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POWERS OF RATIONALS MODULO 1 AND RATIONAL BASE NUMBER SYSTEMS*

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ABSTRACT

A new method for representing positive integers and real numbers in a rational base is considered. It amounts to computing the digits from right to left, least significant first. Every integer has a unique expansion. The set of expansions of the integers is not a regular language but nevertheless addition can be performed by a letter-to-letter finite right transducer. Every real number has at least one such expansion and a countable infinite number of them have more than one. We explain how these expansions can be approximated and characterize the expansions of reals that have two expansions.

The results that we derive are pertinent on their own and also as they relate to other problems in combinatorics and number theory. A first example is a new interpretation and expansion of the constant $K(p)$ from the so-called "Josephus problem." More important, these expansions in the base $\frac{p}{q}$ allow us to make some progress in the problem of the distribution of the fractional part of the powers of rational numbers.

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1. Introduction

The distribution modulo 1 of the powers of a rational number, indeed the problem of proving whether they form a dense set or not, is a frustrating question: "This very old problem of Pisot and Vijayaraghavan is still unanswered" writes Michel Mendés France in [16] and he goes on: "Pisot, Vijayaraghavan and André Weil did however show that there are infinitely many limit points" (cf. [25], for instance.) With this problem as a background, Mahler asked in [15] whether there exists a nonzero real z such that the fractional part of $z(3/2)^n$ for $n = 0, 1, \ldots$ fall into $[0, 1/2[$. It is not known whether such a real, called Z-number, does exist but Mahler showed that the set of Z-numbers is at most countable. His proof is based on the fact that the fractional part of a Z-number (if it exists) has an expansion in base $3/2$ which is entirely determined by its integral part.

In this paper, we introduce and study a new method for representing positive integers and real numbers in the base $\frac{p}{q}$, where $p > q \geqslant 2$ are coprime integers. While this new method does not solve Mahler's original problem, it sheds a new light on the question and allows us to make some progress on the commonly studied generalization of Mahler's problem — as we explain at the end of this introduction.

The idea of nonstandard representation systems of numbers is far from being original and there have been extensive studies of these, from a theoretical standpoint as well as for improving computation algorithms. It is worth (briefly) recalling first the main features of these systems in order to clearly put in perspective and in contrast the results we have obtained on rational base systems.

Many nonstandard numeration systems have been considered in the literature: [13, Vol. 2, Chap. 4] or [14, Chap. 7], for instance, give extensive references. Representation in integer base with signed digits was popularized in computer arithmetic by Avizienis [2] and can even be found earlier in a work of Cauchy [4]. When the base is a noninteger real number $\beta > 1$, any nonnegative real number is given an expansion on the canonical alphabet $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ by the greedy algorithm of Rényi [21]; a number may have several β-representations on the canonical alphabet, but the greedy one is the greatest in the lexicographical order. The set of greedy β-expansions of numbers of $[0, 1[$ is shift-invariant, and its closure forms a symbolic dynamical system called the **β-shift**. The properties of the β-shift are well-understood, using the so-called "β-expansion of 1", see [18, 14].

When β is a Pisot number[1], the β number system shares many properties with the integer base case: the set of greedy representations is recognizable by a finite automaton; the conversion between two alphabets of digits (in particular addition) is realized by a finite transducer [10].

In this work, we first define the **$\frac{p}{q}$-expansion** of an integer N: it is a way of writing N in the **base $\frac{p}{q}$** by an algorithm which produces least significant digits first. We prove

THEOREM 1: *Every non-negative integer N has a* **$\frac{p}{q}$-expansion** *which is an* integer *representation. It is the* **unique finite $\frac{p}{q}$-representation** *of N.*

The $\frac{p}{q}$-expansions are not the $\frac{p}{q}$-representations that would be obtained by the classical "greedy algorithm" in base $\frac{p}{q}$. They are written on the alphabet $A = \{0, 1, \ldots, p-1\}$, and not every word of A^* is admissible. These $\frac{p}{q}$-expansions share some properties with the expansions in an integer base — digit set conversion is realized by a finite automaton, for instance — and are completely different as far as other aspects are concerned. Above all, the set $L_{\frac{p}{q}}$ of all $\frac{p}{q}$-expansions is not a regular language (not even a context-free one).

By construction, the set $L_{\frac{p}{q}}$ is prefix-closed and any element can be extended (to the right) in $L_{\frac{p}{q}}$. Hence, $L_{\frac{p}{q}}$ is the set of labels of the finite paths in an **infinite subtree $T_{\frac{p}{q}}$** of the infinite full p-ary tree of the free monoid A^*. The tree $T_{\frac{p}{q}}$ contains a maximal infinite word $\mathbf{t}_{\frac{p}{q}}$ — maximal in the lexicographic ordering — whose numerical value is $\boldsymbol{\omega}_{\frac{p}{q}}$. We consider the set of infinite words $W_{\frac{p}{q}}$, subset of $A^{\mathbb{N}}$, that label the infinite paths of $T_{\frac{p}{q}}$ as the admissible $\frac{p}{q}$-expansions of real numbers and we prove

THEOREM 2: *Every real in $[0, \boldsymbol{\omega}_{\frac{p}{q}}]$ has exactly one $\frac{p}{q}$-expansion, but for an infinite countable subset of reals which have more than one such expansion.*

If $p \geqslant 2q-1$ then no real has more than two $\frac{p}{q}$-expansions. It is noteworthy as well that no $\frac{p}{q}$-expansion is eventually periodic and thus in particular — and in contrast with the expansion of reals in an integer base — no $\frac{p}{q}$-expansion ends with 0^ω or, which is the same, is finite. This is a very remarkable feature

[1] An algebraic integer > 1 whose Galois conjugates are all less than 1 in modulus.

of the $\frac{p}{q}$ number system for reals and we explain how the $\frac{p}{q}$-expansion of a real number can be computed (in fact, approximated).

We shall give here two examples of the relations of the $\frac{p}{q}$-expansions of reals with other problems in combinatorics and number theory. The first one is the so-called "Josephus problem" in which a certain constant $K(p)$ is defined (cf. [17, 11, 24]) which is a special case of our constant $\boldsymbol{\omega}_{\frac{p}{q}}$ (with $q = p - 1$) and this definition yields a new method for computing $K(p)$.

The connection with the second problem, namely, the distribution of the powers of a rational number modulo 1 with which we opened this introduction, is even more striking. In order to describe this connection, let us first set the framework of this deeply intriguing problem.[2]

Koksma proved that for almost every real number $\theta > 1$ the sequence $\{\theta^n\}$ is uniformely distributed in $[0,1]$, but very few results are known for specific values of θ. One of these is that if θ is a Pisot number, then the above sequence converges to 0 if we identify $[0,1[$ with $\mathbb{R}/\mathbb{Z}$.

Experimental results show that the distribution of $\left\{\left(\frac{p}{q}\right)^n\right\}$ for coprime positive integers $p > q \geqslant 2$ looks more "chaotic" than the distribution of the fractional part of the powers of a transcendental number like e or π (cf. [26]).

The next step in attacking this problem has been to fix the rational $\frac{p}{q}$ and to study the distribution of the sequence

$$f_n(z) = \left\{ z \left(\frac{p}{q}\right)^n \right\}$$

according to the value of the Once again, the sequence $f_n(z)$ is uniformly distributed for almost all $z > 0$, but nothing is known for specific values of z.

In the search for z's for which the sequence $f_n(z)$ is not uniformly distributed, and as already explained, Mahler considered those for which the sequence is eventually contained in $\left[0, \frac{1}{2}\right[$. Mahler's notation is generalized as follow: let I be a (strict) subset of $\left[0,1\right[$ — indeed I will be a finite union of semi-closed intervals — and write

$$\mathbf{Z}_{\frac{p}{q}}(I) = \left\{ z \in \mathbb{R} : \left\{ z\left(\frac{p}{q}\right)^n \right\} \text{ stays eventually in } I \right\}.$$

Mahler's problem is to ask whether $\mathbf{Z}_{\frac{3}{2}}\left(\left[0, \frac{1}{2}\right[\right)$ is empty or not.

[2] This presentation is based on the introduction of [3]. The fractional part of a number x is denoted by $\{x\}$.

Mahler's work has been developed in two directions: the search for subsets I as large as possible such that $\mathbf{Z}_{\frac{p}{q}}\big(I\big)$ is empty and conversely the search for subsets I as small as possible such that $\mathbf{Z}_{\frac{p}{q}}\big(I\big)$ is nonempty.

Along the first line, remarkable progress has been made by Flatto et al. ([8]) who proved that the set of reals s such that $\mathbf{Z}_{\frac{p}{q}}\big(\big[s, s+\frac{1}{p}\big[\big)$ is empty, is **dense** in $[0, 1-\frac{1}{p}]$. Recently Bugeaud [3] proved that its complement is of Lebesgue measure 0. Along the other line, Pollington [20] showed that $\mathbf{Z}_{\frac{3}{2}}\big(\big[\frac{4}{65}, \frac{61}{65}\big[\big)$ is nonempty.

Our contribution to the problem can be seen as an improvement of this last result.

Theorem 3: *If $p \geqslant 2q-1$, there exists a subset $Y_{\frac{p}{q}}$ of $\big[0, 1\big[$, of Lebesgue measure $\frac{q}{p}$, such that $\mathbf{Z}_{\frac{p}{q}}\big(Y_{\frac{p}{q}}\big)$ is countably infinite.*

The elements of $\mathbf{Z}_{\frac{p}{q}}\big(Y_{\frac{p}{q}}\big)$ are indeed the reals which have two $\frac{p}{q}$-expansions (cf. Theorem 49) and this is the reason why the consideration of the $\frac{p}{q}$ number system allowed to make some progress in Mahler's problem. Coming back to the historical 3/2 case, we have

Corollary 4: *The set of positive numbers z such that*

$$\left\{ z\left(\frac{3}{2}\right)^n \right\} \in [0, 1/3[\cup[2/3, 1[\quad \textit{for } n = 0, 1, 2, \ldots$$

is countably infinite.

It is noteworthy that the expansion 'computed' by Mahler for his Z-numbers happens to be exactly one of our $\frac{3}{2}$-expansions — if it exists. Another way to state Corollary 4 is the following. Let us denote by $\|x\|$ the distance between x and the closest integer. Corollary 4 assures that there are (infinitely many) positive numbers x such that $\|x(3/2)^n\| < 1/3$ for $n = 0, 1, \ldots$. This is to be compared with a recent result of Dubickas [6] who showed that $\|x(3/2)^n\| < 0.238117\ldots (n = 0, 1, \ldots)$ implies that $x = 0$ — hence extending his result [5] on the distribution of $\{x\alpha^n\}$ which works basically for any algebraic number α. Though there is a distance between $1/3$ and $0.238117\ldots$, we expect that our $Y_{\frac{p}{q}}$ is minimal in the sense that for any proper subset X of $Y_{\frac{p}{q}}$ which is a finite union of half open intervals, $\mathbf{Z}_{\frac{p}{q}}\big(X\big)$ is empty, an even stronger statement than Mahler's conjecture (cf. (17) at the very end of this paper). Further study on the connection between Mahler's problem and $\frac{p}{q}$ number system is carried out in [1].

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We have introduced and studied here a fascinating object which can be seen from many sides, which raises many difficult questions and whose further study will certainly mix techniques from word combinatorics, automata theory, and number theory.

2. Preliminaries

2.1. Finite and infinite words. An **alphabet** A is a finite set. Here we consider alphabets of digits, that is, subsets of the integers. A finite sequence of elements of A is called a **word**, and the set of words on A, equipped with concatenation is the free monoid A^*. The **empty word**, denoted by ε, is the identity element of A^*.

The **length** of a word w is equal to the length of the **sequence** w and is denoted by $|w|$. The set of words on A of length n (resp., of length smaller than or equal to n) is denoted by A^n (resp., $A^{\leqslant n}$); the concatenation of w repeated n times is denoted by w^n. A word u is a **factor** of a word w if there exist words x and y such that $w = x\,u\,y$. If x (resp., y) is the empty word, then u is a **prefix** (resp., a **suffix**) of w. A subset of A^* is **prefix-closed** (resp., **suffix-closed**) if it contains all prefixes (resp., all suffixes) of any of its elements.

Let us suppose that A is ordered by a **total** order written $\leqslant$, which is rather natural as our alphabets are subsets of $\mathbb{N}$ or $\mathbb{Z}$. The set A^* is totally ordered by the **radix order** $\preccurlyeq$ defined as follows[3]: $v \prec w$ if $|v| < |w|$, or $|v| = |w|$ and there exist letters $a < b$ such that $v = u\,a\,v'$ and $w = u\,b\,w'$. The set A^* is also totally ordered by the **lexicographic order** $\sqsubseteq$ defined as follows: $v \sqsubset w$ if v is a prefix of w, or there exist letters $a < b$ such that $v = u\,a\,v'$ and $w = u\,b\,w'$. The radix order is a well order whereas the lexicographic order is not (if A has more than one letter). Both orders coincide for pair of words of equal length.

