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REDUCIBILITY OF RATIONAL FUNCTIONS IN SEVERAL VARIABLES

BY

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ABSTRACT

We prove an analogous of Stein theorem for rational functions in several variables: we bound the number of reducible fibers by a formula depending on the degree of the fraction.

1. Introduction

Let K be an algebraically closed field. Let $f = p/q \in K(\underline{x})$, with $\underline{x} = (x_1, \ldots, x_n)$, $n \geqslant 2$ and $\gcd(p,q) = 1$, the **degree** of f is

$$\deg f = \max\{\deg p, \deg q\}.$$

We associate to a fraction $f = p/q$ the pencil $p - \lambda q$, $\lambda \in \hat{K}$ (where we denote $\hat{K} = K \cup \{\infty\}$ and by convention if $\lambda = \infty$ then $p - \lambda q = q$).

For each $\lambda \in \hat{K}$ write the decomposition into irreducible factors

$$p - \lambda q = \prod_{i=1}^{n_\lambda} F_i^{r_i}.$$

The **spectrum** of f is $\sigma(f) = \{\lambda \in \hat{K} : n_\lambda > 1\}$, and the **order of reducibility** is $\rho(f) = \sum_{\lambda \in \hat{K}} (n_\lambda - 1)$.

A fraction f is **composite** if it is the composition of a univariate rational fraction of degree more than 1 with another rational function.

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THEOREM 1.1: *Let K be an algebraically closed field of characteristic* 0. *Let $f \in K(\underline{x})$ be noncomposite then*

$$\rho(f) < (\deg f)^2 + \deg f.$$

A theorem of Bertini and Krull implies that if f is noncomposite then $\sigma(f)$ is finite and we should notice that $\#\sigma(f) \leqslant \rho(f)$. Later on, for an algebraically closed field of characteristic zero and for a polynomial $f \in K[x,y]$, Stein [St] proved the formula $\rho(f) < \deg f$. This formula has been generalized in several directions, see [Na1] for references. For a rational function $f \in \mathbb{C}(x,y)$ a consequence of the work of Ruppert [Ru] on pencil of curves, is that $\#\sigma(f) < (\deg f)^2$. For K algebraically closed (of any characteristic) and $f \in K(x,y)$ Lorenzini [Lo] proved under geometric hypotheses on the pencil $(p - \lambda q)$ that $\rho(f) < (\deg f)^2$. This has been generalized by Vistoli [Vi] for a pencil in several variables for an algebraically closed field of characteristic 0.

Let us give an example extracted from [Lo]. Let $f(x,y) = \frac{x^3+y^3+(1+x+y)^3}{xy(1+x+y)}$, then $\deg(f) = 3$ and $\sigma(f) = \{1, j, j^2, \infty\}$ (where $\{1, j, j^2\}$ are the third roots of unity). For $\lambda \in \sigma(f)$, $(f = \lambda)$ is composed of three lines hence $\rho(f) = 8 = (\deg f)^2 - 1$. Then Lorenzini's bound is optimal in two variables.

The motivation of this work is that we develop the analogous theory of Stein for rational functions: composite fractions, kernels of Jacobian derivatives, groups of divisors, ... The method for the two variables case is inspired from the work of Stein [St] and the presentation of that work by Najib [Na1]. Even the proofs similar to the ones of Stein have been included for completeness. Another motivation is that with a bit more effort we get the case of several variables by following the ideas of [Na1] (see the articles [Na2], [Na3]).

In §2, we prove that a fraction is noncomposite if and only its spectrum is finite. Then in §3, we introduce a theory of Jacobian derivation and compute the kernel. Next, in §4, we prove that for a noncomposite fraction in two variables $\rho(f) < (\deg f)^2 + \deg f$. Finally, in §5, we extend this formula to several variables and we close by stating a result for fields of any characteristic.

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2. Composite rational functions

Let K be an algebraically closed field. Let $\underline{x} = (x_1, \dots, x_n)$, $n \geqslant 2$.

Definition 2.1: A rational function $f \in K(\underline{x})$ is **composite** if there exist $g \in K(\underline{x})$ and $r \in K(t)$ with $\deg r \geqslant 2$ such that

$$f = r \circ g.$$

Theorem 2.2: *Let $f = p/q \in K(\underline{x})$. The following assertions are equivalent:*

1. *f is composite;*
2. *$p - \lambda q$ is reducible in $K[\underline{x}]$ for all $\lambda \in \hat{K}$ such that* $\deg p - \lambda q = \deg f$*;*
3. *$p - \lambda q$ is reducible in $K[\underline{x}]$ for infinitely many $\lambda \in \hat{K}$.*

Before proving this result we give two corollaries.

Corollary 2.3: *f is noncomposite if and only if its spectrum $\sigma(f)$ is finite.*

One aim of this paper is to give a bound for $\sigma(f)$. The hard implication of this theorem (3) $\Rightarrow$ (1) is in fact a reformulation of a theorem of Bertini and Krull.

