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MULTIPLIERS OF PERIODIC ORBITS OF QUADRATIC POLYNOMIALS AND THE PARAMETER PLANE

BY

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In memory of my grandmother Esfir Garbuz

ABSTRACT

We prove a result about an extension of the multiplier of an attracting periodic orbit of a quadratic map as a function of the parameter. This has applications to the problem of geometry of the Mandelbrot and Julia sets. In particular, we prove that the size of p/q-limb of a hyperbolic component of the Mandelbrot set of period n is $O(4^n/p)$, and give an explicit condition on internal arguments under which the Julia set of corresponding (unique) infinitely renormalizable quadratic polynomial is not locally connected.

1. Introduction

Douady–Hubbard–Sullivan (DHS) theorem [4], [18], [2] states that the multiplier ρ of an attracting periodic orbit is a conformal isomorphism from a hyperbolic component of the Mandelbrot set onto the unit disk $\{|\rho| < 1\}$, and it extends homeomorpically to the boundaries.

In Theorem 4, we prove that ρ extends further to an analytic isomorphism from a region containing the hyperbolic component onto a simply connected domain $\tilde{\Omega}_n$ containing $\{|\rho| \leq 1\} \setminus \{1\}$, such that the domain $\tilde{\Omega}_n$ is explicitly

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defined by the period n of the attracting orbit. This follows from the Main Inequality, see Theorem 3 below, which in turn is based on Theorem 2 of [12], see formulation in Subsection 3.1 below. We derive from Theorems 3–4 a few consequences including Theorems 1–2, stated in the Introduction.

Let us be more precise. A hyperbolic component W of period n is a component of the interior of the Mandelbrot set M, such that, for c in W, the quadratic map $f_c(z) = z^2 + c$ has an attracting periodic orbit of period n. Denote by $\rho_W(c)$ the multiplier of this orbit of f_c, $c \in W$. By DHS theorem, ρ_W performs a homeomorphism of the closure of W onto the closed unit disk. A number t is called an internal argument of a point $c \in \partial W$ if and only if $\rho_W(c) = \exp(2\pi i t)$. The point c with $t = 0$ is called the root of W. If $c \in \partial W$ is not the root and has a rational internal argument t, the connected component of $M \setminus \{c\}$ which is disjoint with W is not empty and called the t-limb $L(W, t)$ of W.

The first consequence of Theorem 4 concerns the size of the limbs. It strengthens Yoccoz's bound (off the root), see Section 5.

Theorem 1: *There exists $A > 0$, such that, for every hyperbolic component W of period n and every $t = p/q \in [-1/2, 1/2] \setminus \{0\}$, the diameter of the limb $L(W, t)$ is bounded by:*

$$diam L(W, t) \leq A \frac{4^n}{p}. \tag{1}$$

This bound immediately implies the local connectivity of the Mandelbrot set at some parameters c_*, where f_{c_*} is infinitely renormalizable with prescribed unbounded combinatorics, see Corollary 5.1.

Let us discuss another result. Douady and Hubbard proposed an inductive construction to build an infinitely renormalizable quadratic map with nonlocally connected Julia set, and such that the Mandelbrot set is locally connected at this parameter [23], [19]. Their construction involves choosing a sequence of internal arguments t_m of successive bifurcations step by step, by continuity in the corresponding dynamical planes and in the parameter plane (with the help of Yoccoz's bound). The next statement makes explicit the two sides of Douady–Hubbard's procedure (parameter and dynamical).

Theorem 2: *There exists $B > 0$ as follows. Let $n \geq 1$. Let*

$$t_0, t_1, \ldots, t_m, \ldots$$

be any sequence of rational numbers $t_m = p_m/q_m \in (-1/2, 1/2]$. *Denote* $n_0 = n$, $n_m = nq_0 \cdots q_{m-1}$, $m > 0$. *Assume that,*

$$p_m > B4^{n_m}, \quad Bn_m^2 < p_m^2/q_m, \tag{2}$$

for all m *large enough, and also*

$$\sum_{m=1}^{\infty} |t_m|^{1/q_{m-1}} < \infty. \tag{3}$$

Given a hyperbolic component W *of the Mandelbrot set of period* n*, consider the following sequence of hyperbolic components* W^m: $W_0 = W$*, and, for* $m > 0$, W^m *touches the hyperbolic component* W^{m-1} *at a point* c_{m-1} *with an internal argument* t_{m-1}. *For every* m*, consider the* t_m*-limb* $L(W^m, t_m)$ *of* W^m *(it contains* W^{m+1}*). Then the limbs* $L(W^m, t_m)$ *shrink to a unique point* c_**, the Mandelbrot set is locally connected at* c_**, and the map* f_{c_*} *is infinitely renormalizable with nonlocally connected Julia set.*

This statement can be reformulated in terms of the combinatorial data of an infinitely renormalizable map, see Theorem 8.

Let us comment on the inequalities (2)–(3), above. Conditions (2) guarantee that the corresponding multipliers are local parameters. Namely, if $\rho_{W^m}(c)$ is the multiplier of the periodic orbit of f_c, which is attracting for $c \in W^m$, and ψ_m denotes an inverse to $\log \rho_{W^m}$, then the first inequality in (2) ensures that ψ_m extends as a holomorphic function to a disk of radius proportional to n_m/q_m around $2\pi i t_m$ while the second inequality in (2) implies that this analytic continuation is in fact injective. In turn, condition (3) will guarantee that the bifurcated periodic orbits stay away from the origin. This condition was suggested by Milnor in [19, p. 21]. We confirm his guess.

1.1. Notation. We collect some notation to be used throughout the paper.

$f_c(z) = z^2 + c$, $J_c = J(f_c)$ its Julia set, $D_\infty(c)$ the basin of infinity,

$M = \{c \in \mathbf{C} : J_c \text{ connected}\}$ is the Mandelbrot set,

B_c is the Bottcher coordinate at infinity normalized by $B_c(z) \sim z$ as $z \to \infty$,

$G_c = \lim_{n\to\infty} 2^{-n} \log |f_c^n(z)|$ Green's function of $D_\infty(c)$, $G(z) \sim \log |z|$ at ∞ extended by 0 to the whole plane, so that $G_c(z) = \log |B_c(z)|$ near infinity. $B(a, r) = \{z : |z - a| < r\}$.

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2. Main Inequality

Let $O = \{b_k\}_{k=1}^n$ be a periodic orbit of $f = f_{c_0}$ of exact period n. Its multiplier is the number $\rho = (f^n)'(b_k)$. If $\rho \neq 1$, by the Implicit function theorem, there exist n holomorphic functions $b_k(c)$, $k = 1, \dots, n$ defined in a neighborhood of c_0, such that $O(c) = \{b_k(c)\}_{k=1}^n$ is a periodic orbit of f_c, and $O(c_0) = O$. In particular, if $\rho(c) = (f_c^n)'(b_k(c)) = 2^n b_1(c) \cdots b_n(c)$ denotes the multiplier of $O(c)$, it is holomorphic in c in this neighborhood.

In what follows we assume that the multiplier ρ is not very big, because we are interested in studying the behavior of a multiplier not far from the hyperbolic component where the corresponding periodic orbit is attracting. So, we will always assume that

$$|\rho| < e.$$

THEOREM 3: *There exist λ_* and B_0 as follows. Let $O(c)$ be a repelling periodic orbit of f_c of exact period n, and the multiplier of $O(c)$ is equal to ρ. Then the following inequality holds*

(4)

$$|\rho - 1| \leq \frac{1}{n} K_n(c) \Big\{ \log|\rho(c)| + \frac{|\rho'(c)|}{|\rho(c)|} \frac{1}{\pi} \mathbf{area}(\{z : 0 < G_c(z) < 2^{-n} \log \lambda_*\}) \Big\}.$$

Here

$$K_n(c) = \frac{2\lambda_*^2}{\log \lambda_*} \max\{|(f_c^n)'(z)| : z \in J_c\}.$$

We have:

$$K_n(c) \leq B'(\operatorname{diam} J_c)^n \leq B'(2\beta)^n, \tag{5}$$

where $B' = \frac{2\lambda_*^2}{\log \lambda_*}$ and β is the unique positive solution of the equation $\beta^2 - |c| = \beta$, and also

$$K_n(c) \leq B_0 4^n.$$

Comment 1: Let us note for future use a few bounds related to the number $G_c(0)$.

(a) First, the following fact was established in [7]: for the multiplier ρ of every repelling periodic orbit of f_c of period n, we have

$$G_c(0) \leq \frac{1}{n} \log|\rho|.$$

In particular, since we assume $|\rho| < e$, then c belongs to a neighborhood of M where $G_c(0) \leq n^{-1} \leq 1$.

(b) Second, the transfinite diameter of the set $\{z : 0 \leq G_c(z) < 2^{-n} \log \lambda\}$ $(\lambda > 1)$ is equal to $\lambda^{2^{-n}}$, and, hence, by a theorem of Polya [9],

$$\mathbf{area}(\{z : 0 \leq G_c(z) < 2^{-n} \log \lambda\}) \leq \pi \lambda^{2^{-n+1}}.$$

Consider two particular cases of the inequality (4). One case corresponds to passing to a limit as $|\rho| \to 1$, and in the other put $\rho' = 0$. Then we get the following corollary.