An **infinite word** over A is an infinite sequence of elements of A. In this work, infinite words can be indexed by positive, or negative, integers, depending on the context; in both cases, we denote by $A^{\mathbb{N}}$ the set of infinite words on A

[3] It is easier to describe the nonreflexive part of the order.

and whenever it is possible we denote infinite words by bold letters. The prefix of length n of $\mathbf{a}$ is denoted $\mathbf{a}_{[n]}$, but its n-th letter is more lightly written a_n.

The **lexicographic order** is defined on $A^{\mathbb{N}}$ as follows: $\mathbf{a} \sqsubset \mathbf{b}$ if there exist letters $a < b$ such that $\mathbf{a} = u\,a\,\mathbf{a}'$ and $\mathbf{b} = u\,b\,\mathbf{b}'$. An infinite word is said to be **eventually periodic** if it is of the form $u\,v^{\omega} = u\,v\,v\,v\,v\,v\,\cdots$ where u and v belong to A^*.

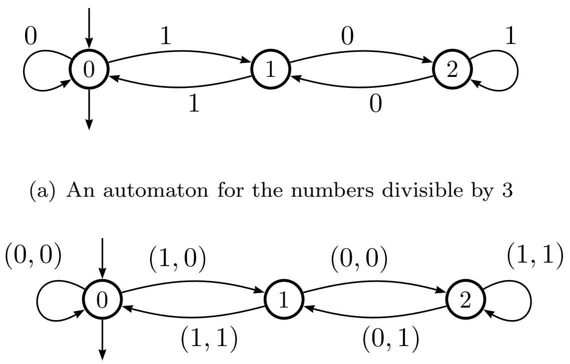
The set $A^{\mathbb{N}}$ is equipped with the distance δ defined by: if $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ and $\mathbf{b} = (b_n)_{n\in\mathbb{N}}$, then $\delta(\mathbf{a},\mathbf{b}) = 2^{-r}$ if $\mathbf{a} \neq \mathbf{b}$ and $r = \min\{n : a_n \neq b_n\}$, and $\delta(\mathbf{a},\mathbf{b}) = 0$ if $\mathbf{a} = \mathbf{b}$. The topology on the set $A^{\mathbb{N}}$ is then the product topology (of the discrete topology on A), and it makes $A^{\mathbb{N}}$ a compact metric space.

2.2. Automata and transducers. An automaton on A, $\mathcal{A} = \langle\, Q, A, E, I, T\,\rangle$, is a labeled graph: Q is the set of vertices, traditionally called **states**; I and T are two subsets of Q, the sets of **initial** and **final** states respectively; and E, the set of edges, traditionally called **transitions**, labeled in A, is (or can be seen as) a subset of $Q \times A \times Q$. The **transposed** of $\mathcal{A}$ is the automaton $\mathcal{A}^{\mathrm{t}} = \langle\, Q, A, E^{\mathrm{t}}, T, I\,\rangle$ where (q,a,p) is in E^{t} if, and only if, (p,a,q) is in E. The automaton $\mathcal{A}$ is **deterministic** if it has only one initial state, and if for every pair (p,a) in $Q \times A$, there exists at most one q such that (p,a,q) is in E; the automaton $\mathcal{A}$ is **co-deterministic** if its transposed is deterministic. A state q is **accessible** if there exists a path from an initial state to q, the **accessible part** of $\mathcal{A}$ is the subgraph induced by the set of accessible states.

A **successful** path in $\mathcal{A}$ is a path whose origin is in I and its end in T. A word in A^* is **accepted** by $\mathcal{A}$ if it is the label of a successful path. Figure 1 (a) shows how automata are depicted; in particular, initial states are marked with incoming arrows and final states with outgoing arrows.

An automaton (on a finite alphabet) is **finite** if it has a finite set of states. A language on A, that is, a subset of A^*, is **regular** if it is the set of words accepted by a finite automaton on A.

Indeed, we shall consider automata whose transitions are labeled in $A^* \times B^*$ — where B is another alphabet — rather than in A, and which we call **transducers**. Pairs of words are multiplied **component wise**, that is, $A^* \times B^*$ is a monoid, and the label of a (successful) path in such a transducer is a pair of words. A transducer realizes then a **relation** from A^* into B^*. Figure 1 (b) shows a transducer $\mathcal{Q}$ that realizes the integral division by 3 on binary representations of numbers: (f,g) is accepted by $\mathcal{Q}$ if f is the binary representation of



(a) An automaton for the numbers divisible by 3

(b) A transducer for the quotient by 3

Figure 1. An automaton on $\{0,1\}$ and another one on $\{0,1\} \times \{0,1\}$ that read, and write, numbers written in the binary system.

an n divisible by 3 and g is the binary representation of $n/3$, possibly prefixed with some 0's.

For more definitions and results on automata theory the reader is referred to [7], [12], or [22], to quote a few. Three more things should be added though. First, the transitions of the transducers we shall consider are labeled in $A \times B$ — we call these transducers **letter-to-letter**. If one retains the first component of the labels of a letter-to-letter transducer $\mathcal{T}$ one gets an automaton (on A): the **underlying input automaton** of $\mathcal{T}$. A transducer is **sequential** (resp., **co-sequential**) if its underlying input automaton is deterministic (resp., co-deterministic).

Second, we shall consider automata where the outgoing arrows are labeled, with pairs of the form (ε, h); this means that if a path in $\mathcal{A}$ from i in I to t in T is labeled with (f, g) and if the outgoing arrow from t is labeled with (ε, h), then f is associated with $g\,h$ by the relation realized by $\mathcal{A}$.

Finally, the label of a path has been implicitly understood as the concatenation **from left to right** of the label of transitions that constitute the path. But one could consider automata which read (and write) words **from right to left**; we call them **right automata**, or **right transducers**. An example of a transducer with these two further characteristics is the one shown at Figure 2 that realizes the addition in the binary system or, more precisely, the conversion of representations written in the digit alphabet $\{0,1,2\}$ into representations written in the classical binary alphabet $\{0,1\}$.

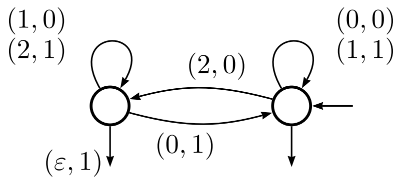


Figure 2. The converter from $\{0,1,2\}^*$ in the binary system.

2.3. Representation of numbers. Let $U = \{u_i : i \in \mathbb{Z}\}$ be a strictly increasing sequence of positive real numbers such that for any $k \geqslant 0$, $\sum_{k\geqslant i\geqslant -\infty} u_i < +\infty$. A **representation in the system** U of a nonnegative real number x on a finite alphabet of digits D is an infinite sequence $(d_i)_{k\geqslant i\geqslant -\infty}$, with k in $\mathbb{Z}$ and every d_i in D, such that

$$x = \sum_{-\infty}^{i=k} d_i u_i.$$

It is denoted by

$$\langle x\rangle_U = d_k \cdots d_0 \mathbf{.} d_{-1} d_{-2} \cdots,$$

most significant digit first.

When a representation ends in infinitely many zeroes, it is said to be **finite**, and the trailing zeroes are omitted. When all the d_i at the right of the radix point are zeroes, the representation is said to be an **integer representation**.

Conversely, the numerical value in the system U of a word on an alphabet of digits D is given by the **evaluation map** π:

$$\pi\colon D^{\mathbb{Z}} \longrightarrow \mathbb{R}, \quad \mathbf{d} = (d_i)_{k\geqslant i\geqslant -\infty} \longmapsto \pi(\mathbf{d}) = \sum_{-\infty}^{i=k} d_i u_i.$$

3. Representation of the integers

3.1. The Modified Division algorithm and the $\frac{p}{q}$ number system. Let $p > q \geqslant 1$ be two co-prime integers. Let N be any positive integer; let us write

$N_0 = N$ and, for $i \geqslant 0$, write

$$qN_i = pN_{i+1} + a_i \tag{1}$$

where a_i is the remainder of the division of qN_i by p, and thus belongs to $A = \{0, \ldots, p-1\}$. Since N_{i+1} is strictly smaller than N_i, the division (1) can be repeated only a finite number of times, until eventually $N_{k+1} = 0$ for some k. The sequence of successive divisions (1) for $i = 0$ to $i = k$ is thus an **algorithm** — that in the sequel is referred to as the Modified Division, or MD algorithm — which, given N, produces the digits a_0, a_1, ..., a_k, and it holds

$$N = \sum_{i=0}^{k} \frac{a_i}{q} \left(\frac{p}{q}\right)^i . \tag{2}$$

We will say that the word $a_k \cdots a_0$, computed from N from right to left, that is to say least significant digit first, is a $\frac{p}{q}$**-representation** of N. Since we will show that this representation is unique in Theorem 1, it will be called **the** $\frac{p}{q}$**-expansion** of N and written $\langle N \rangle_{\frac{p}{q}}$. By convention, the $\frac{p}{q}$-expansion of 0 is the empty word ε.

Example 1: Let $p = 3$ and $q = 2$, then $A = \{0, 1, 2\}$ — this will be our main running example. Table 1 gives the $\frac{3}{2}$-expansions of the eleven first non negative integers.

ε	0
2	1
21	2
210	3
212	4
2101	5
2120	6
2122	7
21011	8
21200	9
21202	10

Table 1.

Following the notations of Section 2.3, let U be the sequence defined by:

$$U = \{u_i = \frac{1}{q}\left(\frac{p}{q}\right)^i : i \in \mathbb{Z}\}.$$

We will say that U, together with the alphabet $A = \{0, \ldots, p-1\}$, is the $\boldsymbol{\frac{p}{q}}$ **number system**. If $q = 1$, it is exactly the classical number system in base p.

It is to be stressed that this definition is not the classical one, the so-called **beta-expansions**, see [21] and [14, Chapter 7], for the numeration system in base $\frac{p}{q}$: U is not the sequence of powers of $\frac{p}{q}$ but rather these powers divided by q and the digits are not the integers smaller than $\frac{p}{q}$ but rather the integers whose quotient by q is smaller than $\frac{p}{q}$. If, on the contrary, $q = 1$ the MD algorithm gives the same expansion as the one given by the classical greedy algorithm, since the expansion is unique.

As stated in the following lemma, one of the main properties of the classical integer base system is nevertheless retained.

Lemma 5: *Let $\pi\colon A^* \to \mathbb{Q}$ be the evaluation map associated with the $\frac{p}{q}$ number system. The restriction of π to A^k, for any k, is* injective.

Proof. Let $u = a_{k-1}a_{k-2}\cdots a_0$ and $v = b_{k-1}b_{k-2}\cdots b_0$ be two words of A^* of length k such that $\pi(u) = \pi(v)$. Hence

$$\sum_{i=0}^{k-1} a_i \left(\frac{p}{q}\right)^i - \sum_{i=0}^{k-1} b_i \left(\frac{p}{q}\right)^i = 0$$

and therefore $\sum_{i=0}^{k-1}(a_i - b_i)X^i$ is a polynomial in $\mathbb{Z}[X]$ vanishing at $X = \frac{p}{q}$. By Gauss Lemma on primitive polynomials, it is then divisible by the minimal polynomial $qX - p$. Contradiction, since the absolute value of the constant term $a_0 - b_0$ is strictly smaller than p. ∎

It is not true that π is injective on the whole A^* since for any u in A^* and any integer h it holds that: $\pi(0^h u) = \pi(u)$. On the other hand, Lemma 5 implies that this is the only possibility and we have:

$$\text{(3)}\qquad \pi(u) = \pi(v) \quad \text{and} \quad |u| > |v| \quad \Longrightarrow \quad u = 0^h v \quad \text{with} \quad h = |u| - |v|.$$

Theorem 1: *Every nonnegative integer N has a $\frac{p}{q}$-*expansion *which is an integer representation. It is the unique finite $\frac{p}{q}$-representation of N.*

Proof. Let $a_{k-1}\cdots a_0$ be the $\frac{p}{q}$-expansion given to N by the MD algorithm, and suppose that there exists another **finite** representation of N in the system U, of the form $e_{\ell-1}e_{\ell-2}\cdots e_0{\boldsymbol\cdot}e_{-1}\cdots e_{-m}$ with $e_{-m}\neq 0$. Then

$$q\left(\frac{p}{q}\right)^m N=\sum_{i=-m}^{\ell} e_i\left(\frac{p}{q}\right)^{m+i}=\sum_{i=0}^{k} a_i\left(\frac{p}{q}\right)^{m+i}$$

and therefore $\pi(e_\ell\cdots e_0e_{-1}e_{-2}\cdots e_{-m})=\pi(a_{k-1}a_{k-2}\cdots a_0 0^m)$. Contradiction between (3) and $e_{-m}\neq 0$.

