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THRESHOLD FOR THE VOLUME SPANNED BY RANDOM POINTS WITH INDEPENDENT COORDINATES*

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ABSTRACT

Let μ be an even compactly supported Borel probability measure on the real line. For every $N > n$ consider N independent random vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ in $\mathbb{R}^n$, with independent coordinates having distribution μ. We establish a sharp threshold for the volume of the random polytope $K_N := \mathrm{conv}\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N\}$, provided that the Legendre transform λ of the cumulant generating function of μ satisfies the condition

$$(*) \qquad \lim_{x\uparrow\alpha} \frac{-\ln \mu([x,\infty))}{\lambda(x)} = 1,$$

where α is the right endpoint of the support of μ. The method and the result generalize work of Dyer, Füredi and McDiarmid on 0/1 polytopes. We verify $(*)$ for a large class of distributions.

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1. Introduction

Our starting point is work of Dyer, Füredi and McDiarmid, establishing a sharp threshold for the expected volume of random ± 1 polytopes. The method they introduced in [7] proved to be extremely useful and accurate; for example, it played a key role in the approach introduced by Bárány and Pór in [2] in order to establish that there exist ± 1 polytopes with a superexponential number of facets, which was further developed in [9] and [10].

We will work in a more general framework which we now describe. Let μ be an even, compactly supported, Borel probability measure on the real line, and consider a random variable X, on some probability space $(\Omega, \mathcal{F}, P)$, with distribution μ, i.e., $\mu(B) := P(X \in B)$, $B \in \mathcal{B}(\mathbb{R})$. To avoid trivialities, we assume that $\mathrm{Var}(X) > 0$. In particular, we then have that

$$p = p(\mu) := \max_{x \in \mathbb{R}} P(X = x) < 1. \tag{1.1}$$

Let also

$$\alpha = \alpha(\mu) := \sup\{x \in \mathbb{R} \colon \mu([x, \infty)) > 0\}) \tag{1.2}$$

be the right endpoint of the support of μ.

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables, defined on the product space $(\Omega^n, \mathcal{F}^{\otimes n}, P^n)$, each with distribution μ. Set $\boldsymbol{X} = (X_1, \ldots, X_n)$ and, for a fixed N satisfying $N > n$, consider N independent copies $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ of $\boldsymbol{X}$, defined on the product space $(\Omega^{nN}, \mathcal{F}^{\otimes nN}, \mathrm{Prob})$. This procedure defines the random polytope

$$K_N := \mathrm{conv}\big\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N\big\}. \tag{1.3}$$

Observe that $K_N \subseteq [-\alpha, \alpha]^n$ almost surely.

Let $\varphi(t) := E\big(e^{tX}\big)$ $(t \in \mathbb{R})$ denote the moment generating function of X, and let $\psi(t) := \ln \varphi(t)$ be its cumulant generating function (or logarithmic moment generating function). By Hölder's inequality, ψ is a convex function on $\mathbb{R}$. Consider the Legendre transform λ of ψ; this is the function

$$\lambda(x) := \sup\{tx - \psi(t) \colon t \in \mathbb{R}\}. \tag{1.4}$$

Define

$$\kappa = \kappa(\mu) := \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} \lambda(x)dx. \tag{1.5}$$

For a large class of distributions μ we will establish the following threshold for the expected volume of K_N: for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \sup\{(2\alpha)^{-n} E(|K_N|) \colon N \leqslant \exp((\kappa-\varepsilon)n)\} = 0 \tag{1.6}$$

and

$$\lim_{n\to\infty} \inf\{(2\alpha)^{-n} E(|K_N|) \colon N \geqslant \exp((\kappa+\varepsilon)n)\} = 1. \tag{1.7}$$

Dyer, Füredi and McDiarmid [7] studied the following two cases:

[DFM 1] If $\mu(\{1\}) = \mu(\{-1\}) = 1/2$ then $\psi(t) = \ln(\cosh t)$. Then, $\lambda : (-1,1) \to \mathbb{R}$ is given by

$$\lambda(x) = \tfrac{1}{2}(1+x)\ln(1+x) + \tfrac{1}{2}(1-x)\ln(1-x), \tag{1.8}$$

and (1.6)–(1.7) hold with $\kappa = \ln 2 - 1/2$. This is the case of ± 1 polytopes.

[DFM 2] If μ is the uniform distribution on $[-1,1]$, then $\psi(t) = \ln(\sinh t/t)$, and (1.6)–(1.7) hold with

$$\kappa = \int_0^\infty \left(\frac{1}{u} - \frac{1}{e^u - 1}\right)^2 du. \tag{1.9}$$

We establish the following result

THEOREM 1.1: *Let μ be an even, compactly supported, Borel probability measure on the real line and assume that $0 < \kappa(\mu) < \infty$. Then (1.6) holds for every $\varepsilon > 0$. Furthermore, (1.7) holds for every $\varepsilon > 0$ whenever the distribution μ satisfies*

$$\lim_{x\uparrow\alpha} \frac{-\ln P(X \geqslant x)}{\lambda(x)} = 1. \tag{1.10}$$

NOTE 1: One always has $\kappa(\mu) > 0$ under our assumptions. Furthermore, our proof will show that in fact (1.6) remains valid even when $\kappa(\mu) = \infty$, in the following sense: $\sup\{(2\alpha)^{-n}E(|K_N|) \colon N \leqslant e^{rn}\} \to 0$ as $n \to \infty$, for any $r > 0$.

NOTE 2: Notice also that, in the presence of (1.10),

$$\kappa(\mu) < \infty \iff \int_{-\alpha}^{\alpha} -\ln P(X \geqslant x)\, dx < \infty, \tag{1.11}$$

giving a criterion for the existence of a threshold for the volume directly in terms of the distribution function of μ.

We next address the question of which probability measures μ satisfy condition (1.10). Of course, as is well-known, one always has that $-\ln P(X \geqslant x) \geqslant \lambda(x)$ for $x \in (0, \alpha)$ under our assumptions on μ (see Proposition 2.6). On the other hand, it is not too hard to see that (1.10) does not hold for arbitrary compactly supported distributions μ — an example is provided in the last section, at the end of the paper. We shall verify it for a large class of compactly supported distributions, however. To begin with, we first recall the following

Definition 1.2 (cf., [8, p. 276]): A measurable function $L\colon (0,\infty) \to (0,\infty)$ is **slowly varying** at zero if, for any $a > 0$, $L(ax)/L(x) \to 1$ as $x \downarrow 0$. As this property is not affected by the values of L on any interval of the form $[b, \infty)$, we shall take it as a requirement of the definition that such a function is bounded on intervals of the form $[b, b']$ with $0 < b < b' < \infty$.

We shall also use the following notation

Notation: For functions $f, g\colon J \to (0, \infty)$, where J is an interval in $\mathbb{R}$, and u_0 in $\bar{J}$, $f(u) \sim g(u)$ as $u \to u_0$ means that $\lim_{u \to u_0} f(u)/g(u) = 1$. In this paper, the notation $f(u) \approx g(u)$ as $u \to u_0$ shall mean that there exist a neighborhood U of u_0 and constants $c_1 > 0$ and $c_2 < \infty$ such that $c_1 g(u) \leqslant f(u) \leqslant c_2 g(u)$ for $u \in U$.

We then have the following

THEOREM 1.3: *Condition* (1.10) *is satisfied in the following cases*:

(i) *When* $P(X = \alpha) > 0$.

(ii) *When* $P(X \geqslant x) \approx (\alpha - x)^\rho L(\alpha - x)$ *as* $x \uparrow \alpha$, *with* $\rho \geqslant 0$ *and* L *slowly varying at zero.*

(iii) *When* $-\ln P(X \geqslant x) \sim \theta(\alpha - x)^{-\rho}$ *as* $x \uparrow \alpha$, *with* $\rho, \theta > 0$.

REMARKS:

1. Note that in fact (i) is subsumed by (ii) in Theorem 1.3 (take $\rho = 0$ and $L(x) = P(X \geqslant \alpha - x)$ for all $x > 0$). Note also that the case [DFM 1] is covered by (i) of Theorem 1.3, while [DFM 2] is covered by (ii) with $\rho = 1$ and $L(x) = 1/2$ for all $x > 0$.
2. It is perhaps noteworthy that case (ii) also covers, for example, the case where the function $x \mapsto P(X \geqslant \alpha - x)$ behaves like the Cantor function near the origin (e.g., when $P(X \leqslant x) = C(x + 1/2)$ for $x \in [-1/2, 1/2]$, where C is the usual ternary Cantor function on $[0, 1]$); in this case $\rho = \log_3 2$, $L \equiv 1$.

3. Finally, note that case (iii) covers the case where $P(X \geqslant \alpha - x)$ behaves, near the origin, like the distribution function of a positive stable random variable with index in $(0,1)$. More precisely, if G_ρ denotes the distribution function of a stable random variable $Y \geqslant 0$ of index $\rho \in (0,1)$, then $-\ln G_\rho(x) \sim \theta x^{-\rho/(1-\rho)}$ as $x \downarrow 0$, as follows from a Tauberian theorem of de Bruijn [3, Theorem 4.12.9].

We end with an observation which may be useful. One can get a more precise information than (1.6) regarding the behavior below the threshold under more stringent assumptions on μ. The following result is a byproduct of the proof of the first part of Theorem 1.1.