We also give a nice application pointed out to us by P. Débes.

Corollary 2.4: *Let $p \in K[\underline{x}]$ irreducible. Let $q \in K[\underline{x}]$ with* $\deg q < \deg p$ *and* $\gcd(p, q) = 1$*. Then for all but finitely many $\lambda \in K$, $p - \lambda q$ is irreducible in $K[\underline{x}]$.*

Convention: When we define a fraction $F = P/Q$ we will assume that $\gcd(P, Q) = 1$.

We start with the easy part of Theorem 2.2:

Proof. (2) $\Rightarrow$ (3) is trivial. Let us prove (1) $\Rightarrow$ (2). Let $f = p/q$ be a composite rational function. There exist $g = u/v \in K(\underline{x})$ and $r \in K(t)$ with $k = \deg r \geqslant 2$ such that $f = r \circ g$. Let us write $r = a/b$. Let $\lambda \in \hat{K}$ such that $\deg a - \lambda b = \deg r$ and factorize $a(t) - \lambda b(t) = \alpha(t - t_1)(t - t_2) \cdots (t - t_k)$, $\alpha \in K^*, t_1, \dots, t_k \in K$. Then,

$$p - \lambda q = q \cdot (f - \lambda) = q \cdot \left(\frac{a - \lambda b}{b}\right)(g) = \alpha q \frac{(g - t_1) \cdots (g - t_k)}{b(g)}.$$

Then by multiplication by v^k at the numerator and denominator we get

$$(p-\lambda q)\cdot(v^k b(g)) = \alpha q(u-t_1 v)\cdots(u-t_k v),$$

which is a polynomial identity. As $\gcd(a,b)=1$, $\gcd(u,v)=1$ and $\gcd(p,q)=1$ then $u-t_1v,\dots,u-t_kv$ divide $p-\lambda q$. Hence $p-\lambda q$ is reducible in $K[\underline{x}]$. ∎

Let us reformulate the Bertini–Krull theorem in our context from [Sc, Theorem 37]. It will enable us to end the proof of Theorem 2.2.

THEOREM 2.5 (Bertini–Krull): *Let $F(\underline{x},\lambda)=p(\underline{x})-\lambda q(\underline{x})\in K[\underline{x},\lambda]$ an irreducible polynomial. Then the following conditions are equivalent:*

(1) *$F(\underline{x},\lambda_0)\in K[\underline{x}]$ is reducible for all $\lambda_0\in K$ such that $\deg_{\underline{x}} F(\underline{x},\lambda_0)=\deg_{\underline{x}} F$.*

(2) (a) *either there exist $\phi,\psi\in K[\underline{x}]$ with $\deg_{\underline{x}} F>\max\{\deg\phi,\deg\psi\}$, and $a_i\in K[\lambda]$, such that*

$$F(\underline{x},\lambda)=\sum_{i=0}^{n} a_i(\lambda)\phi(\underline{x})^{n-i}\psi(\underline{x})^i;$$

(b) *or $\operatorname{char}(K)=\pi>0$ and $F(\underline{x},\lambda)\in K[\underline{x}^\pi,\lambda]$, where $\underline{x}^\pi=(x_1^\pi,\dots,x_n^\pi)$.*

We now end the proof of Theorem 2.2:

Proof. (3) $\Rightarrow$ (1) Suppose that $p-\lambda_0 q$ is reducible in $K[\underline{x}]$ for infinitely many $\lambda_0\in\hat{K}$; then it is reducible for all $\lambda_0\in K$ such that $\deg_{\underline{x}} F(\underline{x},\lambda_0)=\deg_{\underline{x}} F$ (see Corollary 3 of Theorem 32 of [Sc]). We apply Bertini–Krull theorem

CASE (a): $F(\underline{x},\lambda)=p(\underline{x})-\lambda q(\underline{x})$ can be written as

$$p(\underline{x})-\lambda q(\underline{x})=\sum_{i=0}^{n} a_i(\lambda)\phi(\underline{x})^{n-i}\psi(\underline{x})^i.$$

So we may suppose that for $i=1,\dots,n$, $\deg_\lambda a_i=1$, let us write $a_i(\lambda)=\alpha_i-\lambda\beta_i$, $\alpha_i,\beta_i\in K$. Then

$$p(\underline{x})=\sum_{i=0}^{n}\alpha_i\phi(\underline{x})^{n-i}\psi(\underline{x})^i=\phi^n\sum_{i=0}^{n}\alpha_i\Big(\frac{\psi}{\phi}\Big)^i(\underline{x}),$$

and

$$q(\underline{x})=\sum_{i=0}^{n}\beta_i\phi(\underline{x})^{n-i}\psi(\underline{x})^i=\phi^n\sum_{i=0}^{n}\beta_i\Big(\frac{\psi}{\phi}\Big)^i(\underline{x}).$$