Thus the word $a_k\cdots a_0$ of A^* is the unique finite $\frac{p}{q}$-representation of N (with the condition that $a_k\neq 0$) and we denote

$$\langle N\rangle_{\frac{p}{q}}=a_k\cdots a_0. \qquad \blacksquare$$

3.2. The set of $\frac{p}{q}$-expansions of the integers. Let us denote by $L_{\frac{p}{q}}$ the set of **$\frac{p}{q}$-expansions** of the nonnegative integers. If $q=1$, then $L_{\frac{p}{q}}$ is the set of all words of A^* which do not begin with a 0; if we release this last condition, we then get the whole A^*.

3.2.1. *Right contexts.* By construction, $L_{\frac{p}{q}}$ is prefix-closed; the observation of Table 1 shows that it is not suffix-closed if $q\neq 1$. In the sequel we assume that $q\neq 1$, unless it is stated otherwise.

Let n and k be natural integers and let us denote by $RC_k(n)$ the set of words of length smaller than $k+1$ that can be suffixed to the $\frac{p}{q}$-expansion of n and still form words of $L_{\frac{p}{q}}$:

$$RC_k(n)=\{w\in A^{\leqslant k} : \langle n\rangle_{\frac{p}{q}}w\in L_{\frac{p}{q}}\}.$$

Lemma 6: *Let n and m be two nonnegative integers. A word w of length k belongs to both $RC_k(n)$ and $RC_k(m)$ if and only if n and m are congruent modulo q^k and in this case $RC_k(n)=RC_k(m)$. That is:*

$$\left\{\langle n\rangle_{\frac{p}{q}}w\in L_{\frac{p}{q}} \text{ and } \langle m\rangle_{\frac{p}{q}}w\in L_{\frac{p}{q}}\right\}\Rightarrow n\equiv m \pmod{q^k}\Rightarrow RC_k(n)=RC_k(m).$$

Proof. The word $\langle n\rangle_{\frac{p}{q}}w$ belongs to $L_{\frac{p}{q}}$ if and only if $(\frac{p}{q})^k n+\pi(w)$ is in $\mathbb{N}$, and similarly for m. Thus:

$$\left\{\langle n\rangle_{\frac{p}{q}}w\in L_{\frac{p}{q}} \text{ and } \langle m\rangle_{\frac{p}{q}}w\in L_{\frac{p}{q}}\right\}\Rightarrow\left(\frac{p}{q}\right)^k(n-m)\in\mathbb{Z}\Rightarrow n\equiv m \pmod{q^k}$$

since p and q are coprime. Conversely, suppose that $n\equiv m \pmod{q^k}$ then $n\equiv m \pmod{q^h}$ for any $h\leqslant k$. Hence for every word w of length $h\leqslant k$ such that $(\frac{p}{q})^h n+\pi(w)$ is in $\mathbb{N}$, so is $(\frac{p}{q})^k m+\pi(w)$, and $RC_k(n)=RC_k(m)$. $\blacksquare$

Lemma 6 implies immediately that the **coarsest right regular equivalence** that saturates $L_{\frac{p}{q}}$ is the identity, hence, in particular, is not of finite index. A classical statement in formal language theory (see [12]) then implies

Corollary 7: *If $q \neq 1$ then $L_{\frac{p}{q}}$ is not a regular language.*

Along the same line as Lemma 6, one can give a more precise statement on suffixes that are powers of a given word.

Lemma 8: *Let w be in $L_{\frac{p}{q}}$ and $w = uv$ be a proper factorization of w. Then uv^k belongs to $L_{\frac{p}{q}}$ only if $q^{(k-1)|v|}$ divides $\pi(w) - \pi(u)$.*

Proof. The word uv^k belongs to $L_{\frac{p}{q}}$ only if

$$\begin{aligned}\pi(uv^k) - \pi(uv^{k-1}) &= \left(\frac{p}{q}\right)^{|v|} (\pi(uv^{k-1}) - \pi(uv^{k-2})) = \cdots \\ &= \left(\frac{p}{q}\right)^{(k-1)|v|} (\pi(uv) - \pi(u))\end{aligned}$$

is in $\mathbb{Z}$. And this is possible only if $q^{(k-1)|v|}$ divides $\pi(uv) - \pi(u)$. ∎

Lemma 8 will be used in the sequel to show that the closure of $L_{\frac{p}{q}}$ does not contain eventually periodic infinite words; combined with the classical "pumping lemma" (see [12]), it implies another statement related to formal language theory:

Corollary 9: *If $q \neq 1$, then $L_{\frac{p}{q}}$ is not a context-free language.*

3.2.2. *Suffixes.* We observed that $L_{\frac{p}{q}}$ is not suffix-closed. In fact, every word of A^* is a suffix of some words in $L_{\frac{p}{q}}$. More precisely, we have the following statement.

Proposition 10: *For every integer k and every word w in A^k, there exists a unique integer n, $0 \leqslant n < p^k$ such that w is the suffix of length k of the $\frac{p}{q}$-expansion of all integers m congruent to n modulo p^k.*

Proof. Given any integer $n = n_0$, the repetition k times of the division[4] (1) yields

$$q^k n_0 = p^k n_k + q^k \pi(a_{k-1}a_{k-2}\cdots a_0). \tag{4}$$

If we do the same for another integer $m = m_0$ and subtract the equation we get from (4),

$$q^k\,(n_0 - m_0) = p^k\,(n_k - m_k) + q^k\,(\pi(a_{k-1}a_{k-2}\cdots a_0) - \pi(b_{k-1}b_{k-2}\cdots b_0))\,.$$

As q^k is prime with p^k, and using Lemma 5, we get

$$n - m \equiv 0 \pmod{p^k} \iff a_{k-1}a_{k-2}\cdots a_0 = b_{k-1}b_{k-2}\cdots b_0. \tag{5}$$

Since there are exactly p^k words in A^k, each of them must appear once and only once when n ranges from 0 to $p^k - 1$ and (5) gives the second part of the statement. ∎

3.2.3. *The odometer.* Proposition 10 can be interpreted, or reformulated, with the construction of **a machine** that could be called “a boosted Pascal machine” and would be described as follows.

The main feature of the “Pascaline”, the famous adding machine invented by young Blaise Pascal is a series of toothed wheels linked together by a special mechanism: when a wheel finishes a full rotation, it sends to the next wheel on the left an impulse that makes the latter move by one unit.

Think of the Pascal machine as a series — virtually extending infinitely to the left — of wheels with a dial in front of each wheel and every dial is marked with p digits. The original Pascaline used 10 digits, from 0 to 9, since young Pascal was counting in base 10, but any p will be as good. Let us take $p = 3$ as in our running example for the remaining of this description; the digits are 0, 1 and 2. There is an arm, attached to the wheel and moving in front of the corresponding dial.

And let us consider the machine as Pascal designed it, but somewhat turned into a clock. In the beginning, every arm is vertical and points to 0. Imagine that at every second the rightmost wheel moves by one unit: the arm passes in front of 1, 2, 0 again, 1, 2, etc. When that arm comes to 0 again, the arm of the second wheel goes to 1, and so on. After n seconds, one reads on the machine a

[4] If n is large enough, this amounts to the first k steps of the MD algorithm. Otherwise, $n_i = 0$ and $a_i = 0$ for i greater than a certain j and this is not, strictly speaking, the MD algorithm.

certain word written on the alphabet $A = \{0, 1, 2\}$ which is exactly the writing of n in base 3 — if we forget all the 0's on the left — and conversely every word w of A^* appears exactly once, at time $\pi(w)$, where $\pi(w)$ is the number whose writing in base 3 is w.

Imagine now that the machine is so to speak "boosted" and that every quantum of rotation of the wheels is 2 units instead of 1: the rightmost wheel goes every second from 0 to 2, from 2 to 1, from 1 to 0, etc. and every time the arm of one wheel passes in front of 0 of its dial — whether it stops there or not — it sends an impulse to the next wheel to the left that makes it moving by 2 units.

The sequence of words that will then turn up on the dials of the machine is exactly the $\frac{p}{q}$-expansions of integers that is the words of $L_{\frac{p}{q}}$, ordered by length, and within the same length by the lexicographic order. If a finite such machine were constructed (which is more realistic than an infinite one) with, say, k wheels, the above result states that its behavior is periodic, of period p^k, and that every possible configuration of the k wheels will appear once (and only once) during the cycle.

The transformation of words witnessed on the boosted Pascal machine at every impulse is the one that is realized by the machine that is usually called the **odometer** of the number system: it takes as input a word v representing a number n and outputs the word w representing the number $n + 1$.

From the description of the boosted Pascal machine, it is easy to build a digit-to-digit right sequential transducer that realizes the odometer for the $\frac{p}{q}$ number system. It is represented at Figure 3 for our running example.

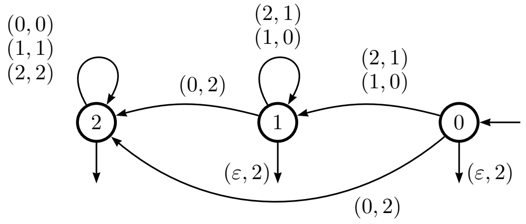


Figure 3. The odometer for the $\frac{3}{2}$ number system.

3.2.4. *Order on numbers, order on words.* In an integer base, the order on integers and the **radix** order on the words (on the canonical alphabet) that represent the numbers coincide, and this is also true of the **lexicographic** order on words of the same length (and with a possible prefix in 0^*). The same properties hold for the $\frac{p}{q}$ number system, provided only the words in $L_{\frac{p}{q}}$ are considered.

Proposition 11: *Let v and w be in $L_{\frac{p}{q}}$. Then $v \preccurlyeq w$ if, and only if, $\pi(v) \leqslant \pi(w)$.*

Proof. Let $v = a_{k-1} \cdots a_0$ and $w = b_{\ell-1} \cdots b_0$ be the $\frac{p}{q}$-expansions of the integers $m = \pi(v)$ and $n = \pi(w)$, respectively. By Theorem 1, we already know that $v = w$ if, and only if, $\pi(v) = \pi(w)$. The proof goes by induction on ℓ, which is (by hypothesis) greater than or equal to k. The proposition holds for $\ell = 1$.

Let us write $v' = a_{k-1} \cdots a_1$ and $w' = b_{\ell-1} \cdots b_1$, and $m' = \pi(v')$ and $n' = \pi(w')$ are integers. It holds:

$$n - m = \frac{p}{q}(n' - m') + \frac{1}{q}(b_0 - a_0)$$

Now $v \prec w$ implies that either $v' \prec w'$ or $v' = w'$ and $a_0 < b_0$. If $v' \prec w'$, then $n' - m' \geqslant 1$ by induction hypothesis and thus $n - m > 0$ since $b_0 - a_0 \geqslant -(p-1)$. If $v' = w'$, then $n - m = \frac{1}{q}(b_0 - a_0) > 0$. ∎

Corollary 12: *Let v and w be in $0^* L_{\frac{p}{q}}$ and of equal length. Then $v \sqsubseteq w$ if, and only if, $\pi(v) \leqslant \pi(w)$.*

It is to be noted also that these statements do not hold without the hypothesis that v and w belong to $L_{\frac{p}{q}}$ (to $0^* L_{\frac{p}{q}}$ respectively). For instance, $\pi(10) = 3/4 < \pi(2) = 1$ and $\pi(2000) = 27/8 < \pi(0212) = 4$.

3.3. Conversion between alphabets. Another property of the integer base systems that carries over to the $\frac{p}{q}$ number system is the fact that the conversion of digits can be realized by a finite (right) transducer.

Let D be a finite alphabet of (positive or negative) digits that contains A. The **digit-set conversion** is a map $\chi_D \colon D^* \to A^*$ which commutes to the evaluation map π, that is, a map which preserves the numerical value:

$$\forall w \in D^* \quad \pi(\chi_D(w)) = \pi(w).$$

PROPOSITION 13: *For any alphabet D the conversion χ_D is realizable by a finite letter-to-letter sequential right transducer $\mathcal{C}_D$.*

Proof. Let $\mathcal{U}_D = \langle \mathbb{Z}, D \times A, E, \{0\}, \omega \rangle$ be the (infinite) transducer whose set of transitions E is defined by:

$$\big(z, (d, a), z'\big) \in E \iff qz + d = pz' + a. \tag{6}$$

As z' and a are uniquely determined for a given z and d, $\mathcal{U}_D$ is sequential.

