ISRAEL JOURNAL OF MATHEMATICS **170** (2009), 253–268
DOI: 10.1007/s11856-009-0028-7

ON THE HYPERPLANE CONJECTURE FOR RANDOM CONVEX SETS

BY

Bo'az Klartag*

Department of Mathematics, Princeton University
Princeton, NJ 08544, USA
e-mail: klartagb@post.tau.ac.il

AND

Gady Kozma

Department of Mathematics, Weizmann Institute of Science
Rehovot 76100, Israel
e-mail: gady.kozma@weizmann.ac.il

ABSTRACT

Let $N \geq n+1$, and denote by $\mathcal{K}$ the convex hull of N independent standard gaussian random vectors in $\mathbb{R}^n$. We prove that with high probability, the isotropic constant of $\mathcal{K}$ is bounded by a universal constant. Thus we verify the hyperplane conjecture for the class of gaussian random polytopes.

1. Introduction

The hyperplane conjecture suggests a positive answer to the following question: Is there a universal constant $c > 0$, such that for any dimension n and for any convex set $\mathcal{K} \subset \mathbb{R}^n$ of volume one, there exists at least one hyperplane $\mathcal{H} \subset \mathbb{R}^n$ with $\mathrm{Vol}_{n-1}(\mathcal{K} \cap \mathcal{H}) > c$? Here, of course, Vol_{n-1} denotes $(n-1)$-dimensional volume.

This seemingly innocuous question, considered two decades ago by Bourgain [4, 5], has not been answered yet. We refer the reader to, e.g., [2], [19] or [15]

* Supported by the Clay Mathematics Institute and by NSF grant #DMS-0456590
Received December 18, 2006 and in revised form July 4, 2007

for partial results, history and for additional literature regarding the hyperplane conjecture. In particular, there are large classes of convex bodies for which an affirmative answer to the above question is known. These include unconditional convex bodies [4, 19], zonoids, duals to zonoids [2], bodies with a bounded outer volume ratio [19], unit balls of Schatten norms [16] and others (e.g., [12, 18]).

A potential counter-example to the hyperplane conjecture could have stemmed from random convex bodies, that typically belong to none of these classes. Recall that, starting with Gluskin's work [9], random polytopes are a major source of counter examples in high-dimensional convex geometry (in addition to the distance problem [9], one has, e.g., the basis problem [24] or the gaussian perimeter problem [20]). The goal of this short note is to show that gaussian random polytopes, and related models of random convex sets, do not constitute a counter-example to the hyperplane conjecture.

Suppose $\mathcal{K} \subset \mathbb{R}^n$ is a convex body. The isotropic constant of $\mathcal{K}$, denoted here by $L_{\mathcal{K}}$, is defined by

$$nL_{\mathcal{K}}^2 = \inf_{T:\mathbb{R}^n \to \mathbb{R}^n} \frac{1}{\mathrm{Vol}_n(\mathcal{K})^{1+2/n}} \int_{\mathcal{K}} |Tx|^2 dx, \tag{1}$$

where the infimum runs over all volume-preserving affine maps $T : \mathbb{R}^n \to \mathbb{R}^n$, and $|\cdot|$ stands for the standard Euclidean norm in $\mathbb{R}^n$. Directly from the definition, the isotropic constant is invariant under affine transformations. It is well-known (see [11] or [19]) that when $\mathrm{Vol}_n(\mathcal{K}) = 1$,

$$c_1/L_{\mathcal{K}} \leq \inf_{T:\mathbb{R}^n \to \mathbb{R}^n} \sup_{\mathcal{H} \subset \mathbb{R}^n} \mathrm{Vol}_{n-1}(T\mathcal{K} \cap \mathcal{H}) \leq c_2/L_{\mathcal{K}}, \tag{2}$$

where the infimum runs over all volume-preserving affine maps $T : \mathbb{R}^n \to \mathbb{R}^n$, the supremum runs over all hyperplanes $\mathcal{H} \subset \mathbb{R}^n$, and $c_1, c_2 > 0$ are some universal constants. Throughout the text, the symbols c, C, c', C', c_1, c_2 etc. denote various positive universal constants, whose value may change from one line to the next.

Thus, according to (2), the hyperplane conjecture is equivalent to the existence of a universal upper bound for the isotropic constant of an arbitrary convex body in an arbitrary dimension. It is well-known that $L_{\mathcal{K}} > c$ for any finite-dimensional convex body $\mathcal{K}$ (see, e.g., [19]). The best known general upper bound is $L_{\mathcal{K}} < Cn^{1/4}$ for a convex body $\mathcal{K} \subset \mathbb{R}^n$ (see [15] and also [6], [7] and [8]).

There are two natural models for random convex bodies (a body is a compact with non-empty interior). In the centrally-symmetric context, the first model is the symmetric convex hull of N random independent points, while the second (its dual), is the intersection of N random strips. For the second model, however, it is quite easy to demonstrate the hyperplane conjecture (say, in the gaussian model). Indeed, if $N \geq 2n$, then a simple calculation shows that it has a bounded outer volume ratio, and hence has a bounded isotropic constant (see [19]). If $n \leq N < 2n$, then the hyperplane conjecture holds deterministically whenever the resulting set is a body, according to [13]. Thus we will focus on the first model.

We say that a random vector $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ is a standard gaussian vector if its coordinates $X_1, \ldots, X_n$ are independent, standard normal variables.

Theorem 1.1: *Let $n \geq 1, N \geq n$ and let $G_0, \ldots, G_N$ be independent standard gaussian vectors in $\mathbb{R}^n$. Denote*

$$\mathcal{K} = \operatorname{conv}\{G_0, \ldots, G_N\} \quad \textit{and} \quad \mathcal{T} = \operatorname{conv}\{\pm G_1, \ldots, \pm G_N\}$$

where conv *denotes convex hull. Then, with probability greater than $1 - Ce^{-cn}$,*

$$L_{\mathcal{K}} < C \quad \textit{and} \quad L_{\mathcal{T}} < C.$$

Here, $C, c > 0$ are universal constants.