Theorem 1.4: *Let μ be an even, compactly supported, Borel probability measure on the real line and assume that $\int_{-\alpha}^{\alpha} \lambda(x)^2 dx < \infty$. Then,*

$$\lim_{n\to\infty} \sup\{(2\alpha)^{-n} E(|K_N|) \colon N \leqslant \exp((\kappa - \varepsilon_n)n)\} = 0 \tag{1.12}$$

for any sequence $\varepsilon_n > 0$ satisfying $\varepsilon_n \sqrt{n} \to \infty$.

The present work is, of course, in the realm of stochastic geometry, the study of randomly generated sets. For discrete aspects of this theory, concerning expectations of geometrically defined random variables or probabilities of events defined by random geometric configurations, we refer the reader to the survey article by Schneider [12] and references therein. In particular, one of the referees pointed out that, in the present context, replacing the uniform (and Gaussian) distribution by more general distributions which fulfill certain regularity conditions has already been considered by Carnal [5], when studying the convex hull of independent random points in the plane with a common rotationally symmetric distribution. He also pointed out that threshold type behaviour in these kind of problems was already considered by Miles [11], who conjectured that the probability that the convex hull of $d+m$ points chosen independently and uniformly from a d-dimensional ball has $d+i$ vertices, tends to one for $i = m$ ($d \to \infty$), and (hence) to zero for all $i = 1, \ldots, m-1$ [11, p. 373]; this conjecture was subsequently verified by Buchta [4, Theorem 1], and relates to estimates for the expected volume of the convex hull of points chosen independently and uniformly from the ball.

We close this introductory section by fixing some notation.

Notation: We work in $\mathbb{R}^n$ which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm and write B_2^n for the Euclidean unit ball. Volume and the cardinality of a finite set will be denoted by $|\cdot|$. All logarithms are natural. The letters c, c', C, c_1, c_2 etc. denote absolute positive constants which may change from line to line.

2. Preliminaries

In this Section we recall some basic facts concerning moment generating and cumulant generating functions. For more information on large deviations techniques the reader may wish to consult the books [6] and [13].

Let μ be an even, compactly supported, Borel probability measure on the real line, and consider a random variable X, on some probability space $(\Omega, \mathcal{F}, P)$, with distribution μ. Set $\alpha := \sup\{x \in \mathbb{R} \colon \mu([x,\infty)) > 0\}$ and $I := (-\alpha, \alpha)$.

Definition 2.1: Let $m : [0,\alpha] \to [0,\infty]$ be defined by

$$m(x) = -\ln \mu([x,\infty)). \tag{2.1}$$

It is clear that m is non-decreasing and that $m(\alpha) < \infty$ if and only if $P(X = \alpha) > 0$. With this definition, condition (1.10) takes the form: $m(x) \sim \lambda(x)$ as $x \uparrow \alpha$.

Recall that

$$\varphi(t) := E\big(e^{tX}\big) \quad (t \in \mathbb{R}) \tag{2.2}$$

is the moment generating function of X, and

$$\psi(t) := \ln \varphi(t) \tag{2.3}$$

is its cumulant generating function. Since X is bounded, φ and ψ are finite for every $t \in \mathbb{R}$. By Hölder's inequality, ψ is a convex function on $\mathbb{R}$. Therefore, φ is also convex. It is easily checked that φ is C^∞ on $\mathbb{R}$. The n-th derivative of φ is given by

$$\varphi^{(n)}(t) = E\big(X^n e^{tX}\big). \tag{2.4}$$

Observe also that, by Markov's inequality, for any $x \in (0,\alpha)$ and any $t \geqslant 0$, one has that

$$\varphi(t) = E\big(e^{tX}\big) \geqslant e^{tx}\mu([x,\infty)), \tag{2.5}$$

and hence,

$$\psi(t) \geqslant tx - m(x). \tag{2.6}$$

Definition 2.2: For every $t \in \mathbb{R}$ define the probability measure P_t on $(\Omega, \mathcal{F})$ by

$$P_t(A) := E\big(e^{tX-\psi(t)}\mathbf{1}_A\big) \quad (A \in \mathcal{F}). \tag{2.7}$$

Define also $\mu_t(A) := P_t(X \in A)$ for $A \in \mathcal{B}(\mathbb{R})$. Then, μ_t has finite moments of all orders, and

$$E_t(X) = \psi'(t) \quad \text{and} \quad \mathrm{Var}_t(X) = \psi''(t). \tag{2.8}$$

Notice that $P_0 = P$ and $\mu_0 = \mu$.

Lemma 2.3: *$\psi' : \mathbb{R} \to I$ is strictly increasing and surjective. In particular,*

$$\lim_{t\to\pm\infty} \psi'(t) = \pm\alpha. \tag{2.9}$$

Proof. Since

$$(\psi')'(t) = \psi''(t) = \mathrm{Var}_t(X) > 0, \tag{2.10}$$

ψ' is strictly increasing. From the inequality $-\alpha e^{tX} \leqslant Xe^{tX} \leqslant \alpha e^{tX}$, which holds with probability one for each fixed $t \in \mathbb{R}$, and the formula $\psi'(t) = E\big(Xe^{tX}\big)/E\big(e^{tX}\big)$, which follows from (2.4), it follows immediately that $\psi'(t) \in (-\alpha, \alpha)$ for every $t \in \mathbb{R}$.

It remains to show that ψ' is onto I. Let $x \in (0, \alpha)$. Consider the function $g_x(t) := tx - \psi(t)$ $(t \in \mathbb{R})$ and fix $y \in (0, \alpha)$. From (2.6) we have that $\psi(t) \geqslant ty - m(y)$ for all $t \geqslant 0$; in particular, $\psi\big(m(y)/(y-x)\big) \geqslant xm(y)/(y-x)$. It follows that g_x satisfies $g_x(0) = 0$ and $g_x\big(m(y)/(y-x)\big) \leqslant 0$. Since g_x is concave and $g_x'(0) = x > 0$, this shows that g_x attains its maximum at some point in the open interval $\big(0, m(y)/(y-x)\big)$, and hence, $\psi'(t) = x$ for some t in this interval. The same argument applies for all $x \in (-\alpha, 0)$. Finally, for $x = 0$ we have that $\psi'(0) = x$. ∎

Definition 2.4: Define $h\colon I \to \mathbb{R}$ by $h := (\psi')^{-1}$.

Remark: Observe that h is a strictly increasing C^∞ function, with

$$h'(x) = \frac{1}{\psi''(h(x))}. \tag{2.11}$$

Definition 2.5: The Legendre transform of ψ is the function

$$\lambda(x) := \sup\{tx - \psi(t) \colon t \in \mathbb{R}\}, \quad x \in \mathbb{R}. \tag{2.12}$$

REMARK: Observe that, since $tx - \psi(t) < 0$ for $t < 0$ when $x \in [0, \alpha)$, one always has that

$$\lambda(x) = \sup\{tx - \psi(t) \colon t \geqslant 0\} \tag{2.13}$$

for $x \in [0, \alpha)$, and similarly $\lambda(x) = \sup\{tx - \psi(t) \colon t \leqslant 0\}$ for $x \in (-\alpha, 0]$.

Maximizing over $t \geqslant 0$ in (2.6) leads to the following fundamental inequality.

PROPOSITION 2.6: *Let μ be an even, compactly supported, Borel probability measure on the line. Then, for any $x \in (0, \alpha)$, one has that*

$$\mu([x, \infty)) \leqslant e^{-\lambda(x)}. \tag{2.14}$$

The basic properties of λ are described in the next Lemma.

LEMMA 2.7: (i) *$\lambda \geqslant 0$, $\lambda(0) = 0$ and $\lambda(x) = \infty$ for $x \in \mathbb{R} \setminus [-\alpha, \alpha]$.*

(ii) *For every $x \in I$ we have $\lambda(x) = tx - \psi(t)$ if and only if $\psi'(t) = x$; hence*

$$\lambda(x) = xh(x) - \psi(h(x)) \quad \text{for } x \in I. \tag{2.15}$$

(iii) *λ is a strictly convex C^∞ function on I, and*

$$\lambda'(x) = h(x). \tag{2.16}$$

(iv) *λ attains its unique minimum on I at $x = 0$.*

The behaviour of μ at the endpoints of I decides whether λ is bounded or not. This is a consequence of the following

LEMMA 2.8: *$\lambda(\alpha) = -\ln P(X = \alpha)$ and $\lambda(x) \to -\ln P(X = \alpha)$ as $x \uparrow \alpha$.*

NOTE: If $P(X = \alpha) = 0$, the convention is that $-\ln P(X = \alpha) = \infty$.

Proof. Since $\psi'(t) \leqslant \alpha$ for all t, the function $t \mapsto t\alpha - \psi(t)$ is non-decreasing. Therefore,

$$\lambda(\alpha) = \sup_{t \in \mathbb{R}} [t\alpha - \psi(t)] = \lim_{t \uparrow \infty} [t\alpha - \psi(t)]. \tag{2.17}$$

However,

$$\lim_{t \uparrow \infty} e^{-t\alpha} \varphi(t) = \lim_{t \uparrow \infty} E\big(e^{t(X-\alpha)}\big) = E\Big(\lim_{t \uparrow \infty} e^{t(X-\alpha)}\Big) = P(X = \alpha), \tag{2.18}$$

by the dominated convergence theorem. It follows that $\lambda(\alpha) = -\ln P(X = \alpha)$.

