A PRIORI ESTIMATES AND LIOUVILLE TYPE RESULTS FOR QUASILINEAR ELLIPTIC EQUATIONS INVOLVING GRADIENT TERMS

By

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Abstract. In this article we study local and global properties of positive solutions of $-\Delta_m u = |u|^{p-1}u + M|\nabla u|^q$ in a domain Ω of $\mathbb{R}^N$, with $m > 1, p, q > 0$ and $M \in \mathbb{R}$. Following some ideas used in [7, 8], and by using a direct Bernstein method combined with Keller–Osserman's estimate, we obtain several a priori estimates as well as Liouville type theorems. Moreover, we prove a local Harnack inequality with the help of Serrin's classical results.

1 Introduction

In this paper, we aim to investigate local and global properties of positive solutions to the following equation

$$-\Delta_m u = |u|^{p-1}u + M|\nabla u|^q \quad \text{in } \Omega, \tag{1}$$

where $m > 1$, $\Delta_m u = \operatorname{div}(|\nabla u|^{m-2}\nabla u)$, $p, q > 0$, $M \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a domain bounded or not and containing 0.

If $M = 0$, then (1) reduces to the generalized Lane–Emden equation

$$-\Delta_m u = |u|^{p-1}u \quad \text{in } \Omega, \tag{2}$$

which has been widely studied in the literature [1, 4, 9, 11, 14, 22, 23, 29, 30, 33, 34, 35, 38], both when Ω is bounded and when Ω is unbounded. Especially, in the semilinear case $m = 2$, one of the celebrated results is given by Gidas and Spruck [22]: if $N > 2$ and $p \in [1, \frac{N+2}{N-2})$, then any nonnegative solution of (2) in $\mathbb{R}^N$ is

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identically zero and the result is sharp. Very surprisingly in Gidas–Spruck's result, there is no a priori information assumption on the behavior of the solutions at infinity. Additional results for the semilinear case, but with a nonlinearity similar to that in (1) can be found in [15] and [21].

For the case of $m > 1$, radially symmetric positive solutions were studied by Ni and Serrin [28, 29, 30], and further results in this direction were obtained by Guedda and Véron [23] and Bidaut-Véron [1].

When one studies the so-called Liouville property of (2), namely whether all positive C^1 solutions of (2) in $\mathbb{R}^N$ are constant, two critical exponents appear

$$m_* = \frac{N(m-1)}{N-m}, \quad m^* = \frac{N(m-1)+m}{N-m}, \tag{3}$$

when $N > m$, known as the Serrin exponent and the Sobolev exponent, respectively. It is well known that the first is optimal for the Liouville property for the inequality

$$-\Delta_m u \geq |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \tag{4}$$

while the second is optimal for the corresponding equality. Indeed, Mitidieri and Pohozaev [26] first proved that if $N > m$ and $p \in (0, m_*]$, or $N \leq m$ and $p \in (0, \infty)$, then any nonnegative solution to (4) is zero. On the other hand, if $N > m$ and $p \in (m_*, \infty)$, then (4) possesses the following bounded positive solution

$$u(x) = C(1 + |x|^{\frac{m}{m-1}})^{-\frac{m-1}{p-m+1}},$$

for some $C > 0$, see [26, Remark 4] or [35]. For equation (2) in $\mathbb{R}^N$, we refer to the marvellous paper by Serrin and Zou [35] (cf. Corollary II), where also nonexistence in the case $N < m$ and $p \in (0, \infty)$ was solved. Of course, if $M \geq 0$, every positive solution of (1) is also a positive solution of the inequality (4).

If we consider the critical case of (2), that is when $p = m^*$, and we restrict our attention to solutions belonging to the space

$$\mathcal{D}^{1,m}(\mathbb{R}^N) := \left\{ u \in L^{m^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^m < \infty \right\},$$

then Damascelli et al. in [14], for $1 < m < 2$, Sciunzi in [33], for $m > 2$, and Vétois in [38], for $m > 1$, showed that all positive solutions are radial and have the following form

$$u(x) = U_{\lambda,x_0}(x) := \left[\frac{\lambda^{\frac{1}{m-1}} N^{\frac{1}{m}} (\frac{N-m}{m-1})^{\frac{m-1}{m}}}{\lambda^{\frac{m}{m-1}} + |x - x_0|^{\frac{m}{m-1}}} \right]^{\frac{N-m}{m}}, \quad \lambda > 0, \ x_0 \in \mathbb{R}^N.$$

Moving to exterior domains, Bidaut-Véron [1] proved that any nonnegative solution of (2) is zero provided that $N > m$ and $p \in (m-1, m_*]$, or $N = m$ and $p \in (m-1, \infty)$, while Bidaut-Véron and Pohozaev [3] showed that (4) admits only the trivial solution $u \equiv 0$ whenever $N > m$ and $p \in (0, m_*]$, or $N = m$ and $p \in (0, \infty)$.