If the final function is defined as $\omega(z) = \langle z \rangle_{\frac{p}{q}}$ for every z in $\mathbb{N}$ (and $\omega(z) = \emptyset$, that is, z is not final, if $z < 0$), it is immediate to verify, by induction on the length of the input words, that $\mathcal{U}_D$, seen as a *right* transducer, realizes the digit conversion of any word w whose numerical value is positive. Thus, it remains to show that the **accessible part** $\mathcal{C}_D$ of $\mathcal{U}_D$ is finite.

Without loss of generality, one can suppose that D is an interval: what matters is the **largest** digit e and the **smallest** digit f in D, $e \geqslant p - 1$ and $f \leqslant 0$, at least one of the two inequalities being strict. It follows from (6) that from a state z it is possible to reach the state $z+1$ (resp., the state $z-1$) in $\mathcal{U}_D$ if there exist d in D and a in A such that $z = \big((d-a)-p\big)/(p-q)$ (resp., such that $z = \big((d-a)+p\big)/(p-q)\big)$.

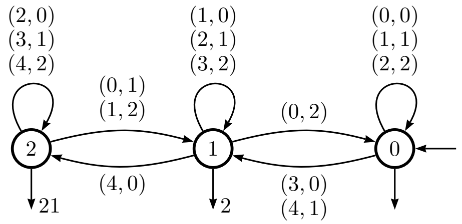
Thus the **largest** accessible positive state and the **smallest** accessible state in $\mathcal{U}_D$ are:

$$z_{\mathsf{max}} = \left\lfloor \frac{e-p}{p-q} \right\rfloor + 1 \quad \text{and} \quad z_{\mathsf{min}} = \left\lceil \frac{f-(p-1)+p}{p-q} \right\rceil - 1 = \left\lceil \frac{f+1}{p-q} \right\rceil - 1$$

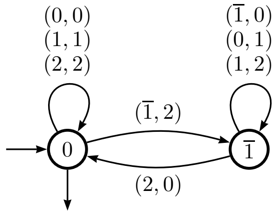
and hence $\mathcal{C}_D$ is finite. ∎

The **integer addition** may be seen — after digit-wise addition — as a particular case of a digit-set conversion χ_D with $D = \{0, 1, \ldots, 2(p-1)\}$ and Figure 4 (a) shows the converter that realizes addition in the $\frac{3}{2}$ number system. For reasons which will be explained in Section 5.2, we also give at Figure 4 (b) the converter on the alphabet $\{-1, 0, 1, 2\}$ in the $\frac{3}{2}$ number system (the signed digit $-d$ is denoted $\bar{d}$).

Remark 14: Let us stress that χ_D is defined on the whole set D^* even for word v such that $\pi(v)$ is not an integer, and also that, if $\pi(v)$ is in $\mathbb{N}$, then $\chi_D(v)$ is the unique $\frac{p}{q}$-representation of $\pi(v)$.



(a) The converter for the addition.



(b) The converter on $\{-1, 0, 1, 2\}$.

Figure 4. Two converters for the $\frac{3}{2}$ number system.

Remark 15: As $\frac{p}{q}$ is not a Pisot number (when $q \neq 1$), the conversion from any representation onto the representation computed by the greedy algorithm is not realized by a finite transducer (see [14, Ch. 7]).

4. The tree $T_{\frac{p}{q}}$

The free monoid A^* is classically represented as the nodes of the (infinite) full p-ary tree: every node is labeled by a word in A^* and has p children, every edge between a node and each of its children is labeled by one of the letter of A and the label of a node is precisely the label of the (unique) path that goes from the root to that node.

As the language $L_{\frac{p}{q}}$ is prefix-closed, it can naturally be seen as a subtree of the full p-ary tree, obtained by cutting some edges. This will form the tree $T_{\frac{p}{q}}$ (after we have changed the label of nodes from words to the numbers represented by these words). This tree, or, more precisely, its infinite paths, will be the basis for the representation of reals in the $\frac{p}{q}$ number system. We give now an 'internal' description of $T_{\frac{p}{q}}$, based on the definition of a family of maps from $\mathbb{N}$ to $\mathbb{N}$, which will proved to be effective for the study of infinite paths.

4.1. Construction of the tree $T_{\frac{p}{q}}$.

Definition 16: (i) For each a in A, let $\tau_a \colon \mathbb{N} \to \mathbb{N}$ be the **partial** map defined by:

$$\forall n \in \mathbb{N} \quad \tau_a(n) = \begin{cases} \frac{1}{q}(pn+a) & \text{if} \quad \frac{1}{q}(pn+a) \in \mathbb{N} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We write $\mathsf{d}(n) = \{a \in A : \tau_a(n) \text{ is defined}\}$, $\mathsf{Md}(n) = \max\{\mathsf{d}(n)\}$ for the largest digit for which $\tau_a(n)$ is defined, and $\mathsf{md}(n) = \min\{\mathsf{d}(n)\}$ for the smallest digit with the same property.

(ii) The tree $T_{\frac{p}{q}}$ is the labeled infinite tree (where both nodes and edges are labeled) constructed as follows. The nodes are labeled in $\mathbb{N}$, and the edges in A, the root is labeled by 0. The children of a node labeled by n are nodes labeled by $\tau_a(n)$ for a in $\mathsf{d}(n)$, and the edge from n to $\tau_a(n)$ is labeled by a.

(iii) We call **path label** of a node s of $T_{\frac{p}{q}}$, and write $\mathrm{p}(s)$, the label of *the* path from the root of $T_{\frac{p}{q}}$ to s. We denote by $I_{\frac{p}{q}}$ the subtree of $T_{\frac{p}{q}}$ made of nodes whose path label does not begin with a 0.

For example, the first six levels of $T_{\frac{3}{2}}$ and $I_{\frac{3}{2}}$ are shown at Figure 5.

The very way $T_{\frac{p}{q}}$ is defined implies that if two nodes have the same label, they are the root of two isomorphic subtrees of $T_{\frac{p}{q}}$ and it follows from Lemma 6 that the converse is true, that is, two nodes which hold distinct labels are the root of two distinct subtrees of $T_{\frac{p}{q}}$. As no two nodes of $I_{\frac{p}{q}}$ have the same label, we have

Proposition 17: *If $q \neq 1$ no two subtrees of $I_{\frac{p}{q}}$ are isomorphic.*

Definition 16 and the MD algorithm imply directly the following facts that will be used in the sequel, most often without explicit reference.

Lemma 18: *For every n in $\mathbb{N}$, it holds:*

(i) $\mathsf{md}(n) = \mathsf{d}(n) \cap \{0, 1, \ldots, q-1\}$ *and* $\mathsf{Md}(n) = \mathsf{d}(n) \cap \{p-q, \ldots, p-1\}$.

(ii) $a \in \mathsf{d}(n)$ *and* $a + q \in A \Longrightarrow a + q \in \mathsf{d}(n)$.

(iii) $a, a+q \in \mathsf{d}(n) \Longrightarrow \tau_{a+q}(n) = \tau_a(n) + 1$.

(iv) $\mathsf{md}(n+1) = \mathsf{Md}(n) + q - p$ *and* $\tau_{\mathsf{md}(n+1)}(n+1) = \tau_{\mathsf{Md}(n)}(n) + 1$.

And finally:

(v) *The label of every node s of $T_{\frac{p}{q}}$ is $\pi(\mathrm{p}(s))$.*

In particular, it follows:

Corollary 19: $\forall n \in \mathbb{N}\ d = \mathsf{md}(n) \iff \tau_d(n) = \left\lceil \frac{p}{q} n \right\rceil$.

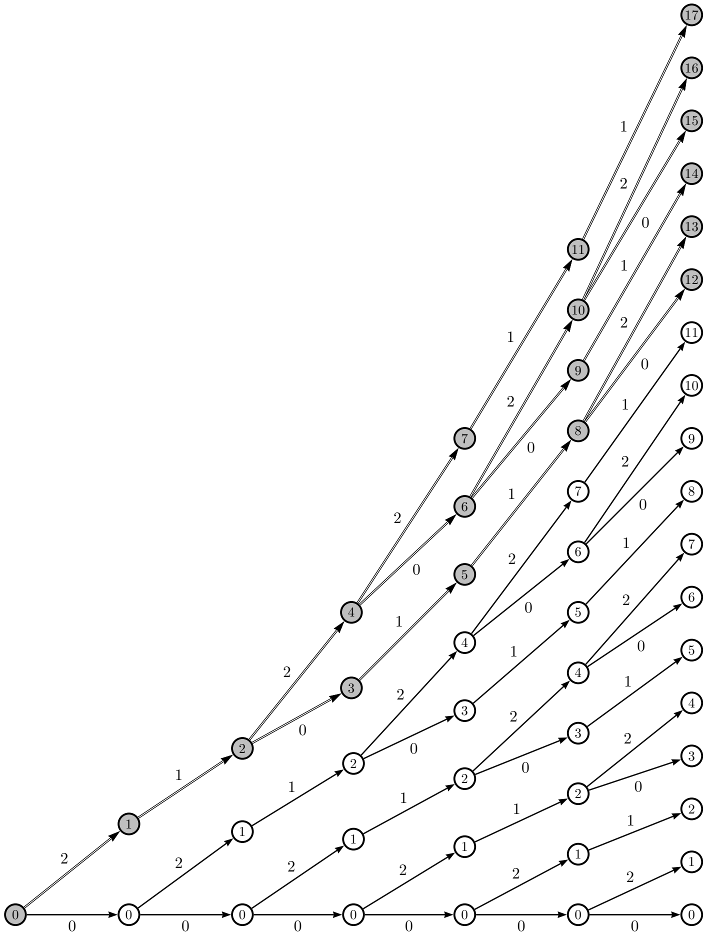


Figure 5. The tree $T_{\frac{3}{2}}$, the tree $I_{\frac{3}{2}}$ in grey and double edge.

This statement induces the definition of the following sequence.

Definition 20: Let $(G_k)_{k\in\mathbb{N}}$ be the sequence of integers defined by:

$$G_0 = 1 \quad \text{and} \quad G_{k+1} = \left\lceil \frac{p}{q} G_k \right\rceil, \quad \forall k \in \mathbb{N}.$$

It then comes, by induction on k:

Proposition 21: *The nodes of depth k in $T_{\frac{p}{q}}$, ordered by their path label in the lexicographic (or radix) order, are labeled by integers from 0 to $G_k - 1$.*

We shall return to the computation of the G_k's at Section 4.4.

4.2. Minimal and maximal words. The infinite paths in the tree $T_{\frac{p}{q}}$ will be used in Section 5 to define the representations of real numbers. Here we consider only some particular infinite paths (or words) in $T_{\frac{p}{q}}$.

We denote by $\mathsf{W}(n)$ (resp., by $\mathsf{w}(n)$) the label of the infinite path that starts from a node with label n and that follows always the edges with the maximal (resp., minimal) digit label. Such a word is said to be a **maximal** word (resp., a **minimal** word) in $T_{\frac{p}{q}}$. The following is a direct consequence of Lemma 18 (i).

Proposition 22:

(i) *For all $n \in \mathbb{N}, \mathsf{W}(n) \in \{p-q, \ldots, p-1\}^{\mathbb{N}}$ and $\mathsf{w}(n) \in \{0, \ldots, q-1\}^{\mathbb{N}}$.*

(ii) *Conversely let $\mathbf{u}$ be the label of an infinite path in $T_{\frac{p}{q}}$.*
If $\mathbf{u}$ is in $\{p-q, \ldots, p-1\}^{\mathbb{N}}$, then there exists an n such that $\mathbf{u} = \mathsf{W}(n)$ and if $\mathbf{u}$ is in $\{0, \ldots, q-1\}^{\mathbb{N}}$, then there exists an n such that $\mathbf{u} = \mathsf{w}(n)$.

(iii) *For every n, the digit-wise difference between $\mathsf{w}(n+1)$ and $\mathsf{W}(n)$ is $(p-q)^{\omega}$.*

Two special cases are worth special notations; we note:

$$\mathbf{t}_{\frac{p}{q}} = \mathsf{W}(0) \quad \text{and} \quad \mathbf{g}_{\frac{p}{q}} = \mathsf{w}(1).$$

The infinite word $\mathbf{t}_{\frac{p}{q}}$ is the maximal element with respect to the lexicographic order of the label of all infinite paths of $T_{\frac{p}{q}}$ that start from the root. Since $\tau_q(0) = 1$, the infinite word $q\,\mathbf{g}_{\frac{p}{q}}$ is the minimal element with respect to the lexicographic order of the label of all infinite paths of $I_{\frac{p}{q}}$ that start from the root. Notice that, for any rational $\frac{p}{q}$, 0^{ω} is the minimal element with respect to the lexicographic order of the label of all infinite paths of $T_{\frac{p}{q}}$ and that, if $q = 1$, that is, in an integer base, $\mathsf{W}(n) = (p-1)^{\omega}$, and $\mathsf{w}(n) = 0^{\omega}$ for every n in $\mathbb{N}$.