For the second assertion, observe that λ is lower semi-continuous on $\mathbb{R}$, being the pointwise supremum of the linear (hence continuous) functions $x \mapsto tx - \psi(t)$, $t \in \mathbb{R}$. ∎

Corollary 2.9: *λ is bounded on I if and only if $P(X = \alpha) > 0$.*

We close this Section with one more elementary observation, which we single out for subsequent use. As already observed, the function φ is C^∞ on $\mathbb{R}$ (cf. (2.4)); since φ is also (strictly) positive, the function $\psi = \ln \varphi$ is also C^∞ on $\mathbb{R}$. By (2.8) we also have that $\psi''(t) > 0$ for all t. Finally, it is also easily seen that the function $t \mapsto E_t\big(|X - \psi'(t)|^3\big)$ is continuous and finite on $\mathbb{R}$. We therefore have the following

Lemma 2.10: *The functions $t \mapsto t^2\psi''(t)$ and $t \mapsto t^3 E_t\big(|X - \psi'(t)|^3\big)$ are bounded away from 0 and infinity, respectively, on any interval $[a, b]$ with $0 < a \leqslant b < \infty$.*

3. The method of Dyer, Füredi and McDiarmid

The method we use for the proof of the Theorem 1.1 generalizes the one introduced in [7]. For $\boldsymbol{x} = (x_1, \dots, x_n) \in I^n$ set

$$\Lambda(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} \lambda(x_i). \tag{3.1}$$

For $0 \leqslant r < \lambda(\alpha)$, define Λ_r by

$$\Lambda_r = \{\boldsymbol{x} \in I^n \colon \Lambda(\boldsymbol{x}) \leqslant r\}. \tag{3.2}$$

Since λ is a convex function on I, Λ_r is a convex body contained in I^n.

Let $U_1, \dots, U_n$ be independent random variables, uniformly distributed in I. Then, for every $0 \leqslant r < \lambda(\alpha)$,

$$(2\alpha)^{-n} |\Lambda_r| = \mathrm{Prob}((U_1, \dots, U_n) \in \Lambda_r) = \mathrm{Prob}\bigg(\frac{1}{n} \sum_{i=1}^{n} \lambda(U_i) \leqslant r\bigg). \tag{3.3}$$

Observe that

$$\kappa = E(\lambda(U_i)). \tag{3.4}$$

By the law of large numbers, we conclude the following

Lemma 3.1: *Assume that* $0 < \kappa(\mu) < \infty$. *For every* $r \in (0, \kappa)$ *we have that*

$$\lim_{n\to\infty} (2\alpha)^{-n}|\mathit{\Lambda}_r| = 0, \tag{3.5}$$

and, similarly, for every $r \in (\kappa, \lambda(\alpha))$ *we have that*

$$\lim_{n\to\infty} (2\alpha)^{-n}|\mathit{\Lambda}_r| = 1. \tag{3.6}$$

Definition 3.2: For $\boldsymbol{x} \in I^n$, define

$$q(\boldsymbol{x}) := \inf\bigl\{\mathrm{Prob}(\boldsymbol{X} \in H)\colon \boldsymbol{x} \in H,\ H \text{ a closed halfspace}\bigr\}. \tag{3.7}$$

Remark: Note that in (3.7), it suffices to consider the infimum only over those halfspaces H for which $\boldsymbol{x} \in \partial H$, where ∂H is the boundary of H.

Lemma 3.3: *For* $\boldsymbol{x} \in I^n$, *one has that*

$$q(\boldsymbol{x}) \leqslant \exp(-n\mathit{\Lambda}(\boldsymbol{x})). \tag{3.8}$$

Proof. Fix $\boldsymbol{x} \in I^n$. Since $q(\boldsymbol{x})$ is determined by those halfspaces H for which $\boldsymbol{x} \in \partial H$, we can write

$$q(\boldsymbol{x}) = \inf\{P^n(\langle \boldsymbol{X} - \boldsymbol{x}, \boldsymbol{u}\rangle \geqslant 0)\colon \boldsymbol{u} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}\}. \tag{3.9}$$

Set $t_i := h(x_i)$, $i \leqslant n$. Then, (3.9), Markov's inequality, the independence of the coordinates of $\boldsymbol{X}$, and Lemma 2.7 (ii), give that

$$\begin{aligned} q(\boldsymbol{x}) &\leqslant P^n\biggl(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\biggr) \leqslant E\Bigl(e^{\sum_{i=1}^n t_iX_i}\Bigr)e^{-\sum_{i=1}^n t_ix_i} = \prod_{i=1}^n e^{\psi(t_i)-t_ix_i} \\ &= \exp\biggl(-\sum_{i=1}^n \lambda(x_i)\biggr) = \exp(-n\mathit{\Lambda}(\boldsymbol{x})). \end{aligned}$$

This proves the lemma. ∎

Lemma 3.4: *Let* $N > n$ *and* $0 < r < \lambda(\alpha)$. *Then*

$$E(|K_N|) \leqslant |\mathit{\Lambda}_r| + N(2\alpha)^n e^{-rn}. \tag{3.10}$$

Proof. First, write

$$E(|K_N|) = E(|K_N \cap \mathit{\Lambda}_r|) + E(|K_N \setminus \mathit{\Lambda}_r|) \leqslant |\mathit{\Lambda}_r| + E(|K_N \setminus \mathit{\Lambda}_r|). \tag{3.11}$$

Next, observe that if H is a closed halfspace containing $\boldsymbol{x}$, and if $\boldsymbol{x} \in K_N$, then there exists $i \leqslant N$ such that $\boldsymbol{X}_i \in H$ (otherwise we would have $\boldsymbol{x} \in K_N \subseteq H^c$, where H^c is the complementary halfspace). It follows that

$$\operatorname{Prob}(\boldsymbol{x} \in K_N) \leqslant N \cdot q(\boldsymbol{x}). \tag{3.12}$$

By Fubini's theorem, Lemma 3.3, and the definition of Λ_r, we then obtain that

$$\begin{aligned} E(|K_N \setminus \Lambda_r|) &= \int_{I^n \setminus \Lambda_r} \operatorname{Prob}(\boldsymbol{x} \in K_N)\, d\boldsymbol{x} \leqslant \int_{I^n \setminus \Lambda_r} N q(\boldsymbol{x})\, d\boldsymbol{x} \\ &\leqslant N \int_{I^n \setminus \Lambda_r} e^{-n\Lambda(\boldsymbol{x})} d\boldsymbol{x} \\ &\leqslant N |I^n| e^{-rn}. \end{aligned}$$

Inserting this into (3.11) yields (3.10). ∎

Proposition 3.5: *Assume that* $0 < \kappa(\mu) < \infty$. *Then, for every* $\varepsilon \in (0, \kappa)$,

$$\lim_{n\to\infty} \sup\{(2\alpha)^{-n} E(|K_N|) \colon N \leqslant \exp((\kappa - \varepsilon)n)\} = 0. \tag{3.13}$$

Proof. Choose $r = \kappa - \varepsilon/2$. From Lemma 3.1 we have that

$$\lim_{n\to\infty} (2\alpha)^{-n} |\Lambda_r| = 0. \tag{3.14}$$

On the other hand, if $N \leqslant \exp((\kappa - \varepsilon)n)$, Lemma 3.4 gives that

$$(2\alpha)^{-n} E(|K_N|) \leqslant (2\alpha)^{-n} |\Lambda_r| + \exp(-\varepsilon n/2), \tag{3.15}$$

and the right-hand side tends to 0 when $n \to \infty$. ∎

In the next section we shall prove that if the distribution μ satisfies $m(x) \sim \lambda(x)$ as $x \uparrow \alpha$, then one has a threshold for the expected volume of K_N at $N^* \sim \exp(\kappa n)$.

We close this section by indicating how to obtain a proof of the statement in Theorem 1.4.

Proof of Theorem 1.4. Assume that $\int_{-\alpha}^{\alpha} \lambda(x)^2 dx < \infty$. We may then use Chebychev's inequality to estimate the probability in (3.3):

$$(2\alpha)^{-n} |\Lambda_r| = \operatorname{Prob}\left(\frac{1}{n} \sum_{i=1}^{n} \lambda(U_i) \leqslant r \right) \leqslant \frac{\int_{-\alpha}^{\alpha} [\lambda(x) - \kappa]^2 dx}{n(\kappa - r)^2 (2\alpha)} \tag{3.16}$$

for any $0 < r < \kappa$.

Let $\varepsilon_n > 0$ be a sequence satisfying $\varepsilon_n \sqrt{n} \to \infty$. Then the choice $r_n := \kappa - \varepsilon_n/2$ in the proof of Proposition 3.5 yields Theorem 1.4. ∎

4. Threshold for the volume

In this section we complete the proof of Theorem 1.1, by showing (1.7) under the assumption that $m(x) \sim \lambda(x)$ as $x \uparrow \alpha$. Our basic strategy is along the lines of Dyer, Füredi and McDiarmid ([7]) again. There are, however, some differences, the most important one being the introduction of condition (1.10) in order to replace the explicit asymptotics of the two functions appearing in Lemma 2.10, used in [7].

Our primary goal will be to show that, under the assumption $m(x) \sim \lambda(x)$ $(x \uparrow \alpha)$, if $N \geqslant \exp((1+\varepsilon)rn + \varepsilon n)$ then $K_N \supseteq \Lambda_r$ with probability close to one (Lemma 4.9); (1.7) will then follow easily from this and (3.6) (Proposition 4.10). To show the aforementioned inclusion in turn, it will be enough to estimate $q_-(\Lambda_r) = \inf_{\boldsymbol{x}\in\Lambda_r} q(\boldsymbol{x})$ from below. This is a consequence of the next lemma (which essentially appears in [7], [2] and [9]).

