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LARGE GROUPS OF DEFICIENCY 1

BY

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ABSTRACT

We prove that if a group possesses a deficiency 1 presentation where one of the relators is a commutator, then it is $\mathbb{Z} \times \mathbb{Z}$, large or is as far as possible from being residually finite. Then we use this to show that a mapping torus of an endomorphism of a finitely generated free group is large if it contains a $\mathbb{Z} \times \mathbb{Z}$ subgroup of infinite index, as well as showing that such a group is large if it contains a Baumslag–Solitar group of infinite index and has a finite index subgroup with first Betti number at least 2. We give applications to free by cyclic groups, 1 relator groups and residually finite groups.

1. Introduction

Recall [45] that a finitely generated group G is large if it has a finite index subgroup possessing a homomorphism onto a non-abelian free group. This is a strong property and implies that G contains a non-abelian free subgroup [42], G is SQ-universal [45] (every countable group is a subgroup of a quotient of G), G has finite index subgroups with arbitrarily large first Betti number [37], G has uniformly exponential word growth [25], as well as having subgroup growth of strict type n^n (which is the largest possible growth for finitely generated groups) [38], and the word problem for G is solvable strongly generically in linear time [32]. Thus on proving that G is large we obtain all these other properties for free.

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There have been a range of results that give criteria for finitely generated or finitely presented groups to be large. Starting with B. Baumslag and S. J. Pride [2] which showed that groups with a presentation of deficiency at least 2 are large, we then have in [23] a condition that implies this result, as well as a proof that a group with a deficiency 1 presentation in which one of the relators is a proper power is large. This latter result was also independently derived by Stöhr in [48] and was followed by conditions for a group with a deficiency 0 presentation where some of the relators are proper powers to be large, due to Edjvet in [19]. Then further conditions for a finitely presented group to be large, all of which imply the Baumslag-Pride result, are by Howie in [29], G. Baumslag in [6] and a characterisation by Lackenby in [34]. In Section 2, we give a criterion, based on the Howie result, for a finitely presented group G to be large which is purely in terms of the Alexander polynomial of G and is straightforward to use in practice. This result is particularly powerful in the case of deficiency 1 groups which are then our focus for much of the rest of the paper. Of course, unlike groups of deficiency 2 or higher, not all groups of deficiency 1 are large: think of $\mathbb{Z}$ or the soluble Baumslag-Solitar groups given by the presentations $\langle x, y | xyx^{-1} = y^m \rangle$ for $m \in \mathbb{Z}\backslash\{0\}$. Other examples of non-large deficiency 1 groups were given by Pride and Edjvet in [20] consisting of those Baumslag-Solitar groups $\langle x, y | xy^l x^{-1} = y^m \rangle$ for $l, m \neq 0$ where l and m are coprime, as well as some HNN extensions of these.

As for large groups of deficiency 1, we have already mentioned those with a relator that is a proper power and we again have examples in [20] with Theorem 6 stating that the group $\langle x, y | x^n y^l x^{-n} = y^m \rangle$ for $l, m, n \neq 0$ is large if $|n| > 1$ or if l and m are not coprime. Further results of a more technical nature which give largeness for some other 2 generator 1 relator presentations are in [18]. At this point it seems difficult to say convincingly either way whether groups of deficiency 1 are generally large. In this paper we hope to offer substantial evidence that largeness is a natural property to expect in a deficiency 1 group. Although we will display a few new groups of deficiency 1 which are not large in Example 3.5(ii), our main results are on establishing families of deficiency 1 groups which are all large. In Section 3 we introduce the concept of a non-abelian residually abelianised (NARA) group and this has a number of equivalent definitions, one of which is that it is finitely generated and non-abelian but has no non-abelian finite quotients; the idea being that a NARA group G is as far from being residually finite as possible because we cannot distinguish G from its abelianisation

G/G' by just looking at finite index subgroups. We obtain Theorem 3.6 which states that if G has a deficiency 1 presentation in which one of its relators is a commutator then $G = \mathbb{Z} \times \mathbb{Z}$ or G is NARA with abelianisation $\mathbb{Z} \times \mathbb{Z}$ or G is large.

In [18] from 1984 it is asked if those groups which are an extension of a finitely generated non-abelian free group by $\mathbb{Z}$ are large. They are certainly torsion free groups with a natural deficiency 1 presentation and are also called mapping tori of finitely generated non-abelian free group automorphisms. These groups appear to make up a sizeable class of deficiency 1 groups but we can expand this class considerably by allowing arbitrary endomorphisms in place of automorphisms to obtain groups which are ascending HNN extensions of finitely generated free groups. Such groups have been the attention of much recent research where significant progress has been made. In particular these groups have been shown to be coherent (every finitely generated subgroup is finitely presented) in [21], Hopfian in [22] and even residually finite in [13]. If largeness were added to this list (on removing the obvious small exceptions), then it would show that such an HNN extension, indeed even a group which is virtually such an HNN extension, has all the nice properties that one could reasonably hope for.

In Section 4, we apply our results to show that for G a mapping torus of a finitely generated free group endomorphism, we have G is large if it contains a $\mathbb{Z} \times \mathbb{Z}$ subgroup of infinite index. Also G is large if it contains a Baumslag-Solitar subgroup and has a finite index subgroup H ($\neq \mathbb{Z} \times \mathbb{Z}$) with $\beta_1(H) \geq 2$. Of course if $\beta_1(H) = 1$ for all H, then we would have an example of such a G which is not large. However, we know of no examples apart from the soluble Baumslag-Solitar groups themselves, and it seems believable that no other G has this property.

In Section 5, we restrict to groups G of the form F-by-$\mathbb{Z}$ where F is free. By Section 4 G is large if it contains $\mathbb{Z} \times \mathbb{Z}$ and $F = F_n$ is of finite rank $n \geq 2$. It is known by [8], [9] and [14] that these are exactly the groups of the form F_n-by-$\mathbb{Z}$ which are not word hyperbolic. We also show that if F is of infinite rank but G is finitely generated then G is, in fact, large. By combining these results with known facts about word hyperbolic groups, this allows us to prove in Theorem 5.4 that if G is any finitely generated group which is virtually free-by-$\mathbb{Z}$ then (apart from the obvious small exceptions) G is SQ-universal, has uniformly exponential growth and has a word problem that is solvable strongly generically in linear time. This is also true for the finitely generated subgroups of G.

Section 6 looks at 1 relator groups G, where we need only consider the case where G has a 2 generator 1 relator presentation. We know that by Section 3 we obtain largeness unless $G = \mathbb{Z} \times \mathbb{Z}$ (which is easily detected in the 1 relator case) or G is NARA. It is true that 2 generator 1 relator groups which are NARA exist, but if we insist that the relator is a product of commutators, then no examples are known; indeed it was only recently that non-residually finite examples of such groups were given in [43] Problem (OR7). Moreover, if the relator is a single commutator, then no examples are known that fail even to be residually finite (this is Open Problem (OR8) in [43]) so in this case not being NARA and hence large seems very likely. We give methods that show this in practice for a given presentation. Finally, in Section 7, we make the straightforward but useful observation that a group G is large if and only if the quotient of G by its finite residual is large, suggesting that the best setting in which to examine largeness is the residually finite case. We prove that a residually finite group with infinitely many ends is large (this is most definitely not true if residual finiteness is removed) and examine finitely presented groups which are LERF, which is a strengthening of being residually finite.

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2. A Condition for Largeness

We quote the following facts that we will need about the Alexander polynomial of a finitely presented group; see [35]. Let G be given by a finite presentation $\langle x_1, \ldots, x_n |\ r_1, \ldots, r_m \rangle$ and let G' be the derived (commutator) subgroup of G. Then the abelianisation $\overline{G} = G/G'$ is a finitely generated abelian group $\mathbb{Z}^{\beta_1(G)} \times T$ for T the torsion subgroup whereas the free abelianisation $ab(G)$ is $\mathbb{Z}^{\beta_1(G)}$. On taking any surjective homomorphism $\chi : G \to \mathbb{Z}$, we have the Alexander polynomial $\Delta_{G,\chi} \in \mathbb{Z}[t^{\pm 1}]$ which is a Laurent polynomial up to the ambiguity of multiplication by the units $\pm t^k$ for $k \in \mathbb{Z}$. It is defined in the following way: on taking t to be an element in G with $\chi(t) = 1$ we have that t acts by conjugation on $H_1(\ker\chi; \mathbb{Z})$, so $H_1(\ker\chi; \mathbb{Z})$ is a module over the group ring $\mathbb{Z}[t^{\pm 1}]$ of the integers. It is easy to see that this is a finitely presented module, for instance we could use the Reidemeister-Schreier rewriting process to obtain a presentation of kerχ from that of G and then abelianise, which would result here in an $(n-1) \times m$ presentation matrix. Thus we have the first elementary ideal which is generated by the maximal minors, these being the determinants of the matrices left over when we cross off the correct number of columns to make the resulting matrix square (here we are assuming there are at least as many columns as rows, or else we let the first elementary ideal and the Alexander polynomial be zero). Note that this ideal is independent of the particular presentation matrix chosen for $H_1(\ker\chi;\mathbb{Z})$. The definition of the Alexander polynomial $\Delta_{G,\chi}(t)$ is then the generator (up to units) of the smallest principal ideal containing the first elementary ideal, or equivalently the highest common factor of the maximal minors.