Let us sketch the proof in the case that $N > n^2$ (for smaller N the argument is only slightly more opaque). It is easy to see that in this case the radius of the inscribed ball is, with high probability, $\geq c\sqrt{\log N}$ — just calculate the probability that in a given direction all points are inside a strip of width $\epsilon\sqrt{\log N}$ and do a union-bound over a dense net in S^{n-1}. On the other hand, with probability 1 all faces of $\mathcal{K}$ are $(n-1)$-dimensional simplices. Further, with high probability all centers of gravity of all simplices are with distance $\leq C\sqrt{\log N}$ from 0 — the center of gravity of each simplex is a gaussian vector whose coordinates have variance $1/n$, and again all that is needed is to do a union-bound over all n-tuples of vertices. The concentration of the volume of a simplex around its center of gravity shows that almost all the mass is within distance of $\approx \sqrt{\log N}$ which implies the required estimate for $L_{\mathcal{K}}$.

On first glance it seems that some miracle is at work — why should we get the same $\sqrt{\log N}$ in the lower and upper bound? However, this is some manifestation of the phenomenon that the maximum of many independent variables is strongly concentrated. Of course, the different faces of our body are not independent, but it turns out that they are sufficiently independent to display a similar concentration phenomenon. Thus our proof is robust and would admit direct generalizations to other types of distributions, in place of the standard gaussian distribution. We will prove it for some other distributions that include the uniform distribution on the cube and on its corners $\{\pm 1\}^n$, see Theorem 3.2 below. The technique should also work for points uniform on S^{n-1}.

Acknowledgement. We would like to thank Jean Bourgain for motivating us to work on this problem. BK would also like to thank Alain Pajor for discussions on the subject.

2. Simplices

In this section, we assume that $N \geq n \geq 1$ are integers, and that $G_0, \ldots, G_N$ are independent random vectors in $\mathbb{R}^n$ which need not be identically distributed. We write $G_i = (G_{i,1}, \ldots, G_{i,n}) \in \mathbb{R}^n$, and we make the following assumptions regarding $G_0, \ldots, G_N$:

- (∗a) The random variables $G_{i,j}$ $(i = 0, \ldots, N,\ j = 1, \ldots, n)$ are independent.
- (∗b) For any $i = 0, \ldots, N$, $j = 1, \ldots, n$,

$$\mathbb{E} G_{i,j} = 0,\ \mathbb{E} G_{i,j}^2 = 1 \quad \text{and} \quad \mathbb{E} \exp\left(G_{i,j}^2/10\right) \leq 10.$$

- (∗c) The G_i $(i = 0, \ldots, N)$ are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$.

The constant 10 plays no special role. Note that (∗a), (∗b) and (∗c) hold when $G_0, \ldots, G_N$ are independent standard gaussian vectors. Our main technical tool is the following Bernstein's inequality for variables with exponential tail ("ψ_1"), see, e. g. [3] or [26, Section 2.2.2].

Theorem 2.1: *Let $L > 0$, let $m \geq 1$ be an integer, and let $X_1, \ldots, X_m$ be independent random variables with zero mean. Assume that*

$$\mathbb{E} \exp(|X_i|/L) \leq 20 \quad \text{for } 1 \leq i \leq m.$$

Then, for any $t > 0$,

$$\mathbb{P}\left\{ \left| \frac{1}{m} \sum_{i=1}^m X_i \right| > t \right\} \leq 2 \exp(-cm \min\{t/L, t^2/L^2\}),$$

where $c > 0$ is a universal constant.

The following lemma is a consequence of Theorem 2.1.

LEMMA 2.2: *Fix* $0 \le k_1 < k_2 < \cdots < k_n \le N$. *Then,*

(i) $\mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n G_{k_i}\right| > C\sqrt{\log\frac{2N}{n}}\right\} < \left(\frac{n}{10N}\right)^n$,

(ii) $\mathbb{P}\left\{\left|\frac{1}{N+1}\sum_{i=0}^N G_i\right| > C\sqrt{\log\frac{2N}{n}}\right\} < \left(\frac{n}{10N}\right)^n$,

(iii) $\mathbb{P}\left\{\frac{1}{n^2}\left(\left|\sum_{i=1}^n G_{k_i}\right|^2 + \sum_{i=1}^n |G_{k_i}|^2\right) > C\log\frac{2N}{n}\right\} < \left(\frac{n}{10N}\right)^n$.

Here, $C > 0$ *is a universal constant.*

Proof. Fix $1 \le i, j \le n$. We have, for an integer $p \ge 1$,

$$\mathbb{E}(G_{k_i,j})^{2p} = (10^p p!) \cdot \mathbb{E}\frac{(G_{k_i,j}^2/10)^p}{p!} \le (10^p p!) \cdot \mathbb{E}e^{G_{k_i,j}^2/10} \overset{(*b)}{\le} p! \cdot 100^p.$$

We use the inequality $x^p/p! + x^{p+1}/(p+1)! \le (p+2)\left(x^p/p! + x^{p+2}/(p+2)!\right)$ for a positive even integer p, and conclude that for any t,

$$\begin{aligned}\mathbb{E}\exp(tG_{k_i,j}) &= 1 + t\mathbb{E}G_{k_i,j} + \sum_{p=2}^{\infty}\frac{\mathbb{E}(tG_{k_i,j})^p}{p!} \le 1 + 2\sum_{p=1}^{\infty}\frac{(2p+2)\mathbb{E}(tG_{k_i,j})^{2p}}{(2p)!} \\ &\le 1 + 4\sum_{p=1}^{\infty}\frac{t^{2p}2^p p! \cdot 100^p}{(2p)!} \\ &\le 1 + 4\sum_{p=1}^{\infty}\frac{t^{2p}100^p}{p!} \\ &\le \exp(400t^2),\end{aligned}$$

where we also used the fact that $\mathbb{E}G_{k_i,j} = 0$, according to (*b). By independence, for any $j = 1, \ldots, n$ and t,

$$\mathbb{E}\exp\left(t\sum_{i=1}^n G_{k_i,j}\right) \le \exp\left(400nt^2\right),$$

so

$$\begin{aligned}\mathbb{E}\exp\left(\left|t\sum_{i=1}^n G_{k_i,j}\right|\right) &\le \mathbb{E}\exp\left(t\sum_{i=1}^n G_{k_i,j}\right) + \exp\left(-t\sum_{i=1}^n G_{k_i,j}\right) \\ &\le 2\exp(400nt^2).\end{aligned} \tag{3}$$