If we set $g(\underline{x}) = \psi(\underline{x})/\phi(\underline{x}) \in K[\underline{x}]$, and $r(t) = \frac{\sum_{i=0}^{n} \alpha_i t^i}{\sum_{i=0}^{n} \beta_i t^i}$ then $\frac{p}{q}(\underline{x}) = r \circ g$. Moreover as $\deg_{\underline{x}} F > \max\{\deg \phi, \deg \psi\}$ this implies $n \geqslant 2$ so that $\deg r \geqslant 2$. Then $p/q = f = r \circ g$ is a composite rational function

Case (b): Let $\pi = \mathrm{char}(K) > 0$ and $F(\underline{x}, \lambda) = p(\underline{x}) - \lambda q(\underline{x}) \in K[\underline{x}^\pi, \lambda]$, For $\lambda = 0$ it implies that $p(\underline{x}) = P(\underline{x}^\pi)$, then there exists $p' \in K[\underline{x}]$ such that $p(\underline{x}) = (p'(\underline{x}))^\pi$. For $\lambda = -1$ we obtain $s' \in K[\underline{x}]$ such that $p(\underline{x}) + q(\underline{x}) = (s'(\underline{x}))^\pi$. Then $q(\underline{x}) = (p(\underline{x}) + q(\underline{x})) - p(\underline{x}) = (s'(\underline{x}))^\pi - (p'(\underline{x}))^\pi = (s'(\underline{x}) - p'(\underline{x}))^\pi$. Then if we set $q' = s' - p'$ we obtain $q(\underline{x}) = (q'(\underline{x}))^\pi$. Now set $r(t) = t^\pi$ and $g = p'/q'$ we get $f = p/q = (p'/q')^\pi = r \circ g$. ∎

3. Kernel of the Jacobian derivation

We now consider the two variables case and K is an uncountable algebraically closed field of characteristic zero.

3.1. Jacobian derivation. Let $f, g \in K(x, y)$, the following formula:

$$D_f(g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x},$$

defines a derivation $D_f : K(x, y) \to K(x, y)$. Notice the $D_f(g)$ is the determinant of the Jacobian matrix of (f, g). We denote by C_f the kernel of D_f:

$$C_f = \{g \in K(x, y) \colon D_f(g) = 0\}.$$

Then C_f is a subfield of $K(x, y)$. We have the inclusion $K(f) \subset C_f$. Moreover, if $g^k \in C_f$, $k \in \mathbb{Z} \setminus \{0\}$ then $g \in C_f$.

Lemma 3.1: *Let $f = p/q$, $g \in K(x, y)$. The following conditions are equivalent:*

1. *$g \in C_f$;*
2. *f and g are algebraically dependent;*
3. *g is constant on irreducible components of the curves $(p - \lambda q = 0)$ for all but finitely many $\lambda \in \hat{K}$;*
4. *g is constant on infinitely many irreducible components of the curves $(p - \lambda q = 0)$, $\lambda \in \hat{K}$.*

Corollary 3.2: *If $g \in C_f$ is not a constant then $C_f = C_g$.*

Proof. • (1) ⇔ (2). We follow the idea of [Na1] instead of [St]. f and g are algebraically dependent if and only if $\mathrm{transc}_K K(f, g) = 1$. And

$\text{transc}_K K(f,g) = 1$ if and only if the rank of the Jacobian matrix of (f,g) is less or equal to 1, which is equivalent to $g \in C_f$.

- (2) $\Rightarrow$ (3). Let f and g be algebraically dependent. Then there exists a two variables polynomial in f and g that vanishes. Let us write

$$\sum_{i=0}^{n} R_i(f)g^i = 0$$

where $R_i(t) \in K[t]$. Let us write $f = p/q$, $g = u/v$ and $R_n(t) = \alpha(t-\lambda_1)\cdots(t-\lambda_m)$. Then

$$\sum_{i=0}^{n} R_i\left(\frac{p}{q}\right)\left(\frac{u}{v}\right)^i = 0, \quad \text{hence } \sum_{i=0}^{n} R_i\left(\frac{p}{q}\right)u^i v^{n-i} = 0.$$

By multiplication by q^d for $d = \max\{\deg R_i\}$ (in order that $q^d R_i(p/q)$ are polynomials) we obtain

$$q^d R_n\Big(p/q\Big)u^n = v\left(-q^d R_{n-1}\Big(p/q\Big)u^{n-1} - \cdots\right).$$

As $\gcd(u,v) = 1$ then v divides the polynomial $q^d R_n(\frac{p}{q})$, then v divides $q^{d-m}(p-\lambda_1 q)\cdots(p-\lambda_m q)$. Then all irreducible factors of v divide q or $p - \lambda_i q$, $i = 1, \ldots, m$.

Let $\lambda \notin \{\infty, \lambda_1, \ldots, \lambda_m\}$. Let V_λ be an irreducible component of $p - \lambda q$, then $V_\lambda \cap Z(v)$ is zero dimensional (or empty). Hence v is not identically equal to 0 on V_λ. Then for all but finitely many $(x,y) \in V_\lambda$ we get

$$\sum_{i=0}^{n} R_i(\lambda)g(x,y)^i = 0.$$

Therefore, g can only reach a finite number of values $c_1, \ldots, c_n$ (the roots of $\sum_{i=0}^{n} R_i(\lambda)t^i$). Since V_λ is irreducible, g is constant on V_λ.