The next point is the crucial fact which allows us to use the Alexander polynomial to detect largeness.

Theorem 2.1: *If G is a finitely presented group which has a homomorphism χ onto $\mathbb{Z}$ such that $\Delta_{G,\chi} = 0$, then G is large.*

Proof. We have seen that $H_1(\ker\chi;\mathbb{Z})$ is a finitely presented module over $\mathbb{Z}[t^{\pm 1}]$ but we can also take rational coefficients and use the fact that $H_1(\ker\chi;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = H_1(\ker\chi;\mathbb{Q})$ is a finitely presented module over $\mathbb{Q}[t^{\pm 1}]$ where t acts in the same way, and we even have the same presentation matrix. Thus we can define the Alexander polynomial over $\mathbb{Q}$ exactly as above in terms of the first elementary ideal, and it will be the same polynomial as for $\mathbb{Z}$, except that now it is only defined up to units of $\mathbb{Q}[t^{\pm 1}]$ which are now $qt^{\pm n}$ for $q \in \mathbb{Q}\backslash\{0\}$. However, note that $\Delta_{G,\chi}$ is zero over $\mathbb{Z}$ if and only if it is zero over $\mathbb{Q}$. The advantage of moving to rational coefficients is that $\mathbb{Q}[t^{\pm 1}]$ is a principal ideal domain, so by the structure theorem it is a direct sum of cyclic modules. Thus the presentation matrix P can be put into canonical form in which all off-diagonal entries are zero and the diagonal entries are $d_1, \ldots, d_k$ for $d_i \in \mathbb{Q}[t^{\pm 1}]$. By evaluating the first elementary ideal we see that the Alexander polynomial over $\mathbb{Q}$ is $d_1 \ldots d_k$ and this is zero if and only if some d_i is zero which happens if and only if $H_1(\ker\chi;\mathbb{Q})$ has a free $\mathbb{Q}[\mathbb{Z}]$-module of at least rank 1 in its decomposition.

Now we invoke Howie's condition for largeness in [29, Section 2]. Adopting that notation, we let K be the standard connected 2-complex obtained from our finite presentation of G, with $N = \ker\chi$ and $\overline{K}$ the 2-complex which is the regular covering of K corresponding to N so that $\pi_1(\overline{K}) = N$. Let $\mathbb{F}$ be a field: on following through the proof of [29, Proposition 2.1], we see that if $H_1(\overline{K};\mathbb{F})$ contains a free $\mathbb{F}[\mathbb{Z}]$-module of rank at least 1 then the conclusion of the proposition holds. But this is the hypothesis of [29, Theorem 2.2] which proves that for any sufficiently large n the finite index subgroup NG^n admits a homomorphism onto the free group of rank 2.

In our case we have on setting $\mathbb{F}=\mathbb{Q}$ that $H_1(\overline{K};\mathbb{Q}) = H_1(\ker\chi;\mathbb{Q})$ so if $\Delta_{G,\chi}=0$ we conclude that G is large. ∎

NOTE: the above also works if we take $\mathbb{F}$ to be $\mathbb{Z}/p\mathbb{Z}$ and $\Delta_{G,\chi}$ vanishes over this field.

COROLLARY 2.2: *If G is a finitely presented group possessing a homomorphism to $\mathbb{Z}$ with kernel having infinite rational first Betti number then G is large.*

Proof. We have by definition that $\beta_1(\ker\chi;\mathbb{Q})$ is the dimension of $H_1(\ker\chi;\mathbb{Q})$ as a vector space over $\mathbb{Q}$. It is also the degree of the Alexander polynomial $\Delta_{G,\chi}$ (where the degree of a Laurent polynomial in t is the degree of the highest non-zero power of t minus the degree of the lowest) by [35, Theorem 6.17] or [40, Section 4]. In particular, $\Delta_{G,\chi}=0$ if and only if $\beta_1(\ker\chi;\mathbb{Q})$ is infinite, so this claim now follows directly from Theorem 2.1. ∎

NOTE: The Corollary is most definitely not true for all finitely generated groups; we do require a finite number of relators too, as can be seen by the example of the restricted wreath product $\mathbb{Z}\wr\mathbb{Z}$.

3. Deficiency 1 groups

The deficiency of a finite presentation is the number of generators minus the number of relators and the deficiency $def(G)$ of a finitely presented group G is the maximum deficiency over all presentations. (It is bounded above by $\beta_1(G)$ so is finite.) We know that groups of deficiency at least 2 are large so it seems reasonable to ask whether we can use our criterion to obtain large groups with lower deficiencies, for instance deficiency 1. In fact this case turns out to be a very fruitful choice, both from the point of view that calculating the Alexander polynomial of a deficiency 1 group is more efficient than for lower deficiencies, and because of the behaviour of deficiency in finite covers. Given a presentation for a group G with n generators and m relators and an index i subgroup H of G, we can use Reidemeister-Schreier rewriting to obtain a presentation for H

of G with $(n-1)i+1$ generators and mi relators, thus the deficiency of H is at least $(def(G)-1)i+1$. So if $def(G)=1$ then either $def(H)=1$ for all $H \leq_f G$ or H, and thus G, is large anyway by [2].

Theorem 3.1: *If G is a group with a deficiency 1 presentation*

$$\langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1} \rangle$$

where one of the relators is of the form $x_i x_j x_i^{-1} x_j^{-1}$ then G is large if the subgroup of $ab(G)$ generated by the images of x_i and x_j has infinite index.

Proof. Without loss of generality we can reorder the generators and so we can assume we have the relator $x_1 x_2 x_1^{-1} x_2^{-1}$. As $ab(G)$ is a free abelian group $\mathbb{Z}^{\beta_1(G)}$ of finite rank, we have that x_1 and x_2 generate a free abelian subgroup of strictly smaller rank. Therefore there must exist a surjective homomorphism $\chi : G \to \mathbb{Z}$ with x_1 and x_2 in the kernel, as well as coprime integers $k_3, \ldots, k_n$ such that $k_3\chi(x_3) + \cdots + k_n\chi(x_n) = 1$. Therefore there exists a matrix $M \in GL(n-2, \mathbb{Z})$ such that its first column is $(k_3, \ldots, k_n)$ and this gives rise to an automorphism β of $\mathbb{Z}^{n-2}$ sending the standard basis $e_3, \ldots, e_n$ (where we think of e_i as the image of the generator x_i in the abelianisation $\mathbb{Z}^n$ of the free group F_n) to a new basis $b_3, \ldots, b_n$. Now by [39] I.4.4, we have an automorphism α of F_n that fixes x_1, x_2 and induces β on $e_3, \ldots, e_n$. On rewriting our presentation in terms of $y_1 = \alpha(x_1), \ldots, y_n = \alpha(x_n)$, we now have $\chi(y_3) = 1$ and so we can regard $H_1(\ker\chi; \mathbb{Z})$ as a $\mathbb{Z}[t^{\pm 1}]$ module where t is equal to y_3 and acts by conjugation. We can obtain a presentation matrix P for this module by performing Reidemeister-Schreier rewriting on G using $\{t^j : j \in \mathbb{Z}\}$ as a Schreier transversal. We find that our original relation $x_1 x_2 x_1^{-1} x_2^{-1}$ becomes the set of group relations $x_{1,j} x_{2,j} x_{1,j}^{-1} x_{2,j}^{-1}$ where $x_{1,j} = t^j x_1 t^{-j}$ and $x_{2,j} = t^j x_2 t^{-j}$. To obtain the equivalent relation for P, we abelianise and regard each of these group relations as the same module relation multiplied by powers of t. But this becomes zero, thus giving us a zero column in P.