Lemma 4.1: *Let $0 < r < \lambda(\alpha)$. Then*

$$1 - \mathrm{Prob}(K_N \supseteq \Lambda_r) \leqslant \binom{N}{n} p^{N-n} + 2\binom{N}{n}[1 - q_-(\Lambda_r)]^{N-n}, \tag{4.1}$$

where $q_-(\Lambda_r) := \inf\{q(\boldsymbol{x})\colon \boldsymbol{x} \in \Lambda_r\}$.

Proof. For every subset $J = \{j_1, \ldots, j_n\}$ of $\{1, \ldots, N\}$, of cardinality n, define the event A_J as follows: $\boldsymbol{X}_{j_1}, \ldots, \boldsymbol{X}_{j_n}$ are affinely independent, and for one of the two closed half-spaces H_1, H_2 they determine, say H_i, we have simultaneously $K_N \subset H_i$ and $P^n(\boldsymbol{X} \notin H_i) \geqslant q_-(\Lambda_r)$. Let also A denote the event that K_N has non-empty interior. We then claim that

$$\{\Lambda_r \nsubseteq K_N\} \subseteq A^c \cup \bigcup_J A_J. \tag{4.2}$$

Indeed, suppose that K_N is full-dimensional and $\Lambda_r \nsubseteq K_N$. Then there exists an $\boldsymbol{x} \in \Lambda_r \setminus K_N$, and consequently a facet F of K_N separating $\boldsymbol{x}$ and K_N. Hence there exist n affinely independent vertices $\boldsymbol{X}_{j_1}, \ldots, \boldsymbol{X}_{j_n}$ of K_N with the property that for one of the two closed half-spaces H_1, H_2 they determine, say H_i, we have simultaneously $K_N \subset H_i$ and $P^n(\boldsymbol{X} \notin H_i) \geqslant q(\boldsymbol{x}) \geqslant q_-(\Lambda_r)$.

From (4.2) we have that

(4.3)
$$\mathrm{Prob}\big(\Lambda_r \nsubseteq K_N\big) \leqslant \mathrm{Prob}(A^c) + \sum_J \mathrm{Prob}(A_J) = \mathrm{Prob}(A^c) + \binom{N}{n}\mathrm{Prob}(A'),$$

where $A' := A_{\{1,\dots,n\}}$. We next show that

$$\text{Prob}(A') \leqslant 2[1 - q_-(\Lambda_r)]^{N-n}. \tag{4.4}$$

Indeed, on the event that $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are affinely independent, denote by $H_i = H_i(\boldsymbol{X}_1, \dots, \boldsymbol{X}_n)$, $i = 1, 2$, the two closed half-spaces determined by $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$. On the event that $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are affinely independent and $P^n(\boldsymbol{X} \notin H_i) \geqslant q_-(\Lambda_r)$ we then have that

$$\text{Prob}(\boldsymbol{X}_{n+1}, \dots, \boldsymbol{X}_N \in H_i \mid \boldsymbol{X}_1, \dots, \boldsymbol{X}_n) \leqslant [1 - q_-(\Lambda_r)]^{N-n},$$

and (4.4) follows.

Finally, to obtain a bound on $\text{Prob}(A^c)$ we argue as follows. If K_N has empty interior, there exists $J = \{j_1, \dots, j_n\} \subset \{1, \dots, N\}$ such that the set $\{\boldsymbol{X}_j : j \notin J\}$ is contained in the affine hull of $\{\boldsymbol{X}_j : j \in J\}$. Now observe that, if S is a fixed affine subspace of dimension smaller than n, then $P^n(\boldsymbol{X} \in S) \leqslant p$. Indeed, fix a hyperplane H containing S. Then $H = \{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{y} - \boldsymbol{x} \rangle = 0\}$ for some $\boldsymbol{u} = (u_1, \dots, u_n) \neq \boldsymbol{0}$ and $\boldsymbol{x} = (x_1, \dots, x_n)$, and suppose that $u_i \neq 0$. Then

$$P^n(\boldsymbol{X} \in S) \leqslant P^n(\boldsymbol{X} \in H) = P^n\Big(X_i = x_i - u_i^{-1} \sum_{j \neq i} u_j (X_j - x_j)\Big),$$

and the latter is $\leqslant p$ because $P(X_i = x) \leqslant p$ for any $x \in \mathbb{R}$. By conditioning on $\{\boldsymbol{X}_j : j \in J\}$, we now see that

$$\text{Prob}(A^c) \leqslant \binom{N}{n} p^{N-n}. \tag{4.5}$$

This completes the proof of the Lemma. ∎

We next estimate the function $q_-(\Lambda_r)$ from below.

Proposition 4.2: *Assume that $m(x) \sim \lambda(x)$ as $x \uparrow \alpha$. Then, for every $\varepsilon > 0$, there exists $n_\mu(\varepsilon) \in \mathbb{N}$, depending only on ε and μ, such that for all $0 < r < \lambda(\alpha)$ and all $n \geqslant n_\mu(\varepsilon)$ we have that*

$$q_-(\Lambda_r) \geqslant \exp(-(1+\varepsilon)rn - \varepsilon n), \tag{4.6}$$

where $q_-(\Lambda_r) := \inf\{q(\boldsymbol{x}) : \boldsymbol{x} \in \Lambda_r\}$.

Proof. First we claim that it suffices to show that, for all n sufficiently large,

$$P^n(\boldsymbol{X} \in H) \geqslant e^{-(1+\varepsilon)rn - \varepsilon n} \tag{4.7}$$

for any closed half-space H whose bounding hyperplane supports Λ_r. Indeed, to see that this is sufficient, simply observe that, if $\boldsymbol{x} \in \Lambda_r$ and H is any closed half-space with $\boldsymbol{x} \in \partial H$, then $P^n(\boldsymbol{X} \in H) \geqslant P^n(\boldsymbol{X} \in H')$ where H' is the closed half-space which is contained in H and whose bounding hyperplane is parallel to that of H and supports Λ_r.

Let H be a closed half-space whose bounding hyperplane supports Λ_r. Then

$$P^n(\boldsymbol{X} \in H) = P^n\biggl(\sum_{i=1}^{n} t_i(X_i - x_i) \geqslant 0\biggr), \tag{4.8}$$

for some $\boldsymbol{x} = (x_1, \ldots, x_n) \in \partial(\Lambda_r)$, where $t_i = \lambda'(x_i)$ $(1 \leqslant i \leqslant n)$. Recall that $\lambda(0) = 0$ and that we are assuming that $m(x) \sim \lambda(x)$ $(x \uparrow \alpha)$. We can thus find $\delta > 0$ with the following properties:

(4.9) If $0 \leqslant x < \delta$ then $0 \leqslant \lambda(x) < \varepsilon$.

(4.10) If $\alpha - \delta \leqslant x < \alpha$ then $P(X \geqslant x) \geqslant \exp(-\lambda(x)(1+\varepsilon))$.

Define then

$$\begin{aligned} I_1 &= I_1(\boldsymbol{x}) := \{i\colon x_i < \delta\}, \\ I_2 &= I_2(\boldsymbol{x}) := \{i\colon \delta \leqslant x_i \leqslant \alpha - \delta\}, \\ I_3 &= I_3(\boldsymbol{x}) := \{i\colon x_i > \alpha - \delta\}, \end{aligned} \tag{4.11}$$

and set

$$P_j = P_j(\boldsymbol{x}) := P^n\biggl(\sum_{i \in I_j} t_i(X_i - x_i) \geqslant 0\biggr) \quad (j = 1, 2, 3). \tag{4.12}$$

By independence we then have that

$$P^n(\boldsymbol{X} \in H) = P^n\biggl(\sum_{i=1}^{n} t_i(X_i - x_i) \geqslant 0\biggr) \geqslant P_1 P_2 P_3. \tag{4.13}$$

We will consider each P_j separately.

Starting with I_1, we write

$$P_1 = P^n\biggl(\sum_{i \in I_1} t_i(X_i - x_i) \geqslant 0\biggr) \geqslant P^n\biggl(\sum_{i \in I_1} t_i(X_i - \delta) \geqslant 0\biggr), \tag{4.14}$$

and use the following fact.

LEMMA 4.3: *For every $\delta \in (0,\alpha)$, there exists $c(\delta) > 0$ depending only on δ and μ, such that for any $k \in \mathbb{N}$ and any $s_1, \dots, s_k \in \mathbb{R}$ with $\sum_{i=1}^k s_i > 0$ we have that*

$$P^k\Big(\sum_{i=1}^k s_i(X_i - \delta) \geqslant 0\Big) \geqslant c(\delta)\, k^{-3/2}\, e^{-k\lambda(\delta)}. \tag{4.15}$$

Proof. The first part of the argument in [2, Lemma 8.2] shows that

$$P^k\Big(\sum_{i=1}^k s_i(X_i - \delta) \geqslant 0\Big) \geqslant \frac{1}{k}\, P^k\Big(\sum_{i=1}^k (X_i - \delta) \geqslant 0\Big) \tag{4.16}$$

for all k. By [1, Theorem 1] on the other hand, there exists a sequence b_k of positive numbers, such that

$$\frac{\sqrt{2\pi k}}{b_k}\, e^{k\lambda(\delta)}\, P^k\Big(\sum_{i=1}^k (X_i - \delta) \geqslant 0\Big) \to 1 \qquad \text{as } k \to \infty, \tag{4.17}$$

with $\ln b_k$ bounded, and hence b_k bounded away from 0. Consequently, there exist $k_0 \in \mathbb{N}$ and $c > 0$ such that (4.15) holds with c in place of $c(\delta)$ and $k \geqslant k_0$. Since also

$$P^k\Big(\sum_{i=1}^k (X_i - \delta) \geqslant 0\Big) \geqslant [P(X \geqslant \delta)]^k = e^{-km(\delta)} \geqslant e^{-k\lambda(\delta)} e^{-k_0|m(\delta)-\lambda(\delta)|} \tag{4.18}$$

for $k < k_0$, (4.15) holds for all k, with $c(\delta) = \min\{c, e^{-k_0|m(\delta)-\lambda(\delta)|}\} > 0$. ∎

Combining Lemma 4.3 with (4.14), and using the facts that $\lambda(x) \leqslant \varepsilon$ on $[0,\delta]$ and that λ is increasing on $(0,\alpha)$, we arrive at the following estimate for P_1:

LEMMA 4.4: *We have that*

$$P_1 \geqslant \exp\Big(-\sum_{i\in I_1}[\lambda(x_i) + \varepsilon] - c_1 \ln|I_1| - c_2\Big), \tag{4.19}$$

where the constants $c_1, c_2 \in [0,\infty)$ depend only on δ and μ.