Denote by $Y_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n G_{k_i,j}$ $(j = 1, \dots, n)$. Then the Y_j are independent random variables of mean zero and variance one. Moreover, by (3), for $j = 1, \dots, n$,

$$\begin{aligned} \mathbb{E} \exp\left(\frac{|Y_j^2 - 1|}{10^6}\right) &\leq 2\mathbb{E} \exp\left(\frac{Y_j^2}{10^6}\right) = 2\int_0^\infty \mathbb{P}\left\{\frac{Y_j^2}{10^6} > \log t\right\} dt \leq \\ &\leq 2 + 2\int_1^\infty t^{-1000} \mathbb{E} e^{\sqrt{\log t}|Y_j|}\, dt \\ &\overset{(3)}{\leq} 2 + 4\int_1^\infty t^{400-1000}\, dt < 3. \end{aligned} \tag{4}$$

Hence, we may apply Theorem 2.1 for the independent random variables $Y_j^2 - 1$, with $L = 10^6$. We conclude that

$$\begin{aligned} \mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^n Y_j^2 > C \log \frac{2N}{n}\right\} &\leq \mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^n [Y_j^2 - 1] > (C/4) \log \frac{2N}{n}\right\} \\ &\leq 2\exp\left(-cn\frac{C}{4 \cdot 10^6} \log \frac{2N}{n}\right) \\ &< \frac{1}{2} \cdot \left(\frac{n}{10N}\right)^n, \end{aligned} \tag{5}$$

for an appropriate choice of a large universal constant C. Consequently,

$$\mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n G_{k_i}\right| > \sqrt{C \log \frac{2N}{n}}\right\} = \mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^n Y_j^2 > C \log \frac{2N}{n}\right\} < \left(\frac{n}{10N}\right)^n.$$

This completes the proof of (i). The argument that leads to prooving (ii) is similar. We define $\tilde{Y}_j = \frac{1}{\sqrt{N+1}} \sum_{i=0}^N G_{i,j}$ $(j = 1, \dots, n)$. Arguing exactly as above, we find that for $j = 1, \dots, n$,

$$\mathbb{E} \exp\left(\frac{|\tilde{Y}_j^2 - 1|}{10^6}\right) < 3.$$

We may invoke Theorem 2.1 for the independent random variables $\tilde{Y}_j^2 - 1$, with $L = 10^6$. This yields

$$\begin{aligned} \mathbb{P}\left\{\left|\frac{1}{N+1}\sum_{i=0}^N G_i\right| > C\sqrt{\log \frac{2N}{n}}\right\} &= \mathbb{P}\left\{\frac{1}{N+1}\sum_{j=1}^n \tilde{Y}_j^2 > C^2 \log \frac{2N}{n}\right\} \\ &\leq \mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^n [\tilde{Y}_j^2 - 1] > (C/2)^2 \log \frac{2N}{n}\right\} \\ &\leq 2\exp\left(-cn\frac{C^2}{2 \cdot 10^6} \cdot \log \frac{2N}{n}\right) < \left(\frac{n}{10N}\right)^n, \end{aligned}$$

provided that C is a sufficiently large universal constant. This proves (ii). It remains to prove (iii). The random variables $G_{i,j}^2 - 1$ ($i = 0, \ldots, N$, $j = 1, \ldots, n$) are independent, have mean zero and satisfy

$$\mathbb{E} \exp \left[|G_{i,j}^2 - 1|/10 \right] \leq 2\mathbb{E} \exp(G_{i,j}^2/10) \leq 20,$$

according to ($*$b). Hence, we may apply Theorem 2.1 for the independent random variables $G_{k_i,j}^2 - 1$, with $L = 10$. We conclude that for any $1 \leq j \leq n$,

$$\text{(6)} \quad \mathbb{P}\left\{ \frac{1}{n} \sum_{i=1}^{n} [G_{k_i,j}^2 - 1] > C \log \frac{2N}{n} \right\} \leq 2 \exp\left(-cn \frac{C}{10} \cdot \log \frac{2N}{n} \right) < \left(\frac{n}{20N} \right)^n,$$

for a large universal constant C. We sum (6) over $j = 1, \ldots, n$ and conclude that

(7)

$$\mathbb{P}\left\{ \frac{1}{n^2} \sum_{i,j=1}^{n} G_{k_i,j}^2 > (C+4) \log \frac{2N}{n} \right\} \leq \sum_{j=1}^{n} \mathbb{P}\left\{ \frac{1}{n} \sum_{i=1}^{n} [G_{k_i,j}^2 - 1] > C \log \frac{2N}{n} \right\}$$
$$\overset{(6)}{<} n \left(\frac{n}{20N} \right)^n < \frac{1}{2} \cdot \left(\frac{n}{10N} \right)^n.$$

The desired conclusion (iii) follows at once from (7) and (5). ∎

The next lemma is a simple, concrete calculation for the regular $(n-1)$-simplex. We write $e_1, \ldots, e_n$ for the standard basis in $\mathbb{R}^n$, and denote

$$\triangle^{n-1} = \operatorname{conv}\{e_1, \ldots, e_n\} \subset \mathbb{R}^n,$$

the $(n-1)$-dimensional regular simplex.