The crucial point about the group presentation having deficiency one is that this makes P a square matrix (of size $n-1$). This means that the Alexander polynomial $\Delta_{G,\chi}$ is merely the determinant of P, which must be zero owing to the zero column, hence we have largeness of G by Theorem 2.1. ∎

Corollary 3.2: *If $G = \langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1} \rangle$ has a deficiency 1 presentation with a relator $r_1 = x_1 x_2 x_1^{-1} x_2^{-1}$ and the abelianisation $\overline{G} = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ for $m \geq 2$, then G is large.*

Proof. We are done by Theorem 3.1 unless the images $\overline{x_1}, \overline{x_2}$ in $\overline{G}$ generate a finite index subgroup S of $\overline{G}$, but if so, then S must have $\mathbb{Z}$-rank equal to that of $\overline{G}$, which is 2. However, S is generated by two elements so in this case S can only be isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Now take a homomorphism θ from G onto $\mathbb{Z}/j\mathbb{Z}$ for some $j \geq 2$ such that S is in the kernel. We require another generator $g \in \{x_3, \ldots, x_n\}$ such that $\theta(g)$ generates $\mathrm{Im}\theta$ but this can be achieved by taking an appropriate automorphism of the free group of rank n that fixes x_1 and x_2, just as in Theorem 3.1. We now perform Reidemeister-Schreier rewriting to obtain from our original presentation of G a deficiency 1 presentation for $\ker\theta$ consisting of $nj + 1$ generators and nj relators. We have g^i, $0 \leq i < j$ as a Schreier transversal for $\ker\theta$ in G and on setting $x_{1,i} = g^i x_1 g^{-i}$ and $x_{2,i} = g^i x_2 g^{-i}$, which will all be amongst the generators for our presentation of $\ker\theta$ given by this process (because $x_1, x_2 \in S \leq \ker\theta$), our original relator r_1 gives rise to j relators $x_{1,i} x_{2,i} x_{1,i}^{-1} x_{2,i}^{-1}$ in the presentation for our subgroup. As these disappear when we abelianise, we see that $\beta_1(\ker\theta)$ is at least $j + 1$ and we are done by Theorem 3.1. ∎

It might be felt that requiring two generators to commute in a deficiency 1 presentation is rather restrictive but most of the rest of our results are based on finding deficiency 1 groups G which have a finite index subgroup H possessing such a presentation. This means $\beta_1(H) \geq 2$ and Corollary 3.2 will apply unless the abelianisation $\overline{H} = \mathbb{Z} \times \mathbb{Z}$. We now discuss a generalisation of the property of being residually finite which allows us to avoid this exception.

Recall that a group G is residually finite if the intersection R_G over all the finite index subgroups $F \leq_f G$ is the trivial group I. Although this works perfectly well as a general definition, it is most useful when G is finitely generated, and that will be our assumption here. Our motivation for the next definition is to ask: how badly can a group fail to be residually finite and what is the worst possible case? The first answer that would come to mind is when G ($\neq I$) has no proper finite index subgroups at all, but we are dealing with groups possessing positive first Betti number and hence infinitely many subgroups of finite index. By noting that elements outside the commutator subgroup G' cannot be in R_G, we obtain our condition.

Definition 3.3: We say that the finitely generated group G is **residually abelianised** if

$$G' = \bigcap_{F \leq_f G} F.$$

If further G is non-abelian then we say it is **NARA (non-abelian residually abelianised)**.

Note that by excluding G being abelian, we have that G residually finite implies G is not NARA. The definition has many equivalent forms but the general idea is that a NARA group cannot be distinguished from its abelianisation if one only uses standard information about its finite index subgroups.

PROPOSITION 3.4: *Let G be finitely generated and non-abelian with commutator subgroup G', abelianisation $\overline{G} = G/G'$ and let R_G be the intersection of the finite index subgroups of G. The following are equivalent:*

(i) *G is NARA.*
(ii) *G has no non-abelian finite quotient.*
(iii) *G has no non-abelian residually finite quotient.*
(iv) *If $a_n(G)$ denotes the number of finite index subgroups of G having index n then $a_n(G) = a_n(\overline{G})$ for all n.*
(v) *For all $F \leq_f G$ we have $F' = G'$.*
(vi) *For all $F \leq_f G$ we have $F \cap G' = G'$.*
(vii) *For all $F \leq_f G$ we have $F' = F \cap G'$.*

Proof. The equivalence of (i) with (ii) is immediate on dropping down to a finite index normal subgroup. We have (iii) implies (ii) and (i) implies (iii) as any residually finite image of G must factor through G/R_G. As for (iv), this is just using the index preserving correspondence between the subgroups of $\overline{G}$ and the subgroups of G containing G'.

As for the rest, we have that $F' \leq F \cap G' \leq G'$ whenever F is a subgroup of G. If (i) holds for G with F a finite index subgroup, then $R_F = R_G = G'$ but R_F is inside F' so F' and G' are equal, giving (v). This immediately implies (vi) and (vii) so we just require that these two in turn imply (i). This is obvious for (vi) and for (vii) we can adopt the proof of [36] Theorem 4.0.8 which states that if Γ is a residually finite group, then for each of its (non-trivial) cyclic subgroups there exists a homomorphism onto another (non-trivial) cyclic group which can be extended to a finite index subgroup of Γ. If (i) fails then take

$F \leq_f G$ and $g \in G'$ but $g \notin F$. Dropping down to $N \leq F$ with $N \trianglelefteq_f G$, we have $H = N\langle g \rangle \leq_f G$ and $H/N \cong \langle g \rangle/(N \cap \langle g \rangle)$. Thus $g \notin H'$ because by being outside N it survives under a homomorphism from H to an abelian group. But g is certainly in $H \cap G'$. ∎

The importance of condition (vii) holding for G is that we fail to pick up extra abelianisation in finite covers $F \leq_f G$ since F/F' is just $F/(F \cap G') \cong FG'/G' \leq_f G/G'$. In particular $\beta_1(F) = \beta_1(G)$, so G is not large.

Example 3.5:

(i) The Thompson group T is NARA. This group has a 2 generator 2 relator presentation with abelianisation $\mathbb{Z} \times \mathbb{Z}$ and its commutator subgroup T' has no proper finite index subgroups as T' is infinite and simple; see [17]. But for $F \leq_f T$ we have $F \cap T' \leq_f T'$ thus $T' \leq F$.

(ii) If G is infinite but has no proper finite index subgroups then G is NARA. Moreover, for any such G and any residually abelianised group A we have $\Gamma = G * A$ is NARA because if $N \trianglelefteq_f \Gamma$, then $N \cap G \trianglelefteq_f G$ so $G \leq N$. This implies that the normal closure C of G is in N so Γ/N must be abelian as it is a finite quotient of $\Gamma/C \cong A$. A famous example that will do for G is the Higman group H with 4 generators and 4 relators as introduced in [27]. It has $\beta_1(G) = 0$ so its deficiency must be zero. Thus $H * H$ is NARA so it too has no proper finite index subgroups, since it is infinite and equals its own commutator subgroup. By repeating this construction we obtain $H * \cdots * H$ using n copies of H which gives us examples of NARA groups G_n which can have arbitrarily many generators (by the Grushko-Neumann theorem) and with $\beta_1(G_n) = 0$ and deficiency zero. In order to obtain examples of deficiency 1 NARA groups we can take the free product of G_n with $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ so that the resulting groups need arbitrarily many generators and have their first Betti number equal to 1 or 2. Of course there are no groups of deficiency two or higher which are NARA because they are all large.

(iii) There are 1 relator groups which are NARA: the first example dates back to a short paper [3] of G. Baumslag in 1969 entitled “A non-cyclic one-relator group all of whose finite quotients are cyclic” with the group in question being

$$\langle a, b | a = a^{-1}b^{-1}a^{-1}bab^{-1}ab \rangle.$$

However Baumslag-Solitar groups are not NARA, as can be seen by taking quotients onto dihedral groups.

In terms of its wide application, the following is our main result on largeness of deficiency 1 groups.

Theorem 3.6: *If G has a deficiency 1 presentation $\langle F_n|R\rangle$ where one of the relators is a commutator in F_n then exactly one of these occurs:*

(i) $G = \mathbb{Z} \times \mathbb{Z}$.

(ii) *G is NARA with abelianisation $\mathbb{Z} \times \mathbb{Z}$.*

(iii) *G is large.*

In particular, if there exists $H \leq_f G$ with $\overline{H} \neq \mathbb{Z} \times \mathbb{Z}$, then G is large.

