A MINIMAX THEOREM FOR LOCALLY LIPSCHITZ FUNCTIONALS AND APPLICATIONS

By

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Abstract. We prove an abstract theorem which provides multiple critical points for locally Lipschtiz functionals under the presence of symmetry. The abstract result is applied to find multiple solutions in $H_0^1(\Omega)$ for the critical semilinear elliptic equation $-\Delta u = f(x, u) + |u|^{4/(N-2)}u$, where f is a discontinuous perturbation and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain.

1 Introduction

It is known that free boundary problems can be reduced to the study of partial differential equations with discontinuous nonlinearities. This is the case in the model of the heat conductivity in eletrical media, where we denote by $K(x, t)$ and $\sigma(x, t)$ the thermal and electrical conductivity, respectively, $x \in \Omega \subset \mathbb{R}^N$ is the position and $t \in \mathbb{R}$ is the temperature. Since we are considering an electrical media, the function σ can be discontinuous on t, and the distribution of the temperature is unknown. This can be described by the PDE

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(K(x, u(x))\frac{\partial u(x)}{\partial x_i}\Big) = \sigma(x, u(x))$$

which is related with a free boundary problem in which the jump surface of the electrical conductivity in unknown. We also learned from J. L. Lions [25] that there is a close connection between PDEs with discontinuos nonlinearities and obstacle problems, since this last one is also a free boudary problem (see [11] for a comprehensive discussion on this subject). Finally, it is worth mentioning that PDEs with discontinuous nonlinearities also arise in the context of nonsmooth mechanics, the seepage surface problem, the Elenbass equation, and other related areas (see [8, 10, 12] for instance).

One of the main techniques to deal with PDEs with discotinuous nonlinearities is the nonsmooth critical point theory. Roughly speaking, it consists in extending for locally Lipschtiz continuous functions the classical variational methods strongly developed since the pioneer work of Ambrosetti and Rabinowitz [3].

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The starting point for the nonsmooth theory comes from the concept of generalized gradients, introduced by Clarke [13, 14, 15]. Let E be a reflexive Banach space with duality given by $\langle \xi, u\rangle$, for any $\xi \in E^*$, $u \in E$, and denote by $\mathrm{Lip}_{loc}(E)$ the set of all locally Lipschitz continuous functionals $I : E \to \mathbb{R}$. The generalized directional derivative of I at $u \in E$ in the direction $v \in E$ is defined as

$$I^0(u; v) := \limsup_{\|h\|_E \to 0,\, \lambda \to 0^+} \frac{I(u + h + \lambda v) - I(u + h)}{\lambda}, \quad \forall v \in E.$$

The generalized gradient of I at u is the following subset of E^*,

$$\partial I(u) := \{\xi \in E^* : I^0(u; v) \geq \langle \xi, v\rangle,\ \forall v \in E\},$$

and we say that $u \in E$ is a critical point of I if $0 \in \partial I(u)$. It can be proved that, if $I \in C^1(E, \mathbb{R})$, then $\partial I(u) = \{I'(u)\}$ and therefore this notion reduces to the classical one in the smooth case.

In a seminal paper [12], Chang used the above concepts to derive minimax theorems for non-differentiable functionals. Among other results, he proved versions of the Mountain Pass Theorem and the Saddle Point Theorem. Since then, many researchers have obtained existence and multiplicity of critical points for non-differentiable functionals under different compactness and geometric conditions. We cite the papers [22, 27, 28, 24, 21, 19, 1], the books [29, 16] and references therein.

Our first interest here is to present conditions that guarantee the existence of multiple critical points for even functionals. Before stating our main theorem we define, for $I \in \mathrm{Lip}_{loc}(E)$ and $u \in E$,

$$\lambda_I(u) := \min\{\, \|\xi\|_{E^*} : \xi \in \partial I(u)\}.$$

We say that I satisfies the nonsmooth $(\mathrm{PS})_c$-condition at level $c \in \mathbb{R}$ if any sequence $(u_n) \subset E$ such that $I(u_n) \to c$ and $\lambda_I(u_n) \to 0$ has a convergent subsequece.

Our main abstract result is the following:

Theorem 1.1. *Let $E = V \oplus X$ be an infinite-dimensional reflexive Banach space with* $\dim V = k < +\infty$. *Suppose that $I \in \mathrm{Lip}_{loc}(E)$ is even, $I(0) = 0$ and*

(I_1) *there exists $\rho > 0$ such that*

$$\inf_{u \in X \cap \partial B_\rho} I(u) \geq 0;$$

(I_2) *for some $M \in \mathbb{R}$ and a subspace $V_0 \subset E$ such that* $\dim V_0 = m > k$, *we have*

$$\sup_{u \in V_0} I(u) < M.$$

If I satisfies the nonsmooth $(PS)_c$-condition for any $c \in [0, M)$, then I has at least $m - k$ pairs of nonzero critical points.

The key point for proving the above result is the correct establishment of deformation arguments. Besides a classical result proved by Chang [12], we use here two new deformation lemmas whose proofs were inspired by results presented for C^1-functionals in the papers of Bartolo, Benci & Fortunato [6] and Silva [32]. Of course, adapting for the Lipsthiz case requires new ideas and some fine calculations.

Theorem 1.1 is the nonsmooth version of [33, Theorem 2.1] (see also [32, Theorem 3.7]) and complements the aforementioned abstract results for locally Lipschitz functionals. It is closely related with previous works of Szulkin [35] and Goeleven, Montreanu and Panagiotopoulos [18] (see also [16, Theorem 2.1.7]). Essentially, they supposed that $I \geq \beta > 0$ on $X \cap \partial B_\rho$, I is anticoercive in V_0 and the nonsmooth $(PS)_c$-condition is satisfied at any level. Besides our weaker geometric conditions, we only requires compactness in the range $[0, M)$. As it is well known, since the paper of Brezis and Nirenberg [7], local compactness conditions are specially important when dealing with PDEs with critical nonlinearities

In the second part of the paper, we apply our abstract theorem to find multiple solutions for the problem

$$(P_\mu) \qquad \begin{cases} -\Delta u = f(x,u) + \mu |u|^{2^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded smooth domain, $\mu > 0$ is a parameter, $2^* := 2N/(N-2)$ is the critical Sobolev exponent and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies:

(f_0) for a.e. $x \in \Omega$, the map $f(x, \cdot)$ is odd and, for any $M > 0$, there holds

$$\sup_{x\in\Omega,\, 0\leq s\leq M} |f(x,s)| < +\infty.$$

There is a vast literature concerning semilinear critical problems with discotinuous nonlinearities. Since it is impossible to give a complet list of references, we just quote the papers [4, 5, 2, 9, 20, 36, 31] and its references.

In order to introduce the class of nonlinearities that we are going to deal with, we define

$$\underline{f}(x,s) := \min\{\lim_{t\to s^-} f(x,t), \lim_{t\to s^+} f(x,t)\},$$

$$\overline{f}(x,s) := \max\{\lim_{t\to s^-} f(x,t), \lim_{t\to s^+} f(x,t)\},$$

and consider the family of index $J := \{0, 1, \ldots, n\}$ or $J := \{0\} \cup \mathbb{N}$. We say that a function satisfying (f_0) belongs to the class $\mathcal{F}$ if there exists an ordered sequence $(s_j)_{j\in J} \subset [0, +\infty)$ such that $s_0 = 0$ and

($\mathcal{F}_1$) if J has infinitely many points, then $\lim_{j\to+\infty} s_j = +\infty$;
($\mathcal{F}_2$) $\lim_{s\to s_j^\pm} f(x,s) \in \mathbb{R}$, for any $j \in J \setminus \{0\}$;
($\mathcal{F}_3$) the map $s \mapsto f(x,s)$ is continuous in (s_{j-1}, s_j) for any $j \in J \setminus \{0\}$ and for a.e. $x \in \Omega$;
($\mathcal{F}_4$) there holds

$$\sup_{x\in\overline{\Omega},\, s>0} |\overline{f}(x,s) - \underline{f}(x,s)| < +\infty;$$

($\mathcal{F}_5$) if $\beta_j := \sup_{x\in\overline{\Omega}} \overline{f}(x,s_j)$, then the set $\mathcal{J} := \{j \in J : \beta_j < 0\}$ is finite;
($\mathcal{F}_6$) if $\overline{f}(x,s_j) > 0$ a.e. in Ω, then $\underline{f}(x,s_j) \geq 0$, a.e. in Ω;
($\mathcal{F}_7$) if $\overline{f}(x,s_j) = 0$ a.e. in Ω, then $\underline{f}(x,s_j) = 0$, a.e. in Ω;
($\mathcal{F}_8$) $f(x,s) = 0$, for any $s \in [0, s_1]$ and a.e. in Ω.

Besides the above structural conditions, we are going to consider the case that f is a locally superlinear with quasicritical growth. More specifically, we shall assume the following:

(f_1) there exist $\sigma \in [0,2)$ and $a_1, a_2 > 0$ such that

$$\frac{1}{2}\underline{f}(x,s)s - F(x,s) \geq -a_1 - a_2 s^\sigma, \quad \forall\, s > 0, \text{ a.e. in } \Omega,$$

where $F(x,s) := \int_0^s f(x,t)\,dt$;

(f_2) there holds

$$\lim_{s\to+\infty} \frac{\overline{f}(x,s)}{s^{2^*-1}} = 0, \quad \text{uniformly a.e. in } \Omega;$$

(f_3) there exist $\theta \in (2, 2^*)$ and $a_3,\ a_4 > 0$ and such that

$$F(x,s) \leq a_3 s^\theta + a_4, \quad \forall\, s > 0, \text{ a.e. in } \Omega;$$

(f_4) there is an open nonempty set $\Omega_0 \subset \Omega$ such that

$$\liminf_{s\to+\infty} \frac{F(x,s)}{s^2} = +\infty, \quad \text{uniformly a.e. in } \Omega_0.$$

As an application of Theorem 1.1, we obtain the following multiplicity result:

Theorem 1.2. *Suppose that $f \in \mathcal{F}$ satisfies (f_1)–(f_4). Then, given $k \in \mathbb{N}$, there exists $\mu_k > 0$ such that, for any $\mu \in (0, \mu_k)$, problem (P_μ) has at least k pairs of nonzero solutions in $W^{2,2N/(N+2)}(\Omega) \cap H_0^1(\Omega)$.*

Denote by $H : \mathbb{R} \to \mathbb{R}$ the Heaviside function given by $H(s) = 0$, if $s \leq 0$, and $H(s) = 1$, otherwise. It is not difficult to see that Theorem 1.2 applies for the model function $f(s) := H(s-a)g(s)$, whenever $g(s)$ behaves like a pure power $|s|^{p-2}s$, with $2 < p < 2^*$. The main point here is that we can consider a larger class of nonlinearities, including that which are not bounded by a polynomial with subcritical growth. Moreover, differently from many of the aforementioned works, it is not necessary for f to have only a finite number of discontinuity points. In the final section of the paper we present two examples of nonlinearities f which seem not to be considered in the literature.