Lemma 2.3: *Let $X = (X_1, \ldots, X_n)$ be a random vector that is distributed uniformly in $\triangle^{n-1}$. Then,*

$$\mathbb{E} X_i X_j = (1 + \delta_{i,j})/(n(n+1))$$

where $\delta_{i,j}$ is Kronecker's delta.

Proof. Examine X without its last coordinate, $(X_1, \ldots, X_{n-1})$. This is distributed uniformly in the simplex

$$\left\{ x \in \mathbb{R}^{n-1} \ ; \ \sum_{i=1}^{n-1} x_i \leq 1, \ \forall i, x_i \geq 0 \right\}.$$

Consequently, the density of the random variable $Y = X_1 + \cdots + X_{n-1}$ is proportional to $t \mapsto t^{n-2}$ in the interval $(0,1)$, and is zero elsewhere. Hence,

$$\mathbb{E}(X_1 + \cdots + X_{n-1})^2 = \mathbb{E}Y^2 = \int_0^1 t^2 \cdot (n-1)t^{n-2}dt = \frac{n-1}{n+1}. \tag{8}$$

Note that $\sum_{i=1}^n X_i \equiv 1$. Therefore

$$1 = \mathbb{E}\left(\sum_{i=1}^n X_i\right)^2 = \sum_{i=1}^n \mathbb{E}X_i^2 + \sum_{i \neq j} \mathbb{E}X_i X_j. \tag{9}$$

From (8) and (9) we get that, when $n \geq 2$,

$$(n-1)\mathbb{E}X_1^2 + (n-1)(n-2)\mathbb{E}X_1X_2 = \frac{n-1}{n+1}, \quad n\mathbb{E}X_1^2 + n(n-1)\mathbb{E}X_1X_2 = 1,$$

and the lemma follows. ∎

Corollary 2.4: *Fix $0 \leq k_1 < k_2 < \cdots < k_n \leq N$, and set*

$$\mathcal{F} = \mathrm{conv}\{G_{k_1}, \ldots, G_{k_n}\}.$$

Denote $Z = \frac{1}{N+1}\sum_{i=0}^N G_i$. Then with probability greater than $1-4\left(n/(10N)\right)^n$, the set $\mathcal{F}$ is an $(n-1)$-dimensional simplex that satisfies

(i) $\frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})}\int_{\mathcal{F}} |x|^2 dx < C \log \frac{2N}{n}$,
(ii) $\frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})}\int_{\mathcal{F}} |x-Z|^2 dx < C \log \frac{2N}{n}$.

Here, $C > 0$ is a universal constant.

Proof. The random vectors G_{k_i} are independent and absolutely continuous according to $(*c)$. Hence, with probability one, the vectors $G_{k_1}, \ldots, G_{k_n}$ span $\mathbb{R}^n$, and the set $\mathcal{F}$ is an $(n-1)$-dimensional simplex in $\mathbb{R}^n$ whose vertices are the points $G_{k_1}, \ldots, G_{k_n}$. Denote $T = (G_{k_i,j})_{i,j=1,\ldots,n}$, an $n \times n$ matrix. Then, $\mathcal{F} = T\left(\triangle^{n-1}\right)$ and hence,

$$\frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})}\int_{\mathcal{F}} |x|^2 dx = \frac{1}{\mathrm{Vol}_{n-1}(\triangle^{n-1})}\int_{\triangle^{n-1}} |Tx|^2\, dx. \tag{10}$$

According to Lemma 2.3,

(11)

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}_{n-1}(\triangle^{n-1})}\int_{\triangle^{n-1}} |Tx|^2\,dx \\
&= \sum_{i,j=1}^{n} \langle G_{k_i}, G_{k_j}\rangle \frac{1}{\mathrm{Vol}_{n-1}(\triangle^{n-1})}\int_{\triangle^{n-1}} x_i x_j\,dx \\
&= \frac{1}{n(n+1)} \sum_{i,j=1}^{n} \langle G_{k_i}, G_{k_j}\rangle (1+\delta_{i,j}) = \frac{1}{n(n+1)}\left(\left|\sum_{i=1}^{n} G_{k_i}\right|^2 + \sum_{i=1}^{n} |G_{k_i}|^2\right).
\end{aligned}$$

Here, of course, $\langle\cdot,\cdot\rangle$ stands for the standard scalar product in $\mathbb{R}^n$. We conclude from (10), (11) and Lemma 2.2(iii) that

$$\mathbb{P}\left\{\frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})}\int_{\mathcal{F}} |x|^2 dx > C\log\frac{2N}{n}\right\} < \left(\frac{n}{10N}\right)^n. \tag{12}$$

As for the second part of the corollary, according to Lemma 2.2(i) and Lemma 2.2(ii) we know that

$$\mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^{n} G_{k_i}\right| + \left|\frac{1}{N+1}\sum_{j=0}^{N} G_j\right| < 2C\sqrt{\log\frac{2N}{n}}\right\} \geq 1 - 2\left(\frac{n}{10N}\right)^n. \tag{13}$$

Additionally,

(14)

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})}\int_{\mathcal{F}} |x-Z|^2\,dx \\
&= |Z|^2 - 2\left\langle \frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})}\int_{\mathcal{F}} x\,dx, Z\right\rangle + \frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})}\int_{\mathcal{F}} |x|^2\,dx \\
&= \left|\frac{1}{N+1}\sum_{j=0}^{N} G_j\right|^2 - 2\left\langle \frac{1}{n}\sum_{i=1}^{n} G_{k_i}, \frac{1}{N+1}\sum_{j=0}^{N} G_j\right\rangle + \frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})}\int_{\mathcal{F}} |x|^2\,dx.
\end{aligned}$$

By combining (14) with (12) and (13), we obtain

$$\mathbb{P}\left\{\frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})}\int_{\mathcal{F}} |x-Z|^2\,dx > 5C\log\frac{2N}{n}\right\} < 3\left(\frac{n}{10N}\right)^n. \tag{15}$$

From (12) and (15) the corollary follows. ∎

For a point $x \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$, we write $d(x, A) = \inf_{y\in A} |x-y|$.