Proof. If our relator $r = uvUV$ for u, v words in the generators for F_n, then we can regard r as the commutator of two generators simply by adding u and v to the generators and their definitions to the relators, noting that this does not change the deficiency. We must have $\beta_1(G) \geq 2$ and if $\beta_1(G) \geq 3$ (or if the subgroup generated by the images of u and v has $\mathbb{Z}$-rank less than 2), then G is large by Theorem 3.1. Moreover, if $\overline{G}$ has non-trivial torsion, then we have largeness by Corollary 3.2 so the only case in which the given presentation for G does not show largeness is when $\overline{G} = \mathbb{Z} \times \mathbb{Z}$ with the subgroup $S = \langle \overline{u}, \overline{v}\rangle$ (where $\overline{u}$ and $\overline{v}$ are the images of u and v in $\overline{G}$) being of finite index. If neither (i) nor (ii) are true, then Proposition 3.4 tells us that none of the given seven conditions hold for G, so the failure of (vii) means that there is a subgroup $L \leq_f G$ with γ in $L \cap G'$ but not in L'. Consequently $\overline{L} = L/L'$ is an abelian group which surjects to $L/(L \cap G')$ with γ in the kernel. But $L/(L \cap G')$ is isomorphic to LG'/G' which is a finite index subgroup of $\overline{G}$ and thus is equal to $\mathbb{Z} \times \mathbb{Z}$. As this is a Hopfian group, we conclude that $\beta_1(L) \geq 2$ but $\overline{L} \neq \mathbb{Z} \times \mathbb{Z}$. We know that L also has a deficiency 1 presentation which we can obtain from G by rewriting and if one of the relations in such a presentation for L were a commutator, then we could conclude by Theorem 3.1 or Corollary 3.2 applied to this presentation that L, and hence G, is large. In fact although we cannot guarantee this, we now show that there is a finite index subgroup of L which has such a presentation, along with the necessary abelianisation. We do this by keeping track of what happens to our original relator r when we perform Reidemeister-Schreier rewriting on dropping to a finite index subgroup. The process of rewriting for a subgroup H in $G = \langle g_1, \ldots, g_n\rangle$ involves taking a Schreier transversal T to obtain a generating set for H of the form $tg_i(\overline{tg_i})^{-1}$, where $t \in T$ and $\overline{x}$ is the element of T in the same coset as x. In particular by taking t equal to the identity we have that if a generator g_i of G is in H, then it becomes a generator of H. Moreover as the relators for H are obtained by expressing the relators tr_jt^{-1} in terms of these generators, any relator made up solely from those g_i which are contained in H will remain unchanged in the presentation for H.

First we take H to be the normal subgroup of finite index in G which is the inverse image of $\langle\overline{u},\overline{v}\rangle$ under the abelianisation map from G to $\overline{G}$. Then the deficiency 1 presentation for H has r as one of its relators thus we have largeness for H and G unless $\overline{H} = \mathbb{Z}\times\mathbb{Z}$. However, if so then we must have $H' = H \cap G'$. This is because otherwise we have a surjective but non-injective homomorphism from H/H' to $H/(H\cap G') \cong \mathbb{Z}\times\mathbb{Z}$, thus we can use Hopficity again. Moreover as $G' \leq H$ we get $H' = G'$.

Let k, l be the minimum positive integers such that $a = u^k$ and $b = v^l$ are in L. We set $N = \langle H', a, b\rangle$ which is a finite index normal subgroup of H and of G, with G/N abelian. We rewrite for N in H in two stages; first we drop to the subgroup with exponent sum of u equal to 0 mod k and rewrite using the transversal $u^i, 0 \leq i < k$, and then we do the same with v. In both of these stages a and b will be amongst the generators for N and our relator $uvUV$ in H gives rise to a relator $ava^{-1}v^{-1}$ after the first rewrite, and then this becomes $abAB$. Thus we have $G' \leq N \trianglelefteq_f G$ with N having a deficiency 1 presentation which includes generators a, b and the relator $abAB$.

Finally we go from N to the subgroup $L\cap N \leq_f G$ which on rewriting will keep a and b because they are generators in the presentation for N which also lie in $L\cap N$, and consequently $abAB$ remains too. Now our $\gamma \in L$ from before which is in $G'\backslash L'$ is also in N as $G' \leq N$. But from above we have a surjective homomorphism from L to $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/j\mathbb{Z}$ for some $j \geq 2$ with γ mapping onto the $\mathbb{Z}/j\mathbb{Z}$ factor. But then we can restrict this surjection to the finite index subgroup $L\cap N$ which also contains γ so $L\cap N$ has the right presentation and the right homology to obtain largeness. ∎

Note that Example 3.5 (ii) shows that case (ii) in Theorem 3.6 can occur but this is the only type of example known to us.

4. Ascending HNN extensions of free groups

A wide and important class of deficiency 1 groups is obtained by taking a free group F_n with free basis $x_1, \ldots, x_n$ and an endomorphism θ of F_n to create the mapping torus

$$G = \langle x_1, \ldots, x_n, t | tx_1t^{-1} = \theta(x_1), \ldots, tx_nt^{-1} = \theta(x_n)\rangle.$$

We call such a presentation a standard presentation for G. We do not assume that θ is injective or surjective. However there is a neat way of sidestepping the non-injective case using [31] where it is noted that G is isomorphic to a mapping torus of an injective free group homomorphism $\tilde{\theta} : F_m \to F_m$ where $m \leq n$. Of course, it might be that F_n is non-abelian but $m = 0$ or 1, however this would mean that $G = \mathbb{Z}$ or $\langle a, t | tat^{-1} = a^k\rangle$ for $k \neq 0$. However in these cases G is soluble and so is definitely not large. Therefore we will assume throughout that θ is injective, whereupon G is also called an ascending HNN extension of the free group F_n, where we conjugate the base F_n to an isomorphic subgroup of itself using the stable letter t.

We have that our base $F_n = \langle x_1, \ldots, x_n\rangle$ embeds in G and we will refer to this copy of F_n in G as Γ. Then $t\Gamma t^{-1} = \theta(\Gamma)$ which is equal to Γ if and only if θ is surjective in which case G is free by $\mathbb{Z}$. Otherwise $\theta(\Gamma) < \Gamma$, with $\theta(\Gamma)$ being isomorphic to F_n meaning that it has infinite index in Γ.

Once an ascending HNN extension G is formed, there is an obvious homomorphism χ from G onto $\mathbb{Z}$ associated with it which is given by $\chi(t) = 1$ and $\chi(\Gamma) = 0$, so that

$$\ker\chi = \bigcup_{i=0}^{\infty} t^{-i}\Gamma t^{i}.$$

In the case of an automorphism $\ker\chi$ is just Γ but otherwise $\theta(\Gamma) = t\Gamma t^{-1} < \Gamma$ so that $\ker\chi$ is a strictly ascending union of free groups, thus is infinitely generated and locally free, but never free because $\beta_1(\ker\chi; \mathbb{Q}) \leq \beta_1(\Gamma; \mathbb{Q})$.

The following result, which is Lemma 3.1 in [22], allows us to recognise ascending HNN extensions "internally".

LEMMA 4.1: *A group G with subgroup Γ is an ascending HNN extension with Γ as base if and only if there exists $t \in G$ with*

(1) $G = \langle \Gamma, t\rangle$;
(2) $t^k \notin \Gamma$ *for any* $k \neq 0$;
(3) $t\Gamma t^{-1} \leq \Gamma$.

A strong property that ascending HNN extensions of free groups possess is that they are coherent by [21], in fact the result is more general and gives us this description which is Theorem 1.2 and Proposition 2.3 in [21].

THEOREM 4.2: *If $G = \langle t, F\rangle$ is an ascending HNN extension of the (possibly infinitely generated) free group F with associated homomorphism χ and H is a finitely generated subgroup of G, then H has a finite presentation of the form*

$$\langle s, a_1, \ldots, a_k, b_1, \ldots, b_l | s a_1 s^{-1} = w_1, \ldots, s a_k s^{-1} = w_k\rangle$$

where $a_i, b_j \in \ker\chi$ and $k, l \geq 0$, with $w_1, \ldots, w_k$ words in the a_i and the b_j.

The next proposition gives us standard but useful properties of ascending HNN extensions.

PROPOSITION 4.3: *Let G be an ascending HNN extension*

$$\langle x_1, \ldots, x_n, t | t x_1 t^{-1} = \theta(x_1), \ldots, t x_n t^{-1} = \theta(x_n)\rangle$$

with respect to the injective endomorphism θ of the finitely generated free group $\Gamma = F_n$ with free basis $x_1, \ldots, x_n$ and let χ be the associated homomorphism.

(i) *Each element g of G has an expression of the form $g = t^{-p}\gamma t^q$ for $p, q \geq 0$ and $\gamma \in \Gamma$.*

(ii) *For each $j \in \mathbb{N}$ we have for $s = t^j$ the normal subgroup $G_j = \langle \Gamma, s\rangle$ of index j in G with $G/G_j \cong \mathbb{Z}/j\mathbb{Z}$ which has presentation*

$$\langle x_1, \ldots, x_n, s | s x_1 s^{-1} = \theta^j(x_1), \ldots, s x_n s^{-1} = \theta^j(x_n)\rangle.$$

(iii) *If $H \leq_f G$ then H is also an ascending HNN extension of a finitely generated free group with respect to the (restriction to H of the) same associated homomorphism χ.*

(iv) *If $\Delta \leq_f \Gamma$ then $H = \langle \Delta, t\rangle$ has finite index in $G = \langle \Gamma, t\rangle$.*

Proof. (i) is [21, Lemma 2.2 (1)].

(ii) This is [30, Lemma 2.2 (1)].