Example 2: For $\frac{p}{q} = \frac{3}{2}$,

$$\mathbf{t}_{\frac{3}{2}} = 2\,1\,2\,2\,1\,1\,1\,2\,2\,1\,2\,1\,1\,2\,2\,1\,2\,1\,2\,1\,1\,2\,2\,1\,\cdots$$
$$\text{and} \quad \mathbf{g}_{\frac{3}{2}} = 1\,0\,1\,1\,0\,0\,0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,1\,0\,1\,0\,0\,1\,1\,0\,\cdots.$$

which illustrates, in particular, Proposition 22 (iii).

In Section 3, words of $L_{\frac{p}{q}}$, and thus labels of finite paths of $T_{\frac{p}{q}}$, were indexed from right to left or, if one prefers, from left to right by decreasing nonnegative integers, always ending with 0; the possibility of extending the indexation to the right after the 'radix' point, and of using decreasing negative integers, up to minus infinity, for indexing the 'fractional' part of a writing was mentioned at Section 2.3. When we deal with infinite words that will correspond to representations of numbers with only fractional part — as it will be the case in the next section — we find it much more convenient to change the convention of **indexing** and use the positive indices in the increasing order (and starting from 1) from left to right.

In particular, we write

$$\mathbf{g}_{\frac{p}{q}} = g_1\, g_2\, g_3\, \cdots ,$$

and by induction and using Corollary 19 and the fact that the label of a node is the value of its path label it then comes:

Corollary 23: *$G_0 = \pi(q) = 1$ and $G_k = \pi(q g_1 g_2 \cdots g_k)$ for all $k \in \mathbb{N}$.*

4.3. Evaluation of infinite words in $T_{\frac{p}{q}}$. According to the convention we have just taken on the indexing of infinite words, the evaluation map takes the following form:

$$\forall \mathbf{a} = a_1\, a_2\, \cdots \in A^{\mathbb{N}} \quad \pi(\boldsymbol{.}\, \mathbf{a}) = \frac{1}{q} \sum_{i \geqslant 1} a_i \left(\frac{q}{p}\right)^i . \tag{7}$$

We use the radix point '$\boldsymbol{.}$' on the left of the infinite word in order to mark the position of the index 0 and distinguish clearly between the use of the evaluation map π in equations such as Corollary 23 and (7). Let $\mathbf{a} = a_1 a_2 \cdots$ be in $A^{\mathbb{N}}$ and $x = \pi(\boldsymbol{.}\, \mathbf{a})$. With these notations we clearly have:

$$\left(\frac{p}{q}\right)^h x = \pi(a_1\, a_2\, \cdots a_h \boldsymbol{.} a_{h+1}\, a_{h+2} \cdots), \quad \text{for all } h \in \mathbb{N} \tag{8}$$

$$x = \lim_{h \to \infty} \left(\frac{q}{p}\right)^h \pi(a_1\, a_2 \cdots a_h) = \lim_{h \to \infty} \left(\frac{q}{p}\right)^h \pi(\mathbf{a}_{[h]}) \tag{9}$$

As in an integer base system, we have:

Proposition 24: [7] *The map $\pi \colon A^{\mathbb{N}} \to \mathbb{R}$ is continuous.*

Notation 25: Let us denote by $W_{\frac{p}{q}}$ the subset of $A^{\mathbb{N}}$ that consists of the labels of infinite paths starting from the root of $T_{\frac{p}{q}}$.

Note that the finite prefixes of the elements of $W_{\frac{p}{q}}$ are the words in $0^*L_{\frac{p}{q}}$. A direct consequence of Lemma 8 is the following.

PROPOSITION 26: *If $q > 1$, then no element of $W_{\frac{p}{q}}$, but 0^ω, is eventually periodic.*

From (8), it then follows:

LEMMA 27: *Let $\mathbf{a} = a_1\, a_2 \cdots$ be in $W_{\frac{p}{q}}$ and $x = \pi(\centerdot\, \mathbf{a})$. Then, for every $k \in \mathbb{N}$,*

$$\left\lfloor \left(\frac{p}{q}\right)^k x \right\rfloor = \pi(a_1\, a_2 \cdots a_k) + \rho_k(x) \quad \textit{with}\, \rho_k(x) = \lfloor \pi(\centerdot\, a_{k+1}\, a_{k+2} \cdots) \rfloor < \frac{p-1}{p-q}.$$

Proof. The statement holds because $\pi(a_1\, a_2 \cdots a_k)$ is an integer as $\mathbf{a}$ is in $W_{\frac{p}{q}}$ and the inequality is strict as no word of $W_{\frac{p}{q}}$ may end in $(p-1)^\omega$. ∎

We call **branching** a node v of $T_{\frac{p}{q}}$ if it has at least two children, that is, if $\mathsf{d}(\pi(\mathrm{p}(v)))$ has at least two elements.

LEMMA 28: *Let v be any branching node in $T_{\frac{p}{q}}$, and $n = \pi(\mathrm{p}(v))$ its label. Let a_1 and $b_1 = a_1 + q$ be in $\mathsf{d}(n)$ and let $m_1 = \tau_{a_1}(n)$ and $m_2 = \tau_{b_1}(n) = m_1 + 1$. Write $\mathsf{W}(m_1) = a_2\, a_3 \cdots$ and $\mathsf{w}(m_2) = b_2\, b_3 \cdots$. Then, it holds*

$$\pi(\centerdot\, a_1\, a_2\, a_3 \cdots) = \pi(\centerdot\, b_1\, b_2\, b_3 \cdots). \tag{10}$$

Proof. Proposition 22 (iii) directly yields the computation:

$$\pi(\centerdot\, a_1\, a_2\, a_3 \cdots) - \pi(\centerdot\, b_1\, b_2\, b_3 \cdots) = \frac{1}{q}\left((-q)\,\frac{q}{p} + (p-q)\sum_{i \geqslant 2}\left(\frac{q}{p}\right)^i\right)$$

and the right member is clearly equal to 0. ∎

We then define the two real numbers $\boldsymbol{\omega}_{\frac{p}{q}}$ and $\boldsymbol{\gamma}_{\frac{p}{q}}$ by:

$$\boldsymbol{\omega}_{\frac{p}{q}} = \pi(\centerdot\, \mathbf{t}_{\frac{p}{q}}) \quad \text{and} \quad \boldsymbol{\gamma}_{\frac{p}{q}} = \pi(\centerdot\, q\, \mathbf{g}_{\frac{p}{q}}), \tag{11}$$

and Lemma 28 implies: $\boldsymbol{\gamma}_{\frac{p}{q}} = \pi(\centerdot\, 0\, \mathbf{t}_{\frac{p}{q}}) = \frac{q}{p}\, \boldsymbol{\omega}_{\frac{p}{q}}$. The next property is a kind of a converse of Lemma 28 but a bit more technical.

LEMMA 29: *Suppose $q \geqslant 2$ and let k and r be two integers, $k > \frac{q-1}{p-q}$ and $r = \left\lceil \frac{q-2}{p-q} \right\rceil$. Let n be any non negative integer and u and v two words of* the same length ℓ *such that* $\pi(u) = n$ *and* $\pi(v) = n + k$. *Then*

$$\pi(\centerdot v\,\mathsf{w}(n+k)) - \pi(\centerdot u\,\mathsf{w}(n)) \geqslant \left(\frac{q}{p}\right)^{\ell+r} \boldsymbol{\omega}_{\frac{p}{q}}.$$

Proof. The following notation, inspired by Corollary 19, will be convenient:

$$\mu(n) = \tau_{\mathsf{md}(n)}(n) = \left\lceil \frac{p}{q}\, n \right\rceil$$

for n in $\mathbb{N}$, and of course $\mu^{i+1}(n) = \mu(\mu^i(n))$. The proof goes in three steps. Since $\left\lceil \frac{p}{q}\, n \right\rceil - \frac{q-1}{q} \leqslant \frac{p}{q}\, n$, the choice of k implies, for every n in $\mathbb{N}$,

$$\left\lceil \frac{p}{q}\,(n+k) \right\rceil - \left\lceil \frac{p}{q}\, n \right\rceil \geqslant \frac{p}{q}\,(n+k) - \left(\frac{p}{q}\, n + \frac{q-1}{q}\right) > k.$$

Thus, since the left handside is an integer,

$$\text{(12)}\quad \forall n \in \mathbb{N} \quad \mu(n+k) \geqslant \mu(n)+k+1 \quad \text{and} \quad \forall i \in \mathbb{N} \quad \mu^i(n+k) \geqslant \mu^i(n)+k+i\,.$$

It then holds, for every n, m, and z in $\mathbb{N}$:

$$\left\lceil \frac{p}{q}\,(n+m+z) \right\rceil \geqslant \left\lceil \frac{p}{q}\, n \right\rceil + \left\lceil \frac{p}{q}\, m \right\rceil + \frac{p}{q}\, z - 2\,\frac{q-1}{q}.$$

The choice of r implies then

$$\frac{p}{q}\,(k+r) - 2\,\frac{q-1}{q} \geqslant k+1+\frac{p}{q}\,r - 2\,\frac{q-1}{q} = k+r+\frac{(p-q)r-(q-2)}{q} \geqslant k+r.$$

And it then follows, by induction on j,

$$\text{(13)}\qquad \mu^j(n+k+r) \geqslant \mu^j(n) + G_{j-1} + k + r,$$

an inequality that follows from (12) for $j = 1$. Indeed, it holds:

$$\begin{aligned}\mu\left(\mu^j(n) + G_{j-1} + k + r\right) &= \left\lceil \frac{p}{q}\,\mu^j(n) + \frac{p}{q}\,G_{j-1} + \frac{p}{q}\,(k+r) \right\rceil \\ &\geqslant \mu^{j+1}(n) + G_j + \frac{p}{q}\,(k+r) - 2\,\frac{q-1}{q} \\ &\geqslant \mu^{j+1}(n) + G_j + k + r.\end{aligned}$$

For sake of brevity let us write now $\mathbf{a} = u\mathsf{w}(n)$ and $\mathbf{b} = v\mathsf{w}(n+k)$. By definition of $\mathsf{w}(n)$ and of $\mu^i(n)$, it comes $\pi(\mathbf{a}_{[h]}\centerdot) = \mu^i(n)$ for $h = \ell+i$. Equation (13) may then be rewritten as

$$\forall j \in \mathbb{N} \quad \pi(\mathbf{b}_{[h]}\centerdot) - \pi(\mathbf{a}_{[h]}\centerdot) \geqslant G_{j-1} + k + r \quad \text{with } h = \ell + r + (j-1),$$

and from (9) it follows

$$\pi(\boldsymbol{.}\mathbf{b}) - \pi(\boldsymbol{.}\mathbf{a}) \geqslant \lim_{j\to+\infty} \left(\frac{q}{p}\right)^{\ell+r+j-1} (G_{j-1} + k + r)$$
$$= \left(\frac{q}{p}\right)^{\ell+r} \left[\frac{p}{q} \lim_{j\to+\infty} \left(\frac{q}{p}\right)^{j} G_{j-1}\right]. \qquad \blacksquare$$

Figure 6 shows the tree $T_{\frac{3}{2}}$ again. But this time every node s is given an ordinate equal to $\pi(\mathrm{p}(s))$; nodes at the same level in the tree are given the same abscissa (as in Figure 5) and the distance between two levels of the tree is multiplied by $\frac{q}{p}$ when the level increases, which gives the fractal aspect.

4.4. The Josephus Problem. The above definitions and notations allow us to give another expression for the integer sequence $(G_k)_{k\in\mathbb{N}}$.

Proposition 30: *For every k in $\mathbb{N}$, there exists an integer e_k, $0 \leqslant e_k < \frac{q-1}{p-q}$, such that*

$$G_k = \left\lfloor \boldsymbol{\gamma}_{\frac{p}{q}} \left(\frac{p}{q}\right)^{k+1} \right\rfloor - e_k.$$

Proof. From the definition of $\boldsymbol{\gamma}_{\frac{p}{q}}$ and as in Lemma 27, it follows:

$$\left\lfloor \boldsymbol{\gamma}_{\frac{p}{q}} \left(\frac{p}{q}\right)^{k+1} \right\rfloor = \pi(q\, g_1 \cdots g_k \boldsymbol{.}) + \left\lfloor \pi(\boldsymbol{.} g_{k+1} g_{k+2} \cdots) \right\rfloor = G_k + e_k,$$

where e_k is an integer strictly smaller than $\frac{q-1}{p-q}$ since g_i is in $\{0, \ldots, q-1\}$ for every $i \geqslant 1$. $\blacksquare$

Corollary 31: *If $p \geqslant 2q-1$ then, for every k in $\mathbb{N}$, $G_k = \left\lfloor \boldsymbol{\gamma}_{\frac{p}{q}} \left(\frac{p}{q}\right)^{k+1} \right\rfloor$.*

Remark 32: Still in the case where $p \geqslant 2q-1$, then for every $k \geqslant 1$, the digit g_k of the $\frac{p}{q}$-expansion of $\boldsymbol{\gamma}_{\frac{p}{q}}$ is obtained as follows:
(i) compute $G_{k+1} = \left\lceil \frac{p}{q} G_k \right\rceil$ (ii) $g_{k+1} = q\, G_{k+1} \pmod{p}$.