Lemma 2.5: *Set*

$$\mathcal{K} = \mathrm{conv}\{G_0, \ldots, G_N\}, \quad \mathcal{T} = \mathrm{conv}\{\pm G_1, \ldots, \pm G_N\},$$

and denote $Z = \frac{1}{N+1}\sum_{i=0}^{N} G_i$. *Then with probability greater than* $1 - Ce^{-cn}$,

(i) $\frac{1}{\mathrm{Vol}_n(\mathcal{T})}\int_{\mathcal{T}} |x|^2 dx < C \log \frac{2N}{n}$,

(ii) $\frac{1}{\mathrm{Vol}_n(\mathcal{K})}\int_{\mathcal{K}} |x - Z|^2 dx < C \log \frac{2N}{n}$.

Here, $c, C > 0$ *are universal constants.*

Proof. The random vectors $G_{i,j}$ are absolutely continuous, by assumption ($*$c). With probability one, the vectors $G_1, \ldots, G_N$ linearly span $\mathbb{R}^n$ and the vectors $G_0, \ldots, G_N$ affinely span $\mathbb{R}^n$, and hence $\mathrm{Vol}_n(\mathcal{T}) > 0, \mathrm{Vol}_n(\mathcal{K}) > 0$. Additionally, with probability one, the points $G_0, \ldots, G_N$ are in general position in $\mathbb{R}^n$; that is, with probability one, no $n+1$ distinct points from $\{G_0, \ldots, G_N\}$ lie in the same affine hyperplane in $\mathbb{R}^n$. Consequently, with probability one, all the $(n-1)$-dimensional facets of the polytopes $\mathcal{K}$ and $\mathcal{T}$ are simplices. Note that, in the case of $\mathcal{T}$ we use the fact that G_i and $-G_i$ could never belong to the same face.

Let $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ be a complete list of the $(n-1)$-dimensional facets of $\mathcal{T}$. Since a facet is determined by n points from $\{\pm G_1, \ldots, \pm G_N\}$, then

$$\ell \le \binom{2N}{n} \le \left(\frac{2eN}{n}\right)^n.$$

According to Corollary 2.4(i), with probability greater than $1 - 4\left(\frac{2e}{10}\right)^n$,

$$\int_{\mathcal{F}_i} |x|^2 dx < C\left(\log \frac{2N}{n}\right) \cdot \mathrm{Vol}_{n-1}(\mathcal{F}_i), \quad \text{for all } i = 1, \ldots, \ell. \tag{16}$$

Each point $x \in \mathcal{T}$ (except for the origin) may be uniquely represented as $x = ty$ with $0 < t \le 1$ and $y \in \partial\mathcal{T}$. We integrate with respect to these standard polar coordinates, and obtain that

$$\frac{1}{\mathrm{Vol}_n(\mathcal{T})}\int_{\mathcal{T}} |x|^2 \, dx = \frac{1}{\mathrm{Vol}_n(\mathcal{T})}\int_{\partial\mathcal{T}}\int_0^1 |ty|^2 t^{n-1}\langle y, \nu_y\rangle \, dt \, dy \tag{17}$$

where ν_y is the unit outward normal to $\partial\mathcal{T}$ at y (ν_y is uniquely defined almost everywhere as $\mathcal{T}$ is convex). When $y \in \mathcal{F}_i$ for some $i = 1, \ldots, \ell$, we have that $\langle y, \nu_y\rangle = d(0, \mathrm{asp}\,\mathcal{F}_i)$ where $\mathrm{asp}\,\mathcal{F}_i$ is the affine subspace spanned by $\mathcal{F}_i$. Hence, from (17),

$$\frac{1}{\mathrm{Vol}_n(\mathcal{T})}\int_{\mathcal{T}}|x|^2dx=\frac{1}{\mathrm{Vol}_n(\mathcal{T})}\sum_{i=1}^{\ell}\frac{d(0,\operatorname{asp}\mathcal{F}_i)}{n+2}\int_{\mathcal{F}_i}|y|^2dy. \tag{18}$$

Recall that $\sum_{i=1}^{\ell} d(0,\operatorname{asp}\mathcal{F}_i)\cdot\mathrm{Vol}_{n-1}(\mathcal{F}_i)=n\,\mathrm{Vol}_n(\mathcal{T})$. We combine (18) with (16), and conclude that with probability greater than $1-4\left(\frac{2e}{10}\right)^n$,

$$\begin{aligned}\frac{1}{\mathrm{Vol}_n(\mathcal{T})}\int_{\mathcal{T}}|x|^2dx&<\frac{1}{\mathrm{Vol}_n(\mathcal{T})}\sum_{i=1}^{\ell}\frac{d(0,\operatorname{asp}\mathcal{F}_i)\cdot\mathrm{Vol}_{n-1}(\mathcal{F}_i)}{n+2}\cdot C\log\frac{2N}{n}\\&<C\log\frac{2N}{n}.\end{aligned}$$

This completes the proof of (i). The proof of (ii) is very similar; we supply some details. Let $\mathcal{G}_1,\dots,\mathcal{G}_k$ denote the $(n-1)$-dimensional facets of $\mathcal{K}$. Observe that $Z\in\mathcal{K}$, and that any $x\in\mathcal{K}$ (except for the point Z) is uniquely represented as $x=Z+t(y-Z)$ with $0<t\le 1$ and $y\in\partial\mathcal{K}$. As before, integration in polar coordinates yields

$$\begin{aligned}\frac{1}{\mathrm{Vol}_n(\mathcal{K})}\int_{\mathcal{K}}|x-Z|^2\,dx&=\frac{1}{\mathrm{Vol}_n(\mathcal{K})}\int_{\partial\mathcal{K}}\int_0^1 t^{n-1}|t(y-Z)|^2\langle y-Z,\nu_y\rangle\,dt\,dy\\&=\frac{1}{n+2}\sum_{i=1}^{k}\frac{d(Z,\operatorname{asp}\mathcal{G}_i)}{\mathrm{Vol}_n(\mathcal{K})}\int_{\mathcal{G}_i}|y-Z|^2\,dy.\end{aligned}$$