(iii) This can be proved directly but we may use the fact that H has a presentation as in Theorem 4.2. We now show that G and all its finite index subgroups have deficiency exactly 1 so we must have $l = 0$ in this presentation and then the result follows. This also demonstrates that in proving certain ascending HNN extensions of free groups are large, we are genuinely finding new examples as opposed to groups that could be proved large by the Baumslag-Pride result [2].

We know that H has a deficiency 1 presentation by Reidemeister-Schreier rewriting the standard presentation for G. By [46, Proposition 3.6 (ii)] we have that the 2-complex C associated to this standard presentation of G is aspherical and the Euler characteristic $\chi(C) = 1 - (n+1) + n = 0$. Therefore the finite cover $\tilde{C}$ of C with fundamental group H has $\chi(\tilde{C}) = 0$ and is also aspherical. Now by the Hopf formula we have that an upper bound for the deficiency of any finitely presented group Γ is $\beta_1(\Gamma) - \beta_2(\Gamma)$. As $\tilde{C}$ is a $K(H,1)$ space we have $\beta_1(H) - \beta_2(H) = \beta_1(\tilde{C}) - \beta_2(\tilde{C}) = 1 - \chi(\tilde{C}) = 1$ and so our deficiency 1 presentation for H is best possible.

(iv) Let $\gamma_1, \ldots, \gamma_d$ be a transversal for Δ in Γ. If we can show that $H \cap K \leq_f K$ for K the kernel of the associated homomorphism then we are done, despite the fact that K and $H \cap K$ are infinitely generated, because $t \in H$ and any $g \in G$ is of the form kt^m for $k \in K$.

The set

$$S = \{t^{-m}\gamma_i t^m : m \in \mathbb{N}, 1 \leq i \leq d\}$$

contains an element of every coset of $H \cap K$ in K. This can be seen by writing $k \in K$ as $t^{-m}\gamma t^m$ for $\gamma \in \Gamma$ using (i). Then there is γ_i such that $\gamma\gamma_i = \delta \in \Delta$. This means that $kt^{-m}\gamma_i t^m = t^{-m}\delta t^m$ which is in H and in K. We now show that the index of $H \cap K$ in K is at most d. Note that for $q > p$, any element of the form $t^{-p}\gamma_i t^p$ is in the same coset as some element of the form $t^{-q}\gamma_j t^q$ because $\theta^{q-p}(\gamma_i)\gamma_j^{-1} = \delta$ for some $\delta \in \Delta$ and some $j \in \{1, \ldots, d\}$, thus giving $t^{-p}\gamma_i t^p (t^{-q}\gamma_j t^q)^{-1} = t^{-q}\delta t^q$ which is in $H \cap K$. Therefore we proceed as follows: S is a set indexed by $(l, i) \in \mathbb{N} \times \mathbb{Z}/d\mathbb{Z}$ and we refer to l as the level. Choose a transversal T for $H \cap K$ in K from S which a priori could be infinite and let g_1 be the element in T with smallest level l_1 (and smallest i if necessary). Then for each level $l_1 + 1, l_1 + 2, \ldots$ above l_1 there is an element in S with this level that is in the same coset of $H \cap K$ as g_1 and so cannot be in T. Cross these elements off from S and now take the next element g_2 in T according to our ordering of S. Certainly g_2 with level l_2 has not been crossed off and we repeat the process of removing one element in each level above l_2; as these are in the same coset as g_2 they too have not been erased already. Now note that we can go no further than g_d because then we will have crossed off all elements from all levels above l_d; thus we must have a transversal for $H \cap K$ in K of no more than d elements. ∎

Let $G = \langle F_n, t\rangle$ be the mapping torus of an injective endomorphism θ of the free group F_n. We say that θ has a periodic conjugacy class if there exists $i > 0$, $k \in \mathbb{Z}$ and $w \in F_n \backslash \{1\}$ such that $\theta^i(w)$ is conjugate to w^k in F_n. If this is so with $\theta^i(w) = vw^k v^{-1}$, then let us take the endomorphism ϕ of F_n such that $\phi = \iota_v^{-1}\theta^i$ where we use ι_v to denote the inner automorphism of F_n that is conjugation by v. We have on setting $\Delta = \langle w\rangle$ and $s = v^{-1}t^i$ that the subgroup $\langle \Delta, s\rangle$ of G is an ascending HNN extension with base Δ and stable letter s by Lemma 4.1. Consequently it has the presentation $\langle s, w | sws^{-1} = w^k\rangle$. These presentations are part of the famous family of 2 generator 1 relator subgroups known as the Baumslag-Solitar groups. We define the Baumslag-Solitar group $BS(j,k) = \langle x, y | xy^j x^{-1} = y^k\rangle$ for $j, k \neq 0$ (and without loss of generality $j > 0$). We have that G contains $BS(1,k)$ for some k if and only if G has a periodic conjugacy class where $\theta^i(w)$ is conjugate to w^k. Furthermore if there exists $i, j > 0$, $k \in \mathbb{Z}$ and $w \in F_n \backslash \{1\}$ with $\theta^i(w^j)$ conjugate in F_n to w^k, then $k = dj$ and $\theta^i(w)$ is conjugate to w^d so that θ has a periodic conjugacy class. Indeed G cannot contain a subgroup isomorphic to a Baumslag-Solitar group $BS(j,k)$ unless $j = 1$ (or $j = k$ in which case G contains $BS(1,1)$ anyway).

We can now deal with ascending HNN extensions of free groups which contain $\mathbb{Z} \times \mathbb{Z}$.

Theorem 4.4: *If θ is an injective endomorphism of the free group Γ of rank n with $w \in F_n \backslash \{1\}$ such that $\theta(w) = w$ then there is a finite index subgroup Δ of Γ and $j \geq 1$ such that Δ has a free basis including w, and such that $\theta^j(\Delta) \leq \Delta$.*

Proof. We use the classic result [24] of Marshall Hall Jnr. that if L is a non-trivial finitely generated subgroup of the non-abelian free group F_n, then there is a finite index subgroup F of F_n such that L is a free factor of F. We just need to put $L = \langle w\rangle$ so that $F = \langle w\rangle * C$ for some $C \leq F_n$ with w a basis element for F. The second condition is the crucial part. The aim is to repeatedly pull back F; although we do not have $F \leq \theta^{-1}(F)$ in general as this is equivalent to $\theta(F) \leq F$ which would mean we are done, we do find that the index is non-increasing. To see this note that $\theta^{-1}(F) = \theta^{-1}(F \cap \theta(\Gamma))$ and $\theta^{-1}\theta(\Gamma) = \Gamma$ as $\theta : \Gamma \to \theta(\Gamma)$ is an isomorphism. Now the index of $F \cap \theta(\Gamma)$ in $\theta(\Gamma)$ is preserved by applying θ^{-1} to both sides, so it is equal to the index of $\theta^{-1}(F)$ in Γ. But the index of $F \cap \theta(\Gamma)$ in $\theta(\Gamma)$ is no more than that of F in Γ, thus $[\Gamma : \theta^{-i}(F)]$ gives us a non-increasing sequence which must stabilise at N with value k. When it does we have for $i \geq 0$ that $\theta^{-(i+N)}(F)$ is just moving around the finitely many index k subgroups. Although it happens that θ^{-1} does not in general permute these index k subgroups, we must land on some such subgroup Δ twice so we have $j \geq 1$ with $\theta^{-j}(\Delta) = \Delta$, giving $\Delta \geq \theta^j(\Delta)$.

We now show that, although the rank of $\theta^{-i}(F)$ reduces whenever the index reduces, we can keep w as an element of a free basis each time we pull back. This time we restrict θ to an injective homomorphism from $\theta^{-1}(F)$ to F with image $\theta\theta^{-1}(F)$. As $\theta\theta^{-1}(F)$ is a finitely generated subgroup of F containing a free basis element w of F, we can ensure w is in a free basis for $\theta\theta^{-1}(F)$ (for instance see [39] Proposition I.3.19). Now $\theta^{-1}(F)$ and $\theta\theta^{-1}(F)$ are isomorphic via θ with inverse ϕ say, so a basis $b_1, \ldots, b_r$ for the latter gives rise to a basis $\phi(b_1), \ldots, \phi(b_r)$ for $\theta^{-1}(F)$ and if $b_1 = w$ then $\phi(b_1) = w$. ∎

Corollary 4.5: *If $G = \langle \Gamma, t\rangle$ is a mapping torus of an injective endomorphism θ of the free group Γ of rank n and $\mathbb{Z} \times \mathbb{Z} \leq G$ then we have $H \leq_f G$ such that H has a deficiency 1 presentation $\langle x_1, \ldots, x_m, s | r_1, \ldots, r_m\rangle$ including a relator of the form $sx_1s^{-1}x_1^{-1}$.*

Proof. As $BS(1,1) \leq G$ we have $w \in \Gamma \backslash \{1\}$ with $\theta^i(w) = vwv^{-1}$ for some $v \in \Gamma$, thus on dropping to the index i subgroup H of G given by $H = \langle \Gamma, t^i\rangle$ and setting ϕ to be $\iota_v^{-1}\theta^i$ where $\iota_v(x) = vxv^{-1}$, we can assume that there is $w \in \Gamma \backslash \{1\}$ with $\phi(w) = w$ and that H is an ascending HNN extension of Γ via the injective endomorphism ϕ and with stable letter t_H say. So by Theorem 4.4 we have $\Delta \leq_f \Gamma$ with Δ having a free basis $w, x_2, \ldots, x_m$ and $j \geq 1$ with $\phi^j(\Delta) \leq \Delta$. Thus by Proposition 4.3 (ii) and (iv) we have that $L = \langle \Delta, s | s = t_H^j\rangle$ has finite index in G and by Lemma 4.1 L is an ascending HNN extension with base Δ and stable letter s. Thus on taking the standard presentation for L given by conjugation of s on this free basis for Δ, we see that it has deficiency 1 with a relator equal to $sws^{-1}w^{-1}$. ∎

We can now gain largeness for a range of mapping tori.

