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PERVERSE SHEAVES ON AFFINE FLAGS AND NILPOTENT CONE OF THE LANGLANDS DUAL GROUP

BY

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ABSTRACT

This paper is a continuation of [2]. In [2], we constructed an equivalence between the derived category of equivariant coherent sheaves on the cotangent bundle to the flag variety of a simple algebraic group and a (quotient of) a category of constructible sheaves on the affine flag variety of the Langlands dual group. Below we prove certain properties of this equivalence related to cells in the affine Weyl group; provide a similar "Langlands dual" description for the category of equivariant coherent sheaves on the nilpotent cone, and link it to perverse coherent sheaves; and deduce some conjectures by Lusztig and Ostrik.

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1. Statements

1.1. Recollection of notation and set-up. We keep the set-up and notation of [2]. In particular, $\mathcal{F}\ell$ is the affine flag variety of a split simple group G over an algebraically closed field k; W_f is the Weyl group of G, and W is the extended affine Weyl group; ${}^fW^f \subset {}^fW \subset W$ are the sets of minimal length representatives of respectively 2-sided and left cosets of W_f in W; $D_I = D_I(\mathcal{F}\ell)$ is the Iwahori equivariant derived category of l-adic sheaves ($l \neq \mathrm{char}(k)$) on $\mathcal{F}\ell$ and $\mathcal{P}_I \subset D_I(\mathcal{F}\ell)$ is the full subcategory of perverse sheaves.

L_w, $w \in W$ are irreducible objects of $\mathcal{P}_I$. The Serre quotient category ${}^f\mathcal{P}_I$ of $\mathcal{P}_I$ is defined by

$${}^f\mathcal{P}_I = \mathcal{P}_I / \langle L_w \mid w \notin {}^fW \rangle,$$

where for an abelian category $\mathcal{A}$, and a set S of irreducible objects of $\mathcal{A}$ we let $\langle S \rangle$ denote the full abelian subcategory of objects obtained from elements of S by extensions.

$\mathcal{N}$ is the variety of nilpotent elements in the Langlands dual Lie algebra $\mathfrak{g}^\vee$; and $p_{Spr} : \tilde{\mathcal{N}} \to \mathcal{N}$ is its Springer resolution. For an algebraic group H acting on a variety X we write $D^H(X)$ instead of $D^b(Coh^H(X))$.

Convolution provides $D_I(\mathcal{F}\ell)$ with a monoidal structure. In [2], we constructed a monoidal functor

$$F : D^{G^\vee}(\tilde{\mathcal{N}}) \to D_I(\mathcal{F}\ell);$$

and used it to define an equivalence

$$ {}^f\Phi : D^{G\check{}}(\tilde{\mathcal{N}}) \xrightarrow{\sim} D^b({}^f\mathcal{P}_I). \tag{1}$$

We now state the results proved in this note.

1.2. CATEGORY ${}^f\mathcal{P}_I^f$ AND THE NIL-CONE. Let us define a further Serre quotient category ${}^f\mathcal{P}_I^f$ of $\mathcal{P}_I$ by

$$ {}^f\mathcal{P}^f{}_I = \mathcal{P}_I/\langle L_w \mid w \notin {}^fW^f \rangle. $$

Let $pr_{ff} : {}^f\mathcal{P}_I \to {}^f\mathcal{P}_I^f$ be the projection functor. (We will use the same notation for the extension of these exact functors to the derived categories. We will also abuse notation by omitting the projection to a quotient category functor from notations; e.g. we will sometimes write "X" or "X considered as an object of ${}^f\mathcal{P}^f$" instead of $pr_{ff}(X)$.)

THEOREM 1: *There exists an equivalence*

$$ {}^f\Phi^f : D^{G\check{}}(\mathcal{N}) \xrightarrow{\sim} D^b({}^f\mathcal{P}^f{}_I), \tag{2}$$

such that

$$ pr_{ff} \circ {}^f\Phi \cong {}^f\Phi^f \circ p_{Spr*}, \tag{3}$$

Remark 1: The functor $({}^f\Phi^f)^{-1}$ is a derived functor of a left exact functor.

Namely, let $\underline{\mathcal{O}}_{G\check{}}$ be the ind-object of $Rep(G\check{})$ corresponding to the module of regular functions on $G\check{}$, where $G\check{}$ acts by left translations. Notice that for an object $\mathcal{F}$ of the derived category of $G\check{}$-equivariant coherent sheaves one has $R^i\Gamma(\mathcal{F}) = \mathrm{Hom}(\mathcal{O}, \mathcal{F} \otimes \underline{\mathcal{O}}[i])$.

For $X \in {}^f\mathcal{P}_I^f$ the space

$$ \mathrm{Hom}_{{}^f\mathcal{P}_I^f}(\delta_e, X * \mathcal{Z}(\underline{\mathcal{O}}_{G\check{}})) = \bigoplus V_\lambda^* \otimes \mathrm{Hom}(\delta_e, X * Z_\lambda); $$

can be given a structure of a $G\check{}$ equivariant $\mathcal{O}(\mathcal{N})$-module. Thus we get a left exact functor $H : {}^f\mathcal{P}_I^f \to Coh^{G\check{}}(\mathcal{N})$; we claim that its derived functor RH is isomorphic to $({}^f\Phi^f)^{-1}$.

A sketch of the proof of this claim is as follows (the claim will not be used below, and details of the proof are omitted). It follows from the theorem and its proof that ${}^f\Phi^f(V \otimes \mathcal{O}_{\mathcal{N}}) \cong \mathcal{Z}(V) \in {}^f\mathcal{P}_I^f$, $V \in Rep(G\check{})$, and that $H \circ {}^f\Phi^f|_{Coh_{fr}^{G\check{}}(\mathcal{N})} \cong \mathrm{id}_{Coh_{fr}^{G\check{}}(\mathcal{N})}$ canonically, where $Coh_{fr}^{G\check{}}(\mathcal{N}) \subset Coh^{G\check{}}(\mathcal{N})$ is the full subcategory consisting of objects of the form $V \otimes \mathcal{O}_{\mathcal{N}}$, $V \in Rep(G\check{})$ (we will call such objects free sheaves).

The proof of Theorem 1 below shows also that for a finite complex $C^\bullet$ of objects of $Coh_{fr}^{G\check{}}(\mathcal{N})$ the object ${}^f\Phi^f(C^\bullet)$ is represented by the complex $({}^f\Phi^f(C^i))$.

Furthermore, it follows from Theorem 7 of [2] that $Ext^{>0}_{{}^f\mathcal{P}_I^f}(\delta_e, Z_\lambda) = 0$, thus for a complex $C^\bullet$ of object of ${}^f\mathcal{P}_I^f$ of the form $C^i = \mathcal{Z}(V_i)$, the object $RH(C^\bullet)$ is represented by the complex $(H(C^i))$. Thus we get a canonical isomorphism

$$RH \circ {}^f\Phi^f|_{D_{fr}^{G\check{}}(\mathcal{N})} \cong \mathrm{id}\,|_{D_{fr}^{G\check{}}(\mathcal{N})}, \tag{4}$$

where $D_{fr}^{G\check{}}(\mathcal{N}) \subset D^{G\check{}}(\mathcal{N})$ represented by a finite complex of free sheaves.

Finally, observe that the functor $RH \circ {}^f\Phi^f$ sends $D^{<0}(Coh^{G\check{}}(\mathcal{N}))$ to itself, because:

$$\begin{aligned}&\mathcal{F} \in D^{<0}(Coh^{G\check{}}(\mathcal{N})) \Rightarrow Ext^{\geq 0}(V \otimes \mathcal{O}, \mathcal{F}) = 0 \ \forall V \in Rep(G\check{}) \Rightarrow \\ &\quad Ext^{>0}(\mathcal{Z}(V), {}^f\Phi^f(\mathcal{F}))\ \forall V \in Rep(G\check{}) \Rightarrow RH({}^f\Phi^f(\mathcal{F})) \in D^{<0}(Coh^{G\check{}}(\mathcal{N})).\end{aligned}$$

Together with (4) this shows that $RH \circ {}^f\Phi^f \cong \mathrm{id}$.

Remark 2: Theorem 1 implies that the functor $Rp_* : D^{G\check{}}(\tilde{\mathcal{N}}) \to D^{G\check{}}(\mathcal{N})$ identifies $D^{G\check{}}(\mathcal{N})$ with the quotient category $D^{G\check{}}(\tilde{\mathcal{N}})/\operatorname{Ker}(Rp_*)$.

Notice that the analogous statement with D^b replaced by D^- is an immediate consequence of the isomorphism $Rp_{Spr*}(\mathcal{O}_{\tilde{\mathcal{N}}}) \cong \mathcal{O}_{\mathcal{N}}$, and the fact that a triangulated functor admitting a left adjoint which is also a right inverse is factorization by a thick subcategory, see, e.g., [14, Proposition II.2.3.3].