For the case with gradient terms, we first recall the Hamilton–Jacobi equation

$$-\Delta_m u = |\nabla u|^q \quad \text{in } \Omega. \tag{5}$$

The Liouville property of (5) was studied by Lions in [25] for $m = 2$, who proved that any C^2 solution to (5) with $q > 1$ in $\mathbb{R}^N$ has to be a constant by using the Bernstein technique. Bidaut-Véron, Garcia-Huidobro and Véron [5] proved that any C^1 solution u of (5) in an arbitrary domain Ω of $\mathbb{R}^N$ with $N \geq m > 1$ and $q > m-1$ satisfies

$$|\nabla u(x)| \leq c_{N,m,q}(\mathrm{dist}(x, \partial\Omega))^{-\frac{1}{q-m+1}} \tag{6}$$

for all $x \in \Omega$. Estimates of this type, not only for the gradient but also for the solutions are called by Serrin and Zou "universal a priori estimates", because they are independent of the solutions and do not need any boundary conditions. In particular, they produce as a direct corollary the Liouville property, since $\mathrm{dist}(x, \partial\Omega)$ can be chosen arbitrarily large when the solution is defined on all $\mathbb{R}^N$. For a detailed discussion in this direction we refer to the paper by Polacik, Quitter and Souplet studied in [31] where new connections between Liouville-type theorems and universal estimates were developed. Here "any solution" means there is no any sign condition on the solution. Estimates of the gradient for more general problems can be found in [24].

For the generalized case of (5) given by

$$-\Delta_m u = u^p|\nabla u|^q \quad \text{in } \Omega, \tag{7}$$

in [6] Bidaut-Véron, Garcia-Huidobro and Véron focused on positive solutions of (7) for $m = 2$, $p \geq 0$ and $0 \leq q < 2$. By using the pointwise Bernstein method and the integral Bernstein method, they determined various regions of (p, q) for which the Liouville property holds. Filippucci, Pucci and Souplet [19] solved the case of $m = 2, p > 0$ and $q > 2$, and they proved that any positive bounded classical solution of (7) in $\mathbb{R}^N$ is identically equal to a constant. Bidaut-Véron [2] obtained the same Liouville-type results for (7) in the case $N > m > 1, p \geq 0$ and $q \geq m$ without the assumption of boundedness on the solution. Recently, the Liouville property of (7) in $\mathbb{R}^N$ for $N \geq 1$, $m > 1$, $p \geq 0$ and $0 \leq q < m$ was studied by Chang, Hu and Zhang [10]. For the case of radial solutions of the coercive vectorial version of (7) in $\mathbb{R}^N$ we refer to [20].

If we consider the inequality version of (7)

$$-\Delta_m u \geq u^p |\nabla u|^q \quad \text{in } \Omega, \tag{8}$$

it was proved in [6] for the case $m = 2$ that any positive solution of (8) in $\mathbb{R}^N$ must be constant if $N > 2, p \geq 0, q \geq 0$ and

$$p(N-2) + q(N-1) < N.$$

The generalization of the above results to the case $m \neq 2$, even in the vectorial case, can be found in [16, 17, 18, 27].

Recently, Sun, Xiao and Xu [36] dealt with (8) when Ω is a geodesically complete noncompact Riemannian manifold, and they obtained the nonexistence and existence of positive solutions to (8) in the range $m > 1$ and $(p, q) \in \mathbb{R}^2$ via the volume growth of geodesical ball.