Corollary 4.6: *If $G = \langle \Gamma, t\rangle$ is the mapping torus of an endomorphism θ of the free group Γ of rank n and $\mathbb{Z} \times \mathbb{Z} \leq G$ then $G = \langle x, y | xyx^{-1} = x^{\pm 1}\rangle$ or is large.*

Proof. We can assume without loss of generality that θ is injective because if not then we can replace θ with $\tilde{\theta}$ which is an injective endomorphism of a free group F_m with $m \leq n$, and then G is still equal to the mapping torus of F_m using $\tilde{\theta}$ and this will contain $\mathbb{Z} \times \mathbb{Z}$. Hence we are in the case of Corollary 4.5 which allows us to apply Theorem 3.6 to $H \leq_f G$. As $\mathbb{Z} \times \mathbb{Z} \leq G$, we do not have $m = 0$ and only the two groups above for $m = 1$. Otherwise G and hence H contain a non-abelian free group for $m \geq 2$ so H is not in case (i) of Theorem 3.6. By Proposition 4.3 (iii) H is an injective mapping torus of a finitely generated free group endomorphism and so the recent result [13] of Borisov and Sapir tells us that H is residually finite, so it is not NARA. Thus H and G are large. ∎

Although it might be said that one only requires the NARA property to apply Theorem 3.6 and not the full force of residual finiteness, it should be pointed out that there are mapping tori G of injective endomorphisms of the free group F_2 such that the abelianisation $\overline{G} = \mathbb{Z} \times \mathbb{Z}$ and such that for any finite index subgroup N which is normal in G with G/N soluble, we have $\overline{N} = \mathbb{Z} \times \mathbb{Z}$.

We finish this section by looking at those mapping tori G of endomorphisms of free groups which contain an arbitrary Baumslag-Solitar subgroup. Our results are not quite definitive because we need $\beta_1(G) \geq 2$ in order to apply our methods and we cannot show that G necessarily has a finite index subgroup with that property. However this is the only obstacle to largeness.

THEOREM 4.7: *If $G = \langle \Gamma, t \rangle$ is a mapping torus of an endomorphism θ of the free group Γ of rank n which contains a Baumslag-Solitar subgroup $BS(j,k)$ then either G is large or $G = BS(1,k)$ or $\beta_1(H) = 1$ for all $H \leq_f G$.*

Proof. As usual we assume that θ is injective. We know that G can only contain Baumslag-Solitar subgroups of type $BS(1,k)$ or $BS(k,k)$ for $k \neq 0$ and as we have already covered those which contain $BS(1,1)$, we need only consider $BS(1,k) \leq G$ for $k \neq \pm 1$. If there is some $H \leq_f G$ with $\beta_1(H) \geq 2$, then we can replace G by H because H is a mapping torus by Proposition 4.3 (iii) and $BS(1,k) \cap H \leq_f BS(1,k)$ so H contains some Baumslag-Solitar group too. Therefore we are looking at the situation where we have a periodic conjugacy class of the form $w \in F_n \backslash \{1\}$ and $i > 0$ with $\theta^i(w)$ conjugate to w^d for some $d \neq \pm 1$. Just as in the $\mathbb{Z} \times \mathbb{Z}$ case, we drop down to a finite index subgroup and change our automorphism by an inner automorphism, so we can assume that $\theta(w) = w^d$. Now we follow the proof of Theorem 4.4 to get $F \leq_f \Gamma$ with $\langle w \rangle$ a free factor of F, observing that $w \in \theta^{-1}(F)$ so that we keep w as we pull back F. Note that we can assume w is not a proper power by the comment before Theorem 4.4 so we can also preserve w in a free basis each time because $w^d \in \theta\theta^{-1}(F)$ and if $w^c \in \theta\theta^{-1}(F)$ for $0 < |c| < |d|$ then the element $u \in \theta^{-1}(F)$ with $\theta(u) = w^c$ cannot be a power of w but $\theta(u^d) = \theta(w^c)$, hence contradicting injectiveness. Thus w^d can be extended to a free basis for $\theta\theta^{-1}(F)$ by [39] Proposition I.3.7 meaning that w will be in the corresponding basis for $\theta^{-1}(F)$.

We can now work to obtain an equivalent version of Corollary 4.5. Having gone from G to the finite index subgroup H which is an ascending HNN extension of Γ via the injective homomorphism θ, we see as before that by repeatedly pulling back F we obtain $\Delta \leq_f \Gamma$ which has a free basis including w and with $\theta^j(\Delta) \leq \Delta$. Hence the HNN extension J of Δ using θ^j with stable letter s has finite index in H, as well as a deficiency 1 presentation that includes the relator $sws^{-1}w^{-e}$ for $e \neq 0, \pm 1$. We will also require later that $e \neq 2$ and this can be obtained by taking the subgroup of J of index 2 as in Proposition 4.3 (ii) so that now the relator would be $sws^{-1}w^{-4}$. Now consider taking a surjective homomorphism χ from J to $\mathbb{Z}$ (which must send w to 0). If $\beta_1(J)$ were 1, then the only available χ would be the homomorphism associated to this HNN extension so it would send s to 1 (or -1). However, if not, then we can find $\chi' \neq \chi$ as we have $\beta_1(H) \geq 2$ and hence $\beta_1(J) \geq 2$ because $J \leq_f H$. Hence we have a non-trivial homomorphism $\chi' - k\chi$ that sends s to 0 (which can be made surjective by multiplying by the right constant) where k is $\chi'(s)$. On evaluating the Alexander polynomial of J with respect to this homomorphism, we proceed as in Theorem 3.1 and discover that the group relation $sws^{-1}w^{-e}$ becomes the module relation $(1-e)w$ when rewritten and abelianised. Thus we have a column in our square presentation matrix consisting of all zeros except $1-e$ in the row corresponding to the generator w. Thus if we apply Theorem 2.1 using the field $\mathbb{Z}/p\mathbb{Z}$ with p a prime dividing $1-e$ then our Alexander polynomial is zero so we have largeness for J, and hence for G. ∎

Although we do not have a proof that a mapping torus of a free group endomorphism containing a Baumslag-Solitar subgroup of infinite index has a finite index subgroup with first Betti number at least two, the statement of Theorem 4.7 is still useful in a practical sense because if we are presented with a particular group G of this form that we would like to prove is large, we can enter the presentation into a computer and ask for the abelianisation of its low index subgroups. As soon as we see one with first Betti number at least two, we can conclude largeness. Note that in [30] it is conjectured that a mapping torus of a free group endomorphism is word hyperbolic if it does not contain Baumslag-Solitar subgroups, and if this and the above question on having a finite index subgroup with first Betti number at least two are both true, then we have proved largeness for all the non-word hyperbolic ascending HNN extensions of finitely generated free groups (with the obvious exception of the soluble Baumslag-Solitar groups).

We can even say something if G is a mapping torus of an injective endomorphism of an infinitely generated free group in the case when G is finitely generated, thanks to the power of [21] by using Theorem 4.2. We immediately see that either G has deficiency at least two and so is large, or $l = 0$ in which case G is also a mapping torus of an endomorphism of a finitely generated free group and so the results of this section apply.

5. Free by Cyclic Groups

If in the previous section we use an automorphism α of a free group F to form our mapping torus, we obtain a semidirect product $F \rtimes_\alpha \mathbb{Z}$ and every free-by-$\mathbb{Z}$ group is of this form. We already have largeness for a range of these groups.