- (3) $\Rightarrow$ (4). Clear.
- (4) $\Rightarrow$ (1). We first give a proof that if g is constant along an irreducible component V_λ of $(p - \lambda q = 0)$ then $D_f(g) = 0$ on V_λ (we suppose that V_λ is not in the poles of g). Let $(x_0, y_0) \in V_\lambda$ and $t \mapsto p(t)$ be a local parametrization of V_λ around (x_0, y_0). By the definition of $p(t)$, we have $f(p(t)) = \lambda$, this implies that

$$\left\langle \frac{dp}{dt} \mid \overline{\text{grad}\, f} \right\rangle = \frac{d(f(p(t))}{dt} = 0,$$

and by hypotheses g is constant on V_λ this implies that $g(p(t))$ is constant and again:

$$\left\langle \frac{dp}{dt} \middle| \overline{\operatorname{grad} g} \right\rangle = \frac{d(g(p(t))}{dt} = 0.$$

Then $\operatorname{grad} f$ and $\operatorname{grad} g$ are orthogonal around (x_0, y_0) on V_λ to the same vector, as we are in dimension 2, this implies that the determinant of Jacobian matrix of (f, g) is zero around (x_0, y_0) on V_λ. By extension $D_f(g) = 0$ on V_λ.

We now end the proof: If g is constant on infinitely many irreducible components V_λ of $(p - \lambda q = 0)$ this implies that $D_f(g) = 0$ on infinitely many V_λ. Then $D_f(g) = 0$ in $K(x, y)$. ∎

3.2. Group of the divisors. Let $f = p/q$, let $\lambda_1, \ldots, \lambda_n \in \hat{K}$, we denote by $G(f; \lambda_1, \ldots, \lambda_n)$ the multiplicative group generated by all the divisors of the polynomials $p - \lambda_i q$, $i = 1, \ldots, n$.

Let

$$d(f) = (\deg f)^2 + \deg f.$$

Lemma 3.3: *Let $F_1, \ldots, F_r \in G(f; \lambda_1, \ldots, \lambda_n)$. If $r \geqslant d(f)$ then there exists a collection of integers $m_1, \ldots, m_r$ (not all equal to zero) such that*

$$g = \prod_{i=1}^{r} F_i^{m_i} \in C_f.$$

Proof. Let $\mu \notin \{\lambda_1, \ldots, \lambda_n\}$, and let S be an irreducible component of $(p - \mu q = 0)$. Let $\bar{S}$ be the projective closure of S. The functions F_i restricted to $\bar{S}$ have their poles and zeroes on the points at infinity of S or on the intersection $S \cap Z(F_i) \subset Z(p) \cap Z(q)$.

Let $n : \tilde{S} \to \bar{S}$ be a normalization of $\bar{S}$. The inverse image under normalisation of the points at infinity are denoted by $\{\gamma_1, \ldots, \gamma_k\}$, their number verifies $k \leqslant \deg S \leqslant \deg f$.

At a point $\delta \in Z(p) \cap Z(q)$, the number of points of $n^{-1}(\delta)$ is the local number of branches of S at δ then it is less or equal than $\operatorname{ord}_\delta(S)$, where $\operatorname{ord}_\delta(S)$ denotes the order (or multiplicity) of S at δ (see e.g., [Sh], paragraph II.5.3). Then

$$\begin{aligned} \# n^{-1}(\delta) \leqslant \operatorname{ord}_\delta(S) \leqslant \operatorname{ord}_\delta Z(p - \mu q) \leqslant \operatorname{ord}_\delta Z(p - \mu q) \cdot \operatorname{ord}_\delta Z(p) \\ \leqslant \operatorname{mult}_\delta(p - \mu q, p) = \operatorname{mult}_\delta(p, q) \end{aligned}$$

where $\mathrm{mult}_\delta(p,q)$ is the intersection multiplicity (see, e.g., [Fu]). Then by Bézout theorem

$$\sum_{\delta\in Z(p)\cap Z(q)} \#n^{-1}(\delta) \leqslant \sum_{\delta\in Z(p)\cap Z(q)} \mathrm{mult}_\delta(p,q) \leqslant \deg p\cdot \deg q \leqslant (\deg f)^2.$$

Then the inverse image under normalisation of $\bigcup_{i=1}^r S\cap Z(F_i)$ denoted by $\{\gamma_{k+1},\ldots,\gamma_\ell\}$ have less or equal than $(\deg f)^2$ elements. Notice that $\ell\leqslant \deg f + (\deg f)^2 = d(f)$.