Next we examine I_3. By independence, we can write

$$P_3 = P^n\Big(\sum_{i\in I_3} t_i(X_i - x_i) \geqslant 0\Big) \geqslant \prod_{i\in I_3} P(X_i \geqslant x_i). \tag{4.20}$$

Since, by our choice of δ,

$$P(X_i \geqslant x_i) \geqslant e^{-\lambda(x_i)(1+\varepsilon)} \tag{4.21}$$

for all $i \in I_3$, we immediately get the following estimate for P_3:

LEMMA 4.5: *We have that*

$$P_3 \geqslant \exp\Big(-(1+\varepsilon)\sum_{i\in I_3} \lambda(x_i)\Big). \tag{4.22}$$

The crux of the proposition is the estimate for I_2. Without loss of generality, we may assume that $I_2 = \{1,\dots,k\}$ for some $k \leqslant n$. Recall that $t_i = \lambda'(x_i) = h(x_i)$ for each i, and that this is equivalent to having $x_i = \psi'(t_i)$ for each i. Define the probability measure $P_{x_1,\dots,x_k}$ on $(\Omega^k, \mathcal{F}^{\otimes k})$, by

$$P_{x_1,\dots,x_k}(A) := E^k\Big[\mathbf{1}_A \cdot \exp\Big(\sum_{i=1}^k [t_iX_i - \psi(t_i)]\Big)\Big] \tag{4.23}$$

for $A \in \mathcal{F}^{\otimes k}$ (E^k denotes expectation with respect to the product measure P^k on $\mathcal{F}^{\otimes k}$). Direct computation shows that, under $P_{x_1,\dots,x_k}$, the random variables $t_1X_1,\dots,t_kX_k$ are independent, with mean, variance and absolute central third moment given by

$$\begin{aligned} E_{x_1,\dots,x_k}(t_iX_i) &= t_i\psi'(t_i) = t_ix_i,\\ E_{x_1,\dots,x_k}\big(|t_i(X_i-x_i)|^2\big) &= t_i^2\psi''(t_i),\\ E_{x_1,\dots,x_k}\big(|t_i(X_i-x_i)|^3\big) &= |t_i|^3E_{t_i}\big(|X-\psi'(t_i)|^3\big), \end{aligned}$$

respectively. Set $\sigma_i^2 := t_i^2\psi''(t_i)$,

$$s_k^2 := \sum_{i=1}^k E_{x_1,\dots,x_k}\big(|t_i(X_i-x_i)|^2\big) = \sum_{i=1}^k t_i^2\psi''(t_i) = \sum_{i=1}^k \sigma_i^2 \tag{4.24}$$

and

$$S_k := \sum_{i=1}^k t_i(X_i - x_i), \tag{4.25}$$

and let $F_k\colon \mathbb{R} \to \mathbb{R}$ denote the cumulative distribution function of the random variable S_k/s_k under the probability law $P_{x_1,\dots,x_k}$:

$$F_k(x) := P_{x_1,\dots,x_k}(S_k \leqslant xs_k)(x \in \mathbb{R}).$$

Write also μ_k for the probability measure on $\mathbb{R}$ defined by $\mu_k(-\infty,x] := F_k(x)$ ($x \in \mathbb{R}$). Notice that $E_{x_1,\dots,x_k}(S_k/s_k) = 0$ and $\mathrm{Var}_{x_1,\dots,x_k}(S_k/s_k) = 1$.

Lemma 4.6: *The following identity holds*:

$$P^k\Big(\sum_{i=1}^k t_i(X_i - x_i) \geqslant 0\Big) = \Big(\int_{[0,\infty)} e^{-u}\, d\mu_k(u)\Big) \exp\Big(-\sum_{i=1}^k \lambda(x_i)\Big). \tag{4.26}$$

Proof. By definition of the measure $P_{x_1,\dots,x_k}$, we have that

$$\begin{aligned} P^k\Big(\sum_{i=1}^k t_i(X_i - x_i) \geqslant 0\Big) &= P^k(S_k \geqslant 0) \\ &= E_{x_1,\dots,x_k}\Big[\mathbf{1}_{[0,\infty)}(S_k)\cdot \exp\Big(-\sum_{i=1}^k [t_iX_i - \psi(t_i)]\Big)\Big]. \end{aligned}$$

It follows that

$$P^k\Big(\sum_{i=1}^k t_i(X_i - x_i) \geqslant 0\Big) = \int_{[0,\infty)} e^{-u}\, d\mu_k(u)\cdot \exp\Big(\sum_{i=1}^k [\psi(t_i) - t_ix_i]\Big), \tag{4.27}$$

and (4.26) now follows from Lemma 2.6 (ii). ∎

We will also use the following consequence of the Berry–Esseen theorem (cf. [8, Theorem XVI.5,2]).

Lemma 4.7: *For any $a, b > 0$, there exist $k_0 \in \mathbb{N}$ and $\eta > 0$ with the following property*: *if $k \geqslant k_0$, and if $Y_1, \dots, Y_k$ are independent random variables with*

$$\mathbb{E}(Y_i) = 0, \quad \sigma_i^2 := \mathbb{E}(Y_i^2) \geqslant a, \quad \mathbb{E}(|Y_i|^3) \leqslant b,$$

then

$$\mathbb{P}\Big(0 \leqslant \sum_{i=1}^k Y_j \leqslant \boldsymbol{s}_k\Big) \geqslant \eta, \tag{4.28}$$

where $\boldsymbol{s}_k^2 = \sigma_1^2 + \cdots + \sigma_k^2$.

We now consider two cases for I_2. Since $\delta \leqslant x_i \leqslant \alpha - \delta$ for all $i \in I_2$, we can find $A, B > 0$, depending only on δ and μ, such that the random variables $Y_i := t_i(X_i - x_i)$, $i \in I_2$, satisfy

$$\sigma_i^2 = E_{x_1,\dots,x_k}(Y_i^2) = t_i^2\psi''(t_i) \geqslant A \tag{4.29}$$

and

$$E_{x_1,\dots,x_k}\big(|Y_i|^3\big) = |t_i|^3 E_{t_i}\big(|X - \psi'(t_i)|^3\big) \leqslant B \tag{4.30}$$

for all $i \in I_2$ (Lemma 2.9). Let k_0 be the constant from Lemma 4.7 corresponding to A and B, and recall that $|I_2| = k$.

Case 1: $|I_2| < k_0$. Then, working as for I_3, we see that

(4.31)
$$P^n\Big(\sum_{i\in I_2} t_i(X_i - x_i) \geqslant 0\Big) \geqslant \prod_{i\in I_2} P(X_i \geqslant x_i) \geqslant e^{-|I_2|m(\alpha-\delta)} \geqslant e^{-k_0 m(\alpha-\delta)}.$$

Case 2: $|I_2| \geqslant k_0$. We may then apply Lemma 4.7. From Lemma 4.6 we obtain

$$P^n\Big(\sum_{i\in I_2} t_i(X_i - x_i) \geqslant 0\Big) \geqslant e^{-s_k}\mu_k([0, s_k])\exp\Big(-\sum_{i\in I_2}\lambda(x_i)\Big), \tag{4.32}$$

and since

$$s_k^2 = \sum_{i\in I_2} E_{x_1,\ldots,x_k}(Y_i^2) \leqslant \sum_{i\in I_2}\big[E_{x_1,\ldots,x_k}\big(|Y_i|^3\big)\big]^{2/3} \leqslant B^{2/3}k, \tag{4.33}$$

Lemma 4.7 yields

$$P^n\Big(\sum_{i\in I_2} t_i(X_i - x_i) \geqslant 0\Big) \geqslant \eta\exp\Big(-\sum_{i\in I_2}\lambda(x_i) - c_3\sqrt{k}\Big), \tag{4.34}$$

where $c_3 = B^{1/3} > 0$ is a constant depending only on μ and δ. Combining Case 1 and Case 2 we finally obtain the following estimate for P_2:

Lemma 4.8: *We have that*

$$P_2 \geqslant \exp\Big(-\sum_{i\in I_2}\lambda(x_i) - c_3\sqrt{|I_2|} - c_4\Big), \tag{4.35}$$

where the constants $c_3, c_4 \in [0,\infty)$ depend only on δ and μ. □

We can now finish the proof of Proposition 4.2. Collecting the estimates from Lemma 4.4, Lemma 4.5 and Lemma 4.8 and inserting them into (4.13) yields the estimate

$$
\begin{aligned}
P^n\Big(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\Big) \geqslant & P_1P_2P_3 \\
\geqslant & \exp\Big(-\sum_{i\in I_1}\lambda(x_i) - c_1\ln|I_1| - c_2\Big) \\
& \times \exp\Big(-\sum_{i\in I_2}\lambda(x_i) - c_3\sqrt{|I_2|} - c_4\Big) \\
& \times \exp\Big(-(1+\varepsilon)\sum_{i\in I_3}\lambda(x_i)\Big) \\
\geqslant & \exp\Big(-(1+\varepsilon)\sum_{i=1}^n\lambda(x_i) - \varepsilon n\Big),
\end{aligned}
$$

provided $n \geqslant n(\mu,\varepsilon)$ for an appropriate $n(\mu,\varepsilon) \in \mathbb{N}$ depending only on ε and μ. This proves (4.7), and hence the result. ∎

We can now show that if N is "a little larger" than e^{rn}, then $K_N \supseteq \Lambda_r$ with probability close to one; we only have to insert the estimate of Proposition 4.2 into Lemma 4.1:

Lemma 4.9: *Let $0 < r < \lambda(\alpha)$ and $\delta > 0$. Then there exists $n_\mu(r,\delta) \in \mathbb{N}$ such that, if $n \geqslant n_\mu(r,\delta)$ and $N \geqslant \exp((1+\delta)rn + \delta n)$, then*

$$\mathrm{Prob}(K_N \supseteq \Lambda_r) \geqslant 1 - 2^{-n+1}. \tag{4.36}$$

Proof. Let $\delta > 0$. By Lemma 4.1 and Proposition 4.2, there exists n_0 depending only on δ and μ, such that for all $r \in (0,\lambda(\alpha))$ and $n \geqslant n_0$ we have that

$$1 - \mathrm{Prob}(K_N \supseteq \Lambda_r) \leqslant \binom{N}{n}p^{N-n} + 2\binom{N}{n}\big[1 - \exp\big(-rn - \tfrac{1}{2}(r+1)\delta n\big)\big]^{N-n}. \tag{4.37}$$

We first claim that

$$\binom{N}{n}p^{N-n} < 2^{-n} \tag{4.38}$$

for n sufficiently large, $n \geqslant n_1$ say. Indeed, since

$$\binom{N}{n} \leqslant e^{-1}\Big(\frac{eN}{n}\Big)^n, \tag{4.39}$$

in order to prove (4.38), it suffices to check that

$$1+\ln\left(\frac{N}{n}\right)+\frac{N-n}{n}\ln p<-\ln 2. \tag{4.40}$$

Set $x:=N/n$. Then, (4.40) is equivalent to

$$-(x-1)\ln p-\ln x>1+\ln 2. \tag{4.41}$$

The claim follows from the facts that the function on the left-hand side increases to infinity as $x\to\infty$, and $x=N/n\geqslant \exp((1+\delta)rn+\delta n)/n\geqslant e^{\delta n}/n\to\infty$ when $n\to\infty$.

Next we check that

$$2\binom{N}{n}\big[1-\exp\big(-rn-(r+1)\delta n/2\big)\big]^{N-n}<2^{-n} \tag{4.42}$$

for all $n\geqslant n_2$ (some n_2). Since $1-x\leqslant e^{-x}$, and using also (4.39) again, it suffices to check that

$$\left(\frac{2eN}{n}\right)^n\exp\Big(-(N-n)e^{-rn-(r+1)\delta n/2}\Big)<1 \tag{4.43}$$

for $n\geqslant n_2$. Setting $x:=N/n$, we see that (4.43) is equivalent to

$$e^{rn+(r+1)\delta n/2}<\frac{x-1}{1+\ln 2+\ln x}. \tag{4.44}$$

Since $N\geqslant\exp(rn+(1+r)\delta n)$, it is readily verified that the right hand side of (4.44) exceeds $e^{rn+2(r+1)\delta n/3}$ when n is large enough, $n\geqslant n_2(r,\delta)$ say, and hence we get (4.42). (4.37), (4.38) and (4.42) prove the result. ∎

We now have all the ingredients to complete the proof of Theorem 1.1.

Proposition 4.10: *Assume that $m(x)\sim\lambda(x)$ as $x\uparrow\alpha$ and that $\kappa(\mu)<\infty$. Then, for every $\varepsilon>0$,*

$$\lim_{n\to\infty}\inf\{(2\alpha)^{-n}E(|K_N|)\colon N\geqslant\exp((\kappa+\varepsilon)n)\}=1. \tag{4.44}$$

Proof. Fix $\varepsilon>0$. Since $(\kappa+x)(1+x)+x\downarrow\kappa$ as $x\downarrow 0$, we can find $\delta>0$ such that, for $r=\kappa+\delta$ we have that $(1+\delta)r+\delta<\kappa+\epsilon$. For this r Lemma 4.9 shows that, if $n\geqslant n_\mu(r,\delta)$, and if $N\geqslant\exp((\kappa+\varepsilon)n)\geqslant\exp((1+\delta)rn+\delta n)$, then

$$E(|K_N|)\geqslant|\Lambda_r|\cdot\operatorname{Prob}\big(K_N\supseteq\Lambda_r\big)\geqslant|\Lambda_r|(1-2^{-n+1}). \tag{4.45}$$

Since $r > \kappa$, Lemma 3.1 shows that

$$\lim_{n\to\infty} (2\alpha)^{-n}|\Lambda_r| = 1. \tag{4.46}$$

This completes the proof. ∎

5. Proof of Theorem 1.3

Assertion (i) of Corollary 1.3 follows immediately from Lemma 2.7. We next prove assertion (ii) in a special case first. We need only consider the case $\rho > 0$.

Proposition 5.1: *Assume that $P(X \geqslant x) \sim (\alpha - x)^\rho L(\alpha - x)$ as $x \uparrow \alpha$, with $\rho > 0$ and L slowly varying at zero. Then,*

$$\lim_{x\uparrow\alpha} \frac{-\ln P(X \geqslant x)}{\lambda(x)} = 1. \tag{5.1}$$

We shall break up the proof into several lemmas. Set

$$G(x) := P(\alpha - X \leqslant x) = P(X \geqslant \alpha - x) \tag{5.2}$$

and

$$\gamma(t) := E\big(e^{t(X-\alpha)}\big) = \int_0^\infty e^{-tx} dG(x). \tag{5.3}$$

Note that then $\varphi(t) = e^{\alpha t}\gamma(t)$, and that $G(x) \sim x^\rho L(x)$ as $x \downarrow 0$ under the assumptions of Proposition 5.1.

Lemma 5.2: *If $G(x) \sim x^\rho L(x)$ as $x \downarrow 0$, with $\rho > 0$ and L slowly varying at zero, then*

$$\gamma(t) \sim \frac{\Gamma(\rho+1)}{t^\rho} L\left(\frac{1}{t}\right) \quad \text{as } t \uparrow \infty. \tag{5.4}$$

Proof. This is the Tauberian theorem on page 445 of [8] (Theorem 2 in conjunction with Theorem 3). ∎

Lemma 5.3: *If $G(x) \sim x^\rho L(x)$ as $x \downarrow 0$, with $\rho > 0$ and L slowly varying at zero, then $t \mapsto -t\gamma'(t)/\gamma(t),\ t \geqslant 0$, is positive and bounded.*

Proof. By definition of γ we have that

$$-\gamma'(t) = \int_0^\infty xe^{-tx} dG(x) = t\int_0^\infty xe^{-tx} G(x)dx - \int_0^\infty e^{-tx} G(x)dx. \tag{5.5}$$

The first equality immediately shows that $\gamma' < 0$, and hence $-t\gamma'(t)/\gamma(t) > 0$ for $t \geqslant 0$. Next, let $\delta > 0$ be such that

$$\frac{1}{2} \leqslant \frac{G(x)}{x^\rho L(x)} \leqslant 2 \quad \text{for } 0 < x \leqslant \delta. \tag{5.6}$$

Then

$$\begin{aligned} -\gamma'(t) &\leqslant 2t \int_0^\delta x^{\rho+1} e^{-tx} L(x)\, dx + t \int_\delta^\infty x e^{-tx} dx \\ &= \frac{2}{t^{\rho+1}} \int_0^{t\delta} x^{\rho+1} e^{-x} L(x/t)\, dx + \left(\delta + t^{-1}\right) e^{-t\delta}. \end{aligned} \tag{5.7}$$

Since L varies slowly at zero, we may, without loss, assume that

$$L\left(\frac{1}{t}\right) = a(t) \exp\left(\int_{1/\delta}^t \frac{b(s)}{s}\, ds \right) \quad (t > 1/\delta), \tag{5.8}$$

with $b(t) \to 0$ and $a(t) \to c$ as $t \to \infty$, $0 < c < \infty$, and a, b measurable and bounded on finite intervals [8, Corollary on page 282]. It follows from (5.7) and (5.8) that

$$-\gamma'(t) \leqslant 2 \frac{L(1/t)}{t^{\rho+1}} \int_0^\infty x^{\rho+1} e^{-x} A e^{\epsilon |\ln x|} dx + \left(\delta + t^{-1}\right) e^{-t\delta} \tag{5.9}$$

for t sufficiently large, with $A < \infty$ and $\epsilon < \rho + 1$, and hence

$$-\gamma'(t) \leqslant C \frac{L(1/t)}{t^{\rho+1}} \tag{5.10}$$

for all sufficiently large t, with $C < \infty$. The result follows by combining (5.10) with Lemma 5.2. ∎

Lemma 5.4: *If $G(x) \sim x^\rho L(x)$ as $x \downarrow 0$, with $\rho > 0$ and L slowly varying at zero, then*

$$-t \frac{\gamma'(t)}{\gamma(t)} \to \rho \quad (t \uparrow \infty). \tag{5.11}$$

Proof. We follow the proof of the Lemma on page 446 of [8] (see also [3, Theorem 1.7.2]). Suppose, to obtain a contradiction, that for some $\epsilon > 0$ and some sequence $t_n \uparrow \infty$

$$\left| -t_n \frac{\gamma'(t_n)}{\gamma(t_n)} - \rho \right| > \epsilon \quad \text{for all } n \in \mathbb{N}. \tag{5.12}$$

Observe that $\gamma''(t) = \int_0^\infty x^2 e^{-tx} dG(x) > 0$. Therefore, the functions $g_t(s) := t\gamma'(ts)/\gamma(t)$, $s > 0$, are non-decreasing. By Lemma 5.3, $g_t(s)$ is also bounded in t for each fixed $s > 0$. It follows from Helly's selection theorem that there exists a subsequence $(t_{n_k})_{k\geqslant 1}$ of $(t_n)_{n\geqslant 1}$, and a right-continuous, non-decreasing function $g\colon (0,\infty) \to \mathbb{R}$, such that $g_{t_{n_k}}(s) \to g(s)$ at points of continuity of g.