THEOREM 5.1: *If the finitely generated group G is free-by-$\mathbb{Z}$, then G is large if the free group F is infinitely generated, or if $\mathbb{Z} \times \mathbb{Z} \leq G$ and F has rank at least 2.*

Proof. If F is infinitely generated, then applying Theorem 4.2 with $H = G$ tells us that G has deficiency at least 2. This is because if $l = 0$ then the kernel of the associated homomorphism χ is

$$\bigcup_{n=0}^{\infty} s^{-n} A s^n \quad \text{where } A = \langle a_1, \ldots, a_k \rangle$$

but then $\beta_1(\ker\chi; \mathbb{Q}) \leq \beta_1(A; \mathbb{Q})$ whereas we have $F = \ker\chi$.

If F has finite rank then this is Corollary 4.6. Note that a semidirect product $A \rtimes B$ is residually finite if both A and B are residually finite and A is finitely generated, so we do not need to use [13] when applying this corollary to G. ∎

In contrast, G. Baumslag gives in [5] an example of an infinitely generated free-by-$\mathbb{Z}$ group with every finite quotient cyclic so that this group is not residually finite. Indeed its finite residual R_G must contain G' and hence it is not large because every finite index subgroup F of G has the property that all of the finite quotients of F are abelian, as $F' \leq G' \leq R_G = R_F$. This is a striking demonstration of how largeness and residual finiteness are best suited to finitely generated groups. He then proves in [4] that finitely generated groups which are F-by-$\mathbb{Z}$ for F an infinitely generated free group are residually finite. As for the residual finiteness of finitely generated groups that are ascending HNN extensions of infinitely generated free groups, this appears to be open (it corresponds to the case $l > 0$ in Theorem 4.2); indeed that they are Hopfian is Conjecture 1.4 in [22].

COROLLARY 5.2: *If G is F_n-by-$\mathbb{Z}$ for F_n the free group of rank n then either G is large or G is word hyperbolic or $G = BS(1, \pm 1)$.*

Proof. It is known by [8], [9] and [14] that such a G being word hyperbolic is equivalent to G containing no subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and also to the automorphism α having no periodic conjugacy classes. ∎

In fact, the equivalence of the last two notions can be proved directly by quoting the classical result of Higman [26] which says that an automorphism of a free group that maps a finitely generated subgroup into itself maps it onto itself. So if α has a periodic conjugacy class we can assume there is $i > 0$ and $w \in F_n \backslash \{1\}$ such that $\theta^i(w)$ is conjugate to $w^{\pm 1}$.

This leaves us with an important question:

QUESTION 5.3: *If G is F_n-by-$\mathbb{Z}$ for $n \geq 2$ and G is a word hyperbolic group then is G large?*

As for whether the six consequences of largeness given in the introduction hold for these groups G, the first is obvious whereas it is unknown if G has superexponential subgroup growth or has infinite virtual first Betti number: Question 12.16 by Casson in [7] is equivalent to asking whether there exists $H \leq_f G$ with $\beta_1(H) \geq 2$. However being word hyperbolic means that the other properties are known to hold, giving us a definitive result for these three cases.

THEOREM 5.4: *If G is finitely generated and is virtually free-by-$\mathbb{Z}$ then for all finitely generated subgroups H of G we have:*

(i) *H has word problem solvable strongly generically in linear time;*
(ii) *H has uniformly exponential growth;*
(iii) *H is SQ-universal;*

unless H is virtually S for $S = \mathbb{Z}$ *or* $\mathbb{Z} \times \mathbb{Z}$.

Proof. We have shown by Theorem 5.1 and Corollary 5.2 that any free-by-$\mathbb{Z}$ group G which is finitely generated (excepting $BS(1, \pm 1)$ and $\mathbb{Z}$) is large or is non-elementary word hyperbolic. This implies SQ-universality (by [44] for the hyperbolic case and [45] when G is large) and uniformly exponential growth (by [33] for the hyperbolic case and [25] when we have largeness). Then Corollary 4.1 in [32] (where we also have the relevant definitions) proves (i) if G has a finite index subgroup with a non-elementary word hyperbolic quotient. Moreover if $G \leq_f \Gamma$, then the properties (i), (ii) and (iii) hold for Γ as well, by [32] for (i), [25] for (ii) and [42] for (iii).

Finally if $H \leq G$ where G is free-by-$\mathbb{Z}$, then $H/(H \cap F) \cong HF/F \leq G/F = \mathbb{Z}$ so either this is trivial with $H \leq F$ hence H is free, or it is isomorphic to $\mathbb{Z}$ and so H is an extension of the free group $H \cap F$ by $\mathbb{Z}$ (although if F is finitely generated, then $H \cap F$ is not necessarily finitely generated if H has infinite index in F). Thus if H is finitely generated then (i), (ii), (iii) or the exceptions hold for H too, and if L is a finitely generated subgroup of the virtually free-by-$\mathbb{Z}$ group Γ with $G \leq_f \Gamma$ then $L \cap G \leq_f L$ so $L \cap G$ is a finitely generated subgroup of G, hence we gain our properties for $L \cap G$ and then also for L. ∎

6. 1 Relator Groups

Groups having a finite presentation with only 1 relator have been much studied. It is known that such a group contains a non-abelian free group unless it is isomorphic to $BS(1, m)$ or is cyclic, see [39, II Proposition 5.27] and [49]. Indeed if the presentation has at least 3 generators, then we know by [2] that the group is large so we need only concern ourselves here with 2 generator 1 relator presentations $\langle a, b | r \rangle$. Largeness is also known by [23] and [48] when r is a proper power, which is exactly when the group has torsion, but for l, m coprime we have by [20] that $BS(l, m)$ is not large as it has virtual first Betti number equal to 1, and similarly Example 3.5 (iii) is not large. Thus another direction in which to go when looking for large 2 generator 1 relator groups is if r is in the commutator subgroup of F_2, as at least that gives first Betti number equal to

2. The starting case to be considered here should be to take r actually equal to a commutator and application of Theorem 3.6 gives us a near definitive result.

Corollary 6.1: *If $G = \langle a, b | uvUV \rangle$ where u and v are any elements of $F_2 = \langle a, b \rangle$ with $uvUV$ not equal to $abAB$, $baBA$ or their cyclic conjugates when reduced and cyclically reduced then G is large or is NARA.*

Proof. It is well-known that $G = \mathbb{Z} \times \mathbb{Z}$ if and only if the relator is of the above form (equivalently if and only if u, v form a free basis for F_2); see, for instance, [41, Theorem 4.11]. Otherwise we are in Theorem 3.6 case (ii) or (iii). ∎

Question 6.2: *If $G = \langle a, b | uvUV \rangle$ then can G be NARA?*

No examples are known to us. This is an important question because a yes answer gives us a non-residually finite 1 relator group with the relator a commutator, the existence of which is Problem (OR8) in the problem list at [43] (however there it is shown that non-residually finite examples exist if the relator is merely in the commutator subgroup) and a no answer gives us largeness. We can prove that we do not have NARA groups in a whole range of cases.

Proposition 6.3: *If $G = \langle F_2 | uvUV \rangle$ then G can only be NARA if $u, v \notin F_2'$ with the images of u and v generating the abelianisation $\mathbb{Z} \times \mathbb{Z}$ of F_2 and such that u is a free basis element for F_2 or $G_u = \langle F_2 | u \rangle$ is NARA, along with the same condition for v.*

Proof. If the images of u and v do not generate the homology of F_2 up to finite index, then we are done by Theorem 3.1. Now suppose that G_u is not NARA or $\mathbb{Z}$ (the latter happening if and only if u is an element of a free basis for F_2) then as G surjects to G_u we see that a non-abelian finite image of G_u is also an image of G. Then swap u and v.

Otherwise we can take a free basis α, β for F_2 such that there are $k, l \geq 1$ with u equivalent to α^k in homology and v to β^l. If $k > 1$, then consider the homomorphism of F_2 into the linear (hence residually finite) group $SL(2, \mathbb{C})$ given by

$$\alpha \mapsto \begin{pmatrix} e^{\pi i/k} & 0 \\ 0 & e^{-\pi i/k} \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This is non-abelian but does make u and v commute so we are done by Proposition 3.4 (iii). ∎

In fact there are other ways to conclude that $G = \langle F_2 | uvUV \rangle$ is not NARA and hence large, for instance the powerful algorithm of K. S. Brown in [15], which determines whether a 2 generator 1 relator group is a mapping torus of an injective endomorphism of a finitely generated group (which must necessarily be free), can be used (along with [13] proving that such groups are residually finite). If all else fails, then there is the option of using the computer to find the abelianisation of some low index subgroups of G and look for one which is not $\mathbb{Z} \times \mathbb{Z}$ in order to obtain largeness. For instance, in [18] it is shown that

$$G = \langle a, b | a^{k_1} b^{l_1} a^{k_2} b^{l_2} a^{k_3} b^{l_3} \rangle$$

for $k_1 k_2 k_3, l_1 l_2 l_3 \neq 0$ and $k_1 + k_2 + k_3 = l_1 + l_2 + l_3 = 0$ is large for all possibilities except for one particular relator (on which we use the computer to find a finite index subgroup H of the form in Theorem 3.6 and with $\beta_1(H) > 2$) and two infinite families of relators (these are of the required form so we can use Brown's algorithm) thus we have shown that the remaining cases are all large.