Corollary 2.1: (A) *There exists C_0, such that, if, for some c, $|\rho(c)| = 1$, then*

$$|\rho'(c)| \geq \frac{n|\rho(c) - 1|}{K_n(c)\pi^{-1}\mathbf{area}(\{z : 0, G_c(z) < 2^{-n} \log \lambda_*\})} \geq \frac{C_0 n|\rho(c) - 1|}{K_n(c)}. \tag{6}$$

(B) *For every n-periodic orbit O of f_c with the multiplier ρ, if $1 < |\rho| < e$ and*

$$|\rho - 1| > K_n(c)\frac{1}{n} \log |\rho|, \tag{7}$$

then $\rho' \neq 0$.

Comment 2: (A) implies, of course, that $\rho' \neq 0$ on the boundary of a hyperbolic component where $|\rho| = 1$, and it was known after [5].

3. Proof of Theorem 3

3.1. Derivative of multiplier and Ruelle operator. The proof is based on the following result, see Theorem 2 of [12]. Consider the Ruelle transfer operator

$$Tg(z) = \sum_{w : f_c(w) = z} \frac{g(w)}{(f_c'(w))^2}.$$

Given a periodic orbit $O = \{b_k\}_{k=1}^n$ of f of exact period n and with the multiplier $\rho \neq 0, 1$, associate to O a function

$$A(z) = \sum_{k=1}^{n} \frac{1}{(z - b_k)^2} + \frac{1}{\rho(1 - \rho)} \sum_{k=1}^{n} \frac{(f^n)''(b_k)}{z - b_k}, \tag{8}$$

where $(f^n)''(z)$ is the second derivative of the n-iterate of f with respect to z. Then

$$A(z) = (TA)(z) + 2\frac{\rho'}{\rho}(T1)(z). \tag{9}$$

Comment 3: In [12] we use this to give another, local proof of DHS theorem.

3.2. Estimate from above. Fix c and denote $f = f_c$, $G = G_c$ etc. Given $\lambda > 1$, denote $U_\lambda = \{z : 0 < G(z) < \log\lambda\}$ and $C_\lambda = U_\lambda \setminus f^{-1}(U_\lambda)$. Note that U_λ is a subset of the basin of infinity D_∞. Introduce also a number associated to the cycle O as follows:

$$M = \liminf_{\delta\to 0} \int_{E_\delta} \frac{d\sigma_z}{|z-b|^2}, \tag{10}$$

where $E_\delta = D_\infty \cap (B(b,\delta) \setminus f_O^{-n}(B(b,\delta)))$, b is a point of the orbit, f_O^{-n} is a branch defined near b that fixes b, and $d\sigma_z$ is the area element in the z-plane. It is easy to see that

$$M \le 2\pi \log|\rho|.$$

Comment 4: One can show (though we will not use this fact below) that $M = 2\pi\alpha\log|\rho|$ where α is the the density of D_∞ at the point b in the logarithmic metric [14].

Lemma 3.1: *For every $\lambda > 1$,*

$$\int_{C_\lambda} |A(z)| d\sigma_z \le M + 2\frac{|\rho'|}{|\rho|}\mathbf{area}\Big(\Big\{z : 0 < G(z) < \frac{1}{2}\log\lambda\Big\}\Big). \tag{11}$$

Proof. Denote by f_O^{-i} a local branch sending a point b of the orbit O to $f^{n-i}(b)$, $i = 1, 2, \dots, n$. For $\delta > 0$, let $V_\delta = U_\lambda \setminus \bigcup_{i=0}^{n-1} f_O^{-i}(B(b,\delta))$. Then A is integrable in V_δ, and one can write

$$\int_{V_\delta} |A| d\sigma = \int_{V_\delta} |TA(z) + 2\frac{\rho'}{\rho}T1(z)| d\sigma \le \int_{f^{-1}(V_\delta)} |A| d\sigma + 2\frac{|\rho'|}{|\rho|}\mathbf{area}(f^{-1}(V_\delta)).$$

Note that $V_\delta = f^{-1}(V_\delta) \cup C_\lambda \cup \Delta \setminus (B(b,\delta) \setminus f_O^{-n}(B(b,\delta)))$, where Δ is an open set, which shrinks to $-O = (-b_1, \dots, -b_n)$ as $\delta \to 0$. Therefore,

$$\int_{C_\lambda} |A| d\sigma \le \liminf_{\delta\to 0} \int_{E_\lambda} |A(z)| d\sigma_z + 2\frac{|\rho'|}{|\rho|}\mathbf{area}(U_{\lambda^{1/2}}).$$

We have: $A(z) = (z-b)^{-2} + r(z)$ where $r(z)$ is integrable at b. Sending $\delta \to 0$, we get the desired inequality. ∎

3.3. Estimate from below. We are left with the problem to estimate the integral

$$I(\lambda) := \int_{C_\lambda} |A(z)| d\sigma_z. \tag{12}$$

from below.

To be able to deal with $I(\lambda)$ when J is disconnected and λ close to 1, let us extend the Bottcher function B of f at ∞ to the following simply connected domain (on the Riemann sphere) D^*_∞: it is obtained from D_∞ by deleting all arcs of external rays (gradient curves of G) starting from 0 and all its preimages up to the Julia set. Denote the extended Bottcher function again by B. It maps D^*_∞ onto a domain A whose boundary is a 'hedgehog', see [15], [16]. Let $\phi : A \to D^*_\infty$ be the inverse map. Note that ϕ' has a singularity at the vertex p of every needle (i.e., the image by B of a critical point of G_c), but it is of the type $|w-p|^{-1/2}$, i.e. integrable against the area.

Let $\lambda > 1$. One writes:

$$I(\lambda) = \int_{C_\lambda} |A(z)| d\sigma_z = \int_{\lambda<|w|<\lambda^2} |A(\phi(w))||\phi'(w)|^2 d\sigma_w.$$

Let us consider the function $p(z) = f^n(z) - z$. Note that it is nonzero in the basin of infinity. By the Fubini theorem, we can continue as follows

$$\int_{\lambda<|w|<\lambda^2} |A(\phi(w))||\phi'(w)|^2 d\sigma_w$$
$$\geq \int_\lambda^{\lambda^2} \min_{|w|=r} \frac{|\phi'(w)|}{|p(\phi(w))|} dr \int_{\{z:G(z)=\log r\}} |A(z)p(z)||dz|.$$

Now,

$$\int_{\{z:G(z)=\log r\}} |A(z)p(z)||dz| \geq \left| \int_{\{z:G(z)=\log r\}} A(z)p(z)dz \right| = 2\pi|R_p|,$$

where

$$R_p = \frac{1}{2\pi i} \int_{\{z:G(z)=\log r\}} A(z)p(z)dz.$$

By the definition of A and the Cauchy formula,

$$R_p = \sum_{k=1}^{n} (p'(b_k) + \gamma_k p(b_k)) = n(\rho - 1),$$

where $\gamma_k = (f^n)''(b_k)/\rho(1-\rho)$. Thus, for every $\lambda > 1$,

$$I(\lambda) \geq 2\pi n|\rho - 1| \int_\lambda^{\lambda^2} \min_{|w|=r} \frac{|\phi'(w)|}{|p(\phi(w))|} dr. \tag{13}$$

Now, from $f \circ \phi(w) = \phi(w^2)$ we conclude that

$$\frac{\phi'(w)}{p(\phi(w))} = \frac{2^n w^{2^n - 1} \phi'(w^{2^n})}{(\phi(w^{2^n}) - \phi(w))(f^n)'(\phi(w))}.$$

There is a choice for $\lambda > 1$. Take $\lambda = \lambda_n$ where $\lambda_n^{2^n} = \lambda_*$ and λ_* is fixed.

Let us estimate $|c|$. By Comment 1, (a), $G_c(0) \le n^{-1}$. If J_c is connected, then $|c| \le 2$. Otherwise the function ϕ which is inverse to the Bottcher coordinate B of f_c, extends in a univalent fashion to the disk $\{|w| > \exp(G_c(0))\}$. Besides, $c = \phi(w_c)$, for some $|w_c| = \exp(2G_c(0))$, and ϕ is odd. Therefore, by a classical distortion theorem, see, e.g., [9]

$$|c| = |\phi(w_c)| \le 2\exp(2G_c(0)) \le 2\exp(2/n), \quad n > 0. \tag{14}$$

Thus c belongs to a bounded neighborhood of the Mandelbrot set. Hence, we can fix λ_* in such a way that for all c in the fixed neighborhood, $|\phi'(w^{2^n})/(\phi(w^{2^n}) - \phi(w))| > 1/(2|w^{2^n}|)$ for all $\lambda_* < |w|^{2^n} < \lambda_*^2$. We define

$$L_n(c) = \frac{2\max\{|(f_c^n)'(z)| : G_c(z) = 2^{-n+1}\log\lambda_*\}}{\log\lambda_*}.$$

Then the inequality (4) holds with $L_n(c)$ instead of $K_n(c)$. By the Bernstein-Walsh inequality,

$$|(f_c^n)'(z)| \le \max\{|(f_c^n)'(z)| : z \in J_c\}\exp\{(2^n - 1)G_c(z)\}. \tag{15}$$

For $G_c(z) = 2^{-n+1}\log\lambda_*$, this implies that

$$L_n(c) \le K_n(c) := \frac{2\lambda_*^2}{\log\lambda_*}\max\{|(f_c^n)'(z)| : z \in J_c\}.$$

3.4. The constant. Let us show that

$$K_n(c) \le B'(2\beta)^n, \tag{16}$$

where β is the unique positive solution of the equation $\beta^2 - |c| = \beta$. (Note that $\beta > 1$ because $c \ne 0$.) Indeed, if, for some $\delta > 0$, $|z| = \beta + \delta$, then $|f_c(z)| > \beta + 2\beta\delta$. Hence, such z lies in D_∞, and

$$\max\{|(f_c^n)'(z)| : z \in J_c\} \le (2\beta)^n.$$

It gives (16).