It is natural to ask whether the equivalence $D^{G\check{}}(\tilde{\mathcal{N}})/\operatorname{Ker}(Rp_*) \cong D^{G\check{}}(\mathcal{N})$ can be deduced directly from the isomorphism $Rp_*(\mathcal{O}_{\tilde{\mathcal{N}}}) \cong \mathcal{O}_{\mathcal{N}}$, and whether a similar equivalence holds for an arbitrary proper morphism p with $Rp_*(\mathcal{O}) \cong \mathcal{O}$. I do not know the answer to these questions.

1.3. DESCRIPTION OF THE t-STRUCTURE ON $D^{G\check{}}(\mathcal{N})$. One can use the equivalences (1), (2) to transport the tautological t-structure on the right-hand side to a t-structure on the left-hand side. Let us call the resulting t-structure on the derived category of equivariant coherent sheaves the *exotic t-structure*. We provide an explicit description of the exotic t-structure on $D^{G\check{}}(\mathcal{N})$.

THEOREM 2: *The exotic t-structure on $D^{G\check{}}(\mathcal{N})$ coincides with the perverse coherent t-structure corresponding to the perversity given by*

$$p(O) = \operatorname{codim}(O)/2; \tag{5}$$

see [1], [4].

1.3.1. Let $\mathbf{O}$ denote the set of $G\check{}$-conjugacy classes of pairs (N,ρ) where $N \in \mathcal{N}$, and ρ is an irreducible representation of the centralizer $Z_{G\check{}}(N)$.

For a pair $(N,\rho) \in \mathbf{O}$ let $\mathcal{L}_\rho$ be the irreducible $G\check{}$-equivariant vector bundle on the orbit $G\check{}(N)$, whose fiber at N is isomorphic to ρ. Let j be the embedding of $G\check{}(N)$ into $\mathcal{N}$. We have the irreducible coherent perverse sheaf $IC_{N,\rho} = j_{!*}(\mathcal{L}_\rho[\frac{-\mathrm{codim} G\check{}(N)}{2}])$, see [1].

Corollary 1: (a) *We have* ${}^f\Phi^f(IC_{N,\rho}) = L_w$ *for some* $w \in {}^fW^f$.

(b) *Identify* $\mathbb{Z}[W]$ *with the Grothendieck group* $K(Coh^{G\check{}}(St))$, *where* $St = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ *is the Steinberg variety, see* [7], [12]. *Let* $\bar{C}_w$ *be the Kazhdan–Lusztig basis of* $\mathbb{Z}[W]$ *(specialization of the Kazhdan–Lusztig basis in the affine Hecke algebra at* $v = 1$*). Let* $pr : \mathbb{Z}[W] \to K(Coh^{G\check{}}(\mathcal{N}))$ *be the map induced by* Rp_*, *where* $p : St \to \mathcal{N}$ *is the projection.*

We have $pr(\bar{C}_w) = 0$ *for* $w \notin {}^fW^f$; *and* $pr(\bar{C}_w)$ *is the class of an irreducible perverse coherent sheaf corresponding to the perversity* (5).

Remark 3: The corollary implies validity of Conjectures 1 and 2, and the first part of Conjecture 3 in [13].

1.4. Dualities. We will denote the Verdier duality functor on various categories by $\mathbb{V}$. Thus $\mathbb{V}$ is an contravariant auto-equivalence of the abelian category $\mathcal{P}_I$, which induces auto-equivalences of the quotient categories ${}^f\mathcal{P}_I$, ${}^f\mathcal{P}_I^f$ and their derived categories.

Define the anti-autoequivalence σ of $D^{G\check{}}(\mathcal{N})$ by $\mathcal{F} \mapsto R\mathrm{Hom}(\mathcal{F}, \mathcal{O})$. It is well-known that $\mathcal{N}$ is a Gorenstein scheme, and the dualizing sheaf for $\mathcal{N}$ is trivial. Thus σ coincides with the Grothendieck–Serre duality up to homological shift. Let $\kappa : G\check{} \to G\check{}$, be an automorphisms which sends an element g to an element conjugate to g^{-1}. We will also use the same letter to denote the induced push-forward functor on the categories of representations and equivariant coherent sheaves.

Theorem 3: *We have*

$$ {}^f\Phi^f \circ \kappa \circ \sigma \cong \mathbb{V} \circ {}^f\Phi^f; \tag{6}$$

1.5. Cells and nilpotent orbits. Recall the notion of a **two-sided cell** in W, and the bijection between the set of two-sided cells in W and nilpotent conjugacy classes in $\mathfrak{g}\check{}$, see [10]; for a two-sided cell $\underline{c}$ let $N_{\underline{c}} \in \mathfrak{g}\check{}$ be a representative of the corresponding conjugacy class.

For a two-sided cell $\underline{c} \subset W$ let $\mathcal{P}_I^{\le \underline{c}} \subset \mathcal{P}_I$ be the Serre subcategory generated by irreducible objects L_w, $w \in \bigcup_{\underline{c}' \le \underline{c}} \underline{c}'$; and let ${}^f\mathcal{P}_I^{\le \underline{c}} \subset {}^f\mathcal{P}_I$, ${}^f\mathcal{P}_I^{f\, \le \underline{c}} \subset {}^f\mathcal{P}_I^f$ be the images of $\mathcal{P}_I^{\le \underline{c}}$. Let also $D^b_{\le \underline{c}}({}^f\mathcal{P}_I) \subset D^b({}^f\mathcal{P}_I)$, $D^b_{\le \underline{c}}({}^f\mathcal{P}_I^f) \subset D^b({}^f\mathcal{P}_I^f)$ be the full triangulated subcategories generated by ${}^f\mathcal{P}_I^{\le \underline{c}}$, ${}^f\mathcal{P}_I^{f\, \le \underline{c}}$ respectively. Replacing the non-strict inequality by the strict one we get the definition of categories ${}^f\mathcal{P}_I^{f\, < \underline{c}}$, $D^b_{< \underline{c}}({}^f\mathcal{P}_I^f)$ etc.

For a closed $G\check{}$-invariant subset $S \subset \mathcal{N}$ or $S \subset \tilde{\mathcal{N}}$ let $D_S^{G\check{}}(\mathcal{N}) \subset D^{G\check{}}(\mathcal{N})$ (respectively, $D_S^{G\check{}}(\tilde{\mathcal{N}})$) be the full subcategory of complexes whose cohomology sheaves are set-theoretically supported on S (i.e., they are supported on some, possibly non-reduced, subscheme with topological space S). We abbreviate $D_{\le N_{\underline{c}}}(\mathcal{N}) = D^{G\check{}}_{\overline{G\check{}(N_{\underline{c}})}}(\mathcal{N})$; $D_{\le N_{\underline{c}}}(\tilde{\mathcal{N}}) = D^{G\check{}}_{p_{Spr}^{-1}(\overline{G\check{}(N_{\underline{c}})})}(\mathcal{N})$.

Theorem 4: a) $D^b_{\le \underline{c}}({}^f\mathcal{P}_I) = {}^f\Phi(D_{\le N_{\underline{c}}}(\tilde{\mathcal{N}}))$;
$D^b_{\le \underline{c}}({}^f\mathcal{P}_I^f) = {}^f\Phi^f(D_{\le N_{\underline{c}}}(\mathcal{N}))$.
b) *We have*

$$c_1 \le c_2 \iff N_{\underline{c}_1} \in \overline{G\check{}(N_{\underline{c}_2})},$$

where the inequality in the left hand side refers to the standard partial order on the set of 2-sided cells.

Remark 4: Part (b) of the theorem was conjectured by Lusztig, see [10].

1.6. Duflo involutions. Recall the notion of a **Duflo** (or **distinguished**) **involution** in an affine Weyl group. We quote two of several available equivalent definitions. On the one hand, an element $w \in W$ is a Duflo involution if and only if the corresponding element in the asymptotic Hecke algebra (which is the Grothendieck ring of the truncated convolution category, see the next subsection) is an idempotent. Moreover, the sum of all these idempotents over all Duflo involutions is the unit element in the asymptotic Hecke algebra.

On the other hand, an element $w \in W$ is a Duflo involution if and only if the degree of the Kazhdan-Lusztig polynomial $P_{e,w}$ is equal to $(l(w) - a(w))/2$, where $a : W \to \mathbb{Z}$ is Lusztig's a-function and l denotes the length (recall that for any w the degree of $P_{e,w}$ is at most $(l(w) - a(w))/2$). The latter characterization will be used in the proof of Lemma 8 below.

It is known that for each two sided cell $\underline{c} \subset W$ the set $\underline{c} \cap {}^fW^f$ contains a unique Duflo involution, it will be denoted by $d_{\underline{c}}$.

For a $G\check{}$-orbit $O \subset \mathcal{N}$ let $\hat{\mathcal{O}}_O$ denote the sheaf $j_*(\mathcal{O})$, where j is the embedding $O \hookrightarrow \mathcal{N}$, and j_* denotes the **non-derived** direct image.