As a final comment, let u be one of the solutions given by Theorem 1.2 and suppose that the set $\{x \in \Omega : |u(x)| > s_1\}$ has zero Lebesgue measure. In this case, it follows from $(\mathcal{F}_8)$ that $f(\cdot, u) \equiv 0$, and therefore $-\Delta u = \mu |u|^{2^*-2}u$ in Ω. So, if $\lambda_1(\Omega) > 0$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$, we can use Poincaré's inequality to get

$$\lambda_1(\Omega)\|u\|^2_{L^2(\Omega)} = \|u\|^2 = \mu\|u\|^{2^*}_{L^{2^*}(\Omega)} \leq \mu s_1^{2^*-2}\|u\|^2_{L^2(\Omega)},$$

from which we conclude that

$$\mu \geq \frac{\lambda_1(\Omega)}{s_1^{2^*-2}}.$$

Hence, for $\mu > 0$ small, the solutions are such that the range of $f(\cdot, u)$ "crosses" the first point of discontinuity. Actually, one of the examples presented in the last section is such that $f(\cdot, u)$ crosses any prescribed number of points where f is not continuous.

The rest of the paper is organized in the following way: in the next section we present some basic results about nonsmooth critical point theory and our deformation results. Theorems 1.1 and 1.2 are proved in Section 3 and 4, respectively.

2 Abstract framework and basic results

For completeness, we start this section by recalling some elements of the critical point theory for locally Lipschitz functionals developed by Chang [12]. From now on $(E, \|\cdot\|)$ is a reflexive Banach space with the duality in its dual space E^* denoted by $\langle \xi, u\rangle$, for any $\xi \in E^*$, $u \in E$. By $\mathrm{Lip}_{loc}(E)$ we mean the set of all functions $I : E \to \mathbb{R}$ which are locally Lipschitz continuous, i.e., for each $u \in E$, there exist a neighbourhood N_u of u and a constant $C_u > 0$ such that

$$|I(u_1) - I(u_2)| \leq C_u\|u_1 - u_2\|, \quad \forall\, u_1, u_2 \in N_u.$$

It is clear that $I \in \mathrm{Lip}_{loc}(E)$ may not be differentiable at a point $u \in E$. However, we can define the generalized directional derivative by setting

$$I^0(u; v) := \limsup_{\|h\|\to 0,\, \lambda\to 0^+} \frac{I(u + h + \lambda v) - I(u + h)}{\lambda}, \quad \forall v \in E.$$

It can be proved that the above quantity is well defined. Moreover, the map $v \mapsto I^0(u; v)$ is sub-additive and positively homogeneous, and therefore convex.

Since it is also continuous, we can follow Clarke [13] and introduce the subdifferential at $u \in E$ as

$$\partial I^0(u;z) := \{\xi \in E^* \, : I^0(u;v) \geq I^0(u;z) + \langle \xi, v - z\rangle, \, \forall v \in E\}.$$

Finally, the generalized gradient of I at u is the subset of E^* given by

$$\partial I(u) := \{\xi \in E^* \, : I^0(u;v) \geq \langle \xi, v\rangle, \, \forall v \in E\}.$$

We collect below the main properties of the generalized gradient.

Proposition 2.1. *For any $u \in E$ and $I \in \mathrm{Lip}_{loc}(E)$, the following hold:*

(a) *the set $\partial I(u) \subset E^*$ is convex and compact in the weak-$*$ topology;*

(b) *the function $\lambda_I : E \to \mathbb{R}$ given by*

$$\lambda_I(u) := \min\{\, \|\xi\|_{E^*} \, : \, \xi \in \partial I(u)\}$$

is well defined and lower semi-continuous;

(c) *if $J \in C^1(E, \mathbb{R})$, then*

$$\partial J(u) = \{J'(u)\}, \qquad \partial(I + J)(u) = \partial I(u) + \{J'(u)\}.$$

According to the last item above, it is natural to say that $u \in E$ is a critical point of $I \in \mathrm{Lip}_{loc}(E)$ if $0 \in \partial I(u)$. This means that $I^0(u_0; v) \geq 0$, for any $v \in E$. We say that c is a critical level if there exists a critical point $u_0 \in E$ of I such that $I(u_0) = c$.

From the above definition we easily get $\partial(-I)(u) = -\partial I(u)$. Hence, it follows from [12, Propostion (9), p. 106] that

Proposition 2.2. *If $I \in \mathrm{Lip}_{loc}(E)$ and $\phi \in C^1([0,1], E)$, then the composition $h := I \circ \phi$ is differentiable almost everywhere and*

$$\min_{\xi \in \partial I(\phi(t))} \langle \xi, \phi'(t)\rangle \leq h'(t) \leq \max_{\xi \in \partial I(\phi(t))} \langle \xi, \phi'(t)\rangle, \quad \textit{for a.e. } t \in [0,1].$$

We say that $I \in \mathrm{Lip}_{loc}(E)$ satisfies the nonsmooth $(\mathrm{PS})_c$-condition if any sequence $(u_n) \subset E$ such that

$$\lim_{n\to+\infty} I(u_n) = c, \qquad \lim_{n\to+\infty} \lambda_I(u_n) = 0$$

has a convergent subsequece. It is clear that this condition implies that the set $K_c := \{u \in E : I(u) = c, \; 0 \in \partial I(u)\}$ is compact.

Given $\alpha, \; \beta \in \mathbb{R}$, we define the sets

$$I^\beta := \{u \in E \, : I(u) \leq \beta\}, \quad K_\alpha^\beta := \{u \in E : I(u) \in [\alpha, \beta], \; 0 \in \partial I(u)\}.$$

For any given set $A \subset E$ and $\delta > 0$, we denote by $N_\delta(A)$ the closed δ-neighborhood of A. We finally set, for any $\varepsilon,\ \delta > 0$,

$$\mathcal{B}(\varepsilon, \alpha, \beta, \delta) := I^{\beta+\varepsilon} \setminus (I^{\alpha-\varepsilon} \cup N_\delta(K_\alpha^\beta)).$$

As is well known, in order to prove minimax results we need to construct a useful homeomorphism deforming a level set of the functional in another one. The starting point is defining some kind of pseudo-gradient field.

Lemma 2.3. *Let E be a reflexive Banach space and $I \in \mathrm{Lip}_{loc}(E)$ satisfying the nonsmooth (PS)$_c$-condition for any $c \in [\alpha, \beta]$. Then, for any given $\delta > 0$, there exist $d_0,\ \varepsilon_0 > 0$ such that*

$$\lambda_I(u) \geq d_0, \quad \forall\, u \in \mathcal{B}(\varepsilon_0, \alpha, \beta, \delta),$$

and a locally Lipschitz vector field $v : \mathcal{B}(\varepsilon_0, \alpha, \beta, \delta) \to E$ satisfying $\|v(u)\| \leq 1$ and

$$\langle \xi, v(u) \rangle > \frac{d_0}{2}, \quad \forall\, \xi \in \partial I(u). \tag{2.1}$$

Moreover, if I is even, the above vector field can be constructed as an odd function.

Proof. Suppose that the first statement is not true. Then we can obtain sequences (ε_n), $(c_n) \subset \mathbb{R}$, $(u_n) \in \mathcal{B}(\varepsilon_n, \alpha, \beta, \delta)$ such that $\varepsilon_n \to 0$, $c_n \to 0$ and $\lambda_I(u_n) \to 0$. Since $I(u_n) \in [\alpha - \varepsilon_n, \beta + \varepsilon_n]$, up to a subsequence we have that $I(u_n) \to c_0 \in [\alpha, \beta]$. It follows from the nonsmooth (PS)$_{c_0}$-condition that, up to a subsequence again, $u_n \to u_0$ strongly in E. Since λ_I is lower semi-continuous, we have that $\lambda_I(u_0) = 0$. Hence, $u_0 \in K_\alpha^\beta$, which contradicts $u_n \notin N_\delta(K_\alpha^\beta)$.

Now we have proved that $\mathcal{B}(\varepsilon, \alpha, \beta, \delta)$ has no critical points, the construction of the vector field v can be done arguing along the same lines of [12, Lemma 3.3]. We omit the details. □

The following results are the keystone for the proof of our abstract result. They are variants, in the nonsmooth setting, of the deformations presented in [32, 6]. In the first one, we require compactness only at level $c = 0$.

Lemma 2.4. *Let $E = V \oplus X$ be a reflexive Banach space with* $\dim V < +\infty$. *Suppose that $I \in \mathrm{Lip}_{loc}(E)$ satisfies $I(0) = 0$, (I_1) and the nonsmooth (PS)$_0$-condition. Then, for each $\delta > 0$, there exist $r, \nu > 0$ and a homeomorphism $\psi : E \to E$ such that:*

(ψ_1) $\psi(u) = u$, for any $u \in E \setminus \overline{B_{\rho+r}}$;

(ψ_2) if $u \in \partial B_\rho \cap X \setminus N_\delta(K_0 \cap \partial B_\rho)$, then $I(\psi(u)) \geq \nu$;

(ψ_3) ψ is odd when I is even.

Proof. Suppose first that $K_0 \cap \partial B_\rho \neq \emptyset$ and set, for any $\overline{\varepsilon} \in (0,1)$,

$$A_{\overline{\varepsilon}} := \{u \in E : d(u, X \cap \partial B_\rho) \le \overline{\varepsilon},\ d(u, K_0 \cap \partial B_\rho) \ge \delta/2,\ |I(u)| \le \overline{\varepsilon}\}$$

and

$$C_{\overline{\varepsilon}} := \{u \in E : d(u, K_0) > 4\overline{\varepsilon},\ \lambda_I(u) \ge \overline{\varepsilon}\},$$

where $d(u, K)$ denotes the distance of the point u for the set $K \subset E$.