Again, $\sum_{i=1}^{k} d(Z,\operatorname{asp}\mathcal{G}_i)\,\mathrm{Vol}_{n-1}(\mathcal{G}_i)=n\,\mathrm{Vol}_n(\mathcal{K})$. Thus, in order to prove (ii), we may simply reproduce the argument from the proof of (i), with Corollary 2.4(ii) replacing the role of Corollary 2.4(i). This completes the proof. ∎

3. Random Polytopes

We summarize the results of Section 2 in the following corollary. Note that the convex bodies discussed in this corollary have diameter that is larger than $c\sqrt{n}$ with high probability. Nevertheless, it is still possible to prove a much better estimate regarding the second moment of the Euclidean norm.

Corollary 3.1: *Let $N\ge n\ge 1$ and suppose that $G_0,\dots,G_N$ are independent random vectors in $\mathbb{R}^n$ that satisfy conditions $(*a)$ and $(*b)$ above. Set*

$$\mathcal{K}=\operatorname{conv}\{G_0,\dots,G_N\},\quad \mathcal{T}=\operatorname{conv}\{\pm G_1,\dots,\pm G_N\},$$

and denote $Z = \frac{1}{N+1}\sum_{i=0}^{N} G_i$*. Then with probability greater than* $1 - Ce^{-cn}$,

$$\frac{1}{\mathrm{Vol}_n(\mathcal{T})}\int_{\mathcal{T}} |x|^2 dx < C\log\frac{2N}{n} \quad \textit{and} \quad \frac{1}{\mathrm{Vol}_n(\mathcal{K})}\int_{\mathcal{K}} |x - Z|^2 dx < C\log\frac{2N}{n},$$

where $C, c > 0$ *are universal constants.*

Proof. Suppose first that $G_0, \ldots, G_N$ satisfy also ($*$c); that is, assume that they are absolutely continuous random variables. Then the desired conclusion follows from Lemma 2.5. For the general case, note that the quantity

$$\mathbb{P}\left\{\frac{1}{\mathrm{Vol}_n(\mathcal{T})}\int_{\mathcal{T}} |x|^2 dx < C\log\frac{2N}{n}, \quad \frac{1}{\mathrm{Vol}_n(\mathcal{K})}\int_{\mathcal{K}} |x - Z|^2 dx < C\log\frac{2N}{n}\right\} \tag{19}$$

depends continuously on the distribution of $G_0, \ldots, G_N$ in the weak topology at measures where $\mathbb{P}(\mathrm{Vol}_n(\mathcal{K}) = 0) = \mathbb{P}(\mathrm{Vol}_n(\mathcal{T}) = 0) = 0$. At other measures (19) may have a discontinuity of no more than $\mathbb{P}(\mathrm{Vol}_n(\mathcal{K}) = 0) + \mathbb{P}(\mathrm{Vol}_n(\mathcal{T}) = 0)$. The corollary follows by approximating $G_0, \ldots, G_N$ with absolutely continuous random vectors that satisfy ($*$a), ($*$b) and ($*$c) and noting that by ($*$a) and ($*$b) we have $\mathbb{P}(\mathrm{Vol}_n(\mathcal{K}) = 0) + \mathbb{P}(\mathrm{Vol}_n(\mathcal{T}) = 0) < Ce^{-cn}$, according to [23]. ∎

The next theorem is concerned with non-gaussian analogs of Theorem 1.1. The main new case covered by that theorem is that of random sign vectors, i.e., independent random vectors whose coordinates are independent, symmetric Bernoulli variables. We remark in passing that in the Bernoulli case the probability of $\mathcal{K}$ or $\mathcal{T}$ to be degenerate was known before [23]. See [14, 25].

Theorem 3.2: *Let* $n \geq 1$ *and* $n \leq N \leq 2^n$*. Suppose that* $G_1, \ldots, G_N$ *are independent random vectors in* $\mathbb{R}^n$ *that satisfy conditions* ($*$a) *and* ($*$b) *above. Assume further that* $G_{i,j}$ $(i = 1, \ldots, N,\ j = 1, \ldots, n)$ *are symmetric random variables. Set* $\mathcal{T} = \mathrm{conv}\{\pm G_1, \ldots, \pm G_N\}$*. Then with probability greater than* $1 - Ce^{-cn}$,

$$L_{\mathcal{T}} < C$$

where $C, c > 0$ *are universal constants.*

Proof. We may assume that $N > n$ (otherwise, $\mathcal{T}$ is a cross-polytope whenever it is non-degenerate. The isotropic constant of the cross-polytope is well-known to be bounded). Recall that $L_{\mathcal{T}} < Cn^{1/4}$ and hence we may assume that n exceeds a certain universal constant. It was proved in [17, Theorem 4.8] that, under the assumptions of the present theorem,

$$(\mathrm{Vol}_n(\mathcal{T}))^{1/n} > C\sqrt{\frac{\log(2N/n)}{n}} \tag{20}$$

with probability greater than $1 - Ce^{-cn}$. From Corollary 3.1 we know that with probability larger than $1 - Ce^{-c'n}$,

$$\frac{1}{\mathrm{Vol}_n(\mathcal{T})}\int_{\mathcal{T}} |x|^2 dx < C \log \frac{2N}{n} \tag{21}$$

The theorem follows by substituting the estimates (20) and (21) into the definition (1). ∎

Generally speaking, the restrictions on N in Theorem 3.2 are typically quite easy to work around. For example, in the case of Bernoulli variables, if $2^n \leq N < 3^n$ then (20) and hence the conclusion of Theorem 3.2 hold with a different constant, while if $N \geq 3^n$, then with very high probability $\mathcal{T}$ is a hypercube (whose isotropic constant equals $1/\sqrt{12}$). The assumption that the $G_{i,j}$ are symmetric is probably redundant; we believe that (20) may be proved, along the lines of the argument from [17], without having to rely on this assumption.