For $0 < a < b$,

$$\frac{\gamma(ta) - \gamma(tb)}{\gamma(t)} = \int_a^b -\frac{\gamma'(ts)t}{\gamma(t)}\,ds = \int_a^b -g_t(s)\,ds. \tag{5.13}$$

The left-hand side in (5.13) tends to $a^{-\rho} - b^{-\rho}$ as $t \uparrow \infty$, by Lemma 5.2, while the right-most side tends to $\int_a^b -g(s)\,ds$ as t runs through the sequence $(t_{n_k})_{k\geqslant 1}$, by the bounded convergence theorem. As this is true for any $0 < a < b$, we must have that $g(s) = -\rho s^{-\rho-1}$ almost everywhere on $(0,\infty)$, and as g is right continuous, this equality must after all prevail everywhere on $(0,\infty)$. For $s = 1$ we then have that

$$-t_{n_k}\frac{\gamma'(t_{n_k})}{\gamma(t_{n_k})} \to \rho \quad (k \uparrow \infty), \tag{5.14}$$

which contradicts (5.12). ∎

The following Corollary implies Proposition 5.1:

Corollary 5.5: *If $G(x) \sim x^\rho L(x)$ as $x \downarrow 0$, with $\rho > 0$ and L slowly varying at zero, then*

$$\lim_{x\uparrow\alpha} e^{\lambda(x)} P(X \geqslant x) = \frac{(\rho/e)^\rho}{\Gamma(\rho+1)}. \tag{5.15}$$

Proof. By Lemma 2.3 and Lemma 2.6 it suffices to show that

$$\lim_{t\uparrow\infty} e^{t\psi'(t)-\psi(t)} P(X \geqslant \psi'(t)) = \frac{(\rho/e)^\rho}{\Gamma(\rho+1)}. \tag{5.16}$$

From the relation $\varphi(t) = e^{\alpha t}\gamma(t)$ it follows that $\psi(t) = \alpha t + \ln\gamma(t)$ and $\psi'(t) = \alpha + \gamma'(t)/\gamma(t)$. Therefore (5.16) is equivalent to

$$\lim_{t\uparrow\infty} e^{t\gamma'(t)/\gamma(t)-\ln\gamma(t)} G\big(-\gamma'(t)/\gamma(t)\big) = \frac{(\rho/e)^\rho}{\Gamma(\rho+1)}. \tag{5.17}$$

This however, follows immediately from Lemma 5.2, Lemma 5.4, our assumption that $G(x) \sim x^\rho L(x)$ as $x \downarrow 0$ and the representation (5.8) for L. ∎

We now show how to modify the above arguments in order to prove Theorem 1.3 (ii) in the general case.

PROPOSITION 5.6: *Assume $P(X \geqslant x) \approx (\alpha - x)^\rho L(\alpha - x)$ as $x \uparrow \alpha$, with $\rho > 0$ and L slowly varying at zero. Then,*

$$\lim_{x \uparrow \alpha} \frac{-\ln P(X \geqslant x)}{\lambda(x)} = 1. \tag{5.18}$$

Proof. First, choose $\delta > 0$ and $c_1, c_2 \in (0, \infty)$ so that

$$c_1 \leqslant \frac{G(x)}{x^\rho L(x)} \leqslant c_2 \quad \text{for } 0 < x \leqslant \delta. \tag{5.19}$$

Then, as in (5.7),

$$\frac{c_1}{t^\rho} \int_0^{t\delta} x^\rho e^{-tx} L(x/t)\, dx \leqslant \gamma(t) \leqslant \frac{c_2}{t^\rho} \int_0^{t\delta} x^\rho e^{-tx} L(x/t)\, dx + \int_\delta^\infty x e^{-tx} dx, \tag{5.20}$$

and, using the representation (5.8), one sees that

$$\gamma(t) \approx \frac{\Gamma(\rho+1)}{t^\rho} L\left(\frac{1}{t}\right) \quad (t \uparrow \infty); \tag{5.21}$$

in particular, $\ln \gamma(t) \sim -\rho \ln t + \ln L(1/t)$ $(t \uparrow \infty)$, and since $\ln L(1/t) = o(\ln t)$, by the representation (5.8), it follows that

$$\ln \gamma(t) \sim -\rho \ln t \quad (t \uparrow \infty). \tag{5.22}$$

We now claim that

$$0 < \liminf_{t \to \infty} -t \frac{\gamma'(t)}{\gamma(t)} \leqslant \limsup_{t \to \infty} -t \frac{\gamma'(t)}{\gamma(t)} < \infty. \tag{5.23}$$

This and the assumption that $G(x) \approx x^\rho L(x)$ as $x \downarrow 0$ imply that

$$-\ln G\big(-\gamma'(t)/\gamma(t)\big) \sim \rho \ln t \quad (t \uparrow \infty). \tag{5.24}$$

By (5.22), (5.23) and (5.24), we then have that

$$\lim_{t \uparrow \infty} \frac{-\ln G\big(-\gamma'(t)/\gamma(t)\big)}{t\gamma'(t)/\gamma(t) - \ln \gamma(t)} = 1, \tag{5.25}$$

which, as explained in Corollary 5.5, implies the result.

It remains to show (5.23). As in Lemma 5.3, one sees that

$$0 \leqslant -t \frac{\gamma'(t)}{\gamma(t)} \leqslant C \tag{5.26}$$

for all t, for some $C < \infty$. From this, the right-most inequality in (5.23) follows at once. Assume next that, for some sequence $t_n \uparrow \infty$, we have that

$$-t_n \frac{\gamma'(t_n)}{\gamma(t_n)} \to 0 \quad (n \to \infty). \tag{5.27}$$

Using (5.26), we see as in Lemma 5.4 that there exist a non-decreasing, right-continuous function $g\colon (0,\infty) \to \mathbb{R}$, such that $-t\gamma'(ts)/\gamma(t) \to -g(s)$ as t tends to infinity along a subsequence $(t_{n_k})_{k\geqslant 1}$ of $(t_n)_{n\geqslant 1}$. Since $-\gamma$ is decreasing, (5.27) implies that

$$-t_{n_k}\frac{\gamma'(st_{n_k})}{\gamma(t_{n_k})} \leqslant -t_{n_k}\frac{\gamma'(t_{n_k})}{\gamma(t_{n_k})} \to 0 \quad (k \uparrow \infty) \tag{5.28}$$

for $s > 1$. By (5.13) and the bounded convergence theorem we then conclude that

$$\frac{\gamma(at_{n_k}) - \gamma(bt_{n_k})}{\gamma(t_{n_k})} \to 0 \quad (k \uparrow \infty) \tag{5.29}$$

for any $1 < a < b$. By (5.21) however, there exist two constants $0 < C_1 < C_2 < \infty$ such that

$$\liminf_{t\uparrow\infty} \frac{\gamma(at_{n_k}) - \gamma(bt_{n_k})}{\gamma(t_{n_k})} \geqslant \frac{C_1}{C_2}\,\frac{1}{a^\rho} - \frac{C_2}{C_1}\,\frac{1}{b^\rho}, \tag{5.30}$$

and this, for $b/a > (C_2/C_1)^{2/\rho}$, contradicts (5.29). ∎

To prove assertion (iii) of Theorem 1.3, we shall use the following lemma. Set

$$U(t) := -\ln\gamma(t) \quad (t \geqslant 0), \tag{5.31}$$

and

$$u(t) := U'(t) = -\frac{\gamma'(t)}{\gamma(t)} \quad (t > 0). \tag{5.32}$$

The function U is positive, increasing and concave, with $U(0) = 0$. It follows that the function $t \mapsto tu(t)/U(t)$ is bounded between 0 and 1, since

$$U(t) = \int_0^t u(s)\,ds \geqslant tu(t) \tag{5.33}$$

for all $t \geqslant 0$. The following lemma is proved just like Lemma 5.4 (it is the Lemma on page 446 of [8]).

Lemma 5.7: *Assume that $U(t) \sim t^r L(1/t)$ as $t \uparrow \infty$, with $r \in [0,1)$ and L slowly varying at zero. Then*

$$t\,\frac{u(t)}{U(t)} \to r \quad (t \uparrow \infty). \tag{5.34}$$ ∎

We shall also need the following Lemma, which is one direction of a Tauberian theorem of de Bruijn [3, Theorem 4.12.9]. (We decided to retain the proof of the Lemma for the convenience of the reader).