We claim that, for $\overline{\varepsilon} > 0$ small, there holds $A_{\overline{\varepsilon}} \subset C_{\overline{\varepsilon}}$. Indeed, if this is not true, we can obtain a sequence $(u_n) \subset E$ such that

$$d(u_n, X \cap \partial B_\rho) \to 0, \quad d(u_n, K_0 \cap \partial B_\rho) \ge \frac{\delta}{2}, \quad I(u_n) \to 0, \tag{2.2}$$

but $d(u_n, K_0) \to 0$ or $\lambda_I(u_n) \to 0$. If the first alternative holds we can use the compactness of K_0 and the first convergence in (2.2) to guarantee that, along a subsequece, $u_n \to u \in K_0 \cap \partial B_\rho$, which contradicts $d(u_n, K_0 \cap \partial B_\rho) \ge \delta/2$. If $\lambda_I(u_n) \to 0$, we can use the nonsmooth $(PS)_0$-condition to guarantee the convergence of (u_n) and argue as before.

By applying Lemma 2.3 with $\alpha = \beta = 0$ and $\delta = \overline{\varepsilon}$, we obtain $d_0, \varepsilon_0 > 0$ and a locally Lipschtiz vector field v defined in $\mathcal{B} := \mathcal{B}(\varepsilon_0, 0, 0, \overline{\varepsilon})$ verifying (2.1). It is clear that we may suppose that $\varepsilon_0 < \overline{\varepsilon}$. We now pick $0 < \varepsilon < \varepsilon_0$ and choose cutoff Lipscthiz functions $g_1, g_2 : E \to [0,1]$ such that

$$h_1(u) = \begin{cases} 1, & \text{if } I(u) \notin [-\varepsilon_0, \varepsilon_0], \\ 0, & \text{if } I(u) \in [-\varepsilon, \varepsilon], \end{cases} \qquad h_2(u) = \begin{cases} 1, & \text{if } u \notin N_{4\overline{\varepsilon}}(K_0), \\ 0, & \text{if } u \in N_{2\overline{\varepsilon}}(K_0). \end{cases}$$

We can easily verify that the function $V : E \to E$ given by

$$W_0(u) := \begin{cases} h_1(u)h_2(u)v(u), & \text{if } u \in \mathcal{B} \\ 0, & \text{if } u \notin \mathcal{B} \end{cases}$$

is a locally Lipsthiz vector field globally defined. We finally consider Lipscthiz functions $h_3, h_4 : E \to [0,1]$ such that

$$h_3(u) = \begin{cases} 1, & \text{if } d(u, X \cap \partial B_\rho) \le \varepsilon, \\ 0, & \text{if } d(u, X \cap \partial B_\rho) \ge \overline{\varepsilon}, \end{cases} \qquad h_4(u) = \begin{cases} 1, & \text{if } d(u, K_0 \cap \partial B_\rho) \ge 3\delta/4, \\ 0, & \text{if } d(u, K_0 \cap \partial B_\rho) \ge \delta/2, \end{cases}$$

and define

$$W(u) := h_3(u)h_4(u)W_0(u), \quad u \in E.$$

We shall construct our deformation as the solution of the Cauchy problem

$$\frac{d}{dt}\sigma(t,u) = W(\sigma(t,u)), \quad \sigma(0,u) = u. \tag{2.3}$$

Since W is locally Lipschitz and $\|W\| \le 1$, the above EDO has a unique solution globally defined and the map $u \mapsto \sigma(t,u)$ is a homeomorphism for any fixed $t \ge 0$. We consider $0 < t_0 < \min\{\varepsilon, \delta/4\}$ and we shall verify the lemma for the map $\psi(u) := \sigma(t_0, u)$, $r = \overline{\varepsilon}$ and $\nu > 0$ to be chosen later.

Let $u \in E \setminus \overline{B_{\rho+\overline{\varepsilon}}}$ and notice that $h_3(u) = W(u) = 0$. Thus $\sigma(\cdot, u) = u$ and therefore property (ψ_1) clearly holds. In order to verify (ψ_2), we take $u \in \partial B_\rho \cap X \setminus N_\delta(K_0 \cap \partial B_\rho)$. If $W(u) = 0$ then $\sigma(\cdot, u) = u$ and $I(\sigma(\cdot, u))$ is constant. Otherwise, if $W(\sigma(t,u)) \neq 0$, then $\sigma(t,u) \in \mathcal{B}$ and it follows from Proposition 2.2 and (2.1) that

$$\frac{d}{dt} I(\sigma(t,u)) \ge \min_{\xi \in \partial I(\psi(t,u))} \langle \xi, \sigma'(t,u) \rangle = \min_{\xi \in \partial I(\sigma(t,u))} \langle \xi, W(\sigma(t,u)) \rangle \ge \frac{d_0}{2},$$

from which we conclude that the map $t \mapsto I(\sigma(t,u))$ is nondecreasing.

If $I(\sigma(t,u)) > \varepsilon$ for some $t \in [0, t_0]$, then $\psi(u) = \sigma(t_0, u)$ satisfies (ψ_2) with $\nu = \varepsilon$. So, we may assume that $I(\sigma(t,u)) \le \varepsilon$, for any $t \in [0, t_0]$. It follows from (I_1) that

$$0 \le I(u) = I(\sigma(0,u)) \le I(\sigma(t,u)) \le \varepsilon < \varepsilon_0, \quad \forall\, t \in [0, t_0]. \tag{2.4}$$

Since $\|W\| \le 1$, after integrating (2.3) over $[0, t]$ we get $\|\sigma(t,u) - u\| \le t$, for any $t \in [0, t_0]$. Recalling that $t_0 \le \varepsilon$, we obtain

$$d(\sigma(t,u), X \cap \partial B_\rho) \le \varepsilon, \quad \forall\, t \in [0, t_0].$$

Moreover, for any $t \in [0, t_0]$ and $z \in K_0 \cap \partial B_\rho$, there holds

$$\|\sigma(t,u) - z\| \ge \|u - z\| - \|\sigma(t,u) - u\| \ge \delta - t_0,$$

and therefore we can use $t_0 \le \delta/2$ to conclude that

$$d(\sigma(t,u), K_0 \cap \partial B_\rho) \ge \frac{3\delta}{4}, \quad \forall\, t \in [0, t_0]. \tag{2.5}$$

Let $t \in [0, t_0]$ be fixed. Since $\varepsilon < \overline{\varepsilon}$, it follows from (2.4)–(2.5) that $\sigma(t,u) \in A_{\overline{\varepsilon}} \subset C_{\overline{\varepsilon}}$. Actually, from the definition of $C_{\overline{\varepsilon}}$ and (2.4) again, we conclude that $\sigma(t,u) \in \mathcal{B}$. Moreover, all the above considerations and the definition of the cutoff functions show that $h_i(\sigma(t,u)) = 1$, for any $i \in \{1,2,3,4\}$. Thus, integrating

$$\frac{d}{dt} I(\sigma(t,u)) \ge \min_{\xi \in \partial I(\sigma(t,u))} \langle \xi, W(\sigma(t,u)) \rangle = \min_{\xi \in \partial I(\sigma(t,u))} \langle \xi, v(\sigma(t,u)) \rangle \ge \frac{d_0}{2}$$

over $[0, t_0]$ and using (I_1), we obtain

$$I(\sigma(t_0,u)) \ge I(u) + \frac{t_0 d_0}{2} \ge \frac{t_0 d_0}{2}.$$

Hence, we conclude that (ψ_2) holds for $\nu := \min\{\varepsilon, t_0 d_0/2\}$, whenever $K_0 \cap \partial B_\rho \neq \emptyset$. If this set is empty, we can proceed in a similar way just dropping the condition $d(u, K_0 \cap \partial B_\rho) > \delta/2$ in the definition of $A_{\overline{\varepsilon}}$.

The proof of (ψ_3) can be done as in Lemma 2.5. □

We state and prove in the sequel our second deformation result.

Lemma 2.5. *Let E be a reflexive Banach space and $I \in \mathrm{Lip}_{loc}(E)$ satisfying the nonsmooth (PS)$_c$-condition for any $c \in [\alpha, \beta]$. If K_α^β is bounded then, for any given $\overline{\varepsilon} > 0$, there exist c_1, $R_1 > 0$, $\widehat{\varepsilon} \in (0, \overline{\varepsilon})$ and a homeomorphism $\eta : E \to E$ such that*

(η_1) $\|\eta(u) - u\| \leq c_1$, *for any $u \in E$;*
(η_2) $I(\eta(u)) \leq I(u)$, *for any $u \in E$;*
(η_3) $\eta(I^\beta \setminus B_{R_1}) \subset I^{\alpha - \widehat{\varepsilon}}$;
(η_4) *η is odd when I is even.*

Proof. Given $\delta \in (0, 1)$, we can apply Lemma 2.3 to obtain $\varepsilon_0 \in (0, \overline{\varepsilon})$ and $d_0 \in (0, 1/2)$ such that

$$\lambda_I(u) \geq d_0 > 0, \quad \forall\, u \in \mathcal{B} := \mathcal{B}(\varepsilon_0, \alpha, \beta, \delta). \tag{2.6}$$

We set

$$A_1 := \{u \in E : I(u) \leq \alpha - \varepsilon_0\} \cup \{u \in E : I(u) \geq \beta + \varepsilon_0\}$$

and, for any $\widehat{\varepsilon} \in (0, \varepsilon_0)$,

$$A_2 := \{u \in E : \alpha - \widehat{\varepsilon} \leq I(u) \leq \beta + \widehat{\varepsilon}\}.$$

Let $R > 0$ be such that $N_\delta(K_\alpha^\beta) \subset B_R$. Consider Lipschitz functions

$$h_1,\ h_2 : E \to [0, 1]$$

satisfying

$$h_1(u) = \begin{cases} 1, & \text{if } u \in A_2, \\ 0, & \text{if } u \in A_1, \end{cases} \qquad h_2(u) = \begin{cases} 1, & \text{if } u \notin B_{2R}, \\ 0, & \text{if } u \in B_R, \end{cases}$$

and define the locally Lipschtiz vector field

$$W(u) := \begin{cases} h_1(u) h_2(u) v(u), & \text{if } u \in \mathcal{B}, \\ 0, & \text{if } u \notin \mathcal{B}, \end{cases}$$

where $v : \mathcal{B} \to E$ comes from Lemma 2.3.

We now consider the Cauchy problem

$$\frac{d}{dt}\sigma(t,u) = -W(\sigma(t,u)), \quad \sigma(0,u) = u.$$

Since $\|W(u)\| \leq 1$, for any $u \in E$, the above EDO has a unique solution globally defined and the map $u \mapsto \sigma(t,u)$ is a homeomorphism for any fixed $t \geq 0$. We pick $t_0 > 2(\beta - \alpha + \widehat{\varepsilon})/d_0$ and we shall prove the lemma for $\eta(u) := \sigma(t_0, u)$.