The definition of the sequence G_k and the computation of $\boldsymbol{\gamma}_{\frac{p}{q}}$ have been developed not only because they are important for the description of $T_{\frac{p}{q}}$ but also as they relate to a classical problem in combinatorics.

Inspired by the so-called "Josephus problem", Odlyzko and Wilf consider, for a real $\alpha > 1$, the iterates of the function $f(x) = \lceil \alpha\, x \rceil$: $f_0 = 1$ and $f_{n+1} = \lceil \alpha\, f_n \rceil$ for $n \geqslant 0$. They show (in [17]) that in the cases where $\alpha \geqslant 2$, or $\alpha = 2 - 1/q$ for

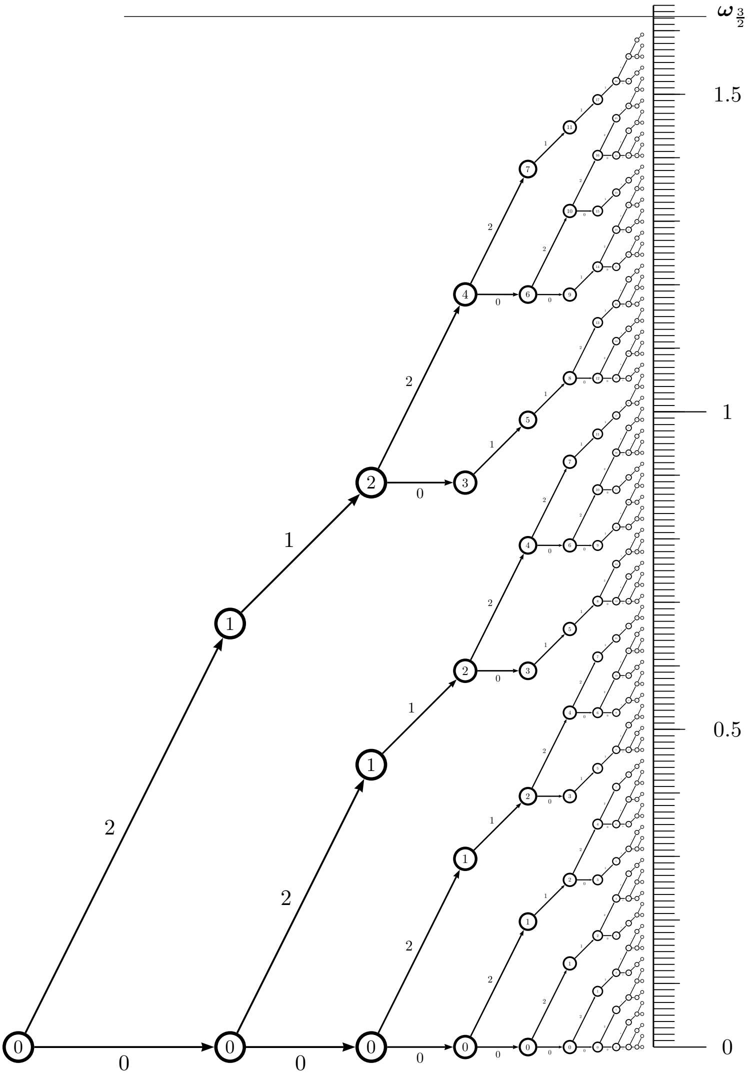


Figure 6. Another view on $T_{\frac{3}{2}}$ (with four more levels)

some integer $q \geqslant 2$, then there exists a constant $H(\alpha)$ such that $f_n = \lfloor H(\alpha)\alpha^n \rfloor$ for all $n \geqslant 0$.

Thus, we obtain the same result as in [17] for any rational $\alpha = \frac{p}{q}$, with $p \geqslant 2q-1$, and we find $H(\frac{p}{q}) = \frac{p}{q}\,\gamma_{\frac{p}{q}} = \boldsymbol{\omega}_{\frac{p}{q}}$. Our method does not yield an "independent" way of computing this constant, as was called for in [17], but the $\frac{p}{q}$-expansion of $\boldsymbol{\omega}_{\frac{p}{q}}$ gives at least an easy algorithm.

In the case where $q = p-1$ (the Josephus case), the constant $\boldsymbol{\omega}_{\frac{p}{q}}$ is the constant $K(p)$ in [17]. In this case the integer e_k of Proposition 30 is less than $p-2$, and this is the same bound as in [17].

Example 3: For $\frac{p}{q} = \frac{3}{2}$, the constant $\boldsymbol{\omega}_{\frac{3}{2}}$ is the constant $K(3)$ already discussed in [17, 11, 24]. Its decimal expansion:

$$\boldsymbol{\omega}_{\frac{3}{2}} = 1.6\,2\,2\,2\,7\,0\,5\,0\,2\,8\,8\,4\,7\,6\,7\,3\,1\,5\,9\,5\,6\,9\,5\,0\,9\,8\,2\,\cdots$$

is recorded as Sequence A083286 in [23]. Observe that, in the same case, the sequence $(G_k)_{k\geqslant 1}$ is Sequence A061419 in [23].

5. Representation of the reals

Every infinite word $\mathbf{a}$ in $A^{\mathbb{N}}$ is given a real value x by π:

$$x = \pi(\boldsymbol{.}\mathbf{a}),$$

and $\mathbf{a}$ is called a $\frac{p}{q}$**-representation** of x. Our purpose here is to associate with every real number a $\frac{p}{q}$-representation which will be as canonical as possible. In contrast with what is done in Pisot base number systems, where the canonical representation, the **greedy expansion**, is defined by an algorithm which computes it for every real, we set these canonical $\frac{p}{q}$-expansions a priori. Then we have to prove: first, that they represent indeed the reals and, second, to what extent they are canonical.

In a second part, we give an algorithmic way to compute a $\frac{p}{q}$-representation which we call the **companion** $\frac{p}{q}$-representation. This representation is not on the alphabet A anymore but on a larger alphabet with negative digits. We investigate then how one can recover the $\frac{p}{q}$-expansion from the companion $\frac{p}{q}$-representation and it is from their relationships that we shall derive in the next section the new results on the power of rational numbers.

5.1. The $\frac{p}{q}$-expansions of real numbers. As announced, the set of $\frac{p}{q}$-expansions is defined a priori and not algorithmically.

Definition 33: The set of expansions in the $\frac{p}{q}$ number systems is $W_{\frac{p}{q}}$.

In other words, an element $\mathbf{a}$ of $W_{\frac{p}{q}}$ is a $\frac{p}{q}$-expansion of the real $x = \pi(\boldsymbol{.}\mathbf{a})$ and conversely any element of $A^{\mathbb{N}}$ which does not belong to $W_{\frac{p}{q}}$ is not a $\frac{p}{q}$-expansion. The following Lemma 34 and Theorem 2 tell respectively that $\frac{p}{q}$-expansions are not too many nor too few and vindicate the definition.

Lemma 34: *The map $\pi\colon W_{\frac{p}{q}} \to \mathbb{R}$ is order preserving.*

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be in $W_{\frac{p}{q}}$. If $\mathbf{a} \sqsubseteq \mathbf{b}$ then, for every k in $\mathbb{N}$, $a_1\, a_2\, \cdots a_k \sqsubseteq b_1\, b_2\, \cdots b_k$ and then, by Corollary 12, $\pi(a_1\, a_2\, \cdots a_k) \leqslant \pi(b_1\, b_2\, \cdots b_k)$. By (9), $\pi(\boldsymbol{.}\, \mathbf{a}) \leqslant \pi(\boldsymbol{.}\, \mathbf{b})$. ∎

By contrast, it follows from the examples given after Corollary 12 that the map $\pi\colon A^{\mathbb{N}} \to \mathbb{R}$ is not order preserving. Let $X_{\frac{p}{q}} = \pi(W_{\frac{p}{q}})$. The elements of $X_{\frac{p}{q}}$ are nonnegative real numbers less than or equal to $\boldsymbol{\omega}_{\frac{p}{q}}$: $X_{\frac{p}{q}} \subseteq [0, \boldsymbol{\omega}_{\frac{p}{q}}]$ (note that $\boldsymbol{\omega}_{\frac{p}{q}} < \frac{p-1}{p-q}$).

Theorem 2: *Every real in $[0, \boldsymbol{\omega}_{\frac{p}{q}}]$ has at least one $\frac{p}{q}$-expansion, that is, $X_{\frac{p}{q}} = [0, \boldsymbol{\omega}_{\frac{p}{q}}]$.*

Proof. By definition, the set $W_{\frac{p}{q}}$ is the set of infinite words w in $A^{\mathbb{N}}$ such that any prefix of w is in $0^*\, L_{\frac{p}{q}}$. As $0^*\, L_{\frac{p}{q}}$ is **prefix-closed** — since $L_{\frac{p}{q}}$ is prefix-closed and the empty word belongs to $L_{\frac{p}{q}}$ — $W_{\frac{p}{q}}$ is closed (see [19]) in the compact set $A^{\mathbb{N}}$, hence compact. Since π is continuous, $X_{\frac{p}{q}}$ is closed.

Suppose that $[0, \boldsymbol{\omega}_{\frac{p}{q}}] \setminus X_{\frac{p}{q}}$ is a nonempty open set, containing a real u. Let $y = \sup\{x \in X_{\frac{p}{q}} : x < u\}$ and $z = \inf\{x \in X_{\frac{p}{q}} : x > u\}$. Since $X_{\frac{p}{q}}$ is closed, y and z both belong to $X_{\frac{p}{q}}$. Let $\mathbf{a} = a_1\, a_2 \cdots$ be the largest $\frac{p}{q}$-expansion of y and $\mathbf{b} = b_1\, b_2 \cdots$ the smallest $\frac{p}{q}$-expansion of z (in the lexicographic order). Of course, $\mathbf{a} \sqsubset \mathbf{b}$ since $\mathbf{a} \neq \mathbf{b}$. Let $a_1 \cdots a_N$ be the **longest common prefix** of $\mathbf{a}$ and $\mathbf{b}$ (with the convention that N can be 0). Set $m = \pi(a_1 \cdots a_N\boldsymbol{.})$, $n = \pi(a_1 \cdots a_N a_{N+1}\boldsymbol{.})$ and $p = \pi(a_1 \cdots a_N b_{N+1}\boldsymbol{.})$. Then

$$\mathbf{a} \sqsubseteq a_1 \cdots a_N a_{N+1} \mathsf{W}(n) \sqsubset a_1 \cdots a_N b_{N+1} \mathsf{w}(p) \sqsubseteq \mathbf{b}.$$

By the choice of $\mathbf{b}$, $\pi(\boldsymbol{.}a_1 \cdots a_N a_{N+1} \mathsf{W}(n)) < z$, and by the choice of $\mathbf{a}$, $\mathbf{a} = a_1 \cdots a_N a_{N+1} \mathsf{W}(n)$. Symmetrically, $\mathbf{b} = a_1 \cdots a_N b_{N+1} \mathsf{w}(p)$.

If $a_{N+1} + q < b_{N+1}$, then there exists a digit c in $\mathsf{d}(m)$ such that $a_{N+1} + q \leqslant c < b_{N+1}$. For any $\mathbf{c}'$ in $A^{\mathbb{N}}$ such that $\mathbf{c} = a_1 \cdots a_N\, c\, \mathbf{c}'$ is in $W_{\frac{p}{q}}$ (and there exist some), we have

$$\mathbf{a} \sqsubset \mathbf{c} \sqsubset \mathbf{b}.$$

Whatever the value of $\pi(\boldsymbol{.}\mathbf{c})$, y or z, we have a contradiction with the extremal choice of $\mathbf{a}$ and $\mathbf{b}$.

If $a_{N+1} + q = b_{N+1}$, then $p = n + 1$ and $z = y$ by Lemma 28, hence a contradiction. And thus $X_{\frac{p}{q}} = [0, \boldsymbol{\omega}_{\frac{p}{q}}]$. ∎

A word in $W_{\frac{p}{q}}$ is said to be **eventually maximal** (resp., **eventually minimal**) if it has a suffix which is a maximal (resp., minimal) word. From Lemma 28, follows

PROPOSITION 35: *If* $\mathbf{a}$ *in* $W_{\frac{p}{q}}$ *is eventually maximal, then* $x = \pi(\boldsymbol{.}\mathbf{a})$ *has another* $\frac{p}{q}$*-expansion which is eventually minimal, and conversely.*

THEOREM 36: *The set of reals in* $X_{\frac{p}{q}}$ *that have more than one* $\frac{p}{q}$*-expansion is countably infinite. The* $\frac{p}{q}$*-expansions of such reals are eventually maximal or eventually minimal.*

We have seen, with Lemma 28, that to every branching node in $T_{\frac{p}{q}}$ corresponds a real with at least two $\frac{p}{q}$-expansions. The theorem will thus be established when the following proposition will be proved.