Proposition 1: *We have*

$$ {}^f\Phi^f\Big(\hat{\mathcal{O}}_{G\check{}(N_{\underline{c}})}\Big[-\frac{\mathrm{codim} G\check{}(N_{\underline{c}})}{2}\Big]\Big) \cong L_{d_{\underline{c}}}. $$

Remark 5: The proposition implies Conjecture 4 in [13].

1.7. Truncated convolution categories. In [11], Lusztig defined for every two sided cell a monoidal category, whose simple objects are L_w, $w \in \underline{c}$. He conjectured a relation between this category and representations of the group $Z_{G\check{}}(N_{\underline{c}})$; these conjectures were partly proved in [5], [6]. More precisely, one of the results of [5] is as follows. Let $\mathcal{P}_I^{\underline{c}}$ denote the Serre quotient category $\mathcal{P}_I^{\leq \underline{c}}/\mathcal{P}_I^{<\underline{c}}$, and $\mathcal{A}_{\underline{c}} \subset \mathcal{P}_I^{\underline{c}}$ be the full subcategory consisting of subquotients of objects of the form $\mathcal{Z}(V) * L_w \mod \mathcal{P}_I^{<\underline{c}}$, $V \in \mathit{Rep}(G\check{})$, $w \in \underline{c}$. Let also $\mathcal{A}_{\underline{c}}^f \subset \mathcal{A}_{\underline{c}}$ be the subcategory consisting of subquotients $\mathcal{Z}(V) * L_w \mod \mathcal{P}_I^{<\underline{c}}$, $V \in \mathit{Rep}(G\check{})$, $w \in \underline{c} \cap {}^fW^f$. Convolution with a central sheaf $\mathcal{Z}(V)$ induces a functor on $\mathcal{A}_{\underline{c}}$, $\mathcal{A}_{\underline{c}}^f$ which is also denoted by $X \mapsto \mathcal{Z}(V) * X$.

Truncated convolution provides $\mathcal{A}_{\underline{c}}$, $\mathcal{A}_{\underline{c}}^f$ with the structure of a monoidal category. In [5] we identified the monoidal category $\mathcal{A}_{\underline{c}}^f$ with the category of representations of a subgroup $H_{\underline{c}} \subset Z_{\underline{c}}$, where $Z_{\underline{c}}$ denotes the centralizer of $N_{\underline{c}}$ in $G\check{}$; in particular, we have the restriction functor $r_{\underline{c}}^f : \mathit{Rep}(Z_{\underline{c}}) \to \mathcal{A}_{\underline{c}}^f$ (see Proposition 3 below for a more detailed statement).

We will compare $r_{\underline{c}}^f$ with a functor arising from ${}^f\Phi^f$. Set $D_{\underline{c}}^b({}^f\mathcal{P}_I^f) := D_{\leq \underline{c}}^b({}^f\mathcal{P}_I^f)/D_{<\underline{c}}^b({}^f\mathcal{P}_I^f)$; let $\mathit{Coh}_{N_{\underline{c}}}^{G\check{}}(\mathcal{N})$ be the category of equivariant coherent sheaves on the formal neighbourhood of the orbit $G\check{}(N_{\underline{c}})$ in $\mathcal{N}$, and $D_{N_{\underline{c}}}^{G\check{}}(\mathcal{N}) \cong D_{\leq N_{\underline{c}}}^{G\check{}}(\mathcal{N})/D_{<N_{\underline{c}}}(\mathcal{N})$ be its bounded derived category.

By Theorem 4(a) the functor ${}^f\Phi^f$ induces an equivalence $D_{N_{\underline{c}}}^{G\check{}}(\mathcal{N}) \xrightarrow{\sim} D_{\underline{c}}^b({}^f\mathcal{P}_I^f)$; we denote this equivalence by $\Phi_{\underline{c}}$.

Proposition 2: *For $\rho \in \mathit{Rep}(Z_{\underline{c}})$ we have a canonical isomorphism in ${}^f\mathcal{P}_I^{f\underline{c}}$*

$$ \Phi_{\underline{c}}(\mathcal{L}_\rho[-m]) \cong r_{\underline{c}}^f(\rho), \tag{7} $$

where $m = \frac{\mathrm{codim} G\check{}(N_{\underline{c}})}{2}$.

Corollary 2: *We have $H_{\underline{c}} = Z_{\underline{c}}$.*

Remark 6: A bijection between the set Λ^+ of dominant weights of $G^{\vee}$ (which is the same as dominant coweights of G) and the set $\mathbf{O}$ was defined in [4]; let ι_1 denote this bijection. From the definition of ι_1 in [4], it follows that

$$ {}^f\Phi^f(IC_{\iota_1(\lambda)}) = L_{w_\lambda}, $$

where $\{w_\lambda\} = {}^fW^f \cap W_f \cdot \lambda \cdot W_f$.

Another bijection between the same sets (which we denote by ι_2) was defined in [5]. ι_2 is characterized as follows. If $(N,\rho) = \iota_2(\lambda)$, and $N = N_{\underline{c}}$ for a two-sided cell $\underline{c}$, then we have an isomorphism in $\mathcal{P}_I^{\underline{c}}$

$$ r_{\underline{c}}(\rho) \circ L_{d_{\underline{c}}} \cong L_{w_\lambda}. $$

Thus Proposition 2 implies that $\iota_1 = \iota_2$.

Remark 7: The equality $\iota_1 = \iota_2$ implies Conjecture 3 in [13].

2. Proofs

2.1. Proof of Theorem 1.

Lemma 1: *For $w \notin {}^fW^f$ we have $p_{Spr*}({}^f\Phi^{-1}(L_w)) = 0$.*

Proof. For $\mathcal{F} \in D^{G^{\vee}}(\tilde{\mathcal{N}})$ we have $p_{Spr*}(\mathcal{F}) = 0$ if and only if $Ext^\bullet(V_\lambda \otimes \mathcal{O}, \mathcal{F}) = 0$ for all $\lambda \in \Lambda^+$. Thus we need to check that for $X \in D^b({}^f\mathcal{P}_I)$ we have

$$ Ext^\bullet_{{}^f\mathcal{P}}(Z_\lambda, L_w) = 0 \tag{8} $$

for $w \notin {}^fW^f$.

We will check the equivalent statement

$$ Ext^\bullet_{D_{\mathcal{IW}}}(\Delta_e * Z_\lambda, \Delta_e * L_w) = 0 $$

for $w \notin {}^fW^f$ [2, Theorem 2].

If $w \in {}^fW$ but $w \notin {}^fW^f$ then for some simple root α, $\alpha \neq \alpha_0$ we have $L_w = \pi_\alpha^!(L'_w)$; here α_0 is the affine simple root, $\pi_\alpha : \mathcal{F}\ell \to \mathcal{F}\ell(\alpha)$ is the projection ($\mathbb{P}^1$ fibration) to the corresponding partial affine flag variety, and L'_w is an **I**-equivariant constructible complex on $\mathcal{F}\ell(\alpha)$ (actually, $L'_w[1]$ is a perverse sheaf). Then $\Delta_e * L_w \cong \pi_\alpha^!(\Delta_e * L'_w)$, and we have

$$ Ext^\bullet(\Delta_e * Z_\lambda, \Delta_e * L_w) = Ext^\bullet(\pi_{\alpha!}(\Delta_e * Z_\lambda), \Delta_e * L'_w) = 0, $$

because $\pi_{\alpha!}(\Delta_e * Z_\lambda) = 0$, [2, proof of Lemma 28]. ∎

2.1.1. The lemma shows that the functor $p_{Spr*}\circ {}^f\Phi^{-1} : D^b({}^f\mathcal{P}_I)\to D^{G^\vee}(\mathcal{N})$ factors through $D^b({}^f\mathcal{P}_I^f)$ (here we use that $D^b(\mathcal{A}/\mathcal{B})\cong D^b(\mathcal{A})/D^b_{\mathcal{B}}(\mathcal{A})$ for an abelian category $\mathcal{A}$, and a Serre subcategory $\mathcal{B}$, where $D^b_{\mathcal{B}}(\mathcal{A})\subset D^b(\mathcal{A})$ is the full subcategory of objects with cohomology in $\mathcal{B}$).

It remains to check that the resulting functor

$$\Upsilon : D^b({}^f\mathcal{P}_I^f)\to D^{G^\vee}(\mathcal{N}),$$

is an equivalence; then ${}^f\Phi^f := \Upsilon^{-1}$ clearly satisfies the conditions of the theorem.