Now let ν_{ij} be the order of F_i at γ_j ($i=1,\ldots,r$; $j=1,\ldots,\ell$). Consider the matrix $M=(\nu_{ij})$. Since the degree of the divisor (F_i) (seen over $\tilde{S}$) is zero we get $\sum_{j=1}^\ell \nu_{ij}=0$, for $i=1,\ldots,r$, that means that columns of M are linearly dependent. Then $\mathrm{rk}\, M<\ell\leqslant d(f)$, by hypothesis $r\geqslant d(f)$. Then the rows of M are also linearly dependent. Let $m_1(\mu,S),\ldots,m_r(\mu,S)$ such that $\sum_{i=1}^r m_i(\mu,S)\nu_{ij}=0$, $j=1,\ldots,\ell$.

Consider the function $g_{\mu,S}=\prod_{i=1}^r F_i^{m_i(\lambda,S)}$. This function is regular and does not have zeroes or poles at the points γ_j, because $\sum_{i=1}^r m_i(\mu,S)\nu_{ij}=0$. Then $g_{\mu,S}$ is constant on S.

This construction gives a map $(\mu,S)\mapsto(m_1(\mu,S),\ldots,m_r(\mu,S))$ from K to $\mathbb{Z}^r$. Since K is uncountable, there exists infinitely many (μ,S) with the same $(m_1,\ldots,m_r)$. Then the function $g=\prod_{i=1}^r F_i^{m_i}$ is constant on infinitely many components of curves of $(p-\mu q=0)$ and by Lemma 3.1 this implies $g\in C_f$. ∎

3.3. Noncomposite rational function. Let $f=p/q$. Let $G(f)$ be the multiplicative group generated by all divisors of the polynomials $p-\lambda q$ for all $\lambda\in\hat{K}$. In fact we have

$$G(f)=\bigcup_{(\lambda_1,\ldots,\lambda_n)\in K^n} G(f;\lambda_1,\ldots,\lambda_n).$$

Definition 3.4: A family $F_1,\ldots,F_r\in G(f)$ is f**-free** if $(m_1,\ldots,m_r)\in\mathbb{Z}^r$ is such that $\prod_{i=1}^r F_i^{m_i}\in C_f$ then $(m_1,\ldots,m_r)=(0,\ldots,0)$.

A f-free family $F_1,\ldots,F_r\in G(f)$ is f**-maximal** if for all $F\in G(f)$, $\{F_1,\ldots,F_r,F\}$ is not f-free.

Theorem 3.5: *Let $f\in K(x,y)$, $\deg f>0$. Then the following conditions are equivalent:*

(1) $\deg f=\min\{\deg g : g\in C_f\setminus K\}$;

(2) *$\sigma(f)$ is finite;*
(3) *$C_f = K(f)$;*
(4) *f is noncomposite.*

Remark 3.6: This does not give a new proof of "$\sigma(f)$ is finite $\Leftrightarrow$ f is noncomposite" because we use Bertini–Krull theorem.

Remark 3.7: The proof (1) $\Rightarrow$ (2) is somewhat easier than in [St], whereas (2) $\Rightarrow$ (3) is more difficult.

Proof.

- (1) $\Rightarrow$ (2). Let us suppose that $\sigma(f)$ is infinite. Set $f = p/q$, with $\gcd(p,q) = 1$. For all $\alpha \in \sigma(f)$, let F_α be an irreducible divisor of $p - \alpha q$, such that $\deg F_\alpha < \deg f$. By Lemma 3.3 there exists a f-maximal family $\{F_1, \ldots, F_r\}$ with $r \leqslant d(f)$. Moreover, $r \geqslant 1$ because $\{F_\alpha\}$ is f-free: if not there exists $k \neq 0$ such that $F_\alpha^k \in C_f$ then $F_\alpha \in C_f$, but $\deg F_\alpha < \deg f$ that contradicts the hypothesis of minimality.

  Now, the collection $\{F_1, \ldots, F_r, F_\alpha\}$ is not f-free, so that there exist integers $\{m_1(\alpha), \ldots, m_r(\alpha), m(\alpha)\}$, with $m(\alpha) \neq 0$, such that

  $$F_1^{m_1(\alpha)} \cdots F_r^{m_r(\alpha)} \cdot F_\alpha^{m(\alpha)} \in C_f.$$

  Since $\sigma(f)$ is infinite then it is equal to $\hat{K}$ minus a finite number of values (see Theorem 2.2), $\sigma(f)$ is uncountable and the map

  $$\alpha \mapsto (m_1(\alpha), \ldots, m_r(\alpha), m(\alpha))$$

  is not injective. Let $\alpha \neq \beta$ such that $m_i(\alpha) = m_i(\beta) = m_i$, $i = 1, \ldots, r$ and $m(\alpha) = m(\beta) = m$. Then $F_1^{m_1} \cdots F_r^{m_r} \cdot F_\alpha^m \in C_f$ and $F_1^{m_1} \cdots F_r^{m_r} \cdot F_\beta^m \in C_f$, it implies that $(F_\alpha/F_\beta)^m \in C_f$, therefore, $F_\alpha/F_\beta \in C_f$.