Furthermore, as we know,

$$|c| \le 2\exp(2/n) \le 2(1 + A_1/n),$$

for some A_1. Therefore,

$$\beta \leq 2 + A_2/n$$

and, hence,

$$K_n(c) \leq B4^n(1 + A_2/(2n))^n \leq 4^n B_0,$$

for some A_2 and B_0.

4. Analytic extension of the multiplier

4.1. A set of domains. Given $C > 1$, consider an open set Ω of points in the punctured ρ-plane defined by the inequality

$$|\rho - 1| > C \log |\rho|. \tag{17}$$

It obviously contains the set $D_* = \{\rho : 0 < |\rho| \leq 1, \rho \neq 1\}$ and is disjoint with an interval $1 < \rho < 1+\epsilon$. Denote by $\Omega(C)$ the connected component of Ω which contains the set D_* completed by 0. Denote also by $\Omega^{\log}(C)$ the set of points

$$L = \log \rho = x + iy, \quad \rho \in \Omega(C),\ |y| \leq \pi.$$

Note that $\Omega^{\log}(C)$ is symmetric with respect to the real axis.

Lemma 4.1: *$\Omega(C)$ is simply-connected. More precisely, the intersection of $\Omega^{\log}(C)$ with any vertical line with $x = x_0 > 0$ is either empty or equal to two (mirror symmetric) intervals. If $C > 4$, then $x < 2/(C-2)$ for all $L = x + iy$ in $\Omega^{\log}(C)$. In particular, $\Omega(C) \subset \{|\rho| < e\}$ for $C > 4$. If C is large enough, $\Omega^{\log}(C)$ contains two (mirror symmetric) domains bounded by the lines $y = \pm(C/2)x$ $(x > 0)$ and $y = \pm\pi$*

Proof. In the log-coordinate $L = \log \rho = x + iy$, $|y| \leq \pi$, the condition (17) with $x > 0$ is equivalent to

$$\sin^2\left(\frac{y}{2}\right) > \frac{C^2x^2 - (\exp(x) - 1)^2}{4\exp(x)}. \tag{18}$$

Given x, the set containing y which satisfies the latter inequality is either empty or a union $[-\pi, -y) \cup (y, \pi]$ with some $y > 0$. We have for $x + iy \in \Omega^{\log}(C)$: $1 > (C^2x^2 - (\exp(x)-1)^2)/4\exp(x)$, and since $x > 0$, $Cx < \exp(x)+1$. If $C > 4$, then it is easy to check that the first positive root of the equation $Cx = \exp(x) + 1$ is smaller than $2/(C-2)$. Therefore, $|x| < 2/(C-2)$ for any $x + iy \in \Omega^{\log}(C)$. If C large, then, by this, $x > 0$ must be small, and the boundary curve of $\Omega^{\log}(C)$

can be written in the form $\sin(y/2) = \pm(1/2)(C^2-1)^{1/2}x(1-x/2+xr(x,C))$ where $r(x,C) = o(1/C^2)$ uniformly in x. The statement follows. ∎

4.2. Hyperbolic components and related definitions. Let W be a component of the interior of M. It is called a hyperbolic component if f_c, $c \in W$, has an attracting periodic orbit $O(c)$. Let us call W n-hyperbolic, if the exact period of the latter orbit is n.

Denote by $\rho_W(c)$ the multiplier of $O(c)$. By the DHS theorem, ρ_W is a analytic isomorphism of W onto the unit disk, and it extends homeomorphically to the boundaries. Given a number $t \in (-1/2, 1/2]$, denote by $c(W,t)$ the unique point in ∂W with the **internal argument** t, i.e., ρ_W at this points is equal to $\exp(2\pi t)$. The root of W is the point $c_W = c(W,0)$ with the internal argument zero. W is called primitive if and only if its root is not a point of other hyperbolic component.

If $t = p/q$ is a rational number, we will always assume that p, q are coprimes. For any rational $t \neq 0$, denote by $L(W,t)$ the connected component of $M \setminus \{c(W,t)\}$ which is disjoint with W. It is called the t-limb of W. Also, denote by $W(t)$ a nq-hyperbolic component with the root point $c(W,t)$; it touches W at this point. The limb $L(W,t)$ contains $W(t)$. The root c_W of W is the landing point of precisely two external rays of the Mandelbrot set [5]. In what follows, the notion of the wake of a hyperbolic component W [11] will be important: it is the only component W^* of the plane cut by two external rays to the root of W that contains W. We will also use the following fact (see, e.g., [16, Theorem 7.2]):

Proposition 1: *The points of periodic orbit $O(c)$ as well as its multiplier ρ_W extend as analytic functions to the wake W^*. Moreover, $|\rho_W| > 1$ in $W^* \setminus \overline{W}$.*

The map ρ_W from W onto the unit disk $c \mapsto \rho_W(c)$ has an inverse, denoted by $c = \psi_W(\rho)$. It is defined, so far, in the unit disk.

4.3. Extension of the multiplier. Introduce

$$\Omega_n = \Omega(n^{-1}4^n B_0),$$

the connected component of the set $\{\rho : |\rho - 1| > \frac{4^n B_0}{n} \log|\rho|\}$ that contains the (punctured) unit disk, and

$$\Omega_n^{\log} = \Omega^{\log}(n^{-1}4^n B_0) = \{L = x+iy : \exp(L) \in \Omega_n, |y| \leq \pi\}.$$

THEOREM 4: (a) *The function* $\psi = \psi_W$ *extends to a holomorphic function in the domain* Ω_n.

(b) *The function* ψ *is univalent in a subset* $\tilde{\Omega}_n$ *of* Ω_n *defined by its log-projection* $\tilde{\Omega}_n^{\log} = \{\log \rho : \rho \in \tilde{\Omega}_n\}$ *as follows:*

$$\tilde{\Omega}_n^{\log} = \Omega_n^{\log} \setminus \{L : |L - R_n| < R_n\},$$

where R_n *depends on* n *only and has an asymptotics*

$$R_n = (2 + O(2^{-n}))n \log 2$$

as $n \to \infty$.

Finally, the image of $\tilde{\Omega}_n$ *by* ψ *is contained in the wake* W^*.

Comment 5: The disk $\{L : |L - R_n| < R_n\}$ cuts off from $\Omega_n^{\log}$ an asymptotically negligible portion: the deleted part is contained in the disk $\{|L| < r_n\}$, where $r_n \sim (4 \log 2/B_0)(n^2/4^n)$.

Proof. Let ρ be a multiplier of some repelling orbit of period n for some f_c. By Corollary 2.1, part (B), if $\rho \in \Omega_n$, then $\rho' \neq 0$.

(a) Since Ω_n is simply connected, it is enough to show that ψ has an analytic extension along any curve in Ω_n (which starts at 0).

Firstly, c is an algebraic function of ρ. Indeed, consider two functions $Q(c, b) = f_c^n(b) - b$ and $P(c, b, \rho) = b f_c(b) \dots f_c^{n-1}(b) - \rho/2^n$. They are polynomials in c, b and ρ, which are of degree 2^n and $2^n - 1$, respectively with respect to b and with leading coefficients 1. Hence, resultant $R(c, \rho)$ of Q and P with respect to b is a polynomial such that $R(c, \rho) = 0$ if and only if Q and P have a joint root b, which means that b is a fixed point of f_c^n and $\rho = (f_c^n)'(b)$.

Now assume that ψ does not have an analytic continuation along a (simple) curve β in Ω_n starting in 0. By the above, it means that β contains a singular point ρ_0 of the algebraic function c, that is, when we make a small loop around ρ_0, we get a different value of c. Let ρ_0 be the first such point when we move from 0. Denote $c_0 = \psi(\rho_0)$ the limit value of ψ when ρ approaches ρ_0 along β. Since c is algebraic, c and ρ can be written in a form:

$$c - c_0 = u^j, \quad \rho - \rho_0 = (\phi(u))^k,$$

where the integers $j, k > 0$ and ϕ is holomorphic near $u = 0$ such that $\phi(0) = 0, \phi'(0) \neq 0$. Note that $\rho_0 \neq 1$, hence ρ is holomorphic in c in a neighborhood of c_0. By the same reason, the period of the corresponding periodic orbit of f_{c_0}

(=limit of the orbit of $f_{\psi(\rho)}$ when ρ tends to ρ_0 along β) is exactly n. $\rho \in \Omega_n$, hence, $\rho'(c_0) \neq 0$. But $dc/d\rho \sim (j/k)u^{j-k}/\phi'(0)$, which implies $j = k$. Since ρ is holomorphic in c near c_0, we get

$$(\phi(u))^k = \tilde{\phi}(u^k)$$

for another holomorphic near $u = 0$ function $\tilde{\phi}$. Here $\tilde{\phi}'(0) \neq 0$, because $\phi'(0) \neq 0$. Thus in a new local coordinate $\tilde{u} = u^k$ we have $c - c_0 = \tilde{u}$, $\rho - \rho_0 = \tilde{\phi}(\tilde{u})$. It follows that ψ can be extended through ρ_0, a contradiction.