2.1.2. Let us check that Υ is a full embedding.

First we claim that

$$\mathrm{Hom}_{D^b({}^f\mathcal{P}_I^f)}(X,Y)\xrightarrow{\Upsilon}\mathrm{Hom}_{D^{G^\vee}(\mathcal{N})}(\Upsilon(X),\Upsilon(Y)) \tag{9}$$

is an isomorphism for $X=Z_\lambda$. Indeed, (8) implies that $\mathrm{Hom}_{D^b({}^f\mathcal{P}_I)}(Z_\lambda,Y)\xrightarrow{\sim}\mathrm{Hom}_{D^b({}^f\mathcal{P}_I^f)}(Z_\lambda,Y)$. Also, the equality $Rp_{Spr*}(\mathcal{O}_{\tilde{\mathcal{N}}})=\mathcal{O}_{\mathcal{N}}$ implies that

$$\begin{aligned}\mathrm{Hom}_{D^{G^\vee}(\tilde{\mathcal{N}})}(V_\lambda\otimes\mathcal{O}_{\tilde{\mathcal{N}}},\mathcal{F})&\xrightarrow{\sim}\mathrm{Hom}_{D^{G^\vee}(\tilde{\mathcal{N}})}(Rp_{Spr*}(V_\lambda\otimes\mathcal{O}_{\tilde{\mathcal{N}}}),Rp_{Spr*}(\mathcal{F}))\\&=\mathrm{Hom}_{D^{G^\vee}(\mathcal{N})}(V_\lambda\otimes\mathcal{O}_{\mathcal{N}},Rp_{Spr*}(\mathcal{F}))\end{aligned}$$

for $\mathcal{F}\in D^{G^\vee}(\tilde{\mathcal{N}})$. Thus validity of (9) for $X=Z_\lambda$ follows from ${}^f\Phi^{-1}$ being an equivalence.

We now want to deduce that (9) is an isomorphism for all X. The argument is a version of the proof of the fact that an effaceable δ-functor is universal.

Lemma 2: *Let D be a triangulated category, $F=(F^i)$, $F'=({F'}^i)$ be cohomological functors from D to an abelian category, and $\phi: F\to F'$ be a natural transformation. Let $S\subset D$ be a set of objects. Assume that*

i) *There exists $d\in\mathbb{Z}$ such that $F^i(X)=0={F'}^i(X)$ for $i<d$, $X\in S$.*

ii) *For any $X\in S$ there exists an exact triangle $X\to\tilde{X}\to Y$ where $Y\in S$, and $\phi: F^i(\tilde{X})\to {F'}^i(\tilde{X})$ is an isomorphism for all i.*

Then $\phi: F^i(X)\to {F'}^i(X)$ is an isomorphism for all $X\in S$.

Proof. We go by induction in i. Condition (i) provides the base of induction. Applying the 5-lemma to

$$\begin{tikzcd}
F^{i-1}(\tilde{X}) \arrow[r] \arrow[d, "||"'] & F^{i-1}(Y) \arrow[r] \arrow[d, "||"'] & F^{i}(X) \arrow[r] \arrow[d] & F^{i}(\tilde{X}) \arrow[r] \arrow[d, "||"'] & F^{i}(Y) \arrow[d] \\
F'^{i-1}(\tilde{X}) \arrow[r] & F'^{i-1}(Y) \arrow[r] & F'^{i}(X) \arrow[r] & F'^{i}(\tilde{X}) \arrow[r] & F'^{i}(Y)
\end{tikzcd}$$

we see that $\phi : F^i(X) \hookrightarrow F'^i(X)$. Since $Y \in S$ we have $\phi : F^i(Y) \hookrightarrow F'^i(Y)$ which implies $\phi : F^i(X) \xrightarrow{\sim} F'^i(X)$. ∎

To exhibit a generating set for $D^b({}^f\mathcal{P}_I^f)$ satisfying the conditions of Lemma 2 we need another Lemma.

A filtration on the object of ${}^f\mathcal{P}_I$ (respectively, ${}^f\mathcal{P}_I^f$) will be called **costandard** if its associated graded is a sum of objects j_{w*}, $w \in {}^fW$ (respectively, $w \in {}^fW^f$). Such a filtration will be called **standard** if its associated graded is a sum of objects $j_{w!}$, $w \in {}^fW$ (respectively, $w \in {}^fW^f$).

LEMMA 3: a) *If $w_1, w_2 \in W$ and $w_2 \in W_f \cdot w_1 \cdot W_f$, then j_{w_1*} and j_{w_2*} are isomorphic in ${}^f\mathcal{P}_I^f$.*

b) *Let $X \in {}^f\mathcal{P}_I^f$ be an object with a costandard filtration. Then there exists a short exact sequence $0 \to Y \to Z \to X \to 0$ in ${}^f\mathcal{P}_I^f$ where Z is a (finite) sum of objects Z_λ, and Y has a costandard filtration.*

Proof. (a) We can assume that $w_2 = sw_1$ or $w_2 = w_1 s$ for a simple reflection $s = s_\alpha \in W_f$, and that $\ell(w_2) > \ell(w_1)$. Assume first that $w_2 = sw_1$. We have $j_{w_2*} = j_{s*} * j_{w_1*}$. The short exact sequence

$$0 \to \delta_e \to j_{s*} \to L_s \to 0$$

(where $e \in W$ is the identity, and $\delta_e = j_{e*} = j_{e!} = L_e$ is the unit object of the monoidal category $D_I(\mathcal{F}\ell)$) yields an exact triangle

$$j_{w_1*} \to j_{w_2*} \to L_s * j_{w_1*}.$$

It is easy to see that $L_s * j_{w_1*}$ is a perverse sheaf; this object is equivariant with respect to the parahoric group scheme $\mathbf{I}_\alpha$. It follows that any of its irreducible subquotient is also equivariant under this group; hence such a subquotient is isomorphic to L_w for some w satisfying $\ell(sw) < \ell(w)$. This shows that $L_s * j_{w_1*}$ is zero in ${}^f\mathcal{P}_I^f$, hence j_{w_2*} and j_{w_1*} are isomorphic in ${}^f\mathcal{P}_I^f$.

In the case $w_2 = w_1 s$ the proof is parallel, with the words "is equivariant under $\mathbf{I}_\alpha$" replaced by "lies in the image of the functor π_α^*." Thus (a) is proved.

(b) Theorem 7 of [2] implies that Z_λ (considered as an object of ${}^f\mathcal{P}_I$) has both a standard and a costandard filtration. The top of the latter is a surjection $f_\lambda : Z_\lambda \to j_{\lambda *}$ whose kernel, subsequently, has a costandard filtration. Taking the image of f_λ in ${}^f\mathcal{P}_I^f$ we get an arrow $\bar{f}_\lambda : Z_\lambda \to j_{w_\lambda *}$ in ${}^f\mathcal{P}_I^f$ whose kernel has a costandard filtration by (a); recall that $\{w_\lambda\} = {}^fW^f \cap W_f \cdot \lambda \cdot W_f$.

Let now $X \in {}^f\mathcal{P}_I^f$ be an object with a costandard filtration; let $0 \to X' \to X \to j_{w*} \to 0$ be the top of the filtration. By induction in the length of the filtration we can assume the existence of an exact sequence

$$0 \to Y' \to Z' \xrightarrow{f'} X' \to 0$$

of the required form. We have $w = w_\lambda$ for some $\lambda \in \Lambda^+$. We claim that the surjection $\bar{f}_\lambda : Z_\lambda \to j_{w*}$ factors through a map $Z_\lambda \to X$. Indeed, the obstruction lies in $Ext^1_{{}^f\mathcal{P}_I^f}(Z_\lambda, X')$. We claim that

$$Ext^1_{{}^f\mathcal{P}_I^f}(Z_\lambda, j_{w*}) = Ext^1_{{}^f\mathcal{P}_I}(Z_\lambda, j_{w*}) = 0.$$

Here the first equality follows from (8). The second one is a consequence of the existence of a standard filtration on Z_λ (considered as an object of ${}^f\mathcal{P}_I$), and the equality

$$Ext^\bullet_{{}^f\mathcal{P}_I}(j_{w_1 !}, j_{w_2 *}) = \overline{\mathbb{Q}_l}^{\delta_{w_1,w_2}}.$$

The latter is a consequence of [2, Theorem 2 and Lemma 1], which identify the left-hand side with an Ext space in the derived category of l-adic complexes on $\mathcal{F}\ell$ (more precisely, with $Ext^\bullet(\Delta_{w_1}, \nabla_{w_2})$ in the notation of [2]).

Now let $\tilde{f}_\lambda : Z_\lambda \to X$ be some map, such that the composition $Z_\lambda \to X \to j_{w*}$ equals $\bar{f}_\lambda$. Then we set $Z = Z' \oplus Z_\lambda$, and the map $f : Z \to X$ is set to be $f := f' \oplus \tilde{f}_\lambda$. The exact sequence

$$0 \to \mathrm{Ker}(f') \to \mathrm{Ker}(f) \to \mathrm{Ker}(\bar{f}_\lambda) \to 0$$

shows that f satisfies the requirements of (b). ∎

2.1.3. We can now finish the proof of Υ being an equivalence. We apply Lemma 2 to the following data:

$D = D^b({}^f\mathcal{P}_I^f)^{op}$,

$F : X \mapsto \mathrm{Hom}^\bullet(X, X_0)$ for some fixed $X_0 \in D^b({}^f\mathcal{P}_I^f)$;

$F' : X \mapsto \mathrm{Hom}^\bullet(\Upsilon(X), \Upsilon(X_0))$;

the transformation ϕ comes from functoriality of Υ;
the set S consists of all objects of ${}^f\mathcal{P}_I^f$ which have a costandard filtration.