  Now, $\deg F_\alpha/F_\beta < \deg f$, then by the hypothesis of minimality, F_α/F_β is a constant. Let $a \in K^*$ such that $F_\alpha = aF_\beta$, by definition F_α divides $p - \alpha q$, but, moreover, F_α divides $p - \beta q$ (as F_β do). Then as F_α divides both $p - \alpha q$ and $p - \beta q$, F_α divides p and q, that contradicts $\gcd(p,q) = 1$.
- (2) $\Rightarrow$ (3). Let $f = p/q$, $\sigma(f)$ finite and $g \in C_f$, we aim at proving that $g \in K(f)$. The proof will be done in several steps:

(a) REDUCTION TO THE CASE $g = u/q^\ell$. Let $g = u/v \in C_f$, then f and g are algebraically dependent, then there exists a polynomial in f and g that vanishes. As before let us write

$$\sum_{i=0}^{n} R_i(f) g^i = 0,$$

where $R_i(t) \in K[t]$. As $f = p/q$, $g = u/v$ then

$$\sum_{i=0}^{n} R_i\left(\frac{p}{q}\right)\left(\frac{u}{v}\right)^i = 0, \quad \text{hence } \sum_{i=0}^{n} R_i\left(\frac{p}{q}\right) u^i v^{n-i} = 0.$$

By multiplication by q^d for $d = \max\{\deg R_i\}$ (in order that all $q^d R_i(p/q)$ are polynomials) we get

$$q^d R_n\Big(p/q\Big) u^n = v\left(-q^d R_{n-1}\Big(p/q\Big) u^{n-1} - \cdots\right).$$

Since $\gcd(u, v) = 1$, v divides the polynomial $q^d R_n(p/q)$; we write $vu' = q^d R_n(p/q)$, then

$$g = \frac{u}{v} = \frac{uu'}{q^d R_n(p/q)}.$$

But $R_n(p/q) \in K(p/q)$ then $(uu')/q^d \in C_f$, but we also have that $g \in K(f)$ if and only if $(uu')/q^d \in K(f)$. This proves the reduction.

(b) REDUCTION TO THE CASE $g = qu$. Let $g = u/q^\ell \in C_f$, $\ell \geqslant 0$. As $\sigma(f)$ is finite by Lemma 3.1 we choose $\lambda \in K$ such that $p - \lambda q$ is irreducible and $g \in C_f$ is constant (equal to c) on $p - \lambda q$. As $g = \frac{u}{q^\ell}$,we have $p - \lambda q$ divides $u - cq^\ell$. We can write

$$u - cq^\ell = u'(p - \lambda q).$$

Then

$$\frac{u}{q^\ell} = \frac{u'}{q^{\ell-1}}\left(\frac{p}{q} - \lambda\right) + c.$$

As u/q^ℓ and $f = p/q$ are in C_f we get $u'/q^{\ell-1} \in C_f$; moreover, $u/q^\ell \in K(f)$ if and only if $u'/q^{\ell-1} \in K(f)$. By induction on $\ell \geqslant 0$ this proves the reduction.

(c) REDUCTION TO THE CASE $g = q$. Let $g = qu \in C_f$. g is constant along the irreducible curve $(p - \lambda q = 0)$. Then $qu = u_1(p - \lambda q) + c_1$. Let $\deg p = \deg q$. Then $q^h u^h = u_1^h(p^h - \lambda q^h)$ (where P^h denotes the homogeneous part of higher degree of the polynomial P). Then

$p^h - \lambda q^h$ divides $q^h u^h$ for infinitely many $\lambda \in K$. As $\gcd(p,q) = 1$, this gives a contradiction.

Hence $\deg p \neq \deg q$. We may assume $\deg p > \deg q$ (otherwise $qu \in C_f$ and $p/q \in C_f$ implies $pu \in C_f$). Then we write

$$qu = qu_1\Big(p/q - \lambda\Big) + c_1,$$

that proves that $qu_1 \in C_f$ and that $qu \in K(f)$ if and only if $qu_1 \in K(f)$. The inequality $\deg p > \deg q$ implies that $\deg u_1 < \deg u$. We continue by induction, $qu_1 = qu_2(p/q - \lambda) + c_2$, with $\deg u_2 < \deg u_1, \ldots$, until we get $\deg u_n = 0$ that is $u_n \in K^*$. Thus first we have to prove that $qu_n \in C_f$, that is to say $q \in C_f$, and secondly that $qu \in K(f)$ if and only if $q \in K(f)$.