By integrating the EDO over $[0, t_0]$ and using $\|W\| \leq 1$, we easily get

$$\|\eta(u) - u\| = \|\sigma(t_0, u) - u\| \leq t_0,$$

and therefore (η_1) holds for $c_1 := t_0$. In order to prove (η_2) we first notice that, if $W(u) = 0$, then $\sigma(t,u) = u$ and there is nothing to do. If $W(\sigma(t,u)) \neq 0$, then $\sigma(t,u) \in \mathcal{B}$ and we can use Proposition 2.2 and (2.6) to get

$$\begin{aligned} \frac{d}{dt}I(\sigma(t,u)) &\leq \max_{\xi \in \partial I(\sigma(t,u))} \langle \xi, \sigma'(t,u)\rangle \\ &= -\min_{\xi \in \partial I(\sigma(t,u))} \langle \xi, W(\sigma(t,u))\rangle \\ &\leq -g_1(\sigma(t,u))h_2(\sigma(t,u))\frac{d_0}{2}. \end{aligned} \tag{2.7}$$

Thus, the map $t \mapsto I(\sigma(t,u))$ is nonincreasing and (η_2) is verified.

Let $R_1 > 2R + t_0$ and $u \in I^{\beta} \setminus B_{R_0}$. If $I(u) \leq \alpha - \widehat{\varepsilon}$, it is clear from the monotonicity of $t \mapsto I(\sigma(t,u))$ that $\eta(u) \in I^{\alpha - \widehat{\varepsilon}}$. So, we need only to consider $u \in I^{\beta} \setminus (I^{\alpha-\widehat{\varepsilon}} \cup B_{R_1})$. In this case, we have that

$$\|\sigma(t,u)\| \geq \|u\| - \|\sigma(t,u) - u\| \geq R_1 - c_1 = R_1 - t_0 > 2R,$$

in such a way that $g_2(\sigma(t,u)) = 1$ on $[0, t_0]$. Suppose by contradiction that $\eta(u) \notin I^{\alpha-\widehat{\varepsilon}}$. Then

$$\alpha - \widehat{\varepsilon} < I(\sigma(t,u)) \leq \beta, \quad \forall\, t \in [0, t_0],$$

which shows that $g_1(\sigma(t,u)) = 1$. Moreover, from the defintion of $\mathcal{B}$ we see that $\sigma(t,u) \in \mathcal{B}$, for any $t \in [0, t_0]$. Thus, integrating (2.7) over this interval and using $t_0 > 2(\beta - \alpha + \widehat{\varepsilon})/c_0$, we obtain

$$I(\sigma(t_0,u)) \leq I(u) - \frac{t_0 c_0}{2} \leq \beta - \frac{t_0 c_0}{2} < \alpha - \widehat{\varepsilon},$$

which is a contradiction. Hence, $\eta(u) = \sigma(t_0, u) \in I^{\alpha-\widehat{\varepsilon}}$ and (η_3) holds.

In order to prove (η_4) we consider the above argument with all the cutoff functions being even and recall that the map v can be taken odd when I is even, according to Lemma 2.3. □

3 The abstract result

We devote this section to the proof of our abstract theorem. From now on, we shall assume that the hypotheses of Theorem 1.1 hold. Let

$$\mathcal{E} := \{Y \subset E \setminus \{0\} : Y \text{ is closed}, \ Y = -Y\}$$

be the class of all closed and symmetric sets of E which do not contain the origin. Recall that the genus [23] of a nonempty set $A \subset \mathcal{E}$ is defined as

$$\gamma(A) := \inf\{k \in \mathbb{N} : \text{there exists } \varphi : A \to \mathbb{R}^k \setminus \{0\} \text{ continuous and odd}\}.$$

If no such map φ exists we define $\gamma(A) = +\infty$ and we also set $\gamma(\emptyset) = 0$. We refer to [30, Chapter 7] for the main properties of the genus.

Since I satisfies the $(\mathrm{PS})_0$-condition the set

$$\widetilde{K}_0 := K_0 \cap B_\rho$$

is compact. Thus, for some $\delta > 0$ small, we have that $\gamma(N_{2\delta}(\widetilde{K}_0)) = \gamma(\widetilde{K}_0)$. For this $\delta > 0$, we apply Lemma 2.4 to obtain $r, \ \nu > 0$ and an odd function $\psi : E \to E$ such that

(ψ_1) $\psi(u) = u$, for any $u \in E \setminus \overline{B_{\rho+r}}$;

(ψ_2) if $u \in \partial B_\rho \cap X \setminus N_\delta(K_0 \cap \partial B_\rho)$, then $I(\psi(u)) \geq \nu$.

We also pick $\beta > \nu$ verifying

$$0 < \sup_{u \in V_0} I(u) < \beta < M \tag{3.1}$$

and fix $\alpha \in (0, \nu)$. By applying now Lemma 2.5 with $0 < \overline{\varepsilon} < \beta - \alpha$, we obtain $c_1, \ R_1 > 0, \widehat{\varepsilon} \in (0, \overline{\varepsilon})$ and an odd function $\eta : E \to E$ such that

(η_1) $\|\eta(u) - u\| \leq c_1$, for any $u \in E$;

(η_2) $I(\eta(u)) \leq I(u)$, for any $u \in E$;

(η_3) $\eta(I^\beta \setminus B_{R_1}) \subset I^{\alpha - \widehat{\varepsilon}}$.

We now recall that $\dim V_0 = m > k = \dim V$, choose $R > 0$ such that

$$R > \max\{R_1, \ \rho + c_1 + r\} \tag{3.2}$$

and set

$$D := \overline{V_0 \cap B_R}, \quad G := \{h \in C(D, E) : h \text{ is odd}, \ h \equiv \eta \text{ on } \partial D\}.$$

Moreover, for any $j \in, \{1, 2, \ldots, m\}$, we define

$$\Gamma_j := \{h(\overline{D \setminus Y}) : h \in G, \ Y \in \mathcal{E}, \ \gamma(Y) \leq m - j\}.$$

We claim that the following properties hold:

(Γ^1) $\Gamma_j \neq \varnothing$;

(Γ^2) $\Gamma_{j+1} \subset \Gamma_j$, if additionally $j \neq m$;

(Γ^3) if $\varphi \in C(E, E)$ is odd and $\varphi \equiv \mathrm{Id}$ on $I^{\alpha-\widehat{\varepsilon}}$ or on $E \setminus \overline{B_{\rho+r}}$, then $\varphi(A) \in \Gamma_j$, for any $A \in \Gamma_j$;

(Γ^4) if $A \in \Gamma_j$, $Y \in \mathcal{E}$, $\gamma(Y) \leq l < j$, then $\overline{A \setminus Y} \in \Gamma_{j-l}$.

The first two statements are clear and (Γ^4) can be proved as in [30, Proposition 9.18], in such a way that we shall only verify (Γ^3). Let $A = h(\overline{D \setminus Y}) \in \Gamma_j$ and notice that, since $\varphi(A) = (\varphi \circ h)(\overline{D \setminus Y})$, it is sufficient to show that $\varphi \circ h = \eta$ on ∂D. Given $u \in \partial D$, we may use $u \in V_0$, (3.1) and (3.2) to get $u \in I^{\beta} \setminus B_{R_1}$. Hence, by ($\eta_3$), we have that $\eta(u) \in I^{\alpha-\widehat{\varepsilon}}$ and therefore $\varphi(\eta(u)) = \eta(u)$ if $\varphi \equiv \mathrm{Id}$ on $I^{\alpha-\widehat{\varepsilon}}$. In the case that $\varphi \equiv \mathrm{Id}$ on $E \setminus \overline{B_{\rho+r}}$, the same conclusion follows from (η_1) and (3.2).

After introducing all this notation, we can define the minimax levels

$$c_j := \inf_{A \in \Gamma_j} \max_{u \in A} I(u), \quad j = 1, 2, \ldots, m.$$

Since $0 \in A$, for any $A \in \Gamma_1$ and $I(0) = 0$, we have that $c_1 \geq 0$. Moreover, (η_2) implies that $\eta|_D \in G$ and $\gamma(\varnothing) = 0$, and therefore we conclude that $\eta(D) \in \Gamma_m$. Hence,

$$c_m \leq \max_{z \in \eta(D)} I(z) = \max_{u \in D} I(\eta(u)) \leq \max_{u \in D \subset V_0} I(u) < M.$$

The above consideration and property (Γ^1) provide

$$0 \leq c_1 \leq \cdots \leq c_j \leq c_{j+1} \leq \cdots \leq c_m < M.$$

The next result is a variant of [30, Proposition 9.23].

Proposition 3.1. *If $m \geq j > k$ and $A \in \Gamma_j$, then $A \cap \partial B_\rho \cap X \neq \varnothing$.*

Proof. Let $A = h(\overline{D \setminus Y}) \in \Gamma_m$, where $Y \in \mathcal{E}$ is such that $\gamma(Y) \leq m - j$. Let $\Omega := h^{-1}(B_\rho)$ and notice that, if $u \in \partial D$, then (η_1) implies that

$$\|h(u)\| = \|\eta(u)\| \geq R - c_1 > \rho.$$

Hence $\Omega \subset \operatorname{int} D$ is an open set of the m-dimensional space V_0, bounded, symmetric and $0 \in \Omega$. The Borsuk–Ulam theorem implies that $\gamma(\partial\Omega) = m$. Since $h(\partial\Omega) \subset \partial B_\rho$, we have that $\partial\Omega \subset \Upsilon := h^{-1}(\partial B_\rho)$ and we can use [30, Proposition 7.5] to obtain

$$\gamma(h(\overline{\Upsilon \setminus Y})) \geq \gamma(\overline{\Upsilon \setminus Y}) \geq \gamma(\Upsilon) - \gamma(Y) \geq \gamma(\partial\Omega) - \gamma(Y) = m - \gamma(Y) > k.$$

By using [30, Proposition 7.8] we conclude that $h(\overline{\Upsilon \setminus Y}) \cap X \neq \varnothing$. The result is a consequence of $h(\Upsilon) \subset A \cap \partial B_\rho$. □

Corollary 3.2. *If $l \geq j > k$, $A \in \Gamma_l$ and $\gamma(\widetilde{K}_0) \leq l - j$, then*

$$A \cap \psi(X \cap \partial B_\rho \setminus N_\delta(\widetilde{K}_0)) \neq \varnothing.$$

Proof. Since ψ^{-1} is even and continuous, it follows from (ψ_1) and (Γ^3) that $\psi^{-1}(A) \in \Gamma_l$. Thus, $\gamma(N_{2\delta}(\widetilde{K}_0)) = \gamma(\widetilde{K}_0) \leq l - j$ and (Γ^4) imply that

$$\mathcal{A} := \overline{\psi^{-1}(A) \setminus N_{2\delta}(\widetilde{K}_0)} \in \Gamma_j.$$