We claim that the conditions of Lemma 2 are satisfied.

In fact, condition (i) is satisfied for any d such that $X_0 \in D^{\geq -d}({}^f\mathcal{P}_I^f)$, $\Upsilon(X_0) \in D^{\geq -d}(Coh^{G\check{}}(\mathcal{N}))$. Vanishing of $F^i(X)$, $X \in S$, $i < d$ follows then from vanishing of negative Ext's between objects of ${}^f\mathcal{P}_I^f$, while vanishing for $(F')^i(X)$, $X \in S$, $i < d$ follows from vanishing of negative Ext's in $Coh^{G\check{}}(\mathcal{N})$, in view of the fact that $\Upsilon(S) \subset Coh^{G\check{}}(\mathcal{N})$. The latter inclusion amounts to the fact that $Ext^{>0}_{{}^f\mathcal{P}_I}(\mathcal{Z}(V), X) = 0$, $X \in S$, $V \in Rep(G\check{})$, which follows from the tilting property of central sheaves, Theorem 7 of [2].

Since (9) has been proven to be an isomorphism for $X = Z_\lambda$, Lemma 3(b) shows that condition (ii) of Lemma 2 is satisfied. Hence (9) is an isomorphism whenever X has a costandard filtration; in particular, for $X = j_{w*}$. But $D^b({}^f\mathcal{P}_I^f)$ is generated as a triangulated category by j_{w*}, $w \in {}^fW^f$; hence (9) holds for all X, i.e., Υ is a full embedding.

It remains to show that Υ is essentially surjective; since it is a full embedding it suffices to see that the image of Υ contains a set of objects generating $D^{G\check{}}(\mathcal{N})$ as a triangulated category. This is done in Lemma 7 of [4]. ∎

2.2. Proof of Theorem 2. It follows from the results of [4] that $D^{G\check{}}(\mathcal{N})$ carries a unique t-structure such that the objects A_λ lie in the heart of this t-structure for all λ, where $A_\lambda = p_{Spr*}(\mathcal{O}(\lambda))$; and this t-structure coincides with the perverse coherent t-structure corresponding to the perversity function $p(O) = \mathrm{codim} O/2$ (which coincides with the middle perversity up to a total shift by $\dim \mathcal{N}/2$). Recall that the objects $J_\lambda \in \mathcal{P}_I$ (the Wakimoto sheaves), satisfy ${}^f\Phi(\mathcal{O}(\lambda)) \cong J_\lambda$; hence

$$ {}^f\Phi^f(A_\lambda) \cong pr_{ff}(J_\lambda). \tag{10} $$

Thus the heart of the t-structure obtained by transport of the tautological t-structure on $D^b({}^f\mathcal{P}_I^f)$ under the equivalence $({}^f\Phi^f)^{-1}$ contains the objects A_λ in its heart, so it coincides with the perverse coherent t-structure. ∎

2.2.1. *Proof of Corollary 1.* a) is immediate from Theorem 2, because $IC_{N,\rho}$ is an irreducible object in the heart of the perverse coherent t-structure, so ${}^f\Phi^f(IC_{N,\rho})$ is an irreducible object of ${}^f\mathcal{P}_I^f$. Let us prove (b). Let $\mathfrak{p}$ denote the map on Grothendieck groups induced by the composition

$$D^b(\mathcal{P}_I) \to D^b({}^f\mathcal{P}_I^f) \xrightarrow{{}^f\Phi^{f\,-1}} D^{G\check{\ }}(\mathcal{N}).$$

We can identify $\mathbb{Z}[W]$ with $K(D^b(\mathcal{P}_I))$ by means of the isomorphism sending $(-1)^{\ell(w)} \cdot w$ to the class $[j_{w!}] = [j_{w*}]$; it maps $\bar{C}_w$ to the class of L_w. Thus it is clear that $\mathfrak{p}(C_w) = 0$ for $w \notin {}^fW^f$; and (a) shows that $\mathfrak{p}(C_w)$ is the class of an irreducible perverse coherent sheaf. It remains to check that $\mathfrak{p} = pr$. This follows from: $\mathfrak{p}(w) = (-1)^{\ell(w)}[A_\lambda] = pr(w)$ for $\lambda \in \Lambda^+$, $w \in W_f \cdot \lambda \cdot W_f$. Here the first equality follows from (10) for $w = \lambda$ and from Lemma 3(a) in general. The second equality holds by Lemma 2.4 in [13]. ∎

2.3. Proof of Theorem 3. Recall that S denotes the equivalence $Rep(G\check{\ }) \to \mathcal{P}_{\mathbf{G_O}}(\mathcal{G}\mathfrak{r})$. Let $\upsilon : Rep(G\check{\ }) \to Rep(G\check{\ })^{op}$ denote the functor $V \mapsto V^*$.

Recall also that $Coh^{G\check{\ }}_{fr}(\mathcal{N})$ denotes the category of $G\check{\ }$-equivariant vector bundles on $\mathcal{N}$ which have the form $V \otimes \mathcal{O}$, $V \in Rep(G\check{\ })$. Thus $Coh^{G\check{\ }}_{fr}(\mathcal{N})$ is a tensor category under the tensor product of vector bundles. It was shown in [2] that the map $V \otimes \mathcal{O} \mapsto \mathcal{Z}(V)$ extends naturally to a monoidal functor $Coh^{G\check{\ }}_{fr}(\mathcal{N}) \to D_I(\mathcal{F}\ell)$; we denote the resulting monoidal functor by $\tilde{\mathcal{Z}}$ (thus $\tilde{\mathcal{Z}} = F \circ p^*_{Spr}$ in notation of [2]).

Lemma 4: *There exists a tensor isomorphism of functors* $Coh^{G\check{\ }}_{fr}(\mathcal{N})^{op} \to \mathcal{P}_{\mathbf{G_O}}(\mathcal{G}\mathfrak{r})$

$$\tilde{\mathcal{Z}} \circ (\sigma \circ \kappa) \cong \mathbb{V} \circ \tilde{\mathcal{Z}}.$$

Proof. The functor $\tilde{\mathcal{Z}}$ is characterized by the following two conditions (cf. [2, Proposition 4(a)]):

$$\tilde{\mathcal{Z}}|_{Rep(G\check{\ })} \cong \mathcal{Z} \tag{11}$$

$$\tilde{\mathcal{Z}}(N^{taut}_{V\otimes\mathcal{O}}) = \mathcal{M}_{\mathcal{Z}(V)}. \tag{12}$$

More precisely, given a functor $\tilde{\mathcal{Z}}'$ with a functorial tensor isomorphism $\tilde{\mathcal{Z}}'(V \otimes \mathcal{O}) \cong \mathcal{Z}(V)$, $V \in Rep(G\check{\ })$, which intertwines $\tilde{\mathcal{Z}}'(N^{taut}_{V\otimes\mathcal{O}})$ and $\mathcal{M}_{\mathcal{Z}(V)}$, one can construct a canonical isomorphism $\tilde{\mathcal{Z}}' \cong \tilde{\mathcal{Z}}$. Here $N^{taut}_{\mathcal{F}}$ is the "topological" endomorphism of an equivariant sheaf $\mathcal{F} \in Coh^{G\check{\ }}(\mathcal{N})$, whose action on the fiber at a point $x \in \mathcal{N}$ coincides with the action of $x \in Stab_{\mathfrak{g}\check{\ }}(x)$ coming from the equivariant structure; and $\mathcal{M}_{\mathcal{Z}(V)}$ is the logarithm of monodromy endomorphism of $\mathcal{Z}(V)$ (arising from the construction of $\mathcal{Z}(V)$ via the nearby cycles functor). Thus we will be done if we show that (11) can be constructed, so that (12) holds, for $\tilde{\mathcal{Z}}$ replaced by $\mathbb{V} \circ \tilde{\mathcal{Z}} \circ (\sigma \circ \kappa)$.