(d) Case $g = q$. If $q \in C_f$ then q is constant along the irreducible curve $(p - \lambda q = 0)$, then $q = a(p - \lambda q) + c$, $a \in K^*$. Then

$$q = \frac{c}{1 - a(p/q - \lambda)} \in K\Big(p/q\Big) = K(f).$$

- $(3) \Rightarrow (4)$. Let us assume that $C_f = K(f)$ and that f is composite, then there exist $r \in K(t)$, $\deg r \geqslant 2$ and $g \in K(x,y)$ such that $f = r \circ g$. By the formula $\deg f = \deg r \cdot \deg g$ we get $\deg f > \deg g$. Now if $r = a/b$, then we have a relation $b(g)f = a(g)$, then f and g are algebraically dependent, hence by Lemma 3.1, $g \in C_f$. As $C_f = K(f)$, there exists $s \in K(t)$ such that $g = s \circ f$. Then $\deg g \geqslant \deg f$. That yields to a contradiction.
- $(4) \Rightarrow (1)$. Assume that f is noncomposite and let $g \in C_f$ of minimal degree. By Corollary 3.2 we get $C_f = C_g$, then

$$\deg g = \min \{\deg h \colon h \in C_g \setminus K\}.$$

Then by the already proved implication $(1) \Rightarrow (3)$ for g, we get $C_g = K(g)$. Then $f \in C_f = C_g = K(g)$, and there exists $r \in K(t)$ such that $f = r \circ g$, but as f is noncomposite then $\deg r = 1$, hence $\deg f = \deg g = \min \{\deg h \colon h \in C_f \setminus K\}$. ∎

4. Order of reducibility of rational functions in two variables

Let $f = p/q \in K(x,y)$; for all $\lambda \in \hat{K}$, let n_λ be the number of irreducible components of $p - \lambda q$. Let

$$\rho(f) = \sum_{\lambda \in \hat{K}} (n_\lambda - 1).$$

By Theorem 2.2, $\rho(f)$ is finite if and only if f is noncomposite. We give a bound for $\rho(f)$. Recall that we defined:

$$d(f) = (\deg f)^2 + \deg f.$$

THEOREM 4.1: *Let K be an algebraic closed field of characteristic 0. If $f \in K(x,y)$ is noncomposite then*

$$\rho(f) < d(f).$$

Proof. First notice that K can be supposed uncountable, otherwise it can be embedded into an uncountable field L and the spectrum in K would be included in the spectrum in L.

Let us assume that f is noncomposite, then by Theorem 2.2 and its corollary we have that $\sigma(f)$ is finite: $\sigma(f) = \{\lambda_1, \ldots, \lambda_r\}$. We suppose that $\rho(f) \geqslant d(f)$. Let $f = p/q$. We decompose the polynomials $p - \lambda_i q$ in irreducible factors, for $i = 1, \ldots, r$

$$p - \lambda_i q = \prod_{j=1}^{n_i} F_{i,j}^{k_{i,j}},$$

where n_i stands for n_{λ_i}. Notice that since $\gcd(p,q) = 1$ then $F_{i,j}$ divides $p - \lambda_i q$ but does not divide any of $p - \mu q$, $\mu \neq \lambda_i$. The collection

$$\{F_{1,1}, \ldots, F_{1,n_1-1}, \ldots, F_{r,1}, \ldots, F_{r,n_r-1}\},$$

is included in $G(f, \lambda_1, \ldots, \lambda_r)$ and contains $\rho(f) \geqslant d(f)$ elements, then Lemma 3.3 provides a collection $\{m_{1,1}, \ldots, m_{1,n_1-1}, \ldots, m_{r,1}, \ldots, m_{r,n_r-1}\}$ of integers (not all equal to 0) such that

$$g = \prod_{i=1}^{r} \prod_{j=1}^{n_i-1} F_{i,j}^{m_{i,j}} \in C_f. \tag{1}$$

By Theorem 3.5 it implies that $g \in K(f)$, then $g = u(f)/v(f)$, where $u, v \in K[t]$. Let $\mu_1, \dots, \mu_k$ be the roots of u and $\mu_{k+1}, \dots, \mu_\ell$ the roots of v. Then

$$g = \frac{u(p/q)}{v(p/q)} = \alpha \frac{\prod_{i=1}^{k} \frac{p}{q} - \mu_i}{\prod_{i=k+1}^{\ell} \frac{p}{q} - \mu_i}$$

so that

$$g = \alpha q^{\ell - 2k} \frac{\prod_{i=1}^{k} p - \mu_i q}{\prod_{i=k+1}^{\ell} p - \mu_i q}. \tag{2}$$

If $m_{i_0,j_0} \neq 0$ then by the definition of g, by equation (1) and by equation (2), we get that F_{i_0,j_0} divides one of the $p - \mu_i q$ or divides q. If F_{i_0,j_0} divides $p - \mu_i q$ then $\mu_i = \lambda_{i_0} \in \sigma(f)$. If F_{i_0,j_0} divides q then $\mu_i = \infty$, so that $\infty \in \sigma(f)$. In both cases $p - \lambda_{i_0} q$ appears in formula (2) at the numerator or at the denominator of g. Then $F_{i_0,n_{i_0}}$ should appears in decomposition (1), that gives a contradiction. Then $\rho(f) < d(f)$. ∎