Hence, we can use Proposition 3.1 to obtain $\mathcal{A} \cap \partial B_\rho \cap X \neq \varnothing$. Since A is closed, we have that $\mathcal{A} \subset \psi^{-1}(A) \setminus N_\delta(\widetilde{K}_0)$ and the result easily follows. □

Corollary 3.3. *If $l \geq j > k$ and $c_j \leq \cdots \leq c_l < \nu$, then $\gamma(\widetilde{K}_0) \geq l - j + 1$.*

Proof. If $\gamma(\widetilde{K}_0) \leq l - j$, it follows from Corollary 3.2 and (ψ_2) that

$$c_l = \inf_{A \in \Gamma_l} \max_A I(u) \geq \inf_{A \in \Gamma_l} \left\{ \max_{A \cap \psi(\partial B_\rho \cap X \setminus N_\delta(\widetilde{K}_0))} I(u) \right\} \geq \nu$$

which is a contradiction. □

Lemma 3.4. *If $l \geq j > k$ and $c_l = c_j = c \geq \alpha > 0$, then $\gamma(K_c) \geq l - j + 1$.*

Proof. By using $I(0) = 0 < c$ and the nonsmooth $(PS)_c$-condition, we get $K_c \in \mathcal{E}$. Suppose, by contradiction, that $\gamma(K_c) \leq l - j$. Then, for some $\delta_1 > 0$ small, there holds $\gamma(N_{\delta_1}(K_c)) \leq l - j$. Since the nonsmooth $(PS)_c$-condition provides compactness for K_c, a simple inspection of the proof of a deformation lemma due to Chang (see [12, Theorem 3.11]) provides $\varepsilon \in (0, \widehat{\varepsilon}/2)$ and an odd homeomorphism $\tau : E \to E$ such that

(τ_1) $\tau(u) = u$, for any $u \in I^{c+\widehat{\varepsilon}/2} \setminus I^{c-\widehat{\varepsilon}/2}$;
(τ_2) $\tau(I^{c+\varepsilon} \setminus N_{\delta_1}(K_c)) \subset I^{c-\varepsilon}$.

Since $c_l = c$, there exists $A \in \Gamma_l$ such that $\max_{u \in A} I(u) \leq c + \varepsilon$. By using (Γ^4) we conclude that $\overline{A \setminus N_{\delta_1}(K_c)} \in \Gamma_j$. From $\alpha \leq c$ and (τ_1), we obtain that $\tau = \mathrm{Id}$ on $I^{\alpha - \widehat{\varepsilon}}$ and therefore we can use (Γ^3) to infer that

$$\mathcal{A} := \tau(\overline{A \setminus N_{\delta_1}(K_c)}) \in \Gamma_j.$$

But (τ_2) provides $\mathcal{A} \subset I^{c-\varepsilon}$ and so

$$c = c_j \leq \max_{u \in \mathcal{A}} I(u) \leq c - \varepsilon,$$

which does not make sense. Hence $\gamma(K_c) \geq l - j + 1$ and we have done. □

We are now ready to present the proof of our abstract theorem.

Proof of Theorem 1.1. Suppose first that $\nu \leq c_{k+1} \leq c_m$. If all the minimax levels $c_{k+1}, c_{k+2}, \ldots, c_m$ are different, it follows from Lemma 3.4 that $\gamma(K_{c_j}) \geq 1$, for each $j \in J := \{k+1, k+2, \ldots, m\}$. So, we have $m-k$ distint critical levels, each of them with a pair of nonzero critical points. On the other hand, if $c_{k+j} = c_{k+j+1}$ for some $j \in J$, then the same lemma implies that $\gamma(K_{c_{k+j}}) \geq 2$, and therefore there are infinitely many pairs of critical points at level c_{k+j}.

We consider now the case that $c_{k+1} \leq c_l < \nu \leq c_{l+1} \leq c_m$. By using Corollary 3.3 we get $\gamma(\widetilde{K}_0) \geq l-k$. As before, if $l-k > 1$ then we have done. Otherwise, $l = k+1$ and there is a pair of nonzero critical points at level 0. Moreover, since $\nu \leq c_{k+2} \leq c_m$, arguing as in the first part of the proof we obtain at least $m-k-1$ nonzero critical points with energy in $[\nu, c_m]$. □

4 Elliptic problem with critical growth

Throughout, we denote by E the Sobolev space $H_0^1(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2}$$

and by $2^\#$ the conjugated exponent of 2^*, namely $2^\# := (2^*)' = 2N/(N+2)$. For any $\mu > 0$, a straightforward computation shows that the functional

$$J_\mu(u) := \frac{1}{2}\|u\|^2 - \frac{\mu}{2^*}\int |u|^{2^*}\mathrm{d}x, \quad u \in E,$$

belongs to $C^1(E, \mathbb{R})$. Concerning the nonsmooth part of the equation we notice that, in view of (f_0), (f_2) and $(\mathcal{F}_4)$, there exists $C \in \mathbb{R}$ such that

$$|f(x,s)| \leq C(1 + |s|^{2^*-1}), \quad \text{for a.e. } x \in \Omega,\ s \in \mathbb{R}.$$

Hence [12, Section 2] the map $F : \Omega \times \mathbb{R} \to \mathbb{R}$ given by $F(x,s) := \int_0^s f(x,t)\mathrm{d}t$ is a Carthéodory function. Moreover, the map

$$\Psi(u) := \int_\Omega F(x,u)\,\mathrm{d}x, \quad u \in E,$$

belongs to $\mathrm{Lip}_{loc}(E)$. We are going to look for solutions of problem (P_μ) as critical points of the functional

$$I_\mu(u) := J_\mu(u) - \Psi(u), \quad u \in E.$$

Given $u \in E$ and $w \in L^1_{loc}(\Omega)$, we write hereafter $w \in [\underline{f}(\cdot,u), \overline{f}(\cdot,u)]$ when $\underline{f}(x,u(x)) \leq w(x) \leq \overline{f}(x,u(x))$, for a.e. $x \in \Omega$. By using this notation, we can characterize the elements of the generalized gradient of I_μ at some point $u \in E$ in the following way:

Lemma 4.1. *If $\xi \in \partial I_\mu(u)$, then there exists $w_\xi \in L^{2^\#}(\Omega)$ such that*

$$\langle \xi, \varphi \rangle = \int_\Omega \nabla u \nabla \varphi \, dx - \mu \int_\Omega |u|^{2^*-2} u\varphi \, \mathrm{d}x - \int_\Omega w_\xi \varphi \, \mathrm{d}x, \quad \forall\, \varphi \in E$$

and $w_\xi \in [\underline{f}(\cdot, u), \overline{f}(\cdot, u)]$.

Proof. According to Proposition 2.1(c), we have that

$$\partial I_\mu(u) = \{J'_\mu(u)\} - \partial \Psi(u).$$

Since $E \subset L^{2^*}(\Omega)$, if $\xi_0 \in \partial\Psi(u)$ there exists a linear functional $\overline{\xi} : L^{2^*}(\Omega) \to \mathbb{R}$ such that $\overline{\xi}|_E = \xi_0$. Recalling that $2^\# = (2^*)'$, we can use Riesz' theorem to obtain $w_\xi \in L^{2^\#}(\Omega)$ such that

$$\langle \xi, \varphi \rangle = \langle \overline{\xi}, \varphi \rangle = \int_\Omega w_\xi \varphi \, \mathrm{d}x, \quad \forall\, \varphi \in E.$$

This and the definition of J_μ prove the first statement. For the last one, we refer to [12, Corollary on page 111]. □

Corollary 4.2. *If $u \in E$ is a critical point of I_μ, then $u \in W^{2,2^\#}(\Omega) \cap H_0^1(\Omega)$ and*

$$-\Delta u - \mu |u|^{2^*-2} u \in [\underline{f}(\cdot, u), \overline{f}(\cdot, u)].$$

Proof. In this case, we have $0 \in \partial I_\mu(u) = \{J'_\mu(u)\} - \partial\Psi(u)$. By picking $\xi = 0$ in Lemma 4.1, we obtain

$$\int_\Omega \nabla u \nabla \varphi \, \mathrm{d}x = \mu \int_\Omega |u|^{2^*-2} u\varphi \, \mathrm{d}x + \int_\Omega w_0 \varphi \, \mathrm{d}x, \quad \forall\, \varphi \in H_0^1(\Omega), \tag{4.1}$$

with $w_0 \in [\underline{f}(\cdot, u), \overline{f}(\cdot, u)]$. Since $\mu |u|^{2^*-2} u + w_0 \in L^{2^\#}(\Omega)$, we obtain from [17, Theorem 9.15] a unique $\overline{u} \in W_0^{1,2^\#}(\Omega) \cap W^{2,2^\#}(\Omega)$ such that $-\Delta \overline{u} = \mu |u|^{2^*-2} u + w_0$ in Ω. Recalling that $\overline{u} \in H_0^1(\Omega)$, we can use (4.1) to conclude that $\overline{u} = u$ and the lemma is proved. □

Lemma 4.3. *There exists $\mu^* > 0$ such that, if $\mu \in (0, \mu^*)$ and $u \in E$ is a critical point of I_μ then u is a strong solution of (P_μ).*

Proof. By Lemma 4.1 and Corollary 4.2, we obtain

$$w_0(x) := -\Delta u(x) - \mu |u(x)|^{2^*-2} u(x) \in [\underline{f}(x, u(x)), \overline{f}(x, u(x))]$$

a.e. in Ω, with $w_0 \in L^{2^\#}(\Omega)$. For any $j \in J \setminus \{0\}$, we define

$$\Omega_j := \{x \in \Omega : u(x) = \pm s_j\},$$

where $s_j > 0$ are points where $f(x,\cdot)$ is discontinuous. If we can prove that any Ω_j has Lebesgue measure zero, it follows from $(\mathcal{F}_3)$ that, for a.e. $x \in \Omega$,

$$[\underline{f}(x,u(x)), \overline{f}(x,u(x))] = \{f(x,u(x))\},$$

in such a way that $w_0(x) = f(x,u(x))$.