This follows from existence of natural isomorphisms satisfying the corresponding equalities:

$$\kappa(V\otimes\mathcal{O})\cong\kappa(V)\otimes\mathcal{O},\qquad \kappa(N^{taut}_{V\otimes\mathcal{O}})=N^{taut}_{\kappa(V)\otimes\mathcal{O}};\tag{13}$$

$$\sigma(V\otimes\mathcal{O})\cong V^{*}\otimes\mathcal{O},\qquad \sigma(N^{taut}_{V\otimes\mathcal{O}})=N^{taut}_{V^{*}\otimes\mathcal{O}};\tag{14}$$

$$\mathbb{V}(\mathcal{Z}(V))\cong\mathcal{Z}(V),\qquad \mathbb{V}(\mathcal{M}_{\mathcal{Z}(V)})=\mathcal{M}_{\mathcal{Z}(V)}.\tag{15}$$

Here (13) and (14) are easy exercises; and (15) follows from the fact that nearby cycles commute with Verdier duality, and the isomorphism $\mathbb{V}\circ\Psi\cong\Psi\circ\mathbb{V}$ (where Ψ is the nearby cycles functor) respects the monodromy action, by inspection of the definition of $\mathcal{Z}$ in [8]. ∎

2.3.1. We are now ready for the proof of the theorem. Using the monoidal functor $\tilde{\mathcal{Z}}$ one can define the action of the monoidal category $Coh^{G\check{}}_{fr}(\mathcal{N})$ on ${}^{f}\mathcal{P}^{f}_{I}$ and on its derived category. It follows from the isomorphisms (3), and the definition of ${}^{f}\Phi^{f}$ in [2] (cf. [2, Theorem 1]) that ${}^{f}\Phi^{f}$ intertwines this action with the action of $Coh^{G\check{}}_{fr}(\mathcal{N})$ on $D^{G\check{}}(\mathcal{N})$ by tensor products; i.e., we have a natural isomorphism

$${}^{f}\Phi^{f}(V\otimes\mathcal{F})\cong\mathcal{Z}(V)*{}^{f}\Phi^{f}(\mathcal{F}).\tag{16}$$

Set $\phi={}^{f}\Phi^{f\,-1}\circ\mathbb{V}\circ{}^{f}\Phi^{f}\circ\sigma\circ\kappa$. We want to construct an isomorphism $\phi\cong\mathrm{id}$.

Lemma 4 shows that ϕ commutes with the action of $Coh^{G\check{}}_{fr}(\mathcal{N})$ by tensor products. Furthermore, it is easy to see that $\phi(\mathcal{O})\cong\mathcal{O}$. Thus we get an isomorphism

$$\phi|_{Coh^{G\check{}}_{fr}(\mathcal{N})}\cong\mathrm{id}\,.\tag{17}$$

Let now $C^{\bullet}$ be a bounded complex where $C^{i}\in Coh^{G\check{}}_{fr}(\mathcal{N})$, and C be the corresponding object of $D^{G\check{}}(\mathcal{N})$. By inspection of the definition of ${}^{f}\Phi^{f}$ one checks that $\phi(C)$ is represented by the complex $(\phi(C^{i}))$. This yields an isomorphism

$$\phi|_{D^{G\check{}}_{fr}(\mathcal{N})}\cong\mathrm{id}\tag{18}$$

(see Remark 1 for notation). As in Remark 1 we see that ϕ preserves $D^{<0}(Coh^{G\check{}}(\mathcal{N}))$, which, together with (18), yields an isomorphism $\phi\cong\mathrm{id}$. ∎

2.4. Proof of Theorem 4. We first recall some results of [6], [5] (see Section 1.7 above for notation).

PROPOSITION 3: *$\mathcal{A}_{\underline{c}}$ carries a natural structure of a rigid monoidal category (given by the truncated convolution $\circ$); $\mathcal{A}^f_{\underline{c}} \subset \mathcal{A}_{\underline{c}}$ is a monoidal subcategory. Let $\mathbf{1}$ be the unit object*[1] *of $\mathcal{A}_{\underline{c}}$. We have a monoidal central*[2] *functor $r_{\underline{c}} : Rep(Z_{\underline{c}}) \to \mathcal{A}_{\underline{c}}$ such that*

- i) *The composition of the restriction functor $res^{\check{G}}_{Z_{\underline{c}}} : Rep(\check{G}) \to Rep(Z_{\underline{c}})$ with $r_{\underline{c}}$ is isomorphic to the functor $V \mapsto \mathcal{Z}(V) * \mathbf{1}$.*
- ii) *The element $N_{\underline{c}}$ yields a tensor endomorphism of the functor $Res^{\check{G}}_{Z_{\underline{c}}}$. The isomorphism of (i) carries this endomorphism into the endomorphism induced by the logarithm of monodromy, see* [8, Theorem 2].
- iii) *For $X \in \mathcal{A}_{\underline{c}}$ we have a canonical isomorphism*

$$Z_\lambda * X \cong r_{\underline{c}}(V_\lambda|_{Z_{\underline{c}}}) \circ X. \tag{19}$$

- iv) *The functor $V \mapsto r_{\underline{c}}(V) \circ X$ from $Rep(Z_{\underline{c}})$ to $\mathcal{A}_{\underline{c}}$ is exact and faithful for all $X \in \mathcal{A}_{\underline{c}}$, $X \neq 0$.*
- v) *The functor $r^f_{\underline{c}}$ defined by $r^f_{\underline{c}}(X) = r_{\underline{c}}(X) \circ L_{d_{\underline{c}}}$ is a monoidal functor $Rep(Z_{\underline{c}}) \to \mathcal{A}^f_{\underline{c}}$.*

There exists an algebraic subgroup $H_{\underline{c}} \subset Z_{\underline{c}}$, and an equivalence $\mathcal{A}^f_{\underline{c}} \cong Rep(H_{\underline{c}})$, which intertwines $r^f_{\underline{c}}$ with the restriction functor $Rep(Z_{\underline{c}}) \to Rep(H_{\underline{c}})$.

Proof. See [6]. ■

We need to spell out compatibility between (19) and equivalence ${}^f\Phi$.

The functor F induces a map

$$\mathrm{Hom}_{Coh^{\check{G}}(\mathcal{N})}(V_1 \otimes \mathcal{O}, V_2 \otimes \mathcal{O}) \to \mathrm{Hom}(\mathcal{Z}(V_1), \mathcal{Z}(V_2));$$

where $V_1, V_2 \in Rep(\check{G})$. For $h \in \mathrm{Hom}_{Coh^{\check{G}}(\mathcal{N})}(V_1 \otimes \mathcal{O}, V_2 \otimes \mathcal{O})$ define $h_X : \mathcal{Z}(V_1) * X \to \mathcal{Z}(V_2) * X$ by $h_X = F(h) * \mathrm{id}_X$.

On the other hand, given $h \in \mathrm{Hom}_{Coh^{\check{G}}(\mathcal{N})}(V_1 \otimes \mathcal{O}, V_2 \otimes \mathcal{O})$ we can consider the induced map of fibers at $N_{\underline{c}}$; we denote this map by $h_{N_{\underline{c}}} \in \mathrm{Hom}_{Z_{\underline{c}}}(V_1, V_2)$.

LEMMA 5: *Let $X \in \mathcal{P}_I^{\leq \underline{c}}$, $X \mod \mathcal{P}_I^{< \underline{c}} \in \mathcal{A}_{\underline{c}}$. Then for*

$$h \in \mathrm{Hom}_{Coh^{\check{G}}(\mathcal{N})}(V_1 \otimes \mathcal{O}, V_2 \otimes \mathcal{O})$$

isomorphism (19) *carries h_X into $r_{\underline{c}}(h_{N_{\underline{c}}}) \circ \mathrm{id}_X$.*

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[1] It follows from the results of Lusztig [5] that $\mathbf{1} \cong L_{d_{\underline{c}}}$; Proposition 1 provides a description of the corresponding object in the derived category of coherent sheaves.

[2] See, e.g., [2, §3.2], or (with more details) [5, §2.1], for a definition.

Proof. We need to enhance (19) to an isomorphism between the two actions of the tensor category $Coh^{G\check{}}_{fr}(\mathcal{N})$ on $\mathcal{A}_{\underline{c}}$, where the first one is given by $\mathcal{F} : X \mapsto \tilde{\mathcal{Z}}(\mathcal{F}) * X$, while the second one is given by $\mathcal{F} : X \mapsto r_{\underline{c}}(\mathcal{F}_{N_{\underline{c}}}) \circ X$, where $\mathcal{F}_{N_c}$ denotes the fiber of $\mathcal{F}$ at $N_{\underline{c}}$. We apply the (easy) uniqueness part of the Proposition 4(a) in [2] to the situation where the target category $\mathcal{C}$ is the category of endo-functors of $\mathcal{A}_{\underline{c}}$. According to that proposition, it suffices to check that (19) is compatible with the image of the tautological endomorphism N^{taut} of $\mathrm{id}_{Coh^{G\check{}}_{fr}(\mathcal{N})}$. In view of Proposition 3 (iii), this compatibility follows by comparing Proposition 3 (ii) with compatibility (12) between N^{taut} and monodromy via $\tilde{\mathcal{Z}}$. ∎

Theorem 4 will be deduced from the next

Lemma 6: a) *For $w \in {}^fW$ we have*

$$w \in \underline{c} \quad \Rightarrow \quad p_{Spr}(\mathrm{supp}({}^f\Phi^{-1}(L_w))) = \overline{G\check{}(N_{\underline{c}})}. \tag{20}$$

b) *For any $X \in D^b({}^f\mathcal{P}_I)$ we have*

$$p_{Spr}(\mathrm{supp}({}^f\Phi^{-1}(X))) = \bigcup_{\underline{c}} \overline{G\check{}(N_{\underline{c}})}, \tag{21}$$

where $\underline{c}$ runs over the set of such 2-sided cells that the multiplicity of L_w in the Jordan–Hoelder series of $H^i(X)$ is non-zero for some $w \in \underline{c} \cap {}^fW$.