We finish this section by mentioning a conjecture of P. M. Neumann in [42] from 1973: that a 1 relator group is either SQ-universal or is isomorphic to $BS(1, m)$ or cyclic (the next comment that "a proof of this by G. Sacredote seems to be almost complete now" turns out with hindsight to be somewhat over optimistic). In addition to presentations with 3 or more generators or with the relator a proper power, at least this can be seen to be true for those 2 generator 1 relator groups G which are free by cyclic, and possibly ascending HNN extensions of free groups if the two questions at the end of Section 4 are true, as well as if G is virtually a group of this type. We make progress on this question from a different direction in the next section.

7. Residual Finiteness

It appears that often when we have a counterexample to statements about largeness, this is achieved by taking a group which is not residually finite. The following straightforward observation suggests why:

Proposition 7.1: *A group G is large if and only if the residually finite group G/R_G is large, where R_G is the intersection of all the finite index subgroups of G.*

Proof. A group is large if any quotient is large, whereas any homomorphism from G to a residually finite group factors through G/R_G and if $H \leq_f G$ then $R_H = R_G$. ∎

Thus perhaps we should take the same approach as those who count finite index subgroups of finitely generated groups by only considering residually finite groups. However the example of $G = BS(2,3)$, where G/R_G is soluble but not finitely presented, means we can lose good properties of our original group. This assumption removes the obvious counterexamples which are SQ-universal but not large, for instance taking free products of groups with no finite index subgroups, and then the two properties begin to look more similar. A recent result of [1] shows that finitely generated groups with infinitely many ends are SQ-universal. Whilst they cannot all be large, as evidenced by these free products, it is straightforward to establish this in the residually finite case by adapting an argument of Lubotzky from [37].

THEOREM 7.2: *A residually finite group with infinitely many ends is large.*

Proof. If $\Gamma = G_1 *_\phi G_2$ where ϕ is an isomorphism from A a finite subgroup of G_1 to B a finite subgroup of G_2 then, as G_1 will be residually finite, we can take $M \trianglelefteq_f G_1$ with $M \cap A = I$ and $[G_1 : M] > 2|A|$, meaning that the subgroup AM/M of G_1/M has index greater than 2 and is isomorphic to A. We can also get $N \trianglelefteq_f G_2$ with $N \cap B = I$ and $[G_2 : N] > 2|B|$. Now we can form $(G_1/M) *_{\overline{\phi}} (G_2/N)$, where $\overline{\phi}(aM) = \phi(a)N$ provides an isomorphism from AM/M to BN/N. This is a quotient of Γ and is virtually free by [47, II Proposition 11], with the index conditions ensuring that it is virtually non-abelian free (see [47, 2.6 Exercise 3]).

As for HNN extensions $\Gamma = G*_\phi$, where ϕ is an isomorphism with domain a finite proper subgroup A of G, and $\phi(A) \leq G$ is conjugate to A in Γ via the stable letter t, we now take $N \trianglelefteq_f \Gamma$ such that there exists $g \in G$ with $ag \notin N$ for all $a \in A$, which implies that $AN \neq GN$. Thus AN/N and $\phi(A)N/N$ are subgroups of GN/N which are conjugate in Γ/N via tN, thus isomorphic, with both of these proper subgroups. Hence the HNN extension $\langle GN/N, s | s(aN)s^{-1} = \phi(a)N \rangle$ can be formed and this is a non-ascending HNN extension of a finite group, thus it is virtually non-abelian free. ∎

A group G is called LERF (equivalently subgroup separable) if every finitely generated subgroup is an intersection of finite index subgroups. An observation in [12] is that G cannot be LERF if there is a finitely generated subgroup H and $t \in G$ with $tHt^{-1} \subset H$, thus proper ascending HNN extensions of finitely generated groups are never LERF. If G is F_n-by-$\mathbb{Z}$ and $\beta_1(G) \geq 2$ then it is possible for G to be simultaneously free-by-cyclic and also to be a proper ascending HNN extension of a finitely generated free group with respect to another associated homomorphism onto $\mathbb{Z}$, so being LERF is quite a lot stronger than merely being residually finite. However if we assume this we can gain some very specific conclusions.

THEOREM 7.3: *If G is finitely presented and LERF then either G has virtual first Betti number equal to 0, or G is large, or G is virtually $L \rtimes \mathbb{Z}$ for L finitely generated.*

Proof. Any group with positive first Betti number is an HNN extension (it is a semidirect product $(\ker\chi) \rtimes \mathbb{Z}$ for χ a homomorphism onto $\mathbb{Z}$) but if it is finitely presented then [11] tells us that it is an HNN extension $L*_\phi$ with stable letter t and with L and the domain A of ϕ both finitely generated. Thus on taking $H \leq_f G$ with $\beta_1(H) > 0$ we have $H = L*_\phi$ with presentation

$$\langle L, t | ta_it^{-1} = \phi(a_i) \text{ for } 1 \leq i \leq m\rangle$$

where $a_1, \ldots, a_m$ is a generating set for A. Now if $A \neq L$, then, as subgroups of LERF groups are also LERF, we have $F \leq_f H$ which contains A but not L. We can take $N \leq F$ which is normal in H and of finite index. This gives us $AN \leq F < LN$ and so we can argue as in Theorem 7.2 to get a non-ascending HNN extension: We have that LN/N is finite and $AN \neq LN$ implies that AN/N is a proper subgroup of LN/N. Now the isomorphism ϕ from A to $\phi(A)$ is induced by conjugation by t, so that in H/N we have that AN/N and $\phi(A)N/N$ are conjugate by the element tN. Hence the induced map $\overline{\phi}(aN) = \phi(a)N$ is well-defined and is an isomorphism between these two subgroups. Therefore we can form $(LN/N)*_{\overline{\phi}}$ with the domain of $\overline{\phi}$ equal to AN/N, and this has presentation

$$\langle LN/N, s | s(a_iN)s^{-1} = \overline{\phi}(a_iN) \text{ for } 1 \leq i \leq m\rangle$$

which is an image of H under the homomorphism $L \mapsto LN/N, t \mapsto s$ and is large by [47, II Proposition 11] so H is large too.

Otherwise the HNN extension is ascending but the LERF condition means that H is in fact a semidirect product. ∎

Going back to deficiency 1 groups, we have [28, Theorem 6] which states that if G has deficiency 1 and is an ascending HNN extension with base the finitely generated subgroup L, then the geometric dimension of G (thus the cohomological dimension) is at most two. But on combining this with [22, Corollary 2.5], which states that if L is of type FP_2 and has cohomological dimension 2, then G has cohomological dimension 3, we see that if G has deficiency 1, then L finitely presented (or even FP_2) implies that L is in fact free. This means that the only way a deficiency 1 group G could fail the Tits alternative of not being virtually soluble and not containing F_2 is for G to be an ascending HNN extension $L*_\phi$ where L is finitely generated but not finitely presented and where L fails the Tits alternative. It is unknown whether this can occur but at least we can conclude the Tits alternative holds for coherent deficiency 1 groups. In [49] it is shown that a soluble deficiency 1 group is $BS(1,m)$ or $\mathbb{Z}$. As these groups are coherent, this result is easily extended to virtually soluble deficiency 1 groups by the above and the result in [11] that a finitely presented group G with $\beta_1(G) > 0$ which does not contain F_2 is an ascending HNN extension of a finitely generated group.

We can use this to obtain some results about LERF groups in particular cases.

Corollary 7.4: *If G is LERF and has deficiency 1, then either G is SQ-universal or G is $BS(1,\pm 1)$ or $\mathbb{Z}$ or $G = L \rtimes \mathbb{Z}$ for L finitely generated but not finitely presented.*

Proof. Certainly $\beta_1(G) > 0$ so by the proof of Theorem 7.3 G is large or equals $L \rtimes \mathbb{Z}$ with L finitely generated. If L is finitely presented, then the above comment shows that L is free and Theorem 5.4 (iii) applies if L is non-abelian free. ∎

We can finish by making some progress on P. M. Neumann's conjecture given in the last section.

Corollary 7.5: *If G is a 1 relator group which is LERF then either G is SQ-universal or G is cyclic or $G = BS(1,\pm 1)$.*

Proof. We know that G is finite cyclic or has deficiency at least two or has deficiency one whereupon Corollary 7.4 applies. But if $G = L \rtimes \mathbb{Z}$ for L finitely generated then L must be free by [15, Section 4]. ∎

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