(b) ψ is univalent in some domain if it takes values in a simply connected domain in the c-plane where the function $\rho(c)$ (local inverse to ψ) is well-defined. Let us choose the wake of W to be such a domain in the c-plane. The choice is correct because as we know the function ρ extends to a holomorphic function from W to its wake W^*. Now it is enough to show that, for any $L = \log \rho \in \tilde{\Omega}_n^{\log}$, the value $\psi(\rho)$ cannot belong to the boundary of W^*. The latter consists of two external rays in the c-plane to the root of W (including the root itself). Assume it is not the case. Then we find a curve l in $\tilde{\Omega}_n$, which starts at 0 and ends at some ρ_0, such that $\psi(\rho) \in W^*$ for all $\rho \in l \setminus \{\rho_0\}$ while $c_0 = \psi(\rho_0)$, for $L_0 = \log \rho_0 \in \Omega_n^{\log}$ does belong to such a ray. Consider a continuation of the n-periodic orbit $O(c)$ of f_c along the curve l which is attracting for $c \in W$. Then the rotation number of the periodic orbit $O(c_0)$ of f_{c_0} is zero (see, e.g., [16, Remark 7.2]). Now, by Theorem 5.1 of [15],

$$\frac{|L_0|^2}{Re(L_0)} \leq \frac{2\pi n \log 2}{\pi/2 - \arctan[(2^n - 1)a_0/\pi]}, \tag{19}$$

where $a_0 = G_{c_0}(0)$. Let us estimate a_0 from above. According to [7, Theorem 1.6], $Re(L_0) = \log|\rho_0| \geq na_0$. By Lemma 4.1, $Re(L_0) < 2/(C_n - 2)$, where $C_n = 4^n B_0/n$. Hence, $a_0 < 2/(4^n B_0 - 2n)$. It allows us to define

$$R_n = \frac{\pi n \log 2}{\pi/2 - \arctan[\frac{2(2^n-1)}{\pi(4^n B_0 - 2n)}]}.$$

Note that $R_n = (2 + O(2^{-n}))n \log 2$ as $n \to \infty$. Now, (19) implies that $|L_0 - R_n| < R_n$, i.e., ρ_0 belongs to the part of Ω_n that we delete, a contradiction. ∎

5. Limbs

5.1. Yoccoz's circles. Let W be an n-hyperbolic component. As we know, the multiplier ρ_W is defined and analytic throughout the wake W^*. Let us formulate a result due to Yoccoz, which is a basic point for further considerations. For every $t = p/q \neq 0$, consider the limb $L(W,t)$. Then, for every $c \in L(W,t)$, a branch of $\log \rho_W(c)$ is contained in the following round disk (Yoccoz's circle):

$$Y_n(t) = \left\{L : \left|L - \left(2\pi i t + \frac{n \log 2}{q}\right)\right| < \frac{n \log 2}{q}\right\}, \tag{20}$$

see [11], [21] as well as [22], [13], [14], [8], [7], [16].

As a well-known corollary [11], we have:

Proposition 2: (a) *The intersection of the wake W^* of W (completed by the root) with M is equal to the union of $\overline{W}$ and its limbs.*

(b) *For every hyperbolic component W there exists a constant C_W depending on W, such that,*

$$diam L(W, p/q) \leq C_W/q.$$

5.2. Condition on the internal argument.

Definition 5.1: Given n, let us call a rational numbers $t = p/q \in [-1/2, 1/2]$ n-deep, if and only if for every n-hyperbolic component W, there is a disk $B(2\pi i t, d)$ with $d < \pi$, such that the following conditions hold:

(a) the inverse ψ_W to the map ρ_W extends to a univalent function defined in the union of the unit disk and the domain $\exp(B(2\pi i t, d))$;

(b) $\psi_W(\exp(B(2\pi i t, d/2))$ covers the limb $L(W,t)$.

In order to stress the choice of d, we will call t also (n,d)-deep point.

Comment 6: By the proof of Theorem 4, if ψ_W extends just *holomorphically* into $\exp(B(2\pi i t, d))$ and maps it into the wake W^*, then ψ_W satisfies (a). Therefore, by (20), if $B(2\pi i t, (4n \log 2)/q) \subset \tilde{\Omega}_n^{\log}$, then t is (n,d)-deep with $d = (4n \log 2)/q$.

Proposition 3: (1) *For every fixed n,*

(1a) *n-deep rationals are dense in $(-1/2, 1/2)$;*

(1b) *the set of all t, which are not n-deep has a unique concentration point 0.*

(2) *There exists* $B>0$, *such that, for every* $n>0$, *the point* $t=p/q\in(-1/2,1/2)$ *is* $(n,4n\log 2/q)$*-deep if the following two inequalities hold:*

$$p > B4^n, \tag{21}$$

$$p^2/q > Bn^2. \tag{22}$$

Proof. (1) follows from Comment 6. As for (2), by the same Comment it is enough to check that the inequalities (21)–(22) guarantee that the disk of radius $(4n\log 2)/q$ centered at the point $2\pi it$ is contained in the domain $\tilde{\Omega}_n^{\log}$. Denote $\theta=2\pi t$ and consider the disk $B(i\theta,r)$. Then for $C>0$ and $L=i\theta+w$ where $|w|<r$, $r<1$, one can write:

$$\begin{aligned} |\exp(L)-1|-CRe(L) &\geq |\exp(-i\theta)-1|-|exp(w)-1|-C|w| \\ &\geq 2|\sin(\theta/2)|-(C+2)r. \end{aligned} \tag{23}$$

That is, if $r<2|\sin(\theta/2)|/(C+2)$, then the disk $B(i\theta,r)$ lies inside of $\Omega^{\log}(C)$. Note that $|\sin(\theta)|\geq 2\theta/\pi$. This shows that if we put here $C=C_n=4^nB_0/n$ and $r=r_n=(4n\log 2)/q$, then there is $B>0$, so that the inequality (21) ensures that $B(i\theta,r_n)$ lies in $\Omega_n^{\log}$, $n>0$.

It is also easy to check that, if B is big enough, the inequality (22) means that $B(i\theta,r_n)$ is disjoint with the disk $B(R_n,R_n)$, for $n>0$, so that both inequalities imply that $B(i\theta,r_n)$ is contained in $\tilde{\Omega}_n^{\log}$. ∎

5.3. Uniform bound on the size of some limbs.

THEOREM 5: (1) *Let* W *be an* n*-hyperbolic component, and let* $c\in\partial W$ *have an internal argument* t*, such that* t *is* n*-deep. Then there is a topological disk* $B(c)$*, such that* $B(c)$ *does not contain the root* c_W *of* W *and it is "roughly" a round disk around the point* c*: for some* r*,*

$$B(c,r/4)\subset B(c)\subset B(c,4r).$$

The function $\log\rho_W$ *is univalent in* $B(c)$ *and maps it onto the disk* $B(2\pi it,d/2)$*. Moreover,* $\log\rho_W$ *extends univalently to a topological disk containing* $B(c)$*, and maps this bigger domain onto* $B(2\pi it,d)$*. The limb* $L(W,t)$ *is contained in* $B(c)$*.*

(2) *There exists* $A>0$*, such that, for every* n*-hyperbolic component* W *and every* $t=p/q\in[-1/2,1/2]$*, the diameter of the limb* $L(W,t)$ *is*

bounded by:

$$diamL(W,t) \le A\frac{4^n}{p} = A\frac{4^n}{t}\frac{1}{q}. \tag{24}$$

Comment 7: Part (2) is Theorem 1 stated in the Introduction. It strengthens the bound of part (b) of Proposition 2 (off the root, i.e., if $t = p/q$ is outside of a neighborhood of zero).

Proof. (1) Following the Definition 5.1, one can take

$$B(c) = \psi_W(\exp(B(2\pi it, d/2))).$$

We use here and later on classical distortion bounds for univalent maps, see, e.g., [9]: if g is univalent in a disk $B(0,R)$ and $r < R$, then

$$B(g(0), \alpha(r/R)^{-1} r|g'(0)|) \subset g(B(0,r)) \subset B(g(0), \alpha(r/R) r|g'(0)|), \tag{25}$$

where $\alpha(x) = (1-x)^{-2}$.

Now (1) is an immediate consequence of the definition 5.1 and the latter bound (with $r = R/2$).

(2) Let us prove the second part of the theorem. It is enough to show that there exist M, N, such that if $p > M4^n$, then $diamL(W,t) \le N4^n/p$ for all n, t. Fix n and $t = p/q$. In the course of the proof A_i will denote different constants from a finite collection of numbers. For $r_n = 2n\log 2/q$ and $C_n = n^{-1}4^n B_0$ one can write:

$$\frac{r_n(C_n+2)}{|\sin(\pi p/q)|} \le A_1\frac{4^n}{p}.$$

Hence, if $A_1 4^n/p < 1/4$, then by (23) the disk $B(2\pi ip/q, 2r_n)$ is contained in $\Omega_n^{\log}$. Hence, so is the disk $Y := \{|L - (i\theta + n\log 2/q)| < n\log 2/q\}$ which is contained in $B(2\pi ip/q, 2r_n)$. Let us introduce a map $\Psi = \psi_W \circ \exp$, an inverse to $\log\rho_W$. By Theorem 4, Ψ extends to a holomorphic function in $B(2\pi ip/q, 2r_n)$. For every c in the limb $L(W,t)$ there is $L \in Y$ such that $c = \Psi(L)$. Therefore, the distance between the root of the limb and c is bounded from above by the integral $J := \int_{i\theta}^{L} |\Psi'(w)||dw|$ for some L with $|L| = r_n$. On the other hand, by Theorem 3,

$$|\Psi'(w)| \le \frac{\pi^{-1}\mathbf{area}(\{z : G_c(z) < a2^{-n}\})C_n}{|\exp(w)-1| - C_n Re(w)}.$$

Here the **area**$(\{z : G_c(z) < a2^{-n}\})$ is smaller than $\pi(1+o(1))$ as $n \to \infty$ by Comment 1 (b). Using (23) we can write

$$J \leq (1+o(1))C_n r_n \int_0^1 \frac{d\tau}{2|\sin(\theta/2)| - r_n(C_n+2)\tau},$$

and hence

$$J \leq (1+o(1))C_n \log \frac{2|\sin(\theta/2)|}{2|\sin(\theta/2)| - r_n(C_n+2)}.$$