Let $\mathcal{J} \subset J$ be the finite subset of index given by $(\mathcal{F}_5)$ and set

$$j_0 := \max\{j : j \in \mathcal{J}\}, \quad \beta^* := \max\{\beta_j : j \in \mathcal{J}\} < 0.$$

We are going to prove the lemma for

$$\mu^* := -\beta^* s_{j_0}^{1-2^*} > 0.$$

Pick $\mu \in (0,\mu^*)$ and suppose, by contradiction, that $|\Omega_j| > 0$. By a result due to Stampacchia [34], we have that $-\Delta u = 0$ in Ω_j, and therefore

$$-\mu|s_r|^{2^*-2}s_r \in [\underline{f}(x,s_j), \overline{f}(x,s_j)]. \tag{4.2}$$

We may suppose that $u(x) = s_j > 0$. Since $\mu > 0$, it follows from $(\mathcal{F}_6)$ and $(\mathcal{F}_7)$ that $j \in \mathcal{J}$. Hence, recalling that $s_j \le s_{j_0}$, we get

$$\underline{f}(x,s_j) < \overline{f}(x,s_j) \le \beta^* = -\mu^* s_{j_0}^{2^*-1} \le -\mu s_j^{2^*-1},$$

which contradicts (4.2). □

We prove in the sequel that sequences of almost critical points are bounded.

Lemma 4.4. *If $(u_n) \subset E$ is such that $I_\mu(u_n) \to c > 0$ and $\lambda_{I_\mu}(u_n) \to 0$, then (u_n) is bounded.*

Proof. From Proposition 2.1(b), there exists $\xi_n \in \partial I_\mu(u_n)$ such that

$$\|\xi_n\|_* = \lambda_{I_\mu}(u_n),$$

where $\|\xi\|_*$ stands for the norm of the linear functional $\xi \in E^*$. If

$$w_n = w_{\xi_n} \in L^{2^\#}(\Omega)$$

is the function given by Lemma 4.1, we have that

$$\begin{aligned} c + o_n(1) + \frac{1}{2}\|\xi_n\|_*\|u_n\| &\ge I_\mu(u_n) - \frac{1}{2}w_n(u_n) \\ &= \frac{\mu}{N}\|u_n\|_{2^*}^{2^*} + \int_\Omega \Big[\frac{1}{2}w_n u_n - F(x,u_n)\Big]\mathrm{d}x, \end{aligned}$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$.

If $u_n(x) \geq 0$, we can use $w_n \in [\underline{f}(\cdot, u_n), \overline{f}(\cdot, u_n)]$ and (f_1) to get

$$\frac{1}{2}w_n u_n - F(x, u_n) \geq \frac{1}{2}\underline{f}(x, u_n)u_n - F(x, u_n) \geq -a_1 - a_2|u_n|^{\sigma}.$$

Otherwise, if $u_n(x) < 0$, using $\overline{f}(x, -s) = -\underline{f}(x, s)$, for $s \geq 0$, and (f_1) again, we obtain

$$\begin{aligned}\frac{1}{2}w_n u_n - F(x, u_n) &\geq \frac{1}{2}\overline{f}(x, u_n)u_n - F(x, u_n)\\ &= \frac{1}{2}\underline{f}(x, -u_n)(-u_n) - F(x, -u_n) \geq -a_1 - a_2|u_n|^{\sigma}.\end{aligned}$$

All together, the above expressions yield

$$\|u_n\|_{2^*}^{2^*} \leq c_1 + c_2\|u_n\| + c_3\|u_n\|_{\sigma}^{\sigma}$$

and therefore we infer from $0 \leq \sigma < 2$ that

$$\frac{1}{2}\|u_n\|_{2^*}^{2^*} \leq c_2\|u_n\| + c_4. \tag{4.3}$$

By using $I_{\mu}(u_n) = c + o_n(1)$ and (f_3), we obtain

$$\frac{1}{2}\|u_n\|^2 \leq c + o_n(1) + \frac{\mu}{2^*}\|u_n\|_{2^*}^{2^*} + b_1\|u_n\|_{\theta}^{\theta} + b_2|\Omega|.$$

This inequality, $2 < \theta < 2^*$ and (4.3) imply that

$$\|u_n\|^2 \leq c_5\|u_n\|_{2^*}^{2^*} + c_5 \leq c_6\|u_n\| + c_5,$$

which proves the lemma. □

We define

$$S := \inf\left\{\int_{\Omega}|\nabla u|^2 \mathrm{d}x \,:\, \int_{\Omega}|u|^{2^*}\mathrm{d}x = 1\right\}$$

and state in the sequel the concentration-compactness principle due to Lions [26]. It will be important in the proof of the nonsmooth Palais–Smale condition.

Lemma 4.5. *Let $(u_n) \subset H_0^1(\Omega)$ be such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Then there exist ν, ζ bounded measures on $\overline{\Omega}$, an at most countable index set J, $\{x_j\}_{j\in J} \subset \overline{\Omega}$ and $(\nu_j)_{j\in J}$, $(\zeta_j)_{j\in J} \subset (0, +\infty)$ such that $\zeta_j \geq S\nu_j^{2/2^*}$,*

(a) $|u_n|^{2^*}\mathrm{d}x \rightharpoonup \nu = |u|^{2^*}dx + \sum_{j\in J}\nu_j\delta_{x_j}$;

(b) $|\nabla u_n|^2\mathrm{d}x \rightharpoonup \zeta \geq |\nabla u|^2 dx + \sum_{j\in J}\zeta_j\delta_{x_j}$;

weakly in the sense of measures.

Before proving the next result we notice that, for any $\varepsilon > 0$ given, we can use $f \in \mathcal{F}$ to obtain $C_\varepsilon > 0$ such that, for a.e. $x \in \Omega$ and any $s \in \mathbb{R}$, there hold

$$\max\{|\underline{f}(x,s)|, |\overline{f}(x,s)|\} \le C_\varepsilon + \varepsilon |s|^{2^*-1} \tag{4.4}$$

and

$$\max\{|\underline{f}(x,s)s|, |\overline{f}(x,s)s|\} \le C_\varepsilon + \varepsilon |s|^{2^*}.$$

Moreover, we can obtain a Carathéodory function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{|s|\to+\infty} \frac{g(x,s)}{|s|^{2^*-1}} = 0, \quad \text{uniformly a.e. in } \Omega \tag{4.5}$$

and

$$\max\{|\underline{f}(x,s)|, |\overline{f}(x,s)|\} \le g(x,s), \tag{4.6}$$

for a.e. $x \in \Omega$ and any $s \in \mathbb{R}$.

We prove in what follows a technical convergence result which will be useful to get compactness.

Lemma 4.6. *If $u_n \rightharpoonup u$ weakly in E, $w_n \rightharpoonup w_0$ weakly in $L^{2^\#}(\Omega)$ and $w_n \in [\underline{f}(\cdot, u_n), \overline{f}(\cdot, u_n)]$, then*

$$\lim_{n\to+\infty} \int_\Omega w_n \varphi \, \mathrm{d}x = \int_\Omega w_0 \varphi \, \mathrm{d}x, \quad \forall \varphi \in E$$

and

$$\lim_{n\to+\infty} \int_\Omega w_n u_n \, \mathrm{d}x = \int_\Omega w_0 u \, \mathrm{d}x.$$

Proof. The first statement directly follows from the weak convergenge of (u_n) and the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega) \simeq (L^{2^\#}(\Omega))^*$. For the second one, we notice that

$$\begin{aligned}\int_\Omega (w_n u_n - w_0 u) \, \mathrm{d}x &= \int_\Omega w_n(u_n - u) \, \mathrm{d}x + \int_\Omega (w_n - w_0) u \, \mathrm{d}x \\ &= \int_\Omega w_n(u_n - u) \, \mathrm{d}x + o_n(1)\end{aligned}$$

and therefore it is sufficient to check that the last integral above goes to zero.

Since we may assume that u_n pointwise converges, it is clear that

$$\lim_{n\to+\infty} w_n(x)(u_n(x) - u(x)) = 0, \quad \text{for a.e. } x \in \Omega. \tag{4.7}$$

Due to the compact embeddings we obtain $h_1 \in L^1(\Omega)$ such that $|u_n|^{2^*-1} \le h_1$ a.e. in Ω. Hence, we can use $w_n \in [\underline{f}(\cdot, u_n), \overline{f}(\cdot, u_n)]$ and (4.4) to conclude that $|w_n| \leq C_1 + h_1$ a.e. in Ω. By using (4.5) and [33, Lemma 3.1], we conclude that $g(\cdot, u_n)u_n \to g(\cdot, u)u$ in $L^1(\Omega)$ and therefore there exists $h_2 \in L^1(\Omega)$ such that $|g(\cdot, u_n)u_n|$, $|g(\cdot, u_n)u| \leq h_2$ a.e. in Ω. Thus, using $w_n \in [\underline{f}(\cdot, u_n), \overline{f}(\cdot, u_n)]$ again and (4.6), we get $|w_n u_n - w_n u| \leq 2h_2$ a.e. in Ω and the result follows from (4.7) and the Lebesgue Theorem. □

Lemma 4.7. *Let $(u_n) \subset H_0^1(\Omega)$ be as in Lemma 4.5 and suppose that*

$$\lambda_{I_\mu}(u_n) \to 0.$$

Then the set J is empty or finite.

Proof. It is sufficent to prove that, if $j \in J$, then

$$\nu_j \geq (S/\mu)^{N/2}. \tag{4.8}$$

Indeed, if this is true, then

$$+\infty > \nu(\overline{\Omega}) \geq \sum_{j \in J} \nu_j \geq \sum_{j \in J} (S/\mu)^{N/2},$$

and therefore J cannot have infinitely many elements.