Proof. Let $\mathcal{J} \subset \mathcal{O}_{\mathcal{N}}$ be the ideal sheaf of the closure of a $G\check{}$-orbit O on $\mathcal{N}$. Fix $n > 0$. There exists a surjection of equivariant sheaves $V \otimes \mathcal{O} \twoheadrightarrow \mathcal{J}^n$ for some $V \in Rep(G\check{})$. Let $\phi : V \otimes \mathcal{O} \to \mathcal{O}$ be the composition $V \otimes \mathcal{O} \twoheadrightarrow \mathcal{J}^n \hookrightarrow \mathcal{O}$; we use the same symbol to denote the pull-back of ϕ under p_{Spr}. Then an object $\mathcal{F} \in D^{G\check{}}(\tilde{\mathcal{N}})$ lies in $D^{G\check{}}_{p_{Spr}^{-1}(\overline{O})}(\tilde{\mathcal{N}})$ if and only if the arrow $\phi \otimes \mathrm{id}_{\mathcal{F}} : V \otimes \mathcal{F} \to \mathcal{F}$ equals zero for some (equivalently, for all large) n. Thus to check (20) it is enough to show that for $w \in \underline{c}$ we have

$$\overline{O} \ni N_{\underline{c}} \iff 0 = \phi_{L_w} \in \mathrm{Hom}(\mathcal{Z}(V) * L_w, L_w). \tag{22}$$

If $w \in \underline{c} \cap {}^fW$, then a morphism $\mathcal{Z}(V) * L_w \to L_w$ is zero if and only if the induced arrow in $\mathcal{A}_{\underline{c}}$ is zero; this is also equivalent to the induced arrow in ${}^f\mathcal{P}_I$ being zero. In view of Lemma 5 the induced map $(\phi)_{L_w} \mod \mathcal{P}_I^{<\underline{c}} \in \mathrm{Hom}(\mathcal{Z}(V) * L_w, L_w)$ equals $r_{\underline{c}}(\phi_{N_{\underline{c}}}) \circ \mathrm{id}_{L_w}$. But $\phi_{N_{\underline{c}}} = 0$ if $N_{\underline{c}} \in \overline{O}$, so (22) holds in this case. Conversely, if $\overline{O} \not\ni N_{\underline{c}}$ then $\phi_{N_{\underline{c}}} \neq 0$ for all n. Since the functor $V \mapsto r_{\underline{c}}(V) \circ X$ from $Rep(Z_{\underline{c}})$ to $\mathcal{A}_{\underline{c}}$ is exact and faithful for all $X \in \mathcal{A}_{\underline{c}}$,

$X \neq 0$ we see that ϕ_{L_w} is non-zero in this case. This shows (22), and hence (20).

(20) implies that the left hand side of (21) is contained in the right-hand side. Let us check the other inclusion. Let $\mathcal{J}$ be the ideal sheaf of a proper $G\check{}$-invariant subvariety S in the right-hand side of (21), and $\phi : V \otimes \mathcal{O} \to \mathcal{O}$ satisfy $im(\phi) = \mathcal{J}^n$ as before. We need to verify that $\mathrm{supp}(X) \not\subset S$, which is equivalent to saying that the induced morphism $\mathcal{Z}(V)(X) \to X$ is nonzero. There exists $w \in \underline{c} \subset W$ such that the multiplicity of L_w in the Jordan–Hoelder series of $H^i(X)$ is non-zero for some i but $N_{\underline{c}} \notin S$. We saw in the previous paragraph that the morphism $(\phi)_{L_w} : \mathcal{Z}(V) * L_w \to L_w$ is non-zero. But the latter is a subquotient of $H^i((\phi)_X)$; so $(\phi)_X \neq 0$ as well. ∎

2.4.1. *Proof of Theorem 4 (conclusion).* (a) follows from (b) and (21); so let us prove (b). Let $\underline{c}_1, \underline{c}_2 \subset W$ be two sided cells. Let $\mathcal{J}_i \subset \mathcal{O}_{\mathcal{N}}$ be the ideal sheaf of $\overline{G\check{}(N_{\underline{c}_i})}$, and $\phi_i : V_i \otimes \mathcal{O}_{\mathcal{N}} \to \mathcal{O}_{\mathcal{N}}$ have $\mathcal{J}_i$ as its image ($i = 1, 2$).

Assume that $c_1 \leq \underline{c}_2$; pick $w_1 \in \underline{c}_1 \cap {}^f W$, $w_2 \in \underline{c}_2 \cap {}^f W$. Then L_{w_1} is a direct summand in the convolution $X_1 * L_{w_2} * X_2$ for some semisimple complexes $X_1, X_2 \in D^b_I(\mathcal{F}\ell)$. Hence the arrow $(\phi_2)_{L_{w_1}}$ is a direct summand in

$$X_1 * ((\phi_2)_{L_{w_2}}) * X_2 = (\phi_2)_{X_1 * L_{w_2} * X_2}.$$

But

$$(\phi_2)_{L_{w_2}} = 0;$$

hence

$$(\phi_2)_{L_{w_1}} = 0,$$

which implies

$$N_{\underline{c}_1} \in p_{Spr}(\mathrm{supp}({}^f\Phi^{-1}(L_{w_1}))) \subset \overline{G\check{}(N_{\underline{c}_2})}.$$

Conversely, suppose that $N_{\underline{c}_1} \in \overline{G\check{}(N_{c_2})}$. Let

$$\mathbb{K} = (0 \to \Lambda^d(V) \otimes \mathcal{O}_{\mathcal{N}} \to \cdots \to V \otimes \mathcal{O}_{\mathcal{N}} \to \mathcal{O}_{\mathcal{N}} \to 0)$$

be the Koszul complex of ϕ_1. Pick $w \in \underline{c}_2 \cap^f W$. Then we have

$$\overline{G\check{}(N_{\underline{c}_1})} = p_{Spr}(\mathrm{supp}(\mathbb{K} \otimes_{\mathcal{O}_{\mathcal{N}}} {}^f\Phi^{-1}(L_w))).$$

Hence, according to (21), there exists $w_1 \in \underline{c}_1$ such that L_{w_1} is a subquotient of $H^i({}^f\Phi(\mathbb{K} \otimes_{\mathcal{O}_{\mathcal{N}}} {}^f\Phi^{-1}(L_w)))$ for some i. The object ${}^f\Phi(\mathbb{K} \otimes_{\mathcal{O}_{\mathcal{N}}} {}^f\Phi^{-1}(L_w))$ is represented by the complex

$$0 \to \mathcal{Z}(\Lambda^d(V)) * L_w \to \mathcal{Z}(\Lambda^{d-1}(V)) * L_w \to \cdots \to \mathcal{Z}(V) * L_w \to L_w \to 0.$$

But the Jordan-Hoelder series of $Z_\lambda * L_w$ consists of L_u with $u \underset{LR}{\leq} w$. Hence $\underline{c}_1 \leq \underline{c}_2$. The Theorem is proved. ∎

2.5. Proof of Proposition 1. The proposition will be deduced from the next two lemmas

Lemma 7: *Let $j : O \hookrightarrow \mathcal{N}$ be an orbit of codimension $2m$, and $\mathcal{F} \in D^{G^\vee}(\mathcal{N})$ satisfy the following properties*

i) *$\mathcal{F}$ is an irreducible perverse coherent sheaf with respect to the perversity* (5).
ii) $\operatorname{supp}(\mathcal{F}) = \overline{O}$.
iii) $\operatorname{Hom}_{D^{G^\vee}(\mathcal{N})}(\mathcal{O}, \mathcal{F}[m]) \neq 0$.

Then $\mathcal{F} \cong \hat{\mathcal{O}}_{\overline{O}}[-m]$.

Proof. The condition $\operatorname{Hom}_{D^{G^\vee}(\mathcal{N})}(\mathcal{O}, \mathcal{F}[m]) \neq 0$ is equivalent to the existence of a non-zero $G^\vee$-invariant section of the coherent sheaf $H^m(\mathcal{F})$ (where the cohomology is taken with respect to the usual t-structure on the derived category of coherent sheaves). For a perverse coherent sheaf $\mathcal{F}$ on $\overline{O}$ we have $H^i(\mathcal{F}) = 0$ for $i < m$, and $H^m(\mathcal{F})$ is a torsion free sheaf on $\overline{O}$. (Indeed, otherwise we would have a non-zero morphism defined on a $G^\vee$-invariant open subscheme of $\overline{O}$ from V to $\mathcal{F}[i]$ where $i \leq m$ and V is the non-derived direct image of a vector bundle under the locally closed embedding of an orbit $O' \subset \overline{O}$, $O' \neq O$. Since $V[-d]$ is a perverse coherent sheaf for $d = codim O'/2 > m$ this would give an Ext of degree $i - d < 0$ between perverse coherent sheaves, which is impossible.)