Since $r_n(C_n+2)/|\sin(\theta/2)| < 1/4$ and $\log(1-x)^{-1} \leq 2x$ if $0 < x < 1/4$, we can proceed:

$$J \leq (1+o(1))C_n \frac{r_n}{|\sin(\theta/2)|} \leq A_2 \frac{4^n}{p}$$

for all n. Thus it is enough to put $M = 4A_1$ and $N = A_2$. ∎

Corollary 5.1: *For any sequence*

$$t_0, t_1, \ldots, t_m, \ldots$$

of rational numbers $t_m = p_m/q_m \in [-1/2, 1/2]$ the following holds. Let W by a n-hyperbolic component. Let $W^0 = W$, $W^m = W^{m-1}(t_{m-1})$, $m = 1, 2, \ldots$ (i.e., the hyperbolic component W^m touches the hyperbolic component W^{m-1} at a point with the internal argument t_{m-1}), so that W^m is n_m-hyperbolic component where the periods $n_0 = n$, $n_m = nq_0 \cdots q_{m-1}$. Assume that

$$\limsup_{m\to\infty} p_m/4^{n_m} = \infty. \tag{26}$$

Then the limbs $L(W^m, t_m)$ shrink to a unique point c_*, in particular, the Mandelbrot set is locally connected at c_*.

Proof. By the previous result, $diam L(W^m, t_m) \to 0$ along a subsequence. Hence, $L(W^m, t_m)$ shrink to a point c_*. The local connectivity follows because these limbs form a shrinking sequence of connected neighborhoods of c_*. ∎

6. Selecting internal arguments

6.1. Bifurcations. Throughout this subsection, we consider the following situation. Let W be a n-hyperbolic component, and let $c_0 \in \partial W$ have an internal argument $t_0 = p/q \neq 0$.

Let $O(c) = \{b_j(c)\}_{j=1}^n$ be the periodic orbit of f_c which is attracting when $c \in W$. Then all $b_j(c)$ as well as the multiplier $\rho(c)$ are holomorphic in W and extend to holomorphic functions in c near c_0 (in fact, in the whole wake of W).

As we know, the multiplier $\rho(c)$ is injective near c_0. Denote the inverse function by $\psi(\rho)$, $\psi(\exp(2\pi t_0)) = c_0$. ψ is well-defined in $\tilde{\Omega}_n$, that includes the unit disk and a neighborhood of the point $\rho_0 = \exp(2\pi i t_0)$.

Consider now the wake of the nq-hyperbolic component $W(t_0)$ (by definition, it tangents W at the point c_0). By Douady–Hubbard theory [5], the root c_0 of $W(t_0)$ is the landing point of precisely two external rays of M. Denote their arguments by $\tau(c_0)$, $\tilde{\tau}(c_0)$. For every c in the wake of $W(t_0)$, we have the following picture in the dynamical plane of f_c: two external rays of f_c with the arguments $\tau(c_0)$ and $\tilde{\tau}(c_0)$ land at one point (denote it $b_n(c)$), which is a point of the repelling periodic orbit $O(c)$. It implies also that c lies in a component of the plane minus these two external rays which does not contain the origin. Indeed, by the formula for the uniformization map of the exterior of M [4], this is true for those c in the wake of $W(t_0)$ which are outside of the Mandelbrot set. Hence, it must be true throughout the wake because c cannot cross the external rays as well as their landing point $b_n(c)$.

We will make use also of a well-known formula [1], [16]:

$$|\tau(c_0) - \tilde{\tau}(c_0)| = \frac{(\beta - \alpha)(2^n - 1)}{2^{nq} - 1}, \tag{27}$$

where $\beta, \alpha \in \{0, 1, 2, \ldots, 2^n - 1\}$ are two “digits” determined by W.

Next statement is combinatorial (cf. [3]).

Lemma 6.1: *Let c be a point of a limb $L(W, t')$ with some $t' = p'/q'$ and $q' > 2$. Assume that f_c has a periodic point of period nQ with the multiplier 1. Then*

$$Q \geq q' - 1.$$

Proof. Consider the dynamical plane of f_c. The critical value c belongs to a petal of a. Since c lies in the sector bounded by the two external rays with arguments $\tau(c')$, $\tilde{\tau}(c')$ where c' is the root of $L(W, t')$, then a is in the same sector, too. On the other hand, a is a landing point of two external rays fixed by f_n^{nQ}. Therefore, there must be $|\tau(c') - \tilde{\tau}(c')| > (2^{nQ} - 1)^{-1}$. If we now apply formula (27), we get $nQ \geq nq' - 2n + 1$, that is, $Q \geq q' - 1$. ∎

The next lemma describes the bifurcation near the parameter c_0, cf. [3], [10].

Lemma 6.2: *There exist a small disk U around the origin, a neighborhood V of the cycle $O(c_0)$, and n functions $F_k(s)$, $k = 1, \ldots, n$, which are holomorphic in U, such that $F_k(0) = 0$, $F_k'(0) \neq 0$ and, for every $s \in U$ and every $\rho = \rho_0 + s^q$,* *the points*

$$b_k(c_0) + F_k(s\exp(2\pi j/q)), \quad k = 1, \dots, n, \ j = 0, \dots, q-1$$

are the only fixed points of f_c^{nq} in the neighborhood V different from $O(c)$, where $c = \psi(\rho_0 + s^q)$ and ρ is the multiplier of $O(c)$. They form a periodic orbit of f_c of period nq.

Proof. Introduce

$$p(c,z) = \frac{f_c^{nq}(z) - z}{f_c^n(z) - z}.$$

It is a polynomial in z and c. As we also know, the function $c = \psi(\rho)$ satisfies another polynomial equation $R(c, \rho) = 0$, see the proof of Theorem 4. Hence, periodic points of period n form an algebraic function in ρ: they satisfy a polynomial equation of the form

$$\tilde{p}(\rho, z) = 0. \tag{28}$$

For every k, the point $(\rho_0, b_k(c_0))$ with $\rho_0 = \exp(2\pi i t_0)$ is a singular point of $\tilde{p}$. Hence, there exist a local uniformizing parameter s and co-prime i, j, such that every solution of (28) in a small enough neighborhood of the singular point, has the form

$$\rho - \rho_0 = s^i, z - b_k(c_0) = F_k(s),$$

where

$$F_k(s) = r_k s^j + O(s^{j+1})$$

is a holomorphic function near 0, $r_k \neq 0$. Let us show that, necessarily, $i = q$, $j = 1$.

We introduce a new (local) variable $w = L(z) = z - b_k(c)$ and consider conjugate map

$$g(\rho, w) = L \circ f_c^n \circ L^{-1}(w),$$

where $c = \psi(\rho)$. Then for all ρ near ρ_0 and w near 0,

$$g(\rho, 0) = 0, \quad \frac{\partial g}{\partial w}(\rho, 0) = \rho. \tag{29}$$

Now consider g^q. Then

$$g^q(\rho_0, w) = w + Aw^{q+1} + O(w^{q+2}),$$

where $A \neq 0$. Taking this into account we obtain, using (29),

$$g^q(\rho, w) - w = (\rho^q - 1)w + O((\rho - \rho_0)w^2) + Aw^{q+1} + O(w^{q+2}) \quad A \neq 0. \tag{30}$$

On the other hand, $b_k(c)-b_k(c_0) = O(\rho-\rho_0) = O(s^i)$ so that a pair of functions

$$(\rho - \rho_0 = s^i, w = r_k s^j + O(s^{j+1}) + O(s^i)) \tag{31}$$

is a solution of the equation $(g^q(\rho, w) - w)/w = 0$. Substituting this pair of functions of s into (30) we arrive at the conclusion that $i = jq$.

Thus, locally, (near $(\rho_0,\, b_k(c_0))$) the algebraic function (ρ, z) has the form

$$\rho - \rho_0 = s^q, z - b_k(c_0) = F_k(s) = F_k'(0)s + O(s^2),\ F_k'(0) \neq 0.$$

Now, nq points $b_k(c_0) + F_k(s\exp(2\pi j/q)), k = 1, \ldots, n, j = 0, \ldots, q-1$ are fixed by f_c^{nq} and are close to the n-cycle of f_{c_0} with a multiplier a primitive q-root of 1. Therefore, they form a single nq-periodic orbit. ∎

The following two lemmas allow us to connect the multiplier ρ with the multiplier of a periodic orbit after the bifurcation. In the first one we use an idea of [3] although our proof is more involved. The reason is that the limb $L(W, t_0)$ can contain more than one hyperbolic components of period nq or less (which is impossible in case $n = 1$ considered in [3]).

Lemma 6.3: *There exists Q, as follows. For all $q > n > 0$, $q > Q$, if the disc $B(\rho_0, (9/10)^q)$ is contained in $\tilde{\Omega}_n$, then each function F_k extends to a holomorphic function in the disk $\{|s| < 9/10\}$. Moreover, the domain $\psi(B(\rho_0, (9/10)^q))$ is disjoint with any limb $L(W, p'/q')$ which is different from $L(W, t_0)$ and such that $q' \leq q+1$. The multiplier $\rho_{W(t_0)}$ of the nq-periodic orbit has a well-defined analytic extension from the hyperbolic component $W(t_0)$ to the union of the limb $L(W, t_0)$ and the domain $\psi(B(\rho_0, (9/10)^q))$.*

Proof. Denote $B = B(\rho_0, (9/10)^q)$. Since $B \subset \tilde{\Omega}_n$, then ψ is defined and univalent in B. Denote $\tilde{B} = \psi(B)$.