5. Extension to several variables

We follow the lines of the proof of [Na3]. We will need a result that claims that the irreducibility and the degree of a family of polynomials remains constant after a generic linear change of coordinates. For $\underline{x} = (x_1, \dots, x_n)$ and a matrix $B = (b_{ij}) \in Gl_n(K)$, we denote the new coordinates by $B \cdot \underline{x}$:

$$B \cdot \underline{x} = \left(\sum_{j=1}^{n} b_{1j} x_j, \dots, \sum_{j=1}^{n} b_{nj} x_j \right).$$

Proposition 5.1: *Let K be an infinite field. Let $n \geqslant 3$ and $p_1, \dots, p_\ell \in K[x_1, \dots, x_n]$ be irreducible polynomials. Then there exists a matrix $B \in Gl_n(K)$ such that for all $i = 1, \dots, \ell$ we get:*

- *$p_i(B \cdot \underline{x})$ is irreducible in $\overline{K(x_1)}[x_2, \dots, x_n]$;*
- $\deg_{(x_2,\dots,x_n)} p_i(B \cdot \underline{x}) = \deg_{(x_1,\dots,x_n)} p_i$.

The proof of this proposition can be derived from [Sm, Ch. 5, Theorem 3D] or by using [FJ, Proposition 9.31]. See [Na3] for details.

Now we return to our main result.

Theorem 5.2: *Let K be an algebraically closed field of characteristic 0. Let $f \in K(\underline{x})$ be noncomposite, then $\rho(f) < (\deg f)^2 + \deg f$.*

Proof. We will prove this theorem by induction on the number of variables n. For $n=2$, we proved in Theorem 4.1 that $\rho(f) < (\deg f)^2 + \deg f$.

Let $f = p/q \in K(\underline{x})$, with $\underline{x} = (x_1, \ldots, x_n)$. We suppose that f is noncomposite. For each $\lambda \in \sigma(f)$ we decompose $p - \lambda q$ into irreducible factors:

$$p - \lambda q = \prod_{i=1}^{n_\lambda} F_{\lambda,i}^{r_{\lambda,i}}. \tag{3}$$

We fix $\mu \notin \sigma(f)$. We apply Proposition 5.1 to the polynomials $p - \mu q$ and $F_{\lambda,i}$, for all $\lambda \in \sigma(f)$ and all $i = 1, \ldots, n_\lambda$. Then the polynomials $p(B \cdot \underline{x}) - \mu q(B \cdot \underline{x})$ and $F_{\lambda,i}(B \cdot \underline{x})$ are irreducible in $\overline{K(x_1)}[x_2, \ldots, x_n]$ and their degrees in $(x_2, \ldots, x_n)$ are equal to the degrees in $(x_1, \ldots, x_n)$ of $p - \mu q$ and $F_{\lambda,i}$.

Denote $k = \overline{K(x_1)}$. This is an uncountable field, algebraically closed of characteristic zero. Now $p(B \cdot \underline{x}) - \mu q(B \cdot \underline{x})$ is irreducible, then $f(B \cdot \underline{x})$ is noncomposite in $k(x_2, \ldots, x_n)$.

Now equation (3) becomes

$$p(B \cdot \underline{x}) - \lambda q(B \cdot \underline{x}) = \prod_{i=1}^{n_\lambda} F_{\lambda,i}(B \cdot \underline{x})^{r_{\lambda,i}}.$$

Which is the decomposition of $p(B \cdot \underline{x}) - \lambda q(B \cdot \underline{x})$ into irreducible factors in $k(x_2, \ldots, x_n)$. Then

$$\sigma(f) \subset \sigma(f(B \cdot \underline{x})),$$

where $\sigma(f)$ is a subset of K, and $\sigma(f(B \cdot \underline{x}))$ is a subset of $k = \overline{K(x_1)}$. As n_λ is also the number of distinct irreducible factors of $p(B \cdot \underline{x}) - \lambda q(B \cdot \underline{x})$ we get

$$\rho(f) \leqslant \rho(f(B \cdot \underline{x})).$$

Now suppose that the result is true for $n-1$ variables. Then for $f(B \cdot \underline{x}) \in k(x_2, \ldots, x_n)$ we get

$$\rho(f(B \cdot \underline{x})) < (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x}))^2 + (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x})).$$

Hence:

$$\begin{aligned}
\rho(f) &\leqslant \rho(f(B \cdot \underline{x})) \\
&< (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x}))^2 + (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x})) \\
&= (\deg_{(x_1,\ldots,x_n)} f)^2 + (\deg_{(x_1,\ldots,x_n)} f) \\
&= (\deg f)^2 + (\deg f) \qquad \blacksquare
\end{aligned}$$

If for $n = 2$ we start the induction with Lorenzini's bound $\rho(f) < (\deg f)^2$ we obtain with the same proof the following result for several variables, for K of any characteristic K and a better bound:

THEOREM 5.3: *Let K be an algebraically closed field. Let $f \in K(\underline{x})$ be non-composite then $\rho(f) < (\deg f)^2$.*

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