l V_k$ for all $k \geq 2,\ k \neq l$.

In order to prove the claim, we observe that since $f_{lk} = f_{kl}$, it follows from the hypothesis that we only need to prove that $f_{lk} = cV_l V_k$ for $k \geq l+1$. From (22) we get

$$\varphi_{x_k} = \beta_l(x_1, \ldots, x_p) f_{lk}, \quad \text{for all } k \neq l.$$

Consider this equation for $k = 1$ and $k \geq l+1$ and take the derivative of each equation with respect to x_l, we get that

$$f_{1l,x_l} f_{lk} - f_{lk,x_l} f_{l1} = 0 \quad \text{for all } k \geq l+1.$$

Using (24) we have

(30) $$f_{lk} = V_l \hat{V}_k(x_k), \quad \text{for all } k \geq l+1.$$

We now relate $\hat{V}_k$ with V_k. From (22) we have

(31) $$f_{1k}\varphi_{x_l} = f_{lk}\varphi_{x_1}, \quad \text{for all } k \geq l+1.$$

It follows from this equation and its derivative with respect to x_k, after using (20), (24) and (30), that

(32) $$\hat{V}_k = c_{lk} V_k, \quad \text{for all } k \geq l+1,$$

where $c_{lk} \neq 0$ is a constant. It follows from (30), (32), and (24) that the equality (31) reduces to $V_1 \varphi_{x_l} = c_{lk} V_l \varphi_{x_1}$ for all $k \geq l+1$. Taking the derivative of this equation with respect to x_2, it follows from (20), (28), (30) and (32) that $c_{lk} = c$ for all $k \geq l+1$. We conclude from (29) that

$$\begin{aligned} &f_{1j} = V_1 V_j, \quad \text{for all } j \geq 2, \\ &f_{jk} = cV_j V_k, \quad \text{for all } j \neq k \geq 2 \quad \text{where } c \in R \setminus \{0\}. \end{aligned}$$

If $c > 0$, then we may consider $\tilde{V}_j = \sqrt{c}V_j$ for all $j \geq 2$ and $\tilde{V}_1 = V_1/\sqrt{c}$. Hence $f_{ij} = \tilde{V}_i\tilde{V}_j$, for all $i \neq j$.

If $c < 0$, then we may consider $\tilde{V}_j = -\sqrt{-c}V_j$ for all $j \geq 2$ and $\tilde{V}_1 = V_1/\sqrt{-c}$. Hence $f_{ij} = -\tilde{V}_i\tilde{V}_j$, for all $i \neq j$.

This completes the proof of the lemma. ∎

LEMMA 1.11: *A nonvanishing differentiable function $\varphi(x_1, \ldots, x_p)$, $p \geq 3$, is a solution of*

a) $\varphi_{x_i x_j} - \varphi = 0$, *for all $i \neq j$, if and only if*

$$\varphi = ae^{\sum_{j=1}^p x_j} + be^{-\sum_{j=1}^p x_j} \tag{33}$$

b) $\varphi_{x_i x_j} + \varphi = 0$, *for all $i \neq j$ if and only if*

$$\varphi = a\cos\sum_{j=1}^p x_j + b\sin\Big(-\sum_{j=1}^p x_j\Big), \tag{34}$$

where $a, b \in R$, $a^2 + b^2 \neq 0$.

Proof. Assume that φ is a solution of

$$\varphi_{x_i x_j} - \varphi = 0, \quad \text{for all } i \neq j. \tag{35}$$

Since $p \geq 3$, it follows that, $\varphi_{x_i x_j x_k} - \varphi_{x_k} = 0$ for all i, j, k distinct. Since φ does not vanish, we have

$$\frac{\varphi_{x_i}}{\varphi} = \beta(x_1, \ldots, x_p), \quad \text{for all } i, \tag{36}$$

where β is a differentiable function. Taking the derivative of (36) with respect to x_j, $j \neq i$, it follows from (35) that

$$\beta_{x_j} + \beta^2 - 1 = 0, \quad \text{for all } j.$$

Hence

$$\beta = c\,\frac{ae^{\sum_j x_j} - be^{-\sum_j x_j}}{ae^{\sum_j x_j} + be^{-\sum_j x_j}},$$

where $a, b \in R$ do not vanish simultaneously. Therefore, we conclude from (36) that φ is given by (33). The converse holds trivially.