PROPOSITION 37: *Let* x *be a real in* $[0, \boldsymbol{\omega}_{\frac{p}{q}}]$*, with* more *than one* $\frac{p}{q}$*-expansion. Then* x *has at most* $k+1$ $\frac{p}{q}$*-expansions, where* k *is the least integer strictly greater than* $\frac{q-1}{p-q}$*, and can be associated with at most* k *branching nodes of* $T_{\frac{p}{q}}$*. The smallest of these expansions is eventually maximal and the largest eventually minimal; the others, if any, are both eventually maximal and eventually minimal.*

Proof. Let $R = \pi^{-1}(x) \cap W_{\frac{p}{q}}$ be the (closed) set of $\frac{p}{q}$-expansions of x. Let $\mathbf{a} = a_1\, a_2 \cdots$ be the smallest and $\mathbf{b} = b_1\, b_2 \cdots$ the largest $\frac{p}{q}$-expansion of x (in the lexicographic order) and, as above, let $a_1 \cdots a_N$ be the **longest common prefix** of $\mathbf{a}$ and $\mathbf{b}$ (with the convention that N can be 0).

Let $\mathbf{c} = c_1 c_2 \cdots$ be in R and different from (and thus smaller than) $\mathbf{b}$. We first claim that it does not exist any integer h such that $\pi(\mathbf{b}_{[h]}\boldsymbol{.}) - \pi(\mathbf{c}_{[h]}\boldsymbol{.}) \geqslant k+1$. Suppose the contrary, write $\pi(\mathbf{c}_{[h]}\boldsymbol{.}) = n-1$, $\pi(\mathbf{b}_{[h]}\boldsymbol{.}) = m \geqslant n+k$ and let d be the word of $0^* L_{\frac{p}{q}}$ of length h such that $\pi(d) = n$. It then holds

$$\mathbf{c} \sqsubset d\,\mathsf{w}(n) \sqsubset \mathbf{b}_{[h]}\mathsf{w}(m) \sqsubseteq \mathbf{b},$$

and thus, by Lemma 29,

$$\pi(\boldsymbol{.}\mathbf{c}) \leqslant \pi(\boldsymbol{.}d\,\mathsf{w}(n)) < \pi(\boldsymbol{.}\mathbf{b}_{[h]}\mathsf{w}(m)) \leqslant \pi(\boldsymbol{.}\mathbf{b}),$$

which contradicts $\pi(\boldsymbol{.}\mathbf{c}) = \pi(\boldsymbol{.}\mathbf{b}) = x$. This directly implies that R, which consists of the $\mathbf{c}$ in $W_{\frac{p}{q}}$ such that $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$, contains at most $k+1$ elements and the corresponding subtree of $T_{\frac{p}{q}}$ at most k branching nodes.

Suppose now that for an integer h greater than N, c_{h+1} is not the 'maximal' digit, that is, c_{h+1} is smaller than $\mathsf{Md}(\pi\left(\mathbf{c}_{[h]}\boldsymbol{.}\right))$. Let us write $\mathbf{c}'_h = \mathsf{W}(\pi\left(\mathbf{c}_{[h]}\right))$. It then holds:

$$\mathbf{c} \sqsubset \mathbf{c}_{[h]}\,\mathbf{c}'_h \sqsubset \mathbf{b}.$$

From this we deduce that the sequence of integers h such that c_{h+1} is not the 'maximal' digit is finite (and smaller than k) and thus that $\mathbf{c}$ is eventually maximal. Symmetrically, any $\frac{p}{q}$-expansion in R that is different from (and thus larger than) $\mathbf{a}$ is eventually minimal. ∎

It follows in particular that if $p \geqslant 2\,q$ then no real number has more than two $\frac{p}{q}$-expansions. A simple combinatorial argument allows to widen the condition — and to recover the case $\frac{p}{q} = \frac{3}{2}$.

Corollary 38: *If $p \geqslant 2q - 1$, then no real number has more than two $\frac{p}{q}$-expansions.*

Proof. Suppose $p = 2q - 1$ since the other cases are already settled by Proposition 37. If x has more than two $\frac{p}{q}$-expansions, then by Proposition 37 one is both eventually maximal and eventually minimal and thus eventually written (cf. Proposition 22) on the alphabet:

$$\{0, \ldots, q-1\} \cap \{p-q, \ldots, p-1\} = \{q-1\}$$

reduced to one letter, since $p - q = q - 1$. Contradiction, since no $\frac{p}{q}$-expansion is eventually periodic. ∎

Remark 39: In contrast with the classical representations of reals, the finite prefixes of a $\frac{p}{q}$-expansion of a real number, completed by zeroes, are not $\frac{p}{q}$-expansions of real numbers (though they can be given a value by the function π of course), that is to say, if a non empty word w is in $L_{\frac{p}{q}}$, then the word $w\,0^\omega$ does not belong to $W_{\frac{p}{q}}$.

5.2. The companion $\frac{p}{q}$-representation and the co-converter. A feature of the $\frac{p}{q}$-expansion of the integers is that it is computed 'least significant digit first', that is, from right to left. This is quite a reasonable process for integers, and becomes problematic when it comes to the reals and to the computation from right to left of a representation which is infinite to the right.[5] This difficulty is somewhat overcome with the definition of another $\frac{p}{q}$-representation for the reals; it can be computed with any prescribed precision (provided we can compute in $\mathbb{Q}$ with the same precision) and somehow from left to right. The price we have to pay for this is that we use a larger alphabet of digits, containing **negative digits**, exactly in the same way as the Avizienis representation of reals which uses negative digits and allows to perform addition from left to right (cf. [2]).

Let $\psi\colon \mathbb{R}_+ \to \mathbb{Z}$ be the function defined by:

$$\psi(x) = q \left\lfloor \left(\frac{p}{q}\right) x \right\rfloor - p \lfloor x \rfloor .$$

Lemma 40: *The function ψ is periodic of period q and for all x in $\mathbb{R}_+$, $\psi(x)$ belongs to the digit alphabet*

$$C = \{-(q-1), \ldots, 0, 1, \ldots, p-1\}.$$

Proof. The function ψ is clearly periodic, of period q. It holds:

$$\left(\frac{p}{q}\right) x - \frac{p}{q} < \left(\frac{p}{q}\right) \lfloor x \rfloor \leqslant \left(\frac{p}{q}\right) x < \left\lfloor \left(\frac{p}{q}\right) x \right\rfloor + 1.$$

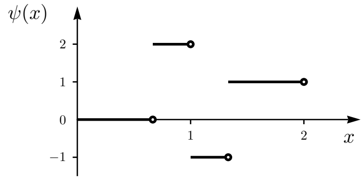
This line being multiplied by q, the two rightmost inequalities give $-q < \psi(x)$ and considering that $q \left\lfloor \left(\frac{p}{q}\right) x \right\rfloor - px$ is nonpositive, the leftmost inequality gives $\psi(x) < p$. ∎

Definition 41: For every x in $\mathbb{R}_+$, the infinite sequence $\varphi(x)$ defined by:

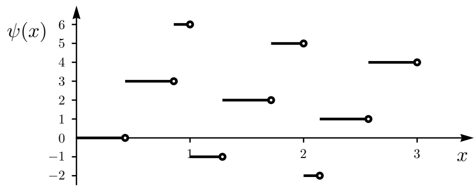
$$\varphi(x) = \mathbf{c} = c_1 c_2 \cdots c_n \cdots \quad \text{with} \quad c_n = \psi\left(\left(\frac{p}{q}\right)^{n-1} x\right) \quad \text{for every } n \geq 1$$

is called the **companion $\frac{p}{q}$-representation** of x.

[5] As W. Allen said: "The infinite is pretty far, especially towards the end."



(a) $\frac{p}{q} = \frac{3}{2}$



(b) $\frac{p}{q} = \frac{7}{3}$

Figure 7. The function ψ

If $q = 1$, c_n is precisely the n-th digit after the radix point in the expansion of x in base p. An obvious computation yields, with the notation of Definition 41,

$$\pi(c_1 \cdots c_n \cdot) = \left\lfloor \left(\frac{p}{q}\right)^n x \right\rfloor - \left(\frac{p}{q}\right)^n \lfloor x \rfloor \tag{14}$$

from which the name 'companion representation' is easily justified (recall that $\{x\}$ denote the **fractional part** of the number x: $\{x\} = x - \lfloor x \rfloor$):

Proposition 42: *For every x in $\mathbb{R}_+$, $\varphi(x)$ is a $\frac{p}{q}$-representation of $\{x\}$.*

Proof. From (9), it comes:

$$\pi(\cdot\varphi(x)) = \lim_{n\to\infty} \left(\frac{q}{p}\right)^n \pi(c_1 \cdots c_n \cdot) = \lim_{n\to\infty} \left(\frac{q}{p}\right)^n \left\lfloor \left(\frac{p}{q}\right)^n x \right\rfloor - \lfloor x \rfloor = x - \lfloor x \rfloor$$

since the limit when n tends to infinity of $\left(\frac{q}{p}\right)^n \left\lfloor \left(\frac{p}{q}\right)^n x \right\rfloor$ is x. ∎

Let x be in $[0, \boldsymbol{\omega}_{\frac{p}{q}}]$, $\langle x \rangle_{\frac{p}{q}} = \mathbf{a} = a_1 a_2 \cdots$ its $\frac{p}{q}$-expansion, and $\varphi(x) = \mathbf{c} = c_1 c_2 \cdots$ its companion representation. As in Lemma 27, we note $\rho_n(x) = \left\lfloor \pi(\cdot a_{n+1} a_{n+2} \cdots) \right\rfloor$ and it holds: $0 \leq \rho_n(x) < \frac{p-1}{p-q}$. From the same lemma, for $k = n$ and $k = n-1$, and from the definition of c_n:

$$c_n = \psi\left(\left(\frac{p}{q}\right)^{n-1} x\right) = q \left\lfloor \left(\frac{p}{q}\right)^n x \right\rfloor - p \left\lfloor \left(\frac{p}{q}\right)^{n-1} x \right\rfloor,$$

it comes, since $q\,\pi(a_1 \cdots a_n) = p\,\pi(a_1 \cdots a_{n-1}) + a_n$:

$$c_n + p\,\rho_{n-1}(x) = a_n + q\,\rho_n(x). \tag{15}$$

Definition 43: Let $\mathcal{A}_{\frac{p}{q}} = \langle\, H, C \times A, F, H, H\,\rangle$ be the letter-to-letter (left) transducer with set of states $H = \{h \in \mathbb{N} \mid 0 \leqslant h < \frac{p-1}{p-q}\}$ and whose set of transitions F is defined by:

$$\bigl(h, (c,a), h'\bigr) \in F \quad\Longleftrightarrow\quad p\,h + c = q\,h' + a. \tag{16}$$

By comparison with (6):

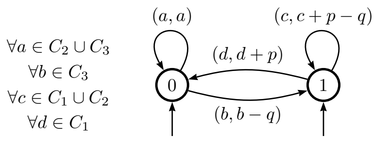
$$\bigl(z, (d,a), z'\bigr) \in E \quad\Longleftrightarrow\quad q\,z + d = p\,z' + a. \tag{6}$$

we recognize that $\mathcal{A}_{\frac{p}{q}}$ is the transposed automaton of the converter $\mathcal{C}_C$ (once the label of the states have been changed to their opposite). Equation (15) amounts then to the proof of the following.

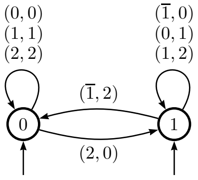
THEOREM 44: *Let x be a real in $[0, \boldsymbol{\omega}_{\frac{p}{q}}]$, $\mathbf{c}$ its companion representation and $\mathbf{a}$ a $\frac{p}{q}$-expansion of x. Then $(\mathbf{c}, \mathbf{a})$ is the label of an infinite path which begins in the state $\rho_0(x) = \lfloor x \rfloor$ in the transducer $\mathcal{A}_{\frac{p}{q}}$.*

Let us write the digit alphabet $C = \{-(q-1), \ldots, 0, 1, \ldots, p-1\}$, the image of the function ψ, as the disjoint union $C = C_1 \cup C_2 \cup C_3$ with $C_1 = \{-(q-1), \ldots, -1\}$, $C_2 = \{0, \ldots, q-1\}$ and $C_3 = \{q, \ldots, p-1\}$.