We prove by several steps that the function $F_k(s)$ extends to a analytic function in $\{|s| < 9/10\}$.

1. $\tilde{B}$ is disjoint with all the limbs $L(W, t')$ of W different from $L(W, t_0)$, such that $t' = p'/q'$ and $q' \leq q+1$. Indeed, let us project the disk B by log. Since $(9/10)^q$ is very small, the projection is (asymptotically) a disk $B(2\pi i t_0, (9/10^q))$. If $q > n$ and q is big enough, $(9/10)^q < r_0 := 1/(nq^2(q+1))$. On the other hand, doing a simple calculation similar to [3] we see that for all n, q, the ball $B(2\pi i t_0, r_0)$ is disjoint with every Yoccoz circle that touches the vertical line at a point $2\pi i p'/q'$ with some $q' \leq q+1$. The claim follows.

2. Assume the contrary: there exists a path γ in B that connects 0 and some s_1, such that F_k cannot be extended analytically through s_1. Since $(\rho(s), a(s))$ where $\rho(s) - \rho_0 = s^q, z(s) - b_k(c_0) = F_k(s)$ satisfy a polynomial equation, the function F_k has an analytic continuation along every curve unless it meets a singular point. Therefore, the point (ρ_1, a), such that $\rho_1 := \rho(s_1)$ and $a := b_k(c_0) + F_k(s_1)$ is so that a is a fixed point of $f_{c_1}^{nq}$ $(c_1 = \psi(\rho(s_1)))$ with the multiplier 1.

Let us see where $c_1 \in \tilde{B}$ can be situated. By Lemma 6.1, c_1 belongs either to $L(W, t_0)$ or to some $L(W, p'/q')$ other than $L(W, t_0)$ and such that $q' \leq q + 1$. The latter possibility is excluded by Step 1. Thus $c_1 \in L(W, t_0)$.

3. Consider the image Γ of the path γ by the map $\psi(\rho_0 + s^q)$. It connects the root $c_0 = c(W, t_0)$ of $L(W, t_0)$ to c_1. The curve Γ cannot belong completely to the wake $W^{**} := W(t_0)^*$ (which contains the limb $L(W, t_0)$). The reason is that, as we know, the point $a(s)$ extends analytically to this wake. Assume for a moment that there is another curve Γ_1 that connects c_1 and c_0 inside the wake W^{**} and such that it lies in $\tilde{B}$. Let us deform (keeping the end points fixed) the curve Γ_1 to Γ inside $\tilde{B}$. If along the way we will not meet another singular point, then it will contradict the fact that the point c_1 is singular for $a(s)$. Hence, we must meet another singular point. It must also belong to the limb $L(W, t_0)$. Then we can replace Γ_1 by another curve that connects c_0 and c_1 inside W^{**} and inside B, such that Γ_1 can be deformed to Γ inside B without meeting singular points, a contradiction.

We conclude that there exists a curve Γ_1 connecting c_0 and c_1 inside the wake W^{**}, such that it leaves $\tilde{B}$ (and then comes back to c_1), such that when we deform Γ_1 to Γ we meet a singular point c_2, which belongs to another limb $L(W, p_2/q_2)$ with $q_2 \leq q + 1$. Then the continuum $L(W, p_2/q_2)$ contains the points c_2 and its root and is disjoint with $L(W, t_0)$, therefore, $L(W, p_2/q_2)$ must cross $\tilde{B}$, a contradiction to the fact that $\tilde{B}$ is disjoint with all such limbs.

This proof also shows that the results of analytic extension of the function $\rho_{W(t_0)}(c)$, $c \in W(t_0)$, along a curve in the limb $L(W, t_0)$ and a curve in the domain $\tilde{B}$ are the same provided the curves have the same ends. ∎

Definition 6.1: The latter lemma shows that for every ρ such that $|\rho - \rho_0| < (9/10)^q$, the points $b_k(c_0) + F_k(s_j), k = 1, \ldots, n, j = 0, \ldots, q-1$, where s_j are the q different roots of the equation $\rho - \rho_0 = s^q$, form a nq-cycle of f_c where

$c = \psi(\rho)$. We denote this periodic orbit by $O^q(c)$. As $c = c_0$, it coincides with the n-cycle $O(c_0)$.

Definition 6.2: If A, B are two sets in the plane, we say that A is δ-close to B and denote it by $d(A, B) < \delta$ if and only if for every point $a \in A$ there exists a point $b \in B$, such that $|a - b| < \delta$.

Comment 8: The function d is not symmetric. On the other hand, it is easy to check the triangle inequality: if $d(A, B) < \delta_1$ and $d(C, A) < \delta_2$, then $d(C, B) < \delta_1 + \delta_2$.

For $t_0 = p/q$, denote

$$B_{t_0} = B(\exp(2\pi i t_0), (9/10)^q). \tag{32}$$

By (14), one can assume that $|c| < 3$. If z is a periodic point of f_c we then have $|z| \le 3$. Now the Schwarz Lemma gives us:

Corollary 6.1: *For every $\rho \in B_{t_0}$, where t_0 is as in Lemma 6.3, and for every s such that $\rho - \rho_0 = s^q$ and $|s| < 9/10$, we have, for $c = \psi(\rho)$:*

$$d(O^q(c), O(c_0)) < 7|s|,$$

$$d(O(c), O(c_0)) < 6\Big(\frac{10|s|}{9}\Big)^q.$$

Denote by $\lambda(\rho)$ the multiplier of the periodic orbit $O^q(c)$ for $c = \psi(\rho)$. In other words, $\lambda = \rho_{W(t_0)} \circ \rho^{-1}$ whenever it is well defined.

It follows from, for instance, Lemma 6.2 that λ is defined and holomorphic near $\rho = \rho_0$. Moreover, by [10],

$$\frac{d\lambda}{d\rho}(\rho_0) = -\frac{q^2}{\rho_0}.$$

This formula can also be derived directly from (30) with help of (31) where $i = q, j = 1$.

Lemma 6.3 tells us that, for $q > n > n_0$, the function λ is holomorphic in B_{t_0}. But it is not necessarily univalent there. On the other hand, to choose next internal argument, we need that the image by λ is not small. This is proved in the next statement.

Lemma 6.4: *For every real $T \in (-1/2, 1/2]$, the equation $\lambda(\rho) = \exp(2\pi i T)$ has at most one solution ρ in B_{t_0}, and for such a solution the corresponding*

$c = \psi(\rho)$ lies on the boundary of the nq-hyperbolic component $W(t_0)$, which tangents W at the point $c_0 = \psi(2\pi i t_0)$. The following covering property holds: for every $r \le (9/10)^q$, the image of the disk $B(\exp(2\pi i t_0), r)$ under the map λ covers the disk $B(1, q^2 r/16)$.

Proof. Let $\lambda(\rho_1) = \exp(2\pi i T_1)$ for some real T_1. Lemma 6.3 tells us that $\psi(B_{t_0})$ is disjoint with any p'/q'-limb of W other than $L(W, t_0)$, where $q' \le q+1$. On the other hand, for $c_1 = \psi(\rho_1)$, the map f_{c_1} has nq-periodic orbit $O^q(c_1)$ with the multiplier $\rho_1 = \exp(2\pi i T_1)$. Therefore, $c_1 \in L(W, t)$ and moreover it lies in the boundary of an nq-hyperbolic component. This hyperbolic component belongs to some limb $L(W, t')$, $t' = p'/q'$, where, by the above, $q' > q+1$. If $t' \ne t_0$, then $L(W, t')$ contains a parameter $\tilde{c}$ on the boundary of this component, such that $f_{\tilde{c}}$ has a nq-periodic orbit with the multiplier 1. By Lemma 6.1, $q \ge q' - 1$, in contradiction with the previous condition $q' > q+1$ or $t' = t_0$. Thus c_1 is in the limb $L(W, t_0)$. Hence, it lies in the boundary of some nq-hyperbolic component belonging this time to $L(W, t_0)$. On the other hand, the multiplier of the periodic orbit $O^q(c)$ is bigger on modulus than 1 off the closure of the component $W(t_0)$. Thus the hyperbolic component containing c_1 in its boundary is just $W(t_0)$. Since the corresponding multiplier is injective on the boundary, it means that ρ_1 is the only solution of the equation $\lambda(\rho) = \exp(2\pi i T_1)$.

To prove the covering property, given $r \le (9/10)^q$, consider a function $m(w) = (q^2 r)^{-1}(\lambda(\rho_0 + rw) - 1)$. It is holomorphic in the unit disk, $m(0) = 0$, $|m'(0)| = 1$ and, by the proven part of the statement, $m(w) = 0$ if and only if $w = 0$. Therefore, by a classical result (Caratheodory–Fekete, see, e.g., [9]), the disk $B(0, 1/16)$ is covered by the image of the unit disk under the map m. This is equivalent to the covering property. ∎

6.2. Nonlocally connected Julia sets. Our aim is to prove Theorem 2 stated in the Introduction. We will do it in a few steps. The main one consists of proving the following result.

Theorem 6: *There exists Q, such that, for every $n > 0$ the following holds. Let $t_0, t_1, \ldots, t_m, \ldots$ be a sequence of rational numbers $t_m = p_m/q_m \in (-1/2, 1/2)$. Denote $n_0 = n$, $n_m = n q_0 q_1 \cdots q_{m-1}$ for $m > 0$.*

(a) *Assume that, for every* m, *we have:* $q_m > n_m$, $q_m > Q$, *and* t_m *is* (n_m, d_m)-*deep, with* $d_m > 1/(n_m q_m^3)$, *see Definition 5.1, and, moreover,* $|t_m|^{1/q_{m-1}} < 1/2$, $m = 1, 2, \ldots$;

Let W *be* n-*hyperbolic components.*

(1) *Denote* $W^0 = W$, $W^m = W^{m-1}(t_{m-1})$, $m = 1, 2, \ldots$ *(i.e., the hyperbolic component* W^m *touches the hyperbolic component* W^{m-1} *at a point with the internal argument* t_{m-1}*). Then (a) implies that the limbs* $L(W^{m-1}, t_{m-1})$ *form a nested sequence of compact sets with the intersection a unique point* c_*.

If, in addition to (a), we also assume:

(b)

$$\sum_{m=1}^{\infty} |t_m|^{1/q_{m-1}} < 4^{-n}/16,$$

then:

(2) *the map* f_{c^*} *is infinitely renormalizable with non locally connected Julia set.*

We begin the proof with some notations.

$$\Delta = 4^{-n}/16, \quad \epsilon_m = |t_{m+1}|^{1/q_m}, \quad \psi_m(w) = \psi_{W^m}(\exp(w)),\ m = 0, 1, \ldots,$$

$$c_m = \psi_m(2\pi i t_m), m = 0, 1, \ldots.$$

Note that c_m is the point in the boundary of W^m with the internal argument t_m.

Since t_m is n_m-deep, the function ψ_m extends in a univalent fashion to $B(2\pi i t_m, d_m)$.

Remind that

$$B(c_m) = \psi_m(B(2\pi i t_m, d_m/2)).$$

By Theorem 5, each $B(c_m)$ is "roughly" a round disk around the point c_m:

$$B(c_m, r_m/4) \subset B(c_m) \subset B(c_m, 4r_m),$$

where $r_m = |\psi_m'(2\pi i t_m)|d_m/2$. Denote

$$B_m = \psi_m(B(2\pi i t_m, (9/10)^{q_m})),$$

$$D_m = \psi_m(B(2\pi i t_m, |t_{m+1}|)),$$

$$D_m' = \psi_m(B(2\pi i t_m, 100|t_{m+1}|)).$$

If $r/d_m \le 1/2$, then $\psi_m(B((2\pi i t_m, r))$ is "roughly" a disk:

$$B(c_m, 4^{-1} r|\psi_m'(2\pi i t_m)|) \subset \psi_m(B((2\pi i t_m, r)) \subset B(c_m, 4r|\psi_m'(2\pi i t_m)|).$$

Note also that by the condition (a) all q_m are big provided Q is big, and $d_m > 1/(n_m q_m^3)$.

It implies $100|t_{m+1}|/(9/10)^{q_m}$ and $(9/10)^{q_m}/d_m$ are small (for Q big enough), B_m, D_m and D'_m are "roughly" disks (around c_m), and for $m = 0, 1, \ldots$,

$$D_m \subset D'_m \subset B_m \subset B(c_m),$$

and, moreover, the diameter of each previous set is much smaller than the diameter of the next one.

LEMMA 6.5: *There is Q, such that, for $q_m > n_m$ and $q_m > Q$ the following holds.*

(i) $c_{m+1} \in D_m$,
(ii) $B(c_{m+1}) \subset D'_m$.

Proof. We can use Lemmas 6.3–6.4 because $(9/10)^{q_m} < 1/(n_m q_m^3)$ if $q_m > n_m$ big enough. Consider the function $\lambda = \rho_{W^{m+1}} \circ \psi_m$. Then λ is holomorphic in $B(2\pi i t_m, (9/10)^{q_m})$ and since $q_m > n_m$ and $|t_{m+1}| = \epsilon_m^{q_m} < (1/2)^{q_m}$, then $\lambda(B(2\pi i t_m, |t_{m+1}|))$ covers the disk $B(1, q_m^2|t_{m+1}|/16)$. If $q_m > 16$ and t_{m+1} is small enough, then

$$\exp(2\pi i t_{m+1}) \sim 1 + 2\pi i t_{m+1} \in B(1, q_m^2|t_{m+1}|/16).$$

Therefore, by the first part of Lemma 6.4, the point

$$c_{m+1} = \psi_{m+1}(2\pi i t_{m+1}) \in \psi_m(B(2\pi i t_m, |t_{m+1}|)) = D_m.$$

To prove that $B(c_{m+1})$ is contained in the bigger disk D'_m, let us notice that by part (1) of Theorem 5, the root c_m of W^{m+1} is outside of $B(c_{m+1})$, and $B(c_{m+1})$ is "roughly" a disk around c_{m+1}. Hence,

$$B(c_{m+1}) \subset B(c_{m+1}, 16|c_{m+1} - c_m|) \quad \text{and} \quad B(c_{m+1}) \subset B(c_m, 17|c_{m+1} - c_m|).$$

Let us estimate $|c_{m+1} - c_m|$. We use the distortion bounds for univalent maps (25). Denote $\delta_m = 2|t_{m+1}|/d_m$. Since $c_{m+1} \in D_m$, then $|c_{m+1} - c_m| < \delta_m \alpha(\delta_m) r_m$ where $\alpha(x) = (1-x)^{-2}$. Thus $B(c_{m+1}) \subset B(c_m, 17\delta_m \alpha(\delta_m) r_m)$. On the other hand, $D'_m = \psi_m(B(2\pi i t_m, 100|t_{m+1}|)$ contains the disk

$$B(c_m, 100\delta_m(\alpha(100\delta_m))^{-1} r_m).$$

We conclude that if $\delta_m < x_0$ where $x_0 > 0$ is the solution of equation

$$100(1 - 100x_0)^2 = 17(1 - x_0)^{-2},$$

then

$$B(c_{m+1}) \subset B(c_m, 17\delta_m \alpha(\delta_m) r_m) \subset B(c_m, 100\delta_m(\alpha(100\delta_m))^{-1} r_m) \subset D'_m.$$

The condition $\delta_m < x_0$ means that $2|t_{m+1}| < d_m x_0$. This holds if $(9/10)^{q_m} < (d_m/2)x_0$ which is apparently always the case if q_m is big enough. Thus if Q is big, the conclusion (ii) holds. The lemma is proved. ∎

The latter lemma implies that for all m,

$$D'_{m+1} \subset B_{m+1} \subset B(c_{m+1}) \subset D'_m.$$

Since the limb $L(W^m, t_m)$ is contained in $B(c_m)$ and the diameters of D'_m is by a definite factor smaller than the diameter of $B(c_m)$, we conclude that the limbs $L(W^m, t_m)$ shrink to a point c_*. This proves conclusion (1) of the theorem. Moreover, we have:

$$\{c_*\} = \bigcap_{m=0}^{\infty} B_m = \bigcap_{m=0}^{\infty} D'_m.$$

Based on this and on Lemma 6.1, it is not difficult to prove the conclusion (2).

Indeed, let us denote by $O_m(c)$ the n_m-periodic orbit of f_c, which is attracting if $c \in W^m$, $m = 0, 1, 2, \ldots$. As we know, O_m extends holomorphically for c in the wake $(W^m)^*$ of W^m. Moreover, Lemma 6.3 tells us that for each $m = 0, 1, 2, \ldots$, the orbit O_{m+1} extends holomorphically to the disk B_m near the root c_m of the wake $(W^{m+1})^*$.

Since $c_* \subset D'_m \subset B_m$ and $c_{m+1} \subset D_m$, Corollary 6.1 allows us to write:

$$d(O_m(c_*), O_m(c_m)) < 6(\frac{10(100)^{1/q_m}\epsilon_m}{9})^{q_m} < 6(2\Delta)^{q_m} < \Delta \tag{33}$$

and

$$d(O_{m+1}(c_{m+1}), O_m(c_m)) < 7\epsilon_m. \tag{34}$$

By the triangle inequality for d, we get from (34):

$$d(O_{m+1}(c_{m+1}), O_0(c_0)) < 7\sum_{k=0}^{m} \epsilon_k$$

and then (33) implies

$$d(O_{m+1}(c_*), O_0(c_0)) < 7\sum_{k=0}^{m} \epsilon_k + \Delta < 8\Delta. \tag{35}$$

On the other hand, since $O_0(c_0)$ is a neutral n-periodic orbit of f_{c_0}, the distance of $O_0(c_0)$ to zero is bigger than 4^{-n}. Hence, we have: if $\Delta = 4^{-n}/16$, then, for all $m = 0, 1, 2, \ldots$, every point of $O_m(c_*)$ is away from the origin by the distance at least $4^{-n}/2$.

It is known that this implies the nonlocal connectivity of J_{c_*} (see [23] for a detailed proof, although our description is closer to [19]). Let us outline a proof. Denote by τ_m, τ'_m two external arguments in the c-plane to the root c_m of the wake of W^m. By the formula (27), $|\tau_{m+1} - \tau'_{m+1}| \leq (2^{n_m})^2/(2^{n_m q_m} - 1) \to 0$ as $m \to \infty$, that is τ_m, τ'_m tend to some τ^*.

On the other hand, for every c in the wake of W^{m+1}, we have the following picture in the dynamical plane of f_c (see beginning of the present section): two external rays of f_c with arguments τ_{m+1} and τ'_{m+1} land at a single point $b_m(c)$, which is a point of the repelling periodic orbit $O_m(c)$. Moreover, c lies in the sector bounded by these two external rays and disjoint with the origin.

Now assume that the Julia set of f_{c_*} is locally connected. Then this discussion implies that, in the dynamical plane of f_{c_*}, the external argument of c_* is equal to τ_* and since $\tau_m \to \tau_*$, also $b_m(c_*) \to c_*$. Taking a preimage by f_{c_*}, it gives us a sequence of points of the sets $O_m(c_*)$ which tends to the origin, a contradiction with the fact that the orbits $O_m(c_*)$ stay away from the origin.

6.3. Proof of Theorem 2. Here we prove a more general Theorem 7. Theorem 2 stated in the Introduction is an immediate corollary of this together with Definition 5.1 and sufficient conditions (21)–(22).

Theorem 7: *Let $n \geq 1$. Let*

$$t_0, t_1, \ldots, t_m, \ldots$$

be any sequence of rational numbers $t_m = p_m/q_m \in (-1/2, 1/2]$ which satisfies the following properties. Denote $n_0 = n$, $n_m = nq_0 \cdots q_{m-1}$, $m > 0$. Assume that, for all m large enough, $q_m > n_m$ and t_m is (n_m, d_m)-deep with $d_m > 1/(n_m q_m^3)$, and also

$$\sum_{m=1}^{\infty} |t_m|^{1/q_{m-1}} < \infty. \tag{36}$$

Given a hyperbolic component W of the Mandelbrot set of period n, consider a sequence of hyperbolic components W^m: $W_0 = W$, and, for $m > 0$, $W^m = W^{m-1}(t_{m-1})$, i.e., W^m touches the hyperbolic component W^{m-1} at a point

c_{m-1} with the internal argument t_{m-1}. For every m, consider the t_m-limb $L(W^m, t_m)$ of W^m (it contains W^{m+1}). Then the sequence of limbs $L(W^m, t_m)$ shrink to a unique point c_*, the Mandelbrot set is locally connected at c_*, and the map f_{c_*} is infinitely renormalizable with nonlocally connected Julia set.

Let m_0 be large enough, so that $|t_{m+1}| < (1/2)^{q_m}$ and t_m is n_m-deep for every $m \geq m_0$. Then the sequence $t_{m_0}, \ldots, t_m, \ldots$ satisfies the conditon (a) of Theorem 6 with $n = n_{m_0}$. Hence, by the conclusion (1) of Theorem 6 the limbs $L(W^m, t_m)$ shrink to a unique point c_*.

It remains to show that the Julia set J_{c_*} of f_{c_*} is not locally connected. We show that *whether J_{c_*} is locally connected or not depends only on the tail of the sequence $t_0, \ldots, t_m, \ldots$.* Let us introduce the following notation. Let W_0 be the 1-hyperbolic component (the main cardioid). Given k, let us start with the component W_0 and the tail $T_k = \{t_k, t_{k+1}, \ldots\}$ in place of W and $\{t_0, t_1, \ldots\}$. Then we get a sequence of hyperbolic components $W_0^{k,m}$, $m \geq k$, where $W_0^{k,k} = W_0$ and, for $m > k$, the component $W_0^{k,m}$ touches $W_0^{k,m-1}$ at the point with internal argument t_{m-1}. We have proved that, for every $k \geq m_0$, the sequence of limbs $L(W_0^{k,m})$, $m = k, k+1, \ldots$ shrinks to a unique point c_*^k. Now, we have

Proposition 4: *For every k large enough, the Julia set of $f_{c_*^k}$ is not locally connected.*

Proof. Choose k big enough so that

$$\sum_{m=k}^{\infty} |t_{m+1}|^{1/q_m} < 4^{-1}/16,$$

and apply Theorem 6 to $n = 1$ and to the sequence $\{t_k, t_{k+1}, \ldots\}$. ∎

Let us come back to the map f_{c_*}. The next statement will finish the proof that J_{c_*} is not locally connected.

Proposition 5: *For every k, there is a restriction $f_{c_*}^{n_k} : U_k \to V_k$, such that this is a polynomial-like map of degree 2, which is quasi-conformally conjugate to $f_{c_*^k}$.*

Proof. Consider first the wake of W^{k-1}. For c in this wake, f_c has a periodic orbit $O_{k-1}(c)$ of period n_{k-1} which is attracting if and only if $c \in W^{k-1}$ and is holomorphic in c. Now consider the wake $(W^k)^*$ of the next component

W^k. Denote also a_k the root of W^k. If $c = a_k$, then $O_{k-1}(a_k)$ is neutral with the multiplier $\exp(2\pi i t_{k-1})$: there are q_{k-1} petals attached to each point of $O_{k-1}(a_k)$, and $f_{a_k}^{n_{k-1}}$ acts (locally) on them as a rotation with the notation number t_{k-1}. If $c \in (W^k)^*$, then $O_{k-1}(c)$ is repelling, and has the rotation number t_{k-1}. There exists a point $b_{k-1}(c)$ of $O_{k-1}(c)$, such that for $c \in (W^k)^* \cup \{a_k\}$, there are two external rays $R_k(c)$, $\tilde{R}_k(c)$ of arguments $\tau(a_k)$, $\tilde{\tau}(a_k)$ landing at $b_{k-1}(c)$, such that the component of the plane bounded by these two rays that contains c contains no other ray to the orbit. There are two topological disks $c \in S'_k(c) \subset S_k(c)$, such that, $S_k(c)$ is bounded by $b_{k-1}(c)$, the rays $R_k(c)$, $\tilde{R}_k(c)$ and by some equipotential, and $f_c^{n_k} : S'_k(c) \to S_k(c)$ is a proper map of degree 2. By a "thickening" of $S_k(c)$, one can turn it into a polynomial-like map $P_{k,c} : U_{k,c} \to V_{k,c}$, see [20] for details.

If $c \in B := \{c_*\} \cup \bigcup_{m=k}^{\infty}(W^m \cup \{a_k\})$, we claim that the Julia set of $P_{k,c} : U_{k,c} \to V_{k,c}$ is connected. Indeed, consider the iterates $f_c^{n_k j}(c)$, $j = 0, 1, \ldots$. If c is close to a_k and in W^k, then they converge to $b_k(c)$. On the other hand, neither of these iterates cannot cross the boundary of $S'_k(c)$ when $c \in B$, because f_c is not Misiurewich map for such c. This proves the claim.

By the Straightening Theorem [6], for every $c \in B$, there is a unique $\nu(c) \in M$, such that $P_{k,c} : U_{k,c} \to V_{k,c}$ is hybrid equivalent to $f_{\nu(c)}$. Moreover, by [6], the map $c \mapsto \nu(c)$ is continuous (it follows essentially from the compactness of K-quasiconformal maps and Proposition 7 of [6]). We need to show that $\nu(c_*) = c_*^k$. By continuity, it is enough to show that, for every root a_m with $m > k$, $\nu(a_m)$ is the root a_m^k of the hyperbolic component $W_0^{k,m}$. Let us prove it by induction on m.

(1) $m = k+1$. Notice that if c is close to a_k and in W^k, then $P_{k,c} : U_{k,c} \to V_{k,c}$ has an attracting fixed point: the point $b_k(c)$ of $O_k(c)$ which lies in $S'_k(c)$ and coincides with $b_{k-1}(c)$ when $c = a_k$. This attracting fixed point persists as $c \in W^k$ (periodic points cannot leave $S'_k(c)$), and when $c = a_{k+1}$, it has the multiplier $\exp(2\pi i t_k)$. Hence, the conjugate map $f_{\nu(a_{k+1})}$ has a neutral fixed point, and it interchanges the petals at this point with the same rotation number t_k. It follows that $\nu(a_{k+1}) = a_{k+1}^k$, the unique point in the boundary of the main cardioid with the internal argument t_k.

(2) Assume that $\nu(a_m) = a_m^k$ holds for some $m \geq k+1$. Then a similar argument shows that $\nu(a_{m+1}) = a_{m+1}^k$. ∎

6.4. Maps with unbounded combinatorics. Let us consider the case $n = 1$, that is we start with the main cardioid. Then we can reformulate Theorem 2 as follows.

Theorem 8: *Let a sequence p_m/q_m, $m = 0, 1, \ldots$, of nonzero rational numbers in $(-1/2, 1/2)$ be such that:*

$$\sum_{m=1}^{\infty} \left|\frac{p_m}{q_m}\right|^{1/q_{m-1}} < \infty,$$

$$p_m > B4^{n_m}, \qquad Bn_m^2 < \frac{p_m^2}{q_m},$$

for all m large enough, where $n_m = q_0 \cdots q_{m-1}$.

Let f_c be an infinitely renormalizable polynomial with the following combinatorial data.

(a) *The renormalization periods consists of the sequence $\{n_m\}_{m=1}^{\infty}$.*

(b) *Denote $J_0 = J(f_c)$, and, for every $m > 0$, J_m be the Julia set of the renormalization of period n_m, which contains 0. Let α_m and β_m be the α (i.e., separating) and β (i.e., nonseparating) fixed points of $f_c^{n_m} : J_m \to J_m$. We assume that, for every m, $\beta_{m+1} = \alpha_m$, and the rotation number of α_m with respect to the map $f_c^{n_m} : J_m \to J_m$ is equal to p_m/q_m.*

Then there exists a unique polynomial which satisfies the conditions (a)–(b). Its Julia set is not locally connected.

Comment 9: Corollary 5.1 can be generalized as follows. Let f admit a sequence of simple renormalizations of periods n_m and with the corresponding rotation numbers p_m/q_m. If $\lim \log p_m/n_m > 0$, then f is rigid.

In turn, by a refinement of the proof of Theorem 2, one can get rid of assumption (2). The conclusion will be that the sequence c_m converges to some c_*, and J_{c^*} is not locally connected. See [17].

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