The most important motivation of the present study is to extend the results obtained for the semilinear equation

$$-\Delta u = |u|^{p-1} u + M |\nabla u|^q \quad \text{in } \Omega, \tag{9}$$

by Bidaut-Véron, Garcia-Huidobro and Véron; see [7, 8]. By using a delicate combination of refined Bernstein techniques and the Keller–Osserman estimate, they obtained a series of a priori estimates for any positive solution of (9) in an arbitrary domain Ω of $\mathbb{R}^N$ in the case $p > 1$, $q \geq \frac{2p}{p+1}$ and $M > 0$ ([7, Theorems A, C, D]). In particular, the nonexistence of positive solutions of (9) in $\mathbb{R}^N$ was obtained for the following cases:

(i) $N \geq 1, p > 1, 1 < q < \frac{2p}{p+1}, M > 0$;

(ii) $N \geq 1, p > 1, q = \frac{2p}{p+1}, M > (\frac{p-1}{p+1})^{\frac{p-1}{p+1}} (\frac{N(p+1)^2}{4p})^{\frac{p}{p+1}}$;

(iii) $N \geq 2, 1 < p < \frac{N+3}{N-1}, 1 < q < \frac{N+2}{N}, M > 0$;

(iv) $N \geq 3, 1 < p < \frac{N+2}{N-2}, q = \frac{2p}{p+1}, |M| \leq \epsilon_0$,

where ϵ_0 is a positive constant given in [7, Theorem E]. They also considered the existence and nonexistence of "large solutions", namely those solutions $u(x) \to \infty$ as $\text{dist}(x, \partial\Omega) \to 0$, and radial solutions of (9).

In this paper, we follow the idea used in [7, 8], based on the Bernstein method, to derive various a priori estimates concerning ∇u for positive solutions of (1) in the cases q is less than, greater than or equal to $\frac{mp}{p+1}$, and consequently we obtain Liouville type theorems.

Our first result is devoted to the case $q > \frac{mp}{p+1}$.

Theorem 1.1. *Let* $\Omega \subset \mathbb{R}^N$, $p > \max\{m-1, 1\}$ *and* $q > \frac{mp}{p+1}$.
Then for any $M > 0$*, there exists a positive constant* $c_{N,m,p,q}$ *such that any positive solution of* (1) *in* Ω *satisfies*

$$|\nabla u(x)| \le c_{N,m,p,q}(M^{-\frac{p+1}{(p+1)q-mp}} + (M\mathrm{dist}(x,\partial\Omega))^{-\frac{1}{q-m+1}}) \tag{10}$$

for all $x \in \Omega$*. Especially, any positive solution of* (1) *in* $\mathbb{R}^N$ *has at most a linear growth at infinity*

$$|\nabla u(x)| \le c_{N,m,p,q}M^{-\frac{p+1}{(p+1)q-mp}}, \quad x \in \mathbb{R}^N. \tag{11}$$

While in the case $q < \frac{mp}{p+1}$, we obtain a nonexistence result.

Theorem 1.2. *Let* $p > \max\{m-1, 1\}$ *and* $\max\{m-1, \frac{m}{2}\} < q < \frac{mp}{p+1}$.
Then for any $M > 0$*, there exists a positive constant* $c_{N,m,p,q}$ *such that* (1) *does not admit positive solutions in* $\mathbb{R}^N$ *satisfying*

$$u(x) \le c_{N,m,p,q}M^{\frac{m}{mp-(p+1)q}}, \quad x \in \mathbb{R}^N. \tag{12}$$

Remark 1.3. Here assumption $q > \frac{m}{2}$ follows from the proof's technique, due to the use of Young's inequality.

For the case $q = \frac{mp}{p+1}$ and M large enough, we have the following nonexistence result in $\mathbb{R}^N$.

Theorem 1.4. *Let* $\Omega \subset \mathbb{R}^N$, $p > \max\{m-1, 1\}$ *and* $q = \frac{mp}{p+1}$.
Then for any

$$M > \frac{N^{\frac{p}{p+1}}(p+1)}{(4p)^{\frac{p}{p+1}}}(p-1)^{\frac{p-1}{p+1}}, \tag{13}$$

there exists a positive constant $c_{N,M,m,p,q}$ *such that any positive solution of* (1) *in* Ω *satisfies*

$$|\nabla u(x)| \le c_{N,M,m,p,q}(\mathrm{dist}(x,\partial\Omega))^{-\frac{p+1}{p-m+1}} \tag{14}$$

for all $x \in \Omega$*. Consequently,* (1) *does not admit positive solutions in* $\mathbb{R}^N$.

When M is allowed to be negative, we derive a nonexistence result for supersolutions of (1) in an exterior domain.