In order to prove (4.8), we consider $\phi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ such that $\phi \equiv 1$ on $\overline{B_1}$ and $\phi \equiv 0$ outside B_2. For any $\varepsilon > 0$, we set $\phi_\varepsilon(x) := \phi((x - x_j)/\varepsilon)$. It is easy to see that $(\phi_\varepsilon u_n) \subset H_0^1(\Omega)$ is bounded. Let $\xi_n \in \partial I_\mu(u_n)$ be such that $\|\xi_n\|_* = \lambda_{I_\mu}(u_n)$ and $w_n = w_{\xi_n} \in L^{2^\#}(\Omega)$ given by Lemma 4.1. Since $\langle \xi_n, \phi_\varepsilon u_n \rangle = o_n(1)$, we can compute

$$I_n^\varepsilon + \int_\Omega |\nabla u_n|^2 \phi_\varepsilon \mathrm{d}x = o_n(1) + \mu \int_\Omega |u_n|^{2^*} \phi_\varepsilon \mathrm{d}x + \int_\Omega w_n u_n \phi_\varepsilon \mathrm{d}x, \tag{4.9}$$

where

$$I_n^\varepsilon := \int_\Omega u_n (\nabla u_n \cdot \nabla \phi_\varepsilon) \mathrm{d}x.$$

Recalling that $w_n \in [\underline{f}(\cdot, u_n), \overline{f}(\cdot, u_n)]$, we can use (4.4) to conclude that the sequence $(w_n) \subset L^{2^\#}(\Omega)$ is bounded. So, we may assume that $w_n \rightharpoonup w_0$ weakly in $L^{2^\#}(\Omega)$ and argue as in the proof of Lemma 4.6 to get

$$\lim_{n \to +\infty} \int_\Omega w_n u_n \phi_\varepsilon \mathrm{d}x = \int_\Omega w_0 u \phi_\varepsilon \mathrm{d}x.$$

It follows from (4.9) and Lemma 4.5 that

$$\limsup_{n \to +\infty} I_n^\varepsilon + \int_{\overline{\Omega}} \phi_\varepsilon \mathrm{d}\zeta = \mu \int_{\overline{\Omega}} \phi_\varepsilon \mathrm{d}\nu + \int_\Omega w_0 u \phi_\varepsilon \mathrm{d}x. \tag{4.10}$$

We now notice that, since we may assume $u_n \to u$ strongly in $L^2(\Omega)$, the change of variables $y = (x - x_j)/\varepsilon$ provides

$$
\begin{aligned}
|I_n^\varepsilon| &\leq \|u_n\| \left(\int_{\Omega \cap B_\varepsilon(x_j)} u_n^2 |\nabla \phi_\varepsilon|^2 \mathrm{d}x \right)^{1/2} \\
&= \frac{c_1}{\varepsilon} \left(\int_{\Omega \cap B_\varepsilon(x_j)} u_n^2 |\nabla \phi((x - x_j)/\varepsilon)|^2 \mathrm{d}x \right)^{1/2} \\
&\leq c_2 \varepsilon^{(N-2)/2} \left(\int_{\Omega \cap B_\varepsilon(x_j)} u^2(\varepsilon y + x_j) \mathrm{d}y + o(1) \right)^{1/2},
\end{aligned}
$$

as $\varepsilon \to 0^+$, from which we conclude that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} I_n^\varepsilon = 0.$$

So, letting $\varepsilon \to 0^+$ in (4.10), using the Lebesgue Theorem and the definition of ϕ_ε, we obtain $\zeta(\{x_j\}) = \mu\nu(\{x_j\})$, that is, $\zeta_j = \mu\nu_j$. This equality together with $S\nu_j^{2/2^*} \leq \zeta_j$ (see Lemma 4.5) imply (4.8). □

Proposition 4.8. *Suppose that $f \in \mathcal{F}$ satisfies (f_1) and (f_2). Then, for any given $M > 0$, there exists $\mu^{**} > 0$ such that I_μ satisfies the nonsmooth $(PS)_c$-condition for all $c < M$ and $\mu \in (0, \mu^{**})$.*

Proof. Let $(u_n) \subset E$ be such that $I_\mu(u_n) \to c < M$ and $\lambda_{I_\mu}(u_n) \to 0$. As before, we consider $\xi_n \in \partial I_\mu(u_n)$ such that $\|\xi_n\|_* = \lambda_{I_\mu}(u_n)$ and $w_n \in L^{2^\#}(\Omega)$ verifies $w_n \in [\underline{f}(\cdot, u_n), \overline{f}(\cdot, u_n)]$ and

$$\langle \xi_n, \varphi \rangle = \int_\Omega (\nabla u_n \cdot \nabla \varphi) \mathrm{d}x - \mu \int_\Omega |u_n|^{2^*-2} u_n \phi \mathrm{d}x - \int w_n \varphi \mathrm{d}x, \tag{4.11}$$

for any $\varphi \in E$. Since (u_n) is bounded we may assume that $u_n \rightharpoonup u$ weakly in E and $w_n \rightharpoonup w_0$ weakly in $L^{2^\#}(\Omega)$.

By using (f_1) as in the proof of Lemma 4.4, we obtain

$$
\begin{aligned}
I_\mu(u_n) - \frac{1}{2}\langle \xi_n, u_n \rangle &= \frac{\mu}{N} \|u_n\|_{2^*}^{2^*} \mathrm{d}x - \int_\Omega \Big[\frac{1}{2} w_n u_n - F(x, u_n) \Big] \mathrm{d}x \\
&\geq \frac{\mu}{N} \|u_n\|_{2^*}^{2^*} - a_1 |\Omega| - a_2 \int_\Omega |u_n|^\sigma \mathrm{d}x \\
&\geq \frac{\mu}{N} \|u_n\|_{2^*}^{2^*} - a_1 |\Omega| - a_2 |\Omega|^{(2^*-\sigma)/2^*} \left(\int_\Omega |u_n|^{2^*} \mathrm{d}x \right)^{\sigma/2^*}.
\end{aligned}
$$

Passing to the limit, using $c < M$ and Lemma 4.5 again, we get

$$\frac{\mu}{N} \int_{\overline{\Omega}} \mathrm{d}\nu \leq M + c_1 + c_2 \left(\int_{\overline{\Omega}} \mathrm{d}\nu \right)^{\sigma/2^*},$$

for $c_1 := a_1|\Omega|$ and $c_2 := a_2|\Omega|^{(2^*-\sigma)/2^*}$. So,

$$\int_{\overline{\Omega}} \mathrm{d}\nu \le \left(\frac{N(M+c_1+c_2)}{\mu}\right)^{2^*/(2^*-\sigma)} =: \left(\frac{c_3}{\mu}\right)^{2^*/(2^*-\sigma)},$$

if $\int_{\overline{\Omega}} \mathrm{d}\nu \ge 1$. From $\sigma \ge 0$, we conclude that $\alpha := N/2 - (2^*-\sigma)/2^* > 0$. By setting

$$\mu^{**} := \min\left\{S, \left(S^{N/2} c_3^{(2^*-\sigma)/2^*}\right)^{1/\alpha}\right\}$$

a straightforward computation shows that, if $\int_{\overline{\Omega}} \mathrm{d}\nu \ge 1$ and $\mu \in (0, \mu^{**})$, then

$$\int_{\overline{\Omega}} \mathrm{d}\nu \le \left(\frac{S}{\mu}\right)^{N/2}.$$

Since $\mu^{**} \le S$, it is clear that the above inequality also holds if $\int_{\overline{\Omega}} \mathrm{d}\nu \le 1$.

The above considerations prove that $0 < \mu < \mu^{**}$ implies that

$$\int_{\overline{\Omega}} \mathrm{d}\nu \le (S/\mu)^{N/2}.$$

It follows from (4.8) that $J = \varnothing$ and therefore Lemma 4.5 yields

$$\lim_{n\to+\infty} \int_{\Omega} |u_n|^{2^*} \,\mathrm{d}x = \int_{\Omega} |u|^{2^*} \,\mathrm{d}x.$$

By using (4.11) with $\varphi = (u_n - u)$, the above convergence, Lemma 4.6 and the weak convergence of u_n, we get

$$\begin{aligned} o_n(1) = \langle \xi_n, u_n - u\rangle &= \int_{\Omega} \nabla u_n \cdot \nabla(u_n - u)\,\mathrm{d}x - \int_{\Omega} w_n(u_n-u)\,\mathrm{d}x \\ &\quad - \mu \int_{\Omega} |u_n|^{2^*}\,\mathrm{d}x + \mu \int_{\Omega} |u_n|^{2^*-2} u_n u \,\mathrm{d}x \\ &= \|u_n\|^2 - \|u\|^2 + o_n(1), \end{aligned}$$

from which we conclude that $\|u_n\|^2 \to \|u\|^2$. Thus, the weak convergence implies that $u_n \to u$ strongly in E. □

In what follows we prove that the functional I_μ verifies the geometric conditions of our abstract critical point theorem. Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be the eigenfunctions of $(-\Delta, H_0^1(\Omega))$ and set, for each $k \in \mathbb{N}$,

$$X_0 := \{0\}, \quad X_k := \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}.$$

The variational characterization of the eigenvalues provides

$$\lambda_{k+1} := \inf_{u \in X_k^{\perp}\setminus\{0\}} \frac{\|u\|^2}{\|u\|_2^2} \to +\infty, \quad \text{as } k \to +\infty. \tag{4.12}$$

Proposition 4.9. *Suppose that* $f \in \mathcal{F}$ *satisfies* (f_3) *and* (f_4)*. Then there exist* $k_0 \in \mathbb{N}$*,* $\rho,\ \mu^{***} > 0$ *such that, for any* $\mu \in (0, \mu^{***})$*,*

$$I_\mu(u) \geq 0, \quad \forall\, u \in X_{k_0}^\perp \cap \partial B_\rho.$$

Moreover, for any given $m \in \mathbb{N}$*, there exists an* m*-dimensional subspace* $V_0 \subset E$ *and a constant* $M > 0$ *such that* $I_\mu(u) \leq M$*, for any* $u \in V_0$*.*

Proof. Let $\theta \in (2, 2^*)$ be given by (f_3) and $\alpha \in (0, 1)$ such that

$$\theta = 2(1-\alpha) + 2^*\alpha.$$

For any $u \in X_k^\perp$, it follows from the definition of λ_{k+1}, (f_3), Sobolev and Hölder's inequalities that

$$\begin{aligned} I_\mu(u) &\geq \frac{1}{2}\|u\|^2 - \mu c_1 \|u\|^{2^*} - a_3 \|u\|_2^{2(1-\alpha)} \|u\|_{2^*}^{2^*\alpha} - a_4|\Omega| \\ &\geq \frac{1}{2}\|u\|^2 - \mu c_1 \|u\|^{2^*} - c_2 \lambda_{k+1}^{\alpha-1} \|u\|^\theta - a_4|\Omega| \\ &= \frac{1}{2}\|u\|^2 (1 - 2c_2\lambda_{k+1}^{\alpha-1}\|u\|^{\theta-2}) - \mu c_1 \|u\|^{2^*} - a_4|\Omega|, \end{aligned}$$

for some $c_1,\ c_2 > 0$. By picking $\rho = \rho(k) > 0$ such that $2c_2\lambda_{k+1}^{\alpha-1}\rho^{\theta-2} = 1/2$, we obtain

$$I_\mu(u) \geq \frac{1}{4}\rho^2 - \mu c_1 \rho^{2^*} - a_4|\Omega|, \quad \forall\, u \in X_k^\perp \cap \partial B_\rho.$$

Since (4.12) implies that $\rho(k) \to +\infty$, as $k \to +\infty$, we can choose $k_0 > 0$ large in such a way that $\rho^2/4 > a_4|\Omega| + \rho^2/8$. Thus

$$I_\mu(u) \geq \rho^2\Big(\frac{1}{8} - 8\mu c_1 \rho^{2^*-2}\Big), \quad \forall\, u \in X_{k_0}^\perp \cap \partial B_\rho$$

and the first statement of the proposition holds for $\mu^{***} := 2^{-7}c_1\rho^{2-2^*}$.