If $p \geqslant 2\,q-1$, the interesting case which we have already considered, $\mathcal{A}_{\frac{p}{q}}$ has then only two states. The transducer $\mathcal{A}_{\frac{p}{q}}$ is drawn at Figure 8 (a) and the case $\frac{p}{q} = \frac{3}{2}$ at Figure 8 (b) (compare with Figure 4 (b) above).



(a) The general case $\mathcal{A}_{\frac{p}{q}}$, $p \geqslant 2\,q-1$.



(b) The transducer $\mathcal{A}_{\frac{3}{2}}$.

Figure 8. The transducer that converts the companion representation into a $\frac{p}{q}$-expansion.

The computation of the companion representation is the first step of the "algorithm" for the computation of $\frac{p}{q}$-expansions of the real numbers. Let x be in $[0, \boldsymbol{\omega}_{\frac{p}{q}}]$, and let $\mathbf{c}$ be its companion representation. Let n be a fixed (large) positive integer and $v = c_1 \cdots c_n$ be the prefix of length n of $\mathbf{c}$. When v is read from right to left by the converter $\mathcal{C}_C$, which is the transposed of $\mathcal{A}_{\frac{p}{q}}$, and taking a state s as initial state, the output is a word $f^{(s)}$ of length n on the alphabet A and which depends upon s. The maximal common prefix of all these words $f^{(s)}$ is the beginning of all the $\frac{p}{q}$-expansions of x.

To get longer prefixes one has to make the computation again with an n' larger than n, but it is not possible to know in advance how large this n' has to be in order to get a better approximation.

A characterization of the companion representation of the reals that have multiple $\frac{p}{q}$-expansions is given now.

Proposition 45: *Suppose that $p \geq 2q-1$. A real x has two $\frac{p}{q}$-expansions if and only if its companion representation is eventually in ${C_2}^{\mathbb{N}}$.*

Proof. Under the hypothesis, $\mathcal{A}_{\frac{p}{q}}$ has only two states, labeled with 0 and 1, and if a digit c belongs to C_1 (resp., to C_3), then a transition labeled (c, a) goes out from state 1 (resp., from state 0).

Let x be a real and $\mathbf{c}$ its companion representation. By Theorem 2, x has at least one $\frac{p}{q}$-expansion $\mathbf{a}$ and by Theorem 44, $(\mathbf{c}, \mathbf{a})$ is the label of an infinite path in $\mathcal{A}_{\frac{p}{q}}$.

If $\mathbf{c}$ is not eventually in $C_2^{\mathbb{N}}$, then there is an increasing sequence of indices (n_i) such that c_{n_i} belongs to $C_1 \cup C_3$. The state from which the transition starts labeled $(c_{n_i}, .)$ is uniquely determined. As the transducer $\mathcal{A}_{\frac{p}{q}}$ is co-deterministic, i.e. input co-deterministic, this implies — by reading backwards from the indices where the state is determined — that the infinite path labeled by $(\mathbf{c}, \mathbf{a})$ is unique and x can have only one $\frac{p}{q}$-expansion.

Assume now that $\mathbf{c}$ is eventually in $C_2^{\mathbb{N}}$, that is, there exists $N > 0$ such that for any $n \geq N$, c_n belongs to C_2. In the transducer $\mathcal{A}_{\frac{p}{q}}$ there are no transitions from state 0 to state 1, or from state 1 to state 0, with input label in C_2. Thus the path with label $(\mathbf{c}, \mathbf{a})$ stays eventually in state 0 or in state 1. Suppose it stays eventually in state 0, that is $\mathbf{a}$ stays eventually in $C_2^{\mathbb{N}}$ as well, hence is a minimal word. By Proposition 35, x has another $\frac{p}{q}$-expansion. Conversely if the path labeled by $(\mathbf{c}, \mathbf{a})$ stays eventually in state 1, then $\mathbf{a}$ stays eventually in $C_3^{\mathbb{N}}$, hence is a maximal word, and for the same reason x has another $\frac{p}{q}$-expansion. ∎

6. On the fractional part of the powers of rational numbers

We are now in a position to explain how the characterization of the $\frac{p}{q}$-expansions of the reals applies to the study of the distribution of the powers of a rational number as presented in the introduction, how it allows to prove Theorem 3 and how close it is from the original description of the 'conjectured' Z-numbers. For that purpose, we first give the description of the inverse of the function ψ — in the case that really interest us, namely, when $p \geqslant 2\,q-1$ — and for easiness of writing, of the function $\psi(q\,x)$ indeed.

Lemma 46: *Suppose $p \geq 2q-1$. For every c in $C_2 = \{0,\ldots,q-1\}$ let k_c be the unique integer in $A = \{0,\ldots,p-1\}$ such that $q\,k_c = c \pmod{p}$. Then $\psi(q\,x) = c$ if, and only if, $\{x\}$ belongs to the interval $\left[\frac{1}{p}k_c, \frac{1}{p}(k_c+1)\right[$.*

Proof. The uniqueness of k_c follows from the fact that p and q are coprime: $q\,(k-k') = 0 \pmod{p}$ implies $k = k' \pmod{p}$ and thus $k = k'$ if k and k' are both in A. For the same reason

$$\psi(q\,x) = q\,\lfloor p\,x\rfloor - p\,\lfloor q\,x\rfloor = c$$

implies that there exists a unique pair (k_c, j_c) such that $\lfloor px\rfloor = k_c$ and $\lfloor qx\rfloor = j_c$ and with k_c in A and j_c in C_2.

Hence $\psi(q\,x) = c$ if, and only if, $x \in \left[\frac{1}{q}j_c, \frac{1}{q}(j_c+1)\right[\bigcap \left[\frac{1}{p}k_c, \frac{1}{p}(k_c+1)\right[$. By hypothesis on c, and on p and q, it holds

$$0 \leqslant q\,k_c - p\,j_c \leqslant q-1 \leqslant p-q.$$

Dividing these inequalities by $p\,q$ it comes $\frac{1}{q}j_c \leqslant \frac{1}{p}k_c$ and $\frac{1}{p}(k_c+1) \leqslant \frac{1}{q}(j_c+1)$ and the lemma holds. ∎

Notation 47: For a fixed rational $\frac{p}{q}$ we denote by $Y_{\frac{p}{q}}$ the subset:

$$Y_{\frac{p}{q}} = \bigcup_{0\leqslant c\leqslant q-1} \left[\frac{1}{p}k_c, \frac{1}{p}(k_c+1)\right[$$

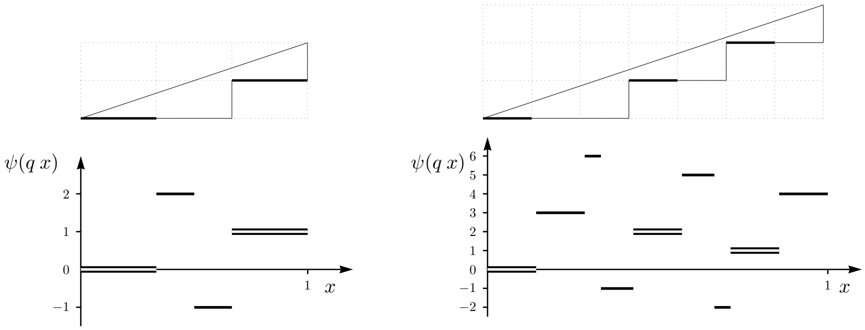
where the k_c's are defined as in Lemma 46.

The set $Y_{\frac{p}{q}}$ is a subset of $\left[0,1\right[$ that consists of the union of q intervals of length $\frac{1}{p}$. For instance:

$$Y_{\frac{3}{2}} = \left[0, \frac{1}{3}\right[\cup \left[\frac{2}{3}, 1\right[.$$

Lemma 46 may be reworded as $\psi(q\,x) \in C_2$ if and only if $\{x\} \in Y_{\frac{p}{q}}$.

Remark 48: Loosely speaking, the set $Y_{\frac{p}{q}}$ corresponds to the way of distributing as evenly as possible q intervals of length $\frac{1}{p}$ inside $\left[0,1\right[$. Another way to describe $Y_{\frac{p}{q}}$ is, as we try to represent at Figure 9, to consider the *Christoffel word* that connects the origin to the point (p,q): this word contains q occurrences of b's and q runs of a's with p a's in total. If the abscissa is scaled by $\frac{1}{p}$, the first a in each of the q runs corresponds to one of the intervals that compose $Y_{\frac{p}{q}}$. (For Christoffel words, see [14], for instance.)



(a) $\frac{p}{q} = \frac{3}{2}$ $k_0 = 0$, $k_1 = 2$. (b) $\frac{p}{q} = \frac{7}{3}$ $k_0 = 0$, $k_1 = 5$, $k_2 = 3$.

Figure 9. The function ψ and the subset $Y_{\frac{p}{q}}$.

The generalized Mahler's notation is then:

$$\mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right) = \left\{ z \geq 0 \;\middle|\; \exists N \in \mathbb{N} \quad \forall n \geqslant N \; \left\{ z \left(\frac{p}{q}\right)^n \right\} \in Y_{\frac{p}{q}} \right\}.$$

These notations being given, Theorem 3 reads then:

THEOREM 3: *If $p \geqslant 2q-1$, $\mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ is countably infinite.*

It is indeed a direct consequence of the following:

THEOREM 49: *Let $p \geq 2q-1$. A positive real z belongs to $\mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ if and only if $q\,z$ has two $\frac{p}{q}$-expansions.*

Proof. From Proposition 45 follows that a real x has two $\frac{p}{q}$-expansions if and only if, there exists $N > 0$ such that for any $n > N$, $c_n = \psi\left(x\left(\frac{p}{q}\right)^{n-1}\right)$ belongs to C_2, and by Lemma 46, if, and only if,

$$\left\{ \left(\frac{p}{q}\right)^{n-1} \frac{x}{q} \right\} \in \bigcup_{0 \leqslant c \leqslant q-1} [\frac{1}{p}k_c, \frac{1}{p}(k_c+1)[= Y_{\frac{p}{q}}$$

and this concludes the proof. ∎

As one can consider arbitrarily large rationals $\frac{p}{q}$ that meet the condition $p \geq 2q - 1$, it then comes:

Corollary 50: *For any $\varepsilon > 0$, there exists a rational $\frac{p}{q}$ and a subset $Y_{\frac{p}{q}} \subseteq \left[0, 1\right[$ of Lebesgue measure less than ε such that $\mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ is countably infinite.*

Let us now come back to the original paper of Mahler. A so-called Z-number is a real number z such that for every n, $\left\{z\ \left(\frac{3}{2}\right)^n\right\} \in \left[0, \frac{1}{2}\right[$. With slight changes in Mahler's original notation, we write the decomposition into integer/fractional parts as

$$z\ \left(\frac{3}{2}\right)^n = h_n + r_n .$$

As r_{n+1} is in $\left[0, \frac{1}{2}\right[$, it follows that if h_n is even, then r_n is in $\left[0, \frac{1}{3}\right[$, and if h_n is odd, then r_n is in $\left[\frac{1}{3}, \frac{1}{2}\right[$. This implies that either

$$\frac{z}{2}\left(\frac{3}{2}\right)^n = \frac{h_n}{2} + \frac{r_n}{2} \quad \text{with } \frac{r_n}{2} \in \left[0, \frac{1}{6}\right[$$

or

$$\frac{z}{2}\left(\frac{3}{2}\right)^n = \frac{h_n - 1}{2} + \frac{r_n + 1}{2} \quad \text{with } \frac{r_n + 1}{2} \in \left[\frac{2}{3}, \frac{3}{4}\right[$$

holds according to the parity of h_n. In other words:

$$\left\{\frac{z}{2}\left(\frac{3}{2}\right)^n\right\} \quad \in \quad \left[0, \frac{1}{6}\right[\bigcup \left[\frac{2}{3}, \frac{3}{4}\right[\tag{17}$$

holds for all n and Theorem 49 implies that z has two $\frac{3}{2}$-expansions.

Furthermore, under the assumption that z is a Z-number, Mahler computes an 'expansion' in base $\frac{3}{2}$ of $r_0 = \{z\}$, which is a sequence of 0's and 1's and shows it is unique for a given $h_0 = \lfloor z \rfloor$. In our setting, this sequence is $\mathsf{w}(h_0)$, the minimal word in $T_{\frac{3}{2}}$ which starts at a node labelled by h_0. Mahler noticed that his expansion of the fractional part of a Z-number must meet further constraints — such as to contain no factor 11. The proof of the nonexistence of Z-numbers is now transferred to the study of the minimal words, which exist, and to the proof that no integer n exists such that $\mathsf{w}(n)$ meets the above mentioned constraints.

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