Similar arguments prove that φ is a nonvanishing solution of $\varphi_{x_i x_j} + \varphi = 0$ if and only if (34) holds. ∎

Proof of Theorem 1.1. Since Ric $g = 0$, we have that $\bar{g} = \frac{1}{\varphi^2} g$, is such that Ric $\bar{g} = T$ if, and only if,

$$T = \text{Ric } \bar{g} = \frac{1}{\varphi^2}\left\{(n-2)\varphi \text{Hess}_g(\varphi) + \left[\varphi\Delta_g\varphi - (n-1)|\nabla_g\varphi|^2\right] g\right\}. \tag{37}$$

This is equivalent to saying that φ is a nonvanishing solution of the following system of equations:

$$\varphi_{x_i x_j} = \frac{f_{ij}}{n-2}\,\varphi, \quad \text{for all } i \neq j, \tag{38}$$

$$f_{ii} = (n-2)\frac{\varphi_{x_i x_i}}{\varphi} + \epsilon_i \frac{\Delta_g \varphi}{\varphi} - \epsilon_i (n-1)\frac{|\nabla_g\varphi|^2}{\varphi^2} \quad \text{for all } i. \tag{39}$$

Since φ is a differentiable function and $n \geq 3$, it follows from (38) that

$$f_{ij}\varphi_{x_k} = f_{ik}\varphi_{x_j} = f_{jk}\varphi_{x_i}, \quad \text{for all } i, j, k \text{ distinct.} \tag{40}$$

T is a nondiagonal symmetric tensor, hence there exists a pair (i_0, j_0) such that $f_{i_0 j_0} = f_{j_0 i_0} \neq 0$ on an open subset $U \subset R^n$. If $f_{i_0 k} \equiv 0$ on U, for all k distinct from i_0 and j_0, then we may assume under a change of the order of the independent variables, if necessary, that $f_{12}(x_1, x_2) \neq 0$ and $f_{1j} \equiv 0$ for $j \geq 3$ on U. Moreover, from (40),

$$f_{12}\varphi_{x_k} = f_{1k}\varphi_{x_2} = f_{2k}\varphi_{x_1},$$

hence, we get that $\varphi_{x_k} = 0$, for all $k \geq 3$, $f_{2k} = 0$ on U. Observe that φ_{x_1} and φ_{x_2} cannot be zero on any open subset of U, otherwise we would have $\varphi_{x_1 x_2} = f_{12}\varphi/(n-2) = 0$. This is a contradiction since φ is a nonvanishing function. Therefore, there exists an open subset $U_1 \subset U$, where $\varphi_{x_1} \neq 0$ and $\varphi_{x_2} \neq 0$ on U_1. Hence, $f_{2k} \equiv 0$ on U_1 for all $k \geq 3$. From (40) we have $f_{2j}\varphi_{x_k} = f_{jk}\varphi_{x_2}$, for $j \neq k \geq 3$ and therefore $f_{jk} \equiv 0$ on U_1. We conclude that φ depends only on x_1, x_2 and it is a solution of the hyperbolic equation (5). Moreover, (39) determines the diagonal elements f_{ii} which will depend only on (x_1, x_2).

Otherwise, there exist indices i, j, k distinct such that f_{ij} and f_{ik} do not vanish on an open subset U of R^n. Observe that φ_{x_k} and φ_{x_j} cannot be zero on any open subset of U, since φ is a nonvanishing differentiable function. Let $U_1 \subset U$ be an open subset where $\varphi_{x_k} \neq 0$ and $\varphi_{x_j} \neq 0$. It follows from (38) $f_{jk} \neq 0$ and $\varphi_{x_i} \neq 0$ on U_1. By reordering the variables, if necessary, we may consider $i = 1$ and $f_{1j} \neq 0$, on an open subset $U_2 \subset U_1$, for all j, such that $2 \leq j \leq p$, where p is an integer $3 \leq p \leq n$ and $f_{1s} \equiv 0$, on U_2 for $p+1 \leq s \leq n$.