Theorem 1.5. *Let* $p > m-1$ *if* $N = m$ *or* $m-1 < p < \frac{N(m-1)}{N-m}$ *if* $N > m$, $q = \frac{mp}{p+1}$ *and* $M > -\mu^*(N)$ *where*

$$\mu^*(N) := (p+1)\Big(\frac{N(m-1)-p(N-m)}{mp}\Big)^{\frac{p}{p+1}}. \tag{15}$$

Then there exist no nontrivial nonnegative supersolutions of (1) *in* $\mathbb{R}^N \setminus \overline{B}_R$ *for any* $R > 0$.

Concerning large solutions, we prove the following.

Theorem 1.6. *Let Ω be an open domain with Lipschitz boundary, $p > m-1$ and $q = \frac{mp}{p+1}$. If $M \geq -\mu^*(m)$, then there exists no positive supersolution of* (1) *in Ω satisfying*

$$\lim_{\operatorname{dist}(x,\partial\Omega)\to 0} u(x) = \infty. \tag{16}$$

Inspired by [6, Theorem A], we derive an a priori estimate for positive solution u of (1) in a neighborhood of 0 as follows. The proof relies on Serrin's classical Harnack inequality [34, Theorem 5] and the fact that every radial solution $u(|x|)$ of (1) is m-superharmonic when $M \geq 0$.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a domain containing* 0*. Assume $1 < m < N$, $m-1 < p < \frac{N(m-1)}{N-m}$, $m-1 < q < \frac{N(m-1)}{N-1}$ and $M \geq 0$. If $u \in C^2(\Omega\backslash\{0\})$ is a positive solution of* (1) *in $\Omega\backslash\{0\}$, then*

$$u(x) + |x||\nabla u(x)| \leq c|x|^{\frac{m-N}{m-1}} \tag{17}$$

holds in a neighborhood of 0 *for some $c > 0$.*

Remark 1.8. Under the assumptions on N, m, p, q and M of Theorem 1.7, we obtain a local Harnack inequality for positive solution u of (1), namely

$$\max_{|x|=r} u(x) \leq K \min_{|x|=r} u(x), \qquad r \in (0, 1/2], \tag{18}$$

for some $K > 0$. The Harnack inequality for a more general model

$$|u|^{p-1}u - M|\nabla u|^q \leq -\Delta_m u \leq c_0|u|^{p-1}u + M|\nabla u|^q, \tag{19}$$

where $c_0 \geq 1$ and $M > 0$, was obtained first by Ruiz [32] in the range

$$m-1 < p < \frac{N(m-1)}{N-m} \quad \text{and} \quad m-1 < q < \frac{mp}{p+1}.$$

Note here $\frac{mp}{p+1} < \frac{N(m-1)}{N-1}$ always holds if p satisfies the assumption of Theorem 1.7.

The final result is a Liouville-type theorem for a positive solution of (1) with a less restrictive assumption on M but a more restrictive assumption on p compared with Theorem 1.4. Actually, as emphasized before [6, Theorem B], the direct Bernstein method allows to obtain pointwise estimates of the gradient without any integration. In particular, in the next result, using cumbersome algebraic manipulations and a rather demanding application of Young's inequality, we obtain an a priori estimate for the norm of the gradient of a power of a positive solution, in the spirit of [6, Theorem B] devoted to elliptic inequality of the Laplacian type with a superlinear absorption term.

Theorem 1.9. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$. Assume $m-1 < p < \frac{(N+3)(m-1)}{N-1}$ and $m-1 < q < \frac{(N+2)(m-1)}{N}$. Then for any $M > 0$, there exist positive constants d and $c_{N,m,p,q}$ such that any positive solution of* (1) *in Ω satisfies*

$$|\nabla u^d(x)| \leq c_{N,m,p,q}(\mathrm{dist}(x, \partial\Omega))^{-1-\frac{md}{p-m+1}}, \quad x \in \Omega. \tag{20}$$

In particular, there exists no nontrivial nonnegative solution of (1) *in $\mathbb{R}^N$.*

As a consequence of (20) the following holds:

Corollary 1.10. *Let Ω be a smooth domain in $\mathbb{R}^N$ $(N \geq 2)$ with a bounded boundary, and under the assumptions of Theorem 1.9. If u is a positive solution of* (1) *in Ω, then there exists a positive constant d_0 depending on Ω and $c_{N,m,p,q} > 0$ such that*

$$u(x) \leq c\Big((\mathrm{dist}(x, \partial\Omega))^{-\frac{m}{p-m+1}} + \max_{\mathrm{dist}(z,\partial\Omega)=d_0} u(z)\Big), \quad x \in \Omega. \tag{21}$$

Remark 1.11. Recently some weak versions of the Bernstein method have been exploited for quasilinear elliptic equations involving gradient terms. It is expected that by using integral or other weak versions of the Bernstein method, the results obtained here may be improved, see [12, 13].