Let $\Omega_0 \subset \Omega$ be given by (f_4) and consider m disjoint open balls contained in Ω_0. We pick m regular functions with support contained in each of these balls and denote by V_0 the subspace spanned by such functions. It is clear that, for some $c_1 > 0$, there holds

$$c_1\|u\|^2 \leq \|u\|_2^2, \quad \forall\, u \in V_0.$$

From $(\mathcal{F}_4)$ and (f_0), we obtain $c_2 > 0$ such that $F(x, s) \geq s^2/(2c_1) - c_2$, for any $x \in \Omega_0$, $s \in \mathbb{R}$. Hence, we obtain

$$I_\mu(u) \leq \frac{1}{2}\|u\|^2 - \int_\Omega F(x, u)\, \mathrm{d}x \leq c_1|\Omega|, \quad \forall\, u \in V_0.$$

This finishes the proof. □

We are ready to present the proof of the last result.

Proof of Theorem 1.2. If $k_0 \in \mathbb{N}$ and $\mu^{***} > 0$ are given by Proposition 4.9, it follows that the even functional I_μ verifies $I_\mu(0) = 0$ and the condition (I_1) of Theorem 1.1 with the decomposition $E = V_{k_0} \oplus V_{k_0}^{\perp}$. The condition (I_2) also follows from the application of Proposition 4.9 with $m := k + k_0$. If we set

$$\mu_k := \min\{\mu^*, \mu^{**}, \mu^{***}\},$$

the nonsomooth $(\mathrm{PS})_c$ condition holds for any $c \in [0, M)$, where M is given by Proposition 4.9. For any $\mu \in (0, \mu_k)$, Theorem 1.1 provides $(k + k_0) - k_0 = k$ pairs of critical points for I_μ which are strong solutions by Lemma 4.3. □

5 Some examples

In this final section we present two applications of Theorem 1.2. For the first one, we pick $0 < \varepsilon < (N+2)/(N-2)$, define

$$g_1(s) := s^{2^*-1-\varepsilon}, \quad g_2(s) := s^{2^*-1-(\log s)^{-1/2}},$$

for $s \geq 1$, and consider an increasing sequence $(t_n) \subset (0, +\infty)$ such that $\log t_1 > \varepsilon^{-1/2}$ and $t_n \to +\infty$. For each $n \in \mathbb{N}$, we pick $\delta_n > 0$ such that

$$2\delta_n[(t_n + \delta_n)^{2^*-1} - (t_n + \delta_n)^{2^*-1-\varepsilon}] \leq \frac{1}{n^2} \tag{5.1}$$

and the intervals $I_n := (t_n - \delta_n, t_n + \delta_n)$ are disjoints. We define

$$g(s) := \begin{cases} 0, & \text{if } 0 \leq s < t_1, \\ g_1(s), & \text{if } s \notin \bigcup_{n=1}^{+\infty} I_n, \\ g_2(s), & \text{if } s \in \bigcup_{n=1}^{+\infty} I_n, \end{cases}$$

and $g(s) := -g(-s)$, if $s < 0$. The function g is even and has infinitely many points of discontinuity. Moreover, if $t_n^* \in (t_n, t_{n+1})$ and $2 < p < 2^*$, we have that

$$\frac{g(t_n^*)}{(t_n^*)^{p-1}} = (t_n^*)^{2^*-p-(\log t_n^*)^{-1/2}} \to +\infty, \quad \text{as } n \to +\infty,$$

and therefore g is not dominated by a subcritical power. For this example, the following holds:

Theorem 5.1. *Let $a > 0$ and $\mu, \lambda > 0$. Then, for any given $k \in \mathbb{N}$, there exists $\mu_k > 0$ such that, for any $\mu \in (0, \mu_k)$, the problem*

$$-\Delta u = \lambda H(|u| - a)g(u) + \mu |u|^{2^*-2}u \text{ in } \Omega, \quad u \in H_0^1(\Omega), \tag{5.2}$$

has at least k pairs of nontrivial solutions in $W^{2,2N/(N+2)}(\Omega) \cap H_0^1(\Omega)$. *Moreover, for any* $M > 0$ *given, there exists* $\overline{\mu} = \overline{\mu}(M) > 0$ *such that, if* $\max\{\lambda, \mu\} < \overline{\mu}$ *and* $u_{\lambda,\mu} \in H_0^1(\Omega)$ *is a weak solution of* (5.2)*, then the set* $\{x \in \Omega : |u_{\lambda,\mu}(x)| \geq M\}$ *has positive measure.*

Proof. We are going to show that the function $f(s) := \lambda H(|s| - a)g(s)$, $s \in \mathbb{R}$, satisfies all the hypotheses of Theorem 1.2. It is clear that $f \in \mathcal{F}$ verifies (f_4). In order to verify (f_1), we fix $s > a$ and call $n_0 \in \mathbb{N}$ the biggest natural number such that $s \in I_{n_0}$. Since

$$t^{2^*-1-\varepsilon} = g_1(t) \leq f(t) \leq g_2(t) = t^{2^*-1-(\log t)^{-1/2}}, \quad \forall t > a, \tag{5.3}$$

we easily get

$$\lim_{s\to+\infty} \frac{\overline{f}(s)}{s^{2^*-1}} \leq \lim_{s\to+\infty} \frac{g_2(s)}{s^{2^*-1}} = \lim_{s\to+\infty} s^{-(\log s)^{-1/2}} = 0,$$

which proves that f satisfies (f_2). It follows from (5.1) and (5.3) that, for some $c_1, c_2 > 0$,

$$\begin{aligned} F(s) &\leq c_1 + \int_0^s t^{2^*-1-\varepsilon}\,\mathrm{d}t + \sum_{n=1}^{n_0} 2\delta_n[g_1(t_n+\delta_n) - g_2(t_n+\delta_n)] \\ &\leq c_1 + \frac{s^{2^*-\varepsilon}}{2^*-\varepsilon} + \sum_{n=1}^{\infty} 2\delta_n[(t_n+\delta_n)^{2^*-1} - (t_n+\delta_n)^{2^*-1-\varepsilon}] \\ &\leq c_2 + \frac{s^{2^*-\varepsilon}}{2^*-\varepsilon}, \end{aligned}$$

which proves (f_3). The above expression, the definition of $\underline{f}$ and (5.3) also imply that

$$\underline{f}(s)s - 2F(s) \geq s^{2^*-\varepsilon} - 2c_2 - \frac{2}{2^*-\varepsilon}s^{2^*-\varepsilon} \geq -2c_2,$$

and therefore f satisfies (f_1). By applying Theorem 1.2, we obtain multiple solutions for problem (5.2).

We prove now the second part of the corollary. Let $u = u_{\lambda,\mu}$ be a nonzero solution of (5.2) and $M > 0$. Of course we may assume that $u \in L^\infty(\Omega)$, since otherwise the result is clearly true. By using that $f(s)s$ is even, we obtain $f(s)s \leq c_1|s|^{2^*}$, for any $s \in \mathbb{R}$ and some $c_1 > 0$. Hence, it follows that

$$\begin{aligned} \lambda_1(\Omega)\|u\|^2_{L^2(\Omega)} &\leq \|u\|^2 = \lambda\int_\Omega f(u)u\,\mathrm{d}x + \mu\int_\Omega |u|^{2^*}\,\mathrm{d}x \\ &\leq \lambda c_1\|u\|^{2^*-2}_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)} + \mu\|u\|^{2^*-2}_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)}, \end{aligned} \tag{5.4}$$

from which we conclude that

$$\|u\|_{L^\infty(\Omega)} \geq \left(\frac{\lambda_1(\Omega)}{\lambda c_1 + \mu}\right)^{(N-2)/4} > M,$$

whenever λ and μ are sufficciently close to zero. □

It is worth noticing that, according to the last part of the above result, the solutions we found are such that the range of $f(\cdot, u)$ crosses any prescribed number of discontinuity of the function f.

In our second example, we consider sequences (a_j), (b_j), $(c_j) \subset L^\infty(\Omega)$ such that

(1) $a_j(x) \geq \Lambda_1 > 0$ for a.e. $x \in \Omega \setminus \Omega_0$;
(2) $c_j(x) \geq \Lambda_1 > 0$ for a.e. $x \in \Omega_0$;
(3) $\|a_j\|_{L^\infty(\Omega)}$, $\|b_j\|_{L^\infty(\Omega)}$, $\|c_j\|_{L^\infty(\Omega)} \leq \Lambda_2$,

for any $j \geq j_0$, some $\Lambda_1, \Lambda_2 > 0$ and $\Omega_0 \subset \Omega$ a proper subset with positive measure. After that, we pick (s_j), (q_j), $(p_j) \subset (0, +\infty)$ and $(\sigma_j) \subset \{0, 1\}$ verifying

(4) $s_j \to +\infty$, $1 < q_j < 2 < p_j < 2^*$, $\sigma_j = 0$, for any $j \geq j_0$,

and define, for each $j \in \mathbb{N}$, the function $f_j : \Omega \times [0, +\infty) \to \mathbb{R}$ as

$$f_j(x, s) := \chi_{\Omega\setminus\Omega_0}(x)[a_j(x)s^{q_j-1} + b_j(x)s] + \chi_{\Omega_0}(x)c_j(x)s^{p_j-1},$$

where χ_A stands for the characteristic function of the set A. For simplicity of notation, we also set $f_0 := 0$.

Arguing as in Theorem 5.1, we can prove the following:

Theorem 5.2. *Consider the problem*

$$-\Delta u = \lambda f(x, u) + \mu|u|^{2^*-2}u \text{ in } \Omega, \quad u \in H_0^1(\Omega), \tag{5.5}$$

where μ, $\lambda > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$f(x, s) := \sum_{j=0}^{\infty}(-1)^{\sigma_j} f_j(x, s)\chi_{[s_j, s_{j+1}]}(s), \quad s \geq 0,$$

and $f(x, s) := -f(x, -s)$, if $s < 0$. Then, for any given $k \in \mathbb{N}$, there exists $\mu_k > 0$ such that, for any $\mu \in (0, \mu_k)$, the same conclusions of Corollary 5.1 hold for the problem (5.5).

We notice that this last nonlinearity is indefinite in sign. Morover, it is not superlinear in all the domain Ω since it is clear from the definition that

$$\limsup_{s\to+\infty} \frac{2F(x, s)}{s^2} \leq \Lambda_1, \quad \text{for a.e. } x \in \Omega \setminus \Omega_0,$$

and therefore the well known Ambrosetti–Rabinowitz superlinear condition (see [3]) is not satisfied.

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