Our next lemma shows the same volume estimates in the Gaussian case for all values of N. It is standard and well-known.

Lemma 3.3: *Let $N > n \geq 1$ and suppose that $G_0, \ldots, G_N$ are independent standard gaussian vectors in $\mathbb{R}^n$. Denote*

$$\mathcal{K} = \mathrm{conv}\{G_0, \ldots, G_N\} \quad \text{and} \quad \mathcal{T} = \mathrm{conv}\{\pm G_1, \ldots, \pm G_N\}.$$

Then, with probability greater than $1 - Ce^{-cn}$,

$$\mathrm{Vol}_n(\mathcal{K})^{1/n} > c\sqrt{\frac{\log(2N/n)}{n}} \quad \text{and} \quad \mathrm{Vol}_n(\mathcal{T})^{1/n} > c\sqrt{\frac{\log(2N/n)}{n}},$$

where $c, C > 0$ are universal constants.

Proof sketch. We start with the lower bound for $\mathrm{Vol}_n(\mathcal{T})$. For the range $N \geq 2n$, it is well-known (see, e.g., [10, 17] and references therein) that with probability greater than $1 - Ce^{-cn}$,

$$c\sqrt{\log \frac{4N}{n}} D^n \subseteq \mathcal{T} \tag{22}$$

where $D^n = \{x \in \mathbb{R}^n ; |x| \leq 1\}$ is the unit Euclidean ball in $\mathbb{R}^n$. Since $\mathrm{Vol}_n^{1/n}(D^n) > c/\sqrt{n}$, the desired lower bound for $\mathrm{Vol}_n(\mathcal{T})$ follows from (22) in this case. In the remaining case $n < N < 2n$, the lower bound for $\mathrm{Vol}_n(\mathcal{T})$ follows from [17, Theorem 4.8].

Regarding $\mathrm{Vol}_n(\mathcal{K})$, consider first the case $n \le N \le Cn$, for a certain large universal constant $C > 0$. It turns out that in this range a single simplex supplies enough volume for our needs. We thus assume that $N = n$. Denote $G_i' = G_i - G_0$ for $i = 1, \ldots, n$, and set $\mathcal{K}' = \mathrm{conv}\{\pm G_1', \ldots, \pm G_n'\}$. Then,

$$\mathrm{Vol}_n(\mathcal{K}) = \mathrm{Vol}_n(\mathrm{conv}\{0, G_1 - G_0, G_2 - G_0, \ldots, G_N - G_0\}) \ge 4^{-n}\,\mathrm{Vol}_n(\mathcal{K}') \tag{23}$$

by the Rogers–Shephard inequality [22]. With probability one, the vectors $G_1, \ldots, G_n$ are linearly independent. Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be the unique linear map that satisfies $S(G_i) = G_i - G_0$ for $i = 1, \ldots, n$. Then $\mathcal{K}' = S(\mathcal{T})$. Hence,

$$\mathrm{Vol}_n(\mathcal{K}') = \det(S) \cdot \mathrm{Vol}_n(\mathcal{T}). \tag{24}$$

Let $v \in \mathbb{R}^n$ be such that $\langle v, G_i \rangle = 1$ for $i = 1, \ldots, n$. The vector v is independent of G_0. Clearly, $Sx = x - \langle x, v \rangle G_0$ for all $x \in \mathbb{R}^n$. Therefore $\det(S) = 1 - \langle v, G_0 \rangle$. Conditioning on v, we see that $\det(S)$ is a gaussian random variable with mean 1 and variance $|v|^2$. Hence,

$$\mathbb{P}\{|\det(S)| < 2^{-n}\} = \mathbb{E}_v \int_{-2^{-n}}^{2^{-n}} \frac{1}{\sqrt{2\pi|v|^2}} \exp\left(-\frac{(t-1)^2}{2|v|^2}\right) dt \le \tilde{C} 2^{-n}, \tag{25}$$

since for all values of $|v|$, the integrand never exceeds $\tilde{C}/2$. The desired lower bound for $\mathrm{Vol}_n(\mathcal{K})$, for the case where $n \le N \le Cn$, follows from (23), (24), (25) and from the lower bound for $\mathrm{Vol}_n(\mathcal{T})$, that was already proved.

The case $N > Cn$, where C is a sufficiently large constant, may be handled as follows: The vector $Z = \frac{1}{N+1}\sum_{i=0}^N G_i \in \mathcal{K}$ satisfies $|Z| < c/2$ with probability greater than $1 - Ce^{-\tilde{c}n}$, where c is the constant from (22). Hence, with probability greater than $1 - Ce^{-\tilde{c}n}$,

$$\mathrm{conv}\{\pm G_0, \ldots, \pm G_N\} \subseteq \frac{c}{2} D^n + (\mathcal{K} - \mathcal{K}) = \{z + x - y; |z| < c/2, x, y \in K\}.$$

From (22) we thus conclude that, with probability greater than $1 - \bar{C}e^{-\bar{c}n}$,

$$\frac{c}{2}\sqrt{\log \frac{4N}{n}}\, D^n \subseteq \mathcal{K} - \mathcal{K}.$$

An additional application of the Rogers–Shephard inequality leads to the desired lower bound for $\mathrm{Vol}_n(\mathcal{K})$. ∎