Since, φ is a nonvanishing function, there is an open subset V of U_2, where $\varphi_{x_j} \neq 0$ for $j = 1, \ldots, p$. It follows from (40) that on V,

$$\begin{aligned} f_{1j}\varphi_{x_k} &= f_{jk}\varphi_{x_1} \quad j \neq k,\ 2 \leq j,k \leq p \\ f_{12}\varphi_{x_s} &= f_{1s}\varphi_{x_2}, \quad p+1 \leq s \leq n \\ f_{kj}\varphi_{x_s} &= f_{sj}\varphi_{x_k}, \quad j \neq k,\ 2 \leq j,k \leq p,\ p+1 \leq s \leq n \\ f_{ks}\varphi_{x_r} &= f_{sr}\varphi_{x_k}, \quad s \neq r,\ p+1 \leq s,r \leq n. \end{aligned}$$

From the first equality we get that $f_{jk} \neq 0$ on V. From the second one we conclude that $\varphi_{x_s} \equiv 0$ on V. It follows from the third one that $f_{sj} \equiv 0$ and from the last equality we conclude that $f_{sr} \equiv 0$ on V. Hence, φ depends on the variables $x_1, \ldots, x_p$, and it satisfies the differential equation (38) for $1 \leq i \neq j \leq p$,where all f_{ij} do not vanish on V.

It follows from Lemma 1.10 that, on each connected component $W \subset V$, where $\Pi_{i \neq j \neq k} f_{ij}\varphi_{x_k} \neq 0$, there exist nonconstant differentiable functions $U_i(x_i)$, $1 \leq i \leq p$ such that

$$\frac{f_{ij}}{n-2} = \epsilon U_i'(x_i)U_j'(x_j), \qquad \text{for } 1 \leq i \neq j \leq p,$$

where $\epsilon = 1$ or $\epsilon = -1$ for all $i \neq j$. We now consider on W the change of variables $y_i = U_i(x_i)$. In this new coordinates $\varphi(y_1, \ldots, y_p)$ satisfies the system

$$\varphi_{y_i y_j} - \epsilon\varphi = 0, \quad \text{for all } i \neq j.$$

Lemma 1.11 implies that φ is given by (6) or (7) on W, according to the value of ϵ. Moreover, the diagonal elements of the tensor T, $f_{ii}(x_1, \ldots, x_p)$ are determined by (39).

In both cases, one can extend the domain of φ to a subset of R^n where the functions U_i are defined and φ does not vanish. The converse in both cases is a straightforward computation. ∎

Proof of Theorem 1.2. Since Ric $g = 0$, we have that $\bar{g} = \frac{1}{\varphi^2}g$, is such that Ric $\bar{g} - \bar{K}\bar{g}/2 = T$ if, and only if,

$$(41)\quad T = \frac{1}{\varphi^2}\Big\{(n-2)\varphi\mathrm{Hess}_g(\varphi) + \Big[-(n-2)\varphi\Delta_g\varphi + \frac{(n-1)(n-2)}{2}|\nabla_g\varphi|^2\Big]g\Big\}.$$

This is equivalent to the following system of equations:

$$\varphi_{x_i x_j} = \frac{f_{ij}}{n-2}\varphi, \quad \text{for all } i \neq j$$

and

$$f_{ii} = (n-2)\Big(\frac{\varphi_{x_i x_i}}{\varphi} - \epsilon_i \,\frac{\Delta_g \varphi}{\varphi} + \epsilon_i (n-1)\frac{|\nabla_g \varphi|^2}{2\varphi^2}\Big) \quad \text{for all } i.$$

The proof now follows by the same arguments as in Theorem 1.1. ∎

Proof of Corollary 1.3. Consider the Euclidean space (R^n, g), $n \geq 3$ and a metric $\bar{g}$ given by Theorems 1.1 or 1.2. If $0 < |\varphi(x)| \leq C$, then the metric $\bar{g}$ is complete, since there exists a constant $m > 0$, such that for any vector $v \in R^n$, $|v|_{\bar{g}} \geq m|v|$. ∎

Proof of Corollaries 1.4 and 1.5. It follows from (8), that for the metric $\bar{g}$ of Theorem 1.1 the scalar curvature is given by (13), (resp., (16)). By defining the function $u^{\frac{-2}{n-2}} = \varphi$, we conclude that u is a solution of (14). ∎

Proof of Corollary 1.6. This result follows immediately from the previous corollaries, since finding a metric $\bar{g} = u^{\frac{4}{n-2}} g$, with scalar curvature $\bar{K}$ is equivalent to solving equation (14). ∎

For the proofs of Theorems 1.8 and 1.9, we consider the function $\psi = \varphi F$. Then arguments similar to those of Theorems 1.1 and 1.2 complete the proofs.

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