Thus a non-zero section of $H^m(\mathcal{F})$ does not vanish on O. Also, $j^*(H^m(\mathcal{F}))$ is an irreducible $G^\vee$-equivariant vector bundle. Such a vector bundle has a non-zero $G^\vee$-invariant section if and only if it is trivial; in which case we have $\mathcal{F} \cong j_{!*}(\mathcal{O}_O[-m]) \cong \hat{\mathcal{O}}_{\overline{O}}[-m]$, where the last equality is proved in [4], Remark 11. ∎

Lemma 8: *We have*

$$Ext^{a(d_{\underline{c}})}_{f\mathcal{P}_I}(L_0, L_{d_{\underline{c}}}) \neq 0,$$

where a stands for Lusztig's a-function on W, see, e.g., [9, Section 1.1].

Proof. The standard definition of a Duflo involution (see, e.g., [9, Section 1.3]) shows that the costalk $j_e^!(L_d)$ has non-zero cohomology in degree $a(d)$. We can think of $j_e^!(L_d)$ as an object in the $\mathbf{I}$-equivariant derived category of

l-adic sheaves on the point. Moreover, it is a pull-back of an object in the **I**-equivariant derived category of l-adic sheaves on the spectrum of a finite field. The latter object is known to be pure (cf., e.g., [8, Appendix, Section A.7]); hence it is isomorphic to the direct sum of its cohomology (notice that Hom between two objects of the bounded **I**-equivariant derived category of the point is identified with Hom between corresponding complexes with constant cohomology on $(\mathbb{P}^n)^{\mathrm{rank}\ (G)}$, $n \gg 0$. Thus any pure object in the **I**-equivariant derived category of the point is isomorphic to the sum of its cohomology by [3, Theorem 5.4.5]).

It follows that $\mathrm{Hom}_{D^b_I(\mathcal{F}\ell)}(L_e, L_d[a(d)]) \neq 0$. In view of Theorem 2 of [2] we will be done if we check that the map

$$\mathrm{Hom}_{D^b_I(\mathcal{F}\ell)}(L_e, L_d[a(d)]) \to \mathrm{Hom}_{D_{\mathcal{IW}}}(\Delta_e, \Delta_e * L_d[a(d)]),$$

sending h to $\mathrm{id}_{\Delta_e} * h$ is injective.

Recall (see, e.g., [11]) that $L_d[a(d)]$ is a direct summand in $L_w * L_{w^{-1}}$ for any $w \in \underline{c} \cap {}^f W$ (e.g., for $w = d$). Thus for any $h \in \mathrm{Hom}_{D_I(\mathcal{F}\ell)}(L_e, L_d[a(d)])$ the composition

$$L_e \xrightarrow{h} L_d[a(d)] \to L_w * L_{w^{-1}} \tag{23}$$

is non-zero for such w.

For $w \in W$ and $X, Y \in D(\mathcal{F}\ell)$; or $X, Y \in D_I(\mathcal{F}\ell)$ we have a canonical isomorphism

$$\mathrm{Hom}(X * L_w, Y) \cong \mathrm{Hom}(X, Y * L_{w^{-1}}). \tag{24}$$

In particular, $\mathrm{Hom}(L_e, L_w * L_{w^{-1}}) \cong \mathrm{Hom}(L_w, L_w)$ is a one dimensional space; thus multiplying $h \in \mathrm{Hom}_{D_I(\mathcal{F}\ell)}(L_e, L_d[a(d)])$ by a constant we can assume that the composition (23) corresponds to $\mathrm{id} \in \mathrm{Hom}(L_w, L_w)$ under the isomorphism (24). Then one can check that the composition

$$\Delta_e \xrightarrow{\mathrm{id}_{\Delta_e} * h} \Delta_e * L_d[a(d)] \to \Delta_e * L_w * L_{w^{-1}}$$

corresponds under (24) to $\mathrm{id} \in \mathrm{Hom}(\Delta_e * L_w, \Delta_e * L_w)$. In particular, it is not equal to zero. ∎

2.5.1. We are now ready to finish the proof of the proposition. It suffices to see that the object $({}^f\Phi^f)^{-1}(L_d)$ satisfies the conditions of Lemma 7. The first condition holds by Theorem 2. The second one holds by Theorem 4(a). Finally, to check condition (iii) notice that by Lemma 8 we have

$$\begin{aligned} ll\,\mathrm{Hom}(\mathcal{O}_{\mathcal{N}}, {}^f\Phi^{f^{-1}}(L_d)[a(d_{\underline{c}})]) &= \mathrm{Hom}(\mathcal{O}_{\mathcal{N}}, p_{Spr*}({}^f\Phi^{-1}(L_d)[a(d_{\underline{c}})]) \\ &= \mathrm{Hom}(\mathcal{O}_{\tilde{\mathcal{N}}}, {}^f\Phi^{-1}(L_d)[a(d_{\underline{c}})]) \\ &= \mathrm{Hom}(L_e, L_d[a(d_{\underline{c}})]) \neq 0. \end{aligned}$$

By [10], Theorem 4.8(c) we have $a(d_{\underline{c}}) = \mathrm{codim}(G^{\vee}(N_{\underline{c}}))/2$, which implies condition (iii). ∎

2.6. Proof of Proposition 2. If ρ is trivial then (7) follows from Proposition 1. Applying (19) we see that (7) holds when $\rho = Res^{G^{\vee}}_{Z_{G^{\vee}}(N_{\underline{c}})}(V)$ for $V \in Rep(G^{\vee})$.

Let now ρ be arbitrary. Let $\mathcal{L} \in Coh^{G^{\vee}}(\mathcal{N})$ be some sheaf supported on the closure of $G^{\vee}(N_{\underline{c}})$, and such that $\mathcal{L}|_{G^{\vee}(N_{\underline{c}})} \cong \mathcal{L}_\rho$. We can choose a short exact sequence

$$W \otimes \mathcal{O} \xrightarrow{\phi} V \otimes \mathcal{O} \to \mathcal{L} \to 0, \tag{25}$$

$V, W \in Rep(G^{\vee})$.

Then we get an exact sequence

$$W|_{Z_{G^{\vee}}(N_{\underline{c}})} \to V|_{Z_{G^{\vee}}(N_{\underline{c}})} \to \rho \to 0$$

in $Rep(Z_{\underline{c}})$, and hence an exact sequence in ${}^f\mathcal{P}_I^{\underline{c}}$:

$$r_{\underline{c}}(W|_{Z_{\underline{c}}}) \circ L_{d_{\underline{c}}} \to r_c(V|_{Z_{\underline{c}}}) \circ L_{d_{\underline{c}}} \to r_c(\rho) \circ L_{d_{\underline{c}}} \to 0;$$

by (19) it can be written as

$$\mathcal{Z}(W) * L_{d_{\underline{c}}} \to \mathcal{Z}(V) * L_{d_{\underline{c}}} \to r_c(\rho) \circ L_{d_{\underline{c}}} \to 0. \tag{26}$$

On the other hand, consider the tensor product of (25) by $j_*(\mathcal{O})$ where j stands for the embedding $G^{\vee}(N_{\underline{c}}) \hookrightarrow \mathcal{N}$ (and j_* is the non-derived direct image). We get a short exact sequence

$$W \otimes j_*(\mathcal{O}) \to V \otimes j_*(\mathcal{O}) \to L' \to 0 \tag{27}$$

where $L'|_{G^{\vee}(N_{\underline{c}})} \cong \mathcal{L}_\rho$. Theorem 2 and the definition of a perverse coherent sheaf show that the functor $\mathcal{F} \mapsto \Phi_{\underline{c}}(\mathcal{F})[-m]$ is exact with respect to the **standard**

t-structure on the category $D^{G^\vee}_{N_{\underline{c}}}(\mathcal{N})$. Applying this functor to (27) we get an exact sequence in ${}^f\mathcal{P}_I^{f\underline{c}}$, which by Theorem 1 has the form

$$\mathcal{Z}(W) * L_{d_{\underline{c}}} \to \mathcal{Z}(V) * L_{d_{\underline{c}}} \to \Phi_{\underline{c}}(\mathcal{L}_\rho[-m]) \to 0. \tag{28}$$

Lemma 5 implies that (28) is isomorphic to (26) (or rather to its image in the quotient category ${}^f\mathcal{P}_I^{f\underline{c}}$); in particular, (7) holds. ∎

2.6.1. *Proof of Corollary 2.* The functor $\Phi_{\underline{c}}$ is an equivalence, thus Proposition 2 implies that the functor $r^f_{\underline{c}} : Rep(Z_{\underline{c}}) \to A^f_{\underline{c}} \cong Rep(H_{\underline{c}})$ is fully faithful. The functor of restriction of a representation to a subgroup can only be fully faithful if the subgroup coincides with the whole group, thus $H_{\underline{c}} = Z_{\underline{c}}$. ∎

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