Notations. In the above and below, the letters $C, C', C_0, C_1, c_0, c_1, \ldots$ denote positive constants whose values are unimportant and may vary at different occurrences, and $C_{x,\ldots,z}$ or $C(x, \ldots, z)$ denotes the positive constant whose value relies on the choices of $x, \ldots, z$.

2 Proof of Theorems 1.1, 1.2 and 1.4

We begin with the following lemma which plays a key role in our proofs.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ and $m > 1$. Assume that v is a C^1 function in Ω such that $|\nabla v| > 0$, and let w be a continuous and nonnegative function in Ω with $w \in C^2(\mathcal{W}_+)$, where $\mathcal{W}_+ = \{x \in \Omega : w(x) > 0\}$. Define the operator*

$$w \to \mathscr{A}_v(w) := -\Delta w - (m-2)\frac{\langle D^2 w \nabla v, \nabla v\rangle}{|\nabla v|^2}.$$

If w satisfies, for some $\xi > 1$ and a real number c_0,

$$\mathscr{A}_v(w) + w^{\xi} \leq c_0 \frac{|\nabla w|^2}{w}$$

on each connected component of $\mathcal{W}_+$, then

$$w(x) \leq c_{N,\xi,c_0}(\mathrm{dist}(x, \partial\Omega))^{-\frac{2}{\xi-1}}, \quad \forall x \in \Omega.$$

In particular, $w \equiv 0$ if $\Omega = \mathbb{R}^N$.

Proof. This proof is a combination of [5, Proposition 2.1], and that of [2, Lemma 3.1] in the special case $\beta(x) = 0$. In particular, the operator $\mathscr{A}_v(w)$ was first introduced in [5, Proposition 2.1]. For the reader's convenience, we list the proof here. First, write $\mathscr{A}_v(w)$ as follows

$$\mathscr{A}_v(w) = -\sum_{i,j=1}^{N} a_{ij} w_{x_i x_j}, \tag{22}$$

where a_{ij} depends on the gradient, indeed $a_{ij} = \delta_{ij} + (m-2)\frac{v_{x_i} v_{x_j}}{|\nabla v|^2}$, and $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$. Noting that $(\frac{v_{x_i} v_{x_j}}{|\nabla v|^2})$ is nonnegative definite, then

$$\min\{1, m-1\}|\eta|^2 \le \sum_{i,j=1}^{N} a_{ij}\eta_i\eta_j \le \max\{1, m-1\}|\eta|^2, \tag{23}$$

for all $\eta = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N$. Therefore, $\mathscr{A}_v$ is uniformly elliptic in $\{|\nabla v| > 0\}$.

Consider a ball $B_R(x_0) \subset \Omega$. Let $r = |x - x_0|$ and set

$$\psi(x) = \lambda(R^2 - r^2)^{-\frac{2}{\xi-1}},$$

where $\lambda > 0$. Let G be a connected component of $\{x \in B_R(x_0) : w(x) > \psi(x)\}$, then $G \subset \mathcal{W}_+$ and $\overline{G} \subset \overline{B}_R(x_0)$.

Let us define $\mathcal{L}(\psi)$ by

$$\mathcal{L}(\psi) := \mathscr{A}_v(\psi) + \psi^{\xi} - c_0 \frac{|\nabla \psi|^2}{\psi}. \tag{24}$$

A standard computation shows that

$$\mathcal{L}(\psi) \ge \lambda(R^2 - r^2)^{-\frac{2\xi}{\xi-1}}(\lambda^{\xi-1} - cR^2), \tag{25}$$

where $c = c(N, \xi, c_0)$. By choosing $\lambda = (cR^2)^{\frac{1}{\xi-1}}$, we derive $\mathcal{L}(\psi) \ge 0$.

If $x_1 \in G$ satisfies

$$w(x_1) - \psi(x_1) = \max