LOCAL SIGN CHANGES OF POLYNOMIALS

By

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Abstract. The trigonometric monomial $\cos(\langle k, x\rangle)$ on $\mathbb{T}^d$, a harmonic polynomial $p : \mathbb{S}^{d-1} \to \mathbb{R}$ of degree k and a Laplacian eigenfunction $-\Delta f = k^2 f$ have a root in each ball of radius $\sim \|k\|^{-1}$ or $\sim k^{-1}$, respectively. We extend this to linear combinations and show that for any trigonometric polynomials on $\mathbb{T}^d$, any polynomial $p \in \mathbb{R}[x_1, \ldots, x_d]$ restricted to $\mathbb{S}^{d-1}$ and any linear combination of global Laplacian eigenfunctions on $\mathbb{R}^d$ with $d \in \{2, 3\}$ the same property holds for any ball whose radius is given by the sum of the inverse constituent frequencies. We also refine the fact that an eigenfunction $-\Delta\phi = \lambda\phi$ in $\Omega \subset \mathbb{R}^n$ has a root in each $B(x, \alpha_n \lambda^{-1/2})$ ball: the positive and negative mass in each $B(x, \beta_n \lambda^{-1/2})$ ball cancel when integrated against $\|x - y\|^{2-n}$.

1 Introduction

The purpose of this paper is to prove same type of result for

(1) trigonometric polynomials on the torus $\mathbb{T}^d$,

(2) the restriction of polynomials $p \in \mathbb{R}[x_1, \ldots, x_d]$ to the unit sphere $\mathbb{S}^{d-1}$,

(3) and global solutions of $(-\Delta + \lambda)\phi = 0$ on $\mathbb{R}^d$ where $d = 2, 3$.

In each of these settings a single basis object (a trigonometric monomial, a harmonic polynomial, a Laplacian eigenfunction) has many roots: each ball with radius inversely proportional to degree/frequency is guaranteed to contain a root. We will extend this to linear combinations and show that they still have many roots on a suitable scale. A result in this style was first proven by Kozma–Oravecz [17].

Theorem (Kozma–Oravecz [17]). *Let $f : \mathbb{T}^d \to \mathbb{R}$ be a real-valued trigonometric polynomial with mean value 0 of the form*

$$f(x) = \sum_{k \in S} a_k \exp(2\pi i \langle x, k\rangle),$$

where $S \subset \mathbb{Z}^d$. Then f has a zero in each ball of radius

$$r(f) = \frac{1}{4} \sum_{k \in S} \frac{1}{\|k\|}.$$

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Note that f being real-valued necessarily entails that $-S = S$ and that $a_{-k} = a_k$. The function having mean value 0 implies $0 \notin S$. In one dimension, $d = 1$, the result is sharp up to constants: [17, Theorem 2] shows that if f has frequencies supported in $[-A - B, -A] \cup [A, A + B]$, then the maximum length of an interval without sign change is $(B + 1)/(2A + B)$. The question dates back at least to a 1965 paper of Taikov [25] with an extremal trigonometric polynomial given by Babenko [1]. The same extremal polynomial also appears in [15, 24].

2 Results

2.1 Trigonometric Polynomials. We start by proving a result in the style of Kozma–Oravecz. We show that instead of counting the number of summands, it suffices to look at the number of contributing frequencies.

Theorem 1. *If $f : \mathbb{T}^d \to \mathbb{R}$ is a real-valued trigonometric polynomial with mean value 0 of the form*

$$f(x) = \sum_{k \in S} a_k \exp(2\pi i \langle x, k \rangle),$$

then, introducing $\Lambda = \{ \|k\| : k \in S\}$, f has a zero in each ball of radius

$$r(f) = d^{3/2} \sum_{\lambda \in \Lambda} \frac{1}{\lambda}.$$

The result is identical (up to the value of the constant) to the result of Kozma–Oravecz in dimension $d = 1$. The improvement is more pronounced in higher dimensions where many different trigonometric polynomials may correspond to the same frequency (in higher dimensions, a sphere can contain many lattice points). The proof indicates that the optimal constant may perhaps be expected to grow linearly (or slower) in the dimension; we comment on this after the proof.

2.2 Spherical Harmonics. There is an analogous result for the restriction of arbitrary polynomials on the unit sphere. If $p_n \in \mathbb{R}[x_1, \ldots, x_d]$ is a polynomial of degree n in $\mathbb{R}^d$, then its restriction onto the unit sphere $\mathbb{S}^{d-1}$ can be expressed as a linear combination of harmonic polynomials of degree at most n

$$p_n(x)\big|_{\mathbb{S}^{d-1}} = \sum_{k=0}^{n} a_k f_k(x) \quad \text{where } f_k \in \mathcal{H}_k^d.$$

We recall that the space of harmonic polynomials of degree k is

$$\mathcal{H}_k^d = \{f \in \mathbb{R}[x_1, \ldots, x_d] : f \text{ homogenenous of degree } k \text{ and } \Delta f = 0\}.$$

There exists an elementary argument that if $f \in \mathcal{H}_k^d$, then f has zero on each ball of radius $c_d k^{-1}$ (see below). This can be extended to sums of harmonic polynomials.

Theorem 2. *If $p \in \mathbb{R}[x_1, \dots, x_d]$ has the restriction*

$$p(x)\big|_{\mathbb{S}^{d-1}} = \sum_{k \in S} a_k f_k(x) \quad \text{where } f_k \in \mathcal{H}_k^d$$

and mean value 0 on $\mathbb{S}^{d-1}$, then $p\big|_{\mathbb{S}^{d-1}}$ has a zero on each (geodesic) ball of radius

$$r = \pi^2 d \sum_{k \in S} \frac{1}{k}.$$

The ball $B(x, r)$ here refers to the set of all points on $\mathbb{S}^{d-1}$ whose (geodesic) distance from $x \in \mathbb{S}^{d-1}$ is at most r. We did not optimize the constant $\pi^2 d$. Our approach will necessarily lead to a linear growth of the constant in the dimension and this dependence could conceivably be optimal.