LEMMA 5.8: *If* $-\ln G(x) \sim \theta x^{-\rho}$ *as* $x \downarrow 0$, *with* $\theta, \rho > 0$, *then*

$$U(t) \sim (\rho+1)\theta^{1/(\rho+1)}\,(t/\rho)^{\rho/(\rho+1)} \quad (t \uparrow \infty). \tag{5.35}$$

Proof. Given $\epsilon > 0$ have $\delta > 0$ such that

$$\exp(-(1+\epsilon)\theta/x^\rho) \leqslant G(x) \leqslant \exp(-(1-\epsilon)\theta/x^\rho) \quad \text{for } 0 < x \leqslant \delta. \tag{5.36}$$

Since $\gamma(t) = t\int_0^\infty e^{-tx}G(x)\,dx$, by (5.3), (5.36) implies that

$$t\int_0^\delta \exp(-tx-(1+\epsilon)\theta/x^\rho)dx \leqslant \gamma(t) \leqslant t\int_0^\delta \exp(-tx-(1-\epsilon)\theta/x^\rho)dx + e^{-t\delta}. \tag{5.37}$$

Set $x_\pm = \big((1\pm\epsilon)\theta\rho/t\big)^{1/(\rho+1)}$. The function $x \mapsto -tx-(1\pm\epsilon)\theta/x^\rho$ is increasing on $(0, x_\pm)$ and decreasing on $(x_\pm, \infty)$. It follows that

$$t\int_0^{x_-} \exp(-tx-(1-\epsilon)\theta/x^\rho)\,dx \leqslant tx_- \exp(-tx_- -(1-\epsilon)\theta/x_-^\rho), \tag{5.38}$$

while integration by parts shows that

$$t\int_{x_-}^\infty \exp(-tx-(1-\epsilon)\theta/x^\rho)\,dx \leqslant \exp(-tx_- -(1-\epsilon)\theta/x_-^\rho) \times\left(1+\int_{x_-}^\infty \frac{(1-\epsilon)\theta\rho}{x^{\rho+1}}\,dx\right). \tag{5.39}$$

It follows from (5.37), (5.38) and (5.39) that

$$\gamma(t) \leqslant p(t)\exp(-(\rho+1)[(1-\epsilon)\theta]^{1/(\rho+1)}(t/\rho)^{\rho/(\rho+1)}) + e^{-t\delta}, \tag{5.40}$$

with

$$p(t) = 1 + (\rho+1)[(1-\epsilon)\theta]^{1/(\rho+1)}(t/\rho)^{\rho/(\rho+1)}. \tag{5.41}$$

For t sufficiently large on the other hand,

$$t\int_0^\delta \exp(-tx-(1+\epsilon)\theta/x^\rho)\,dx \geqslant t\int_{x_+}^\delta \exp(-tx-(1+\epsilon)\theta/x^\rho)\,dx, \tag{5.42}$$

and since $x \mapsto -(1+\epsilon)\theta/x^\rho$ is increasing, this yields

$$\begin{aligned} t\int_0^\delta &\exp(-tx-(1+\epsilon)\theta/x^\rho)\,dx \\ &\geqslant \exp(-(1+\epsilon)\theta/x_+^\rho)\int_{x_+}^\delta te^{-tx}dx \\ &= (1-e^{-t(\delta-x_+)})\,\exp(-(1+\epsilon)\theta/x_+^\rho - tx_+). \end{aligned} \tag{5.43}$$

(5.37) and (5.43) now give the estimate

$$\gamma(t) \geqslant (1 - e^{-t(\delta - x_+)}) \exp(-(\rho+1)[(1+\epsilon)\theta]^{1/(\rho+1)}(t/\rho)^{\rho/(\rho+1)}). \tag{5.44}$$

(5.40) and (5.44) yield the result. ∎

We can now conclude the proof of Theorem 1.3 (iii):

Proposition 5.9: *If* $-\ln P(X \geqslant x) \sim \theta(\alpha - x)^{-\rho}$ *as* $x \uparrow \alpha$, *with* $\theta, \rho > 0$, *then*

$$\lim_{x\uparrow\alpha} \frac{-\ln P(X \geqslant x)}{\lambda(x)} = 1. \tag{5.45}$$

Proof. From Lemma 5.7 and Lemma 5.8 we conclude that

$$\lim_{x\downarrow 0} \frac{-\ln G\big(-\gamma'(t)/\gamma(t)\big)}{t\gamma'(t)/\gamma(t) - \ln\gamma(t)} = 1. \tag{5.46}$$

As in the proof of Corollary 5.5 however, this is the same as assertion (5.45). ∎

6. Concluding Remarks

1) From the proof of Theorem 1.1 it is evident that the assumption for the measure μ to be symmetric is inessential, and was only used to simplify the exposition. If, instead, we only assume that μ is compactly supported, and set

$$\beta = \beta(\mu) := \inf\{x \in \mathbb{R} \colon \mu((-\infty, x]) > 0\} \tag{6.1}$$

to be the left endpoint of the support of μ, then Theorem 1.1 is still valid, provided we supplement (1.10) with the condition that the measure μ also satisfies

$$\lim_{x\downarrow\beta} \frac{-\ln P(X \leqslant x)}{\lambda(x)} = 1. \tag{6.2}$$

One then also has the obvious analogue of Theorem 1.3 relating to condition (6.2).

2) We close with an example of a compactly supported distribution μ which does not satisfy (1.10). Let λ_1 denote the Legendre transform of the log-moment generating function $\psi_1(t) = \ln(\sinh t) - \ln t$ of the uniform distribution μ_1 on $[-1, 1]$. By Theorem 1.3 (ii),

$$\lambda_1(x) \sim -\ln\mu_1([x,1]) \sim -\ln(1-x) \quad \text{as } x \uparrow 1. \tag{6.3}$$

Define λ_2 analogously, with μ_2 the symmetric distribution on $[-1,1]$ given by $\mu_2([x,1]) := \frac{1}{2}e^{1-1/\sqrt{1-x}}$ for $0 < x < 1$. By Theorem 1.3 (iii) this time,

$$\lambda_2(x) \sim -\ln \mu_2([x,1]) \sim \frac{1}{\sqrt{1-x}} \quad \text{as } x \uparrow 1. \tag{6.4}$$

Each λ_i is a strictly increasing function on $(0,1)$ with $\lambda_i((0,1)) = (0,\infty)$ $(i=1,2)$. Thus there exists $x_1 > 0$ such that $\lambda_2(x_1) = \ln 2$. Define the sequence x_n inductively, by defining x_{n+1} to be the unique number in $(0,1)$ satisfying $\lambda_1(x_{n+1}) = \lambda_2(x_n)$ for $n \in \mathbb{N}$. Now define the measure μ as follows. Set

$$m(x) = -\ln \mu([x,1]) := \lambda_1(x_{n+1}) \quad \text{for } x_n < x \leqslant x_{n+1},\ n \in \mathbb{N}, \tag{6.5}$$

and $m(x) = -\ln \mu([x,1]) := \lambda_2(x_1) = \ln 2$ for $0 \leqslant x \leqslant x_1$; this defines a purely atomic measure on $[0,1]$, with atoms at the points $x_2, x_3, \ldots,$ with

$$\mu(\{x_n\}) = e^{-\lambda_1(x_n)} - e^{-\lambda_1(x_{n+1})} \quad (n \geqslant 2), \tag{6.6}$$

and with total mass equal to $e^{-\lambda_1(x_2)} = e^{-\lambda_2(x_1)} = 1/2$, which therefore extends uniquely to a symmetric probability measure μ on $[-1,1]$. By Proposition 2.6,

$$m(x) \geqslant \lambda(x) \tag{6.7}$$

for all $x \in (0,1)$, where λ is the Legendre transform corresponding to μ. By the convexity of λ then,

$$\begin{aligned}\lambda(sx_n + (1-s)x_{n+1}) &\leqslant s\lambda(x_n) + (1-s)\lambda(x_{n+1}) \leqslant sm(x_n) + (1-s)m(x_{n+1}) \\ &= s\lambda_1(x_n) + (1-s)\lambda_1(x_{n+1}) \\ &= s\lambda_1(x_n) + (1-s)\lambda_2(x_n),\end{aligned}$$

for any $s \in (0,1)$. Therefore,

$$\begin{aligned}\frac{m(sx_n + (1-s)x_{n+1})}{\lambda(sx_n + (1-s)x_{n+1})} &= \frac{\lambda_1(x_{n+1})}{\lambda(sx_n + (1-s)x_{n+1})} = \frac{\lambda_2(x_n)}{\lambda(sx_n + (1-s)x_{n+1})} \\ &\geqslant \frac{\lambda_2(x_n)}{s\lambda_1(x_n) + (1-s)\lambda_2(x_n)},\end{aligned}$$

and hence by (6.3)–(6.4),

$$\varliminf_{n\to\infty} \frac{m(sx_n + (1-s)x_{n+1})}{\lambda(sx_n + (1-s)x_{n+1})} \geqslant \frac{1}{1-s}$$

for any $s \in (0,1)$. In fact, by choosing $s_n = 1 - n^{-1}$, we see that along the sequence $y_n := (1-n^{-1})x_n + n^{-1}x_{n+1}$ we have that $m(y_n)/\lambda(y_n) \to \infty$.

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