Proof of Theorem 1.1. We may assume that $N > n$; otherwise, with probability one, $\mathcal{T}$ is a cross-polytope and $\mathcal{K}$ is a simplex, both with a bounded isotropic constant. From Corollary 3.1 and Lemma 3.3 we see that with probability greater than $1 - Ce^{-cn}$,

$$\frac{1}{\mathrm{Vol}_n(\mathcal{T})^{1+\frac{2}{n}}}\int_{\mathcal{T}}|x|^2dx < Cn \quad \text{and} \quad \frac{1}{\mathrm{Vol}_n(\mathcal{K})^{1+\frac{2}{n}}}\int_{\mathcal{K}}|x-Z|^2dx < Cn$$

for some point $Z \in \mathbb{R}^n$ depending on $\mathcal{K}$. The theorem follows from the definition (1). ∎

References

[1] K. Ball, *Logarithmically concave functions and sections of convex sets in* $\mathbb{R}^n$, Studia Mathematica **88** (1988), 69–84.

[2] K. Ball, *Normed spaces with a weak-Gordon-Lewis property*, in *Functional Analysis (Austin, Texas, 1987/1989)*, Lecture Notes in Math., Vol. 1470, Springer, Berlin, 1991, pp. 36–47.

[3] G. Bennett, *Probability Inequalities for the Sum of Independent Random Variables,* Journal of the American Statistical Association **57** (1962), 33–45.

[4] J. Bourgain, *On high-dimensional maximal functions associated to convex bodies,* American Journal of Mathematics **108** (1986), 1467–1476.

[5] J. Bourgain, *Geometry of Banach spaces and harmonic analysis*, in *Proceedings of the International Congress of Mathematicians, (Berkeley, CA, 1986)*, Amer. Math. Soc., Providence, RI, 1987, pp. 871–878.

[6] J. Bourgain, *On the distribution of polynomials on high-dimensional convex sets*, in *Geometric Aspects of Functional Analysis (1989–90)*, Lecture Notes in Math., Vol. 1469, Springer, Berlin, 1991, pp. 127–137.

[7] J. Bourgain, *On the isotropy-constant problem for "PSI-2"-bodies,* in *Geometric Aspects of Functional Analysis (2001–2002)*, Lecture Notes in Math., Vol. 1807, Springer, Berlin, 2003, pp. 114–121.

[8] S. Dar, *Remarks on Bourgain's problem on slicing of convex bodies,* in *Geometric Aspects of Functional Analysis (Israel, 1992–1994)*, Oper. Theory Adv. Appl., 77, Birkhäuser, Basel, 1995, pp. 61–66.

[9] E. D. Gluskin, *The diameter of Minkowski compactum roughly equals to n,* Funktsional'nyj Analiz i ego Prilozheniya **15** (1981), 72–73; English translation in Functional Analysis and its Applications **15** (1981), 57–58.

[10] E. D. Gluskin, *Extremal properties of orthogonal parallelepipeds and their applications to the geometry of Banach spaces* (Russian), Matematicheskiĭ Sbornik. Novaya Seriya **136(178)** (1988), no. 1, 85–96; English translation in Mathematics of the USSR-Sbornik **64** (1989), no. 1, 85–96.

[11] D. Hensley, *Slicing convex bodies—bounds for slice area in terms of the body's covariance*, Proceedings of the American Mathematical Society **79** (1980), 619–625.

[12] M. Junge, *Hyperplane conjecture for quotient spaces of L_p,* Forum Mathematicum **6** (1994), 617–635.

[13] M. Junge, *Proportional subspaces of spaces with unconditional basis have good volume properties*, in *Geometric Aspects of Functional Analysis (Israel, 1992–1994)*, Operator Theory: Advances and Applications, 77, Birkhäuser, Basel, 1995, pp. 121–129.

[14] J. Kahn, J. Komlós and E. Szemerdi, *On the probability that a random* 1*-matrix is singular*, Journal of the American Mathematical Society **8** (1995), 223–240.

[15] B. Klartag, *On convex perturbations with a bounded isotropic constant,* Geometric and Functional Analysis (GAFA) **16** (2006), 1274–1290.

[16] H. König, M. Meyer and A. Pajor, *The isotropy constants of the Schatten classes are bounded*, Mathematische Annalen **312** (1998), 773–783.

[17] A. E. Litvak, A. Pajor, M. Rudelson and N. Tomczak-Jaegermann, *Smallest singular value of random matrices and geometry of random polytopes,* Advances in Mathematics **195** (2005), 491–523.

[18] E. Milman, *Dual mixed volumes and the slicing problem*, Advances in Mathematics **207** (2006),566–598.

[19] V. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, in *Geometric Aspects of Functional Analysis (1987–88)*, Lecture Notes in Math., Vol. 1376, Springer, Berlin, 1989, pp. 64–104.

[20] F. Nazarov, *On the maximal perimeter of a convex set in $\mathbb{R}^n$ with respect to a Gaussian measure*, in *Geometric Aspects of Functional Analysis (2001–02)*, Lecture Notes in Math., Vol. 1807, Springer, Berlin, 2003, pp. 169–187.

[21] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Tracts in Mathematics, Vol. 94, Cambridge University Press, Cambridge, 1989.

[22] C. A. Rogers and G. C. Shephard, *The difference body of a convex body*, Archiv der Mathematik **8** (1957), 220–233.

[23] M. Rudelson and R. Vershynin, *The Littlewood-Offord Problem and invertibility of random matrices,* Advances in Mathematics **218** (2008), 600–633.

[24] S. J. Szarek, *The finite-dimensional basis problem with an appendix on nets of Grassman manifold*, Acta Mathematica **141**, (1983), 153–179.

[25] T. Tao and V. Vu, *On the singularity probability of random Bernoulli matrices*, Journal of the American Mathematical Society **20** (2007), 603–628.

[26] A. W. Van der Vaart and J. A. Wellner, *Weak Convergence and Empirical Processes*, Springer-Verlag, 1996.