2.3 Laplacian eigenfunctions. On a compact, smooth manifold (M, g) a Laplacian eigenfunction is a solution of $-\Delta f = \lambda f$. A basic property of such a function is that f changes sign on each ball of radius $c_M \cdot \lambda^{-1/2}$. A natural problem is whether this can be extended to linear combinations of eigenfunctions [6, 7, 11, 13, 16, 19, 20, 21, 22]. The problem is well-understood in the one-dimensional setting where the answer follows from Sturm–Liouville theory; we refer to recent papers of Bérard–Helffer [3, 4]. The Laplacian eigenfunctions on $\mathbb{T}^d$ are given by the trigonometric polynomials. The eigenfunctions on $\mathbb{S}^{d-1}$ are the harmonic polynomials and

$$\forall f \in \mathcal{H}_k^d \qquad -\Delta_{\mathbb{S}^{d-1}} f = k(k+d-2)f.$$

Theorem 1 and Theorem 2 follow the same basic blueprint.

Question. Let (M, g) be a compact, smooth manifold and let $-\Delta \phi_k = \lambda_k \phi_k$ be the sequence of Laplacian eigenfunctions. Is it true, that for some $0 < c_M < \infty$ depending only on the manifold, that any finite linear combination

$$f(x) = \sum_{k \in S} a_k \phi_k(x) \quad \text{has a root in each ball of radius} \quad r = c_M \sum_{k \in S} \frac{1}{\sqrt{\lambda_k}}?$$

We learned this question from Stefano Decio (see also [10]). Theorem 3 proves it for global eigenfunctions on $\mathbb{R}^2$ and $\mathbb{R}^3$. This result can be seen as being similar in spirit to Theorem 1 for $d = 2, 3$ while allowing for a much larger class of functions.

Theorem 3. *Let $d \in \{2,3\}$ and $n \in \mathbb{N}$. Suppose, for each $1 \leq k \leq n$, the smooth function $\phi_k : \mathbb{R}^d \rightarrow \mathbb{R}$ is a global solution of $-\Delta\phi_k = \lambda_k\phi_k$. Then*

$$f(x) = \sum_{k=1}^{n} a_k\phi_k(x) \quad \textit{has a zero in every ball with radius} \quad r = 2\pi \sum_{k=1}^{n} \frac{1}{\sqrt{\lambda_k}}.$$

We give a proof using the closed-form solution of a linear, non-homogeneous wave equation in Euclidean space. Because of finite speed of propagation, there is some hope of a variant of it also working on a bounded domain $\Omega \subset \mathbb{R}^d$.

2.4 An identity for eigenfunctions. The proof of Theorem 3 suggests an interesting identity for Laplacian eigenfunctions.

Theorem 4. *Suppose $-\Delta\phi = \lambda\phi$ in a neighborhood of $B(x,r) \subset \mathbb{R}^n$ and $n \geq 3$. Then, for an explicit universal function function $Q_n : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ we have*

$$\int_{\|x-y\| \leq r} \frac{\phi(y)}{\|y-x\|^{n-2}} dy = \frac{1}{\lambda} Q_n(\sqrt{\lambda} \cdot r) \cdot \phi(x).$$

In particular, in three dimensions, $n = 3$,

$$\int_{\|x-y\| \leq r} \frac{\phi(y)}{\|y-x\|} dy = 4\pi \frac{1-\cos(\sqrt{\lambda} \cdot r)}{\lambda} \cdot \phi(x).$$

The statement is purely local and does not depend on any boundary conditions which might make it useful in the study of the behavior of eigenfunctions. Moreover, the function Q_n is completely explicit and can be written as

$$Q_n(x) = 2^{\frac{n-2}{2}} \Gamma(n/2) n\omega_n \int_0^x s^{\frac{4-n}{2}} J_{\frac{n-2}{2}}(s) ds.$$

When $n = 3$, we get $J_{1/2}(x) = \sqrt{2/\pi} x^{-1/2} \sin(x)$ and the expression simplifies. An interesting consequence, valid in all dimensions as long as $B(x,r) \subset \Omega$, is

$$\text{if } \phi(x) = 0, \text{ then } \quad \int_{\|x-y\| \leq r} \frac{\phi(y)}{\|y-x\|^{n-2}} dy = 0$$

which says that mass around a root is perfectly balanced with respect to $\|x-y\|^{2-n}$.

Another interesting consequence is with respect to the distribution of roots: for example, any eigenfunction on $\Omega \subset \mathbb{R}^3$ with Dirichlet boundary conditions has a root in each $\pi\lambda^{-1/2}$ ball intersecting the domain and this is the sharp constant. As a consequence of Theorem 4, we see that on a ball twice that size

$$\int_{\|x-y\| \leq 2\pi\lambda^{-1/2}} \frac{\phi(y)}{\|y-x\|} dy = 0$$

which is a way of saying that there is a precise balance between positive and negative mass on each ball of radius $2\pi\lambda^{-1/2}$ with respect to the Coulomb kernel. This is also true (with the smallest positive root of Q_n as constant) in higher dimensions.

3 Proof of Theorem 1

A key ingredient is an asymptotic result for the smallest positive root of the Bessel function $J_{d/2-1}$, sometimes denoted as $j_{\frac{d}{2}-1,1}$. Results on these roots are classical and we only need a relatively simple bound.

Lemma 1. *The smallest positive root of the Bessel function $J_{d/2-1}$ satisfies*

$$\forall\, d \geq 2 \quad j_{\frac{d}{2}-1,1} \leq \frac{j_{0,1}}{2} d.$$

Sketch. Asymptotics of roots of the Bessel function are a classical subject. In our setting, an old 1949 result of Tricomi [26] implies that, for some $\alpha \in \mathbb{R}$ as $d \to \infty$,

$$j_{\frac{d}{2}-1,1} = \frac{d}{2} + \alpha d^{1/3} + \mathcal{O}(d^{-1/3}).$$

Checking the first few values of d, we see that $j_{d/2-1,1}/d$ is maximal when $d = 1$ (and then steadily decaying towards its limit $1/2$). In the case of $d = 1$, there is an explicit closed form expression (being $\pi/2$). Since our result is implied by the result of Kozma–Oravecz when $d = 1$, we are only interested in $d \geq 2$. The largest value is assumed when $d = 2$ corresponding to $j_{0,1} \sim 2.404\ldots$. □

Proof of Theorem 1. We assume $d \geq 2$. The proof is by induction on $\#\Lambda$.

The case $\#\Lambda = 1$. When $\#\Lambda = 1$, then $\Lambda = \{\lambda\}$ and f is a Laplacian eigenfunction $-\Delta f = 4\pi^2\lambda^2$ and we deduce the existence of a sign change in every ball of radius $d^{3/2} \cdot \lambda^{-1}$ as follows. Suppose, without loss of generality, that $f > 0$ on the ball $B(x_0, r)$ where $x_0 = (1/2, 1/2, \ldots, 1/2)$. We consider the largest connected domain $B(x_0, r) \subset \Omega \subset \mathbb{T}^d$ containing x_0 on which f is positive. It is a classical fact that an eigenfunction restricted to a nodal domain Ω is a multiple of the first nontrivial eigenfunction with Dirichlet boundary conditions on that domain (see [2, 8]). This means that, restricting the function f to its nodal domain Ω, we arrive at

$$4\pi^2\lambda^2 = \frac{\int_\Omega |\nabla f|^2 dx}{\int_\Omega f^2\, dx} = \lambda_1(\Omega) = \inf_{\substack{g:\Omega\to\mathbb{R} \\ g|_{\partial\Omega}=0}} \frac{\int_\Omega |\nabla g|^2 dx}{\int_\Omega g^2\, dx}.$$

Domain monotonicity implies that the Laplacian eigenvalue increases when we restrict to a smaller sub-domain. This could also be seen from the variational characterization since the space of functions vanishing at the boundary becomes strictly smaller when restricting to a subset. Since $B(x_0, r) \subset \Omega$, we have

$$4\pi^2\lambda^2 = \lambda_1(\Omega) \le \lambda_1(B(x_0, r)).$$

We now distinguish two cases: if $r > 1/2$, then trivially $B(x_0, 1/2) \subset B(x_0, r)$. In that case we can simply treat $B(x_0, 1/2) \subset [0, 1]^d$ as a subset of Euclidean space. Finding a function with a small Rayleigh–Ritz quotient on $B(x_0, 1/2)$ (vanishing at the boundary) is strictly harder than finding such a function on Ω (because each of the former is also an example for the latter). The first problem, however, can be solved in closed form. In Euclidean space $\mathbb{R}^d$ we have

$$\lambda_1(B(x_0, r)) = r^{-2} j^2_{\frac{d}{2}-1,1},$$

where $j_{d/2-1,1} > 0$ is the smallest positive zero of the Bessel function of index $d/2 - 1$. If $r > 1/2$, then, using the Lemma, we deduce

$$4\pi^2\lambda^2 = \lambda_1(\Omega) \le 4 j^2_{\frac{d}{2}-1,1} \le j^2_{0,1} d^2$$

and thus

$$1 \le \frac{j_{0,1} d}{2\pi\lambda}.$$

In that case, we also conclude that, since f has mean value 0 and vanishes somewhere, $r \le \sqrt{d}/4 = \mathrm{diam}(\mathbb{T}^d)/2$ is certainly an admissible (albeit trivial) inequality. We deduce that

$$r \le \frac{\sqrt{d}}{4} \le \frac{\sqrt{d}}{4}\frac{j_{0,1}}{2\pi}\frac{d}{\lambda} \le \frac{d^{\frac{3}{2}}}{\lambda}.$$

If $r < 1/2$, then, from a direct comparison with the Euclidean setting,

$$4\pi^2\lambda^2 = \lambda_1(\Omega) \le r^{-2}\lambda_1(B) = r^{-2} j^2_{\frac{d}{2}-1,1}.$$

Appealing to the Lemma,

$$r \le \frac{j_{d/2-1,1}}{2\pi}\frac{1}{\lambda} \le \frac{j_{0,1}}{4\pi}\frac{d}{\lambda} \le \frac{1}{2}\frac{d}{\lambda} \le \frac{d^{\frac{3}{2}}}{\lambda}.$$

The case $\#\Lambda \ge 2$. Let us now suppose $\#\Lambda \ge 2$ and that the set Λ is given by $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Suppose now that there exists a function $f : \mathbb{T}^d \to \mathbb{R}$ supported on these frequencies such that, for some ball B of radius $r(f)$, we have, without loss of generality, that $f > 0$. Our goal will be to transform f into a function supported on the frequencies $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ which is positive on a ball of not much smaller radius which then implies the result via induction (note that this overall structure is the same as in [17]). It remains to explain the construction. We consider a new function $g_\delta : \mathbb{R}^d \to \mathbb{R}$

$$g_\delta(x) = \chi_{\|x\| \leq \delta}$$

which we can identify with the periodic function $h_\delta : \mathbb{T}^d \to \mathbb{R}$ via

$$h_\delta(x) = \sum_{k \in \mathbb{Z}^d} g_\delta(x+k).$$

The function g is the characteristic function of a ball of radius δ centered at the origin. There is an explicit formula for the Fourier coefficients of g and

$$\widehat{g_\delta}(\xi) = \alpha_d \frac{J_{d/2}(2\pi\|\xi\|\delta)}{\|2\pi\xi\delta\|^{d/2}},$$

where α_d is some constant depending only on d and $J_{d/2}$ is the Bessel function of order $d/2$. The same formula holds for the Fourier coefficient of h and

$$\forall k \in \mathbb{Z}^d \quad \widehat{h_\delta}(k) = \alpha_d \frac{J_{d/2}(2\pi\|k\|\delta)}{\|2\pi k\delta\|^{d/2}}.$$

Let now $j_{d/2,1} > 0$ be the smallest positive root of $J_{d/2}$, i.e. $J_{d/2}(j_{d/2,1}) = 0$. Then, choosing

$$\delta^* = \frac{j_{d/2,1}}{2\pi\lambda_n}$$

implies that the Fourier transform $\widehat{g_{\delta^*}}$ vanishes on all lattice points of size $\|k\| = \lambda_n$. We now consider the convolution

$$(f * h_{\delta^*})(x) = \int_{\mathbb{T}^d} f(x-y) h_{\delta^*}(y)\, dy.$$

Convolution becomes multiplication on the Fourier side and thus if

$$f(x) = \sum_{k \in S} a_k \exp(2\pi i \langle x, k\rangle),$$

then

$$(f * h_{\delta^*})(x) = \sum_{k \in S} a_k \cdot \widehat{h_{\delta^*}}(k) \cdot \exp(2\pi i \langle x, k\rangle).$$

Now $f * h_{\delta^*}$ is a trigonometric polynomial with frequencies in $\lambda_1 < \cdots < \lambda_{n-1}$. Simultaneously, if for some ball $B(x_0, r(f))$ of radius $r(f)$, we have that $f > 0$, then surely $f * h_{\delta^*}$ has the same property on the ball $B(x_0, r(f) - \delta^*)$. We deduce

$$r(f) \leq r(f * h_{\delta^*}) + \delta^*.$$

Using the Lemma once more, we arrive at

$$\delta^* = \frac{j_{0,1}(d+2)}{4\pi\lambda_n} = \frac{j_{0,1}}{4\pi\lambda_n}\frac{d+2}{d} d \leq \frac{j_{0,1}}{2\pi}\frac{d}{\lambda_n} \leq \frac{1}{2}\frac{d}{\lambda_n} \leq \frac{d^{3/2}}{\lambda_n}. \qquad \square$$

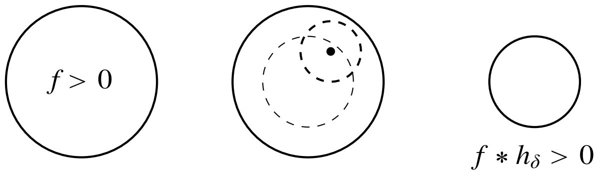

Figure 1. Induction step: if $f > 0$ on a ball of radius r and we convolve f with a positive function supported on a ball of radius δ^*, then the convolution is positive on a ball of radius $r - \delta^*$.

Question. The only time the scaling $d^{3/2}$ appears is when setting up the induction case. This leads to a natural question: if $-\Delta f = \lambda f$ is an eigenfunction (a sum of trigonometric terms corresponding to the same frequency) on $\mathbb{T}^d \cong [0,1]^d$, is there a root in each ball of radius $r = 100d \cdot \lambda^{-1/2}$?

4 Proof of Theorem 2

We start by noting that it suffices to prove the result for $d \geq 3$. The case $d = 2$ follows from Theorem 1 since $\mathbb{S}^{d-1} = \mathbb{S}^1 \equiv \mathbb{T}$ and everything reduces to cosines. An important new ingredient is the Funk–Hecke formula: it describes the effect of convolution on the sphere in terms of having a multiplicative effect on spherical harmonics. We refer to the exposition in Dai–Xu [9] for additional details.

Lemma 2 (Funk–Hecke Formula)**.** *If $g : [-1,1] \to \mathbb{R}$ is integrable and*

$$\int_{-1}^{1} |g(t)|(1-t^2)^{\frac{d-3}{2}} dt < \infty,$$

then for every $q \in \mathcal{H}_k^d$ we have

$$\int_{\mathbb{S}^{d-1}} g(\langle x, y \rangle) q(y) d\sigma(y) = \lambda_k(g) \cdot q(x),$$

where, C_n^λ denoting the Gegenbauer polynomials,

$$\lambda_k(g) = \frac{\omega_{d-1}}{C_k^{\frac{d-2}{2}}(1)} \int_{-1}^{1} g(t) \cdot C_k^{\frac{d-2}{2}}(t) \cdot (1-t^2)^{\frac{d-3}{2}} dt.$$

Proof of Theorem 2. We prove the result with induction on $\#S$.
The case $\#S = 1$. We start with the case where $\#S = 1$ which corresponds to it being a single harmonic polynomial $f \in \mathcal{H}_k^d$. We will show that in that case there is a root in each ball of radius

$$r \leq 2\pi \frac{d}{k}.$$

Let us assume $f \in \mathcal{H}_k^d$ and let us assume it is positive on the (geodesic) ball $B(x_0, r) \subset \mathbb{S}^{d-1}$ and then consider the associated nodal set $B(x_0, r) \subset \Omega$. The same argument as in the proof of Theorem 1 implies

$$k^2 \leq k(k+d-2) = \lambda_1(\Omega) \leq \lambda_1(B(x,r)).$$

Now $B(x,r)$ is a $(d-1)$-dimensional manifold with boundary, a spherical cap, and we are interested in the ground state of the Laplace-Beltrami operator on such a spherical cap. This problem has been considered by Borisov-Freitas [5] who prove

$$\lambda_1(B(x,r)) \leq \begin{cases} \frac{j_{0,1}^2}{r^2} + \frac{1}{3} & \text{on } \mathbb{S}^2 \\ \frac{\pi^2}{r^2} + 1 & \text{on } \mathbb{S}^3 \\ \frac{j_{(d-2)/2,1}^2}{r^2} - \frac{(d-1)^2}{4} + \frac{(d-1)(d-3)}{4}\left[\frac{1}{s(r)^2} - \frac{1}{r^2}\right] & \text{on } \mathbb{S}^d,\ d \geq 4, \end{cases}$$

where $s(r) = \sin r$. Since $0 \leq r \leq \pi$, we can bound the first two terms from above by $2\pi^2/r^2$. This means that in dimension $d \in \{2,3\}$, we have

$$1 \leq k^2 \leq k(k+d-2) = \lambda_1(\Omega) \leq \lambda_1(B(x,r)) \leq \frac{2\pi^2}{r^2}$$

and thus

$$r \leq \frac{2\pi}{k} \leq 2\pi \frac{d}{k}.$$

It remains to deal with the case $d \geq 4$. A little bit of computation shows that either

$$-\frac{(d-1)^2}{4} + \frac{(d-1)(d-3)}{4}\left[\frac{1}{s(r)^2} - \frac{1}{r^2}\right] \leq 0 \quad \text{or} \quad r \geq 2.$$

Using again domain monotonicity and the fact that these spherical caps get bigger as r increases, we conclude that the eigenvalue has to be monotonically decreasing in r and we can thus improve the third upper bound, for $d \geq 4$, to

$$\lambda_1(B(x,r)) \leq \max\left\{\frac{j_{(d-2)/2,1}^2}{r^2}, \frac{j_{(d-2)/2,1}^2}{4}\right\}.$$

Using Lemma 1, this can be further simplified to

$$\lambda_1(B(x,r)) \leq \frac{j_{0,1}^2 \cdot d^2}{4} \max\left\{\frac{1}{r^2}, \frac{1}{4}\right\} \leq \frac{3d^2}{2} \max\left\{\frac{1}{r^2}, \frac{1}{4}\right\}.$$

Thus, combining the previous argument, we arrive at

$$k^2 \leq \lambda_1(B(x,r)) \leq \frac{3d^2}{2} \max\left\{\frac{1}{r^2}, \frac{1}{4}\right\}.$$

We distinguish two cases: if $r \geq 2$, then

$$k^2 \leq \frac{3d^2}{8} \quad \text{then} \quad \frac{d}{k} \geq \sqrt{\frac{3}{8}} \geq \frac{3}{5}$$

and then

$$r \leq \pi \leq 2\pi\frac{3}{5} \leq 2\pi\frac{d}{k}.$$

If $r \leq 2$, then we deduce

$$r \leq \sqrt{\frac{3}{2}}\frac{d}{k} \leq 2\pi\frac{d}{k}$$

which establishes the desired result.

The case $\#S \geq 2$**.** Let us now assume that

$$f(x) = \sum_{k \in S} a_k f_k(x), \qquad \text{where}\, f_k \in \mathcal{H}_k^d,$$

is given and that $\#S \geq 2$ with $\max S = m$. We consider, for a suitable function $g : [-1,1] \to \mathbb{R}$ that remains to be constructed, the new function

$$f^*(x) = \int_{\mathbb{S}^{d-1}} g(\langle x, y\rangle) f(y) d\sigma(y).$$

The Funk–Hecke formula shows that

$$\begin{aligned} f^*(x) &= \int_{\mathbb{S}^{d-1}} g(\langle x,y\rangle) \sum_{k\in S} a_k f_k(y) d\sigma(y) \\ &= \sum_{k\in S} a_k \int_{\mathbb{S}^{d-1}} g(\langle x,y\rangle) f_k(y) d\sigma(y) = \sum_{k \in S} a_k \lambda_k(g) f_k(x). \end{aligned}$$

Motivated by the proof of Theorem 1, it makes sense to design g in such a way that its support is as close as possible to 1 while simultaneously satisfying $\lambda_m(g) = 0$. Recalling that, for some constant $\alpha_{d,m} \in \mathbb{R}$

$$\lambda_m(g) = \alpha_{d,m} \int_{-1}^{1} g(t) \cdot C_m^{\frac{d-2}{2}}(t) \cdot (1-t^2)^{\frac{d-3}{2}} dt,$$

there is a particularly canonical choice: if we define g to be a bump function suitably localized around the largest root of the Gegenbauer polynomial, this is guaranteed to lead to a function that is compactly supported with support close to 1 and $\lambda_m(g) = 0$. A result of Driver–Jordaan [12] (see also Nikolov [18]) shows that the largest root of $C_m^{\lambda}(x)$ satisfies

$$x_1 > 1 - \frac{(\lambda+3)^2}{m^2}.$$

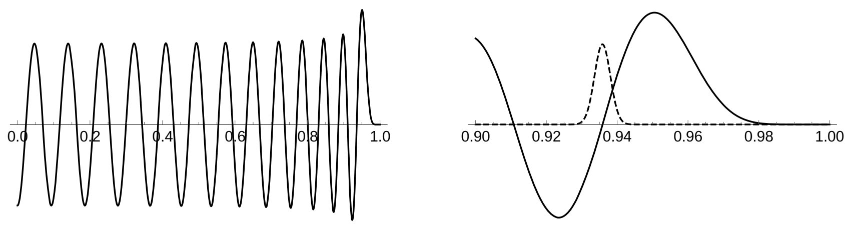

Figure 2. Left: the function $C_{50}^{(20)}(x)(1-x^2)^{10}$ on $[0, 1]$ where $(1-x^2)^{10}$ is multiplied to emphasize the overall sign structure (note that $C_{50}^{(20)}(1) \neq 0$). Right: the same function shown close to 1 with a possible choice for g hinted (dashed).

The bounds in [12, 18] are slightly stronger than that (at the level of constants), we have chosen a slightly algebraically easier form for simplicity of exposition.

The roots of the Gegenbauer polynomials are simple which means that $C_m^{(\lambda)}$ changes sign in x_1. At this point, we define the function $g : [-1, 1] \to \mathbb{R}$ to be a positive bump function compactly supported in a sufficiently small interval J around x_1, where J is chosen such that

$$1 - \frac{(\lambda+3)^2}{m^2} = \inf J < x_1 < \sup J \leq 1$$

and g is chosen in such a way that $g \geq 0$ and

$$\int_J g(t) \cdot C_m^{\frac{d-2}{2}}(t) \cdot (1-t^2)^{\frac{d-3}{2}} dt = 0.$$

Since we have no further requirements on g, this can be done in many different ways: any arbitrary compactly supported bump function can be rescaled to be supported on a sufficiently small interval and then sliding over the root and using the intermediate value theorem produces an example. Recalling that $\lambda = (d-2)/2$,

$$J \subseteq \left(1 - \frac{(d+4)^2}{4m^2}, 1\right).$$

Observe that if $a, b \in \mathbb{S}^{d-1}$ are two points on the sphere with inner product $\langle a, b \rangle = x_1$, then the Euclidean distance between these points satisfies

$$\|a-b\|^2 = 2 - 2\langle a, b\rangle \leq 2 - 2\left(1 - \frac{(d+4)^2}{4m^2}\right) = \frac{(d+4)^2}{2m^2}$$

and thus

$$\|a-b\| \leq \frac{d+4}{\sqrt{2}} \frac{1}{m}.$$

We now return to the new function

$$f^*(x) = \int_{\mathbb{S}^{d-1}} g(\langle x, y \rangle) f(y) d\sigma(y)$$

and conclude, from the computation above and $\lambda_m(g) = 0$, that

$$f^*(x) = \sum_{k \in S \setminus \{m\}} a_k \lambda_k(g) f_k(x).$$

We know that if there is a Euclidean ball $B(x, r(f))$ of radius $r(f)$ such that f does not have a zero in $B(x, r(f)) \cap \mathbb{S}^{d-1}$, then f^* contains a ball of radius at least

$$r(f^*) \geq r(f) - \frac{d+4}{\sqrt{2}} \frac{1}{m}$$

on which the function does not have a zero. By induction hypothesis, we have

$$r(f) \leq r(f^*) + \frac{d+4}{\sqrt{2}} \frac{1}{m} \leq \frac{d+4}{\sqrt{2}} \frac{1}{m} + 2\pi d \sum_{k \in S \setminus \{m\}} \frac{1}{k} \leq 2\pi d \sum_{k \in S} \frac{1}{k}.$$

This constant is with respect to measuring distances using the Euclidean norm in $\mathbb{R}^d$; switching to the geodesic distance incurs another factor of $\pi/2$ which then proves the desired result. □

5 Proof of Theorem 3

Proof. We argue again using induction on n.

The case $n = 1$. We establish this case by proving the Corollary first. Let $-\Delta\phi = \lambda\phi$ be a smooth, global eigenfunction on $\mathbb{R}^d$ where $d \in \{2, 3\}$. The main ingredient in our argument is the inhomogeneous wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta \right) u(t, x) = \phi(x)$$

with vanishing initial conditions

$$u\big|_{t=0} = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}\big|_{t=0} = 0.$$

An explicit computation shows that this equation has the closed-form solution

$$u(t, x) = \frac{\cos(\sqrt{\lambda} t) - 1}{\lambda} \phi(x).$$

We note, in particular, that at time $t^* = 2\pi/\sqrt{\lambda}$ we have $u(t,x) = 0$. However, the inhomogeneous wave equation on $\mathbb{R}^d$ with $d \in \{2,3\}$ and vanishing initial conditions has a nice closed-form solution as well. In $\mathbb{R}^2$ this solution is

$$u(t,x) = \int_0^t \frac{1}{2\pi(t-s)^2} \int_{B(x,t-s)} \frac{(t-s)^2\phi(y)}{((t-s)^2 - |y-x|^2)^{1/2}} dyds.$$

In $\mathbb{R}^3$, the solution is

$$u(t,x) = \frac{1}{4\pi} \int_{B(x,t)} \frac{\phi(y)}{\|y-x\|} dx.$$

We set $t^* = 2\pi/\sqrt{\lambda}$ and see that ϕ has a root in each ball with radius $r = 2\pi\lambda^{-1/2}$.

The case $n \geq 2$. Let now

$$f(x) = \sum_{k=1}^{n} a_k \phi_k(x)$$

and let us assume without loss of generality that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $a_i \neq 0$ for all $1 \leq i \leq n$. We again consider the inhomogeneous wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(t,x) = f(x)$$

with vanishing initial conditions $u|_{t=0} = 0$ and $u_t|_{t=0} = 0$ and see that

$$u(t,x) = \sum_{k=1}^{n} a_k \frac{\cos(\sqrt{\lambda_k} t) - 1}{\lambda_k} \phi_k(x).$$

At time $t^* = 2\pi/\sqrt{\lambda_n}$ the solution can be written as

$$g(x) = u(t^*, x) = \sum_{k=1}^{n-1} a_k \frac{\cos(\sqrt{\lambda_k} t) - 1}{\lambda_k} \phi_k(x).$$

Suppose now that $f(x)$ does not have a zero on the ball $B(z, r(f))$ and is either positive or negative in that region. From the explicit solution formula of the wave equation we see that, for all $0 \leq t \leq r(f)$, the function

$$u(t,x) \quad \text{does not change sign on} \quad B(z, r(f) - t).$$

We set $t^* = 2\pi/\sqrt{\lambda_n}$ and conclude that $u(t^*, x)$ does not change sign on a ball of radius $r(f) - t^*$ (note that if $r(f) \leq t^*$, then the desired result follows automatically). However, by induction assumption we have that

$$u(t^*, x) \quad \text{must change sign on every ball of radius} \quad 2\pi \sum_{k=1}^{n-1} \frac{1}{\sqrt{\lambda_k}}$$

and therefore

$$r(f) - t^* \leq 2\pi \sum_{k=1}^{n-1} \frac{1}{\sqrt{\lambda_k}}$$

and the desired result follows. □

6 Proof of Theorem 4

Proof. We will assume, throughout the argument, that $n \geq 3$. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $-\Delta f = \lambda f$ in some neighborhood of the ball

$$B = B\left(x, \frac{2\pi}{\sqrt{\lambda}}\right).$$

We introduce the average value on a spherical shell of radius r centered around x,

$$\mathrm{Av}(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} f(y) dy,$$

where $|\partial B(x,r)|$ denotes the $(n-1)$-dimensional surface area of the ball. Using the Green identities in $\mathbb{R}^n$ in the formulation (see [14, §2.2.2])

$$\frac{\partial}{\partial r}\left(\frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y) dy\right) = \frac{r}{n} \frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \Delta u(y) dy$$

we deduce that

$$\frac{\partial}{\partial r} \mathrm{Av}(r) = \frac{r}{n} \frac{1}{\omega_n r^n} \int_{B(x,r)} \Delta f \, dy.$$

We note that if ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, then the surface area of a spherical shell is given by $n\omega_n r^{n-1}$ since

$$\omega_n r^n = \int_{B(x,r)} 1 dy = \int_0^r n\omega_n s^{n-1} ds.$$

Since $\Delta f = -\lambda f$, we deduce, rewriting everything in terms of spherical averages,

$$\begin{aligned}
\frac{\partial}{\partial r} \mathrm{Av}(r) &= -\frac{\lambda}{n\omega_n r^{n-1}} \int_{B(x,r)} f(y) dy \\
&= -\frac{\lambda}{n\omega_n r^{n-1}} \int_0^r \omega_n n s^{n-1} \mathrm{Av}(s) dy \\
&= -\frac{\lambda}{r^{n-1}} \int_0^r s^{n-1} \mathrm{Av}(s) dy.
\end{aligned}$$

The final ingredient is the function

$$Q(r)=\int_{B(x,r)}\frac{f(y)}{\|x-y\|^{n-2}}dy.$$

We note that, since f is locally bounded for r small, we have

$$|Q(r)|\lesssim\int_{B(x,r)}\frac{\|f\|_{L^\infty_{\text{loc}}}}{\|x-y\|^{n-2}}dy\lesssim\|f\|_{L^\infty_{\text{loc}}}\cdot r^2$$

and therefore $Q(0)=0$ and $Q'(0)=0$. Using continuity of the eigenfunction, we deduce that, as $r\to 0$,

$$\begin{aligned}Q(r)&=\int_{B(x,r)}\frac{f(y)}{\|x-y\|^{n-2}}dy=(f(x)+\mathcal{O}(r))\int_{B(x,r)}\frac{1}{\|x-y\|^{n-2}}dy\\&=(f(x)+\mathcal{O}(r))\int_0^r\frac{n\omega_n s^{n-1}}{s^{n-2}}ds=(f(x)+\mathcal{O}(r))\frac{n\omega_n}{2}r^2\end{aligned}$$

from which we deduce $Q''(0)=n\omega_n f(x)$. By switching to polar coordinates,

$$Q(r)=\int_0^r n\omega_n s\mathrm{Av}(s)ds.$$

Differentiating on both sides leads to $Q'(r)=n\omega_n r\mathrm{Av}(r)$ and differentiating again

$$\begin{aligned}Q''(r)&=n\omega_n\mathrm{Av}(r)+n\omega_n r\Big(-\frac{\lambda}{n\omega_n r^{n-1}}\int_{B(x,r)}f(y)dy\Big)\\&=n\omega_n\mathrm{Av}(r)-n\omega_n\frac{\lambda r}{n\omega_n r^{n-1}}\int_{B(x,r)}f(y)dy\\&=n\omega_n\mathrm{Av}(r)-n\omega_n\frac{\lambda}{r^{n-2}}\int_0^r s^{n-1}\mathrm{Av}(s)ds.\end{aligned}$$

Therefore

$$rQ''(r)=n\omega_n r\mathrm{Av}(r)-\frac{\lambda}{r^{n-3}}\int_0^r\mathrm{Av}(s)n\omega_n s^{n-1}ds.$$

Using the identity $Q'(r)=n\omega_n r\mathrm{Av}(r)$ we can rewrite this as

$$rQ''(r)=Q'(r)-\frac{\lambda}{r^{n-3}}\int_0^r s^{n-2}Q'(s)ds.$$

Integration by parts shows that

$$\begin{aligned}\int_0^r s^{n-2}Q'(s)ds&=Q(s)s^{n-2}\Big|_0^r-(n-2)\int_0^r Q(s)s^{n-3}ds\\&=Q(r)r^{n-2}-(n-2)\int_0^r Q(s)s^{n-3}ds.\end{aligned}$$

Therefore

$$rQ''(r) = Q'(r) - \lambda Q(r)r + \frac{(n-2)\lambda}{r^{n-3}} \int_0^r Q(s)s^{n-3}ds.$$

At this point we already see that $Q(s)$ is governed by some sort of differential-integral equation that is quite independent of the actual eigenfunction. The remainder of the argument is dedicated to understanding what that equation is. Multiplying with r^{n-3}, we get

$$r^{n-2}Q''(r) = r^{n-3}Q'(r) - \lambda Q(r)r^{n-2} + (n-2)\lambda \int_0^r Q(s)s^{n-3}ds.$$

Differentiating in r leads to

$$\begin{aligned} r^{n-2}Q'''(r) + (n-2)r^{n-3}Q''(r) &= (n-3)r^{n-4}Q'(r) + r^{n-3}Q''(r) - \lambda Q'(r)r^{n-2} \\ &\quad - (n-2)\lambda Q(r)r^{n-3} + (n-2)\lambda Q(r)r^{n-3}. \end{aligned}$$

The last two terms cancel, so the equation simplifies to

$$r^{n-2}Q'''(r) + (n-2)r^{n-3}Q''(r) = (n-3)r^{n-4}Q'(r) + r^{n-3}Q''(r) - \lambda Q'(r)r^{n-2}$$

which then further simplifies to

$$r^{n-2}Q'''(r) + (n-3)r^{n-3}Q''(r) = (n-3)r^{n-4}Q'(r) - \lambda Q'(r)r^{n-2}.$$

At this point we make a case distinction. If $n = 3$, then the system simplifies to $r^{n-2}Q'''(r) = -\lambda Q'(r)r^{n-2}$ and thus $Q'''(r) = -\lambda Q'(r)$ from which we deduce, together with the initial conditions, that

$$Q(r) = 4\pi \frac{1 - \cos(\sqrt{\lambda} \cdot r)}{\lambda} \cdot \phi(x).$$

We can now resume, for the remainder of the argument, that $n \geq 4$ and thus, in particular, divide by r^{n-4} to arrive at

$$r^2Q'''(r) + (n-3)rQ''(r) = (n-3)Q'(r) - \lambda Q'(r)r^2.$$

Working instead with the derivative $R(r) = Q'(r)$, we deduce that we are only interested in solutions that satisfy $R(0) = 0$ as well as $R'(0) = n\omega_n f(x)$ together with the equation

$$r^2R''(r) + (n-3)rR'(r) - (n-3)R(r) + \lambda R(r)r^2 = 0.$$

Two independent solutions of this equation are given in terms of the Bessel functions of the first and the second kind,

$$r^{\frac{4-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} \cdot r) \quad \text{and} \quad r^{\frac{4-n}{2}} Y_{\frac{n-2}{2}}(\sqrt{\lambda} \cdot r).$$

We note that $n \geq 4$ and thus the polynomial powers in r are either 1 or have a singularity in the origin. Bessel functions of the second kind $Y_{(n-2)/2}$ also have a singularity at the origin which tells us, since $R(0) = 0$, that the solution we are interested in has to be a multiple of the first solution which we can thus write as

$$R(r) = \alpha\lambda^{\frac{n-4}{4}}(\sqrt{\lambda}\cdot r)^{\frac{4-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda}\cdot r).$$

We note that

$$S(r) = r^{\frac{4-n}{2}} J_{\frac{n-2}{2}}(r) \quad \text{satisfies} \quad \lim_{r\to 0} S'(r) = \frac{2^{-\frac{n-2}{2}}}{\Gamma(n/2)}.$$

Therefore

$$n\omega_n f(x) = R'(0) = \alpha\frac{2^{-\frac{n-2}{2}}}{\Gamma(n/2)}\lambda^{\frac{n-2}{4}}$$

from which it follows that

$$\alpha = \frac{2^{\frac{n-2}{2}}\Gamma(n/2)n\omega_n}{\lambda^{\frac{n-2}{4}}}.$$

Therefore

$$\begin{aligned} R(r) &= \alpha\lambda^{\frac{n-4}{4}}(\sqrt{\lambda}\cdot r)^{\frac{4-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda}\cdot r) \\ &= \frac{2^{\frac{n-2}{2}}\Gamma(n/2)n\omega_n}{\lambda^{\frac{n-2}{4}}}\lambda^{\frac{n-4}{4}}(\sqrt{\lambda}\cdot r)^{\frac{4-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda}\cdot r) \\ &= 2^{\frac{n-2}{2}}\Gamma(n/2)n\omega_n\frac{1}{\sqrt{\lambda}}(\sqrt{\lambda}\cdot r)^{\frac{4-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda}\cdot r) \\ &= \frac{1}{\sqrt{\lambda}}S_n(\sqrt{\lambda}\cdot r), \end{aligned}$$

where

$$S_n(s) = 2^{\frac{n-2}{2}}\Gamma(n/2)n\omega_n s^{\frac{4-n}{2}} J_{\frac{n-2}{2}}(s).$$

Introducing the antiderivative

$$T_n(s) = \int_0^s S_n(z)dz,$$

we deduce

$$Q(r) = \frac{1}{\sqrt{\lambda}}T_n(\sqrt{\lambda}\cdot r)\frac{1}{\sqrt{\lambda}} = \frac{1}{\lambda}\cdot T_n(\sqrt{\lambda}\cdot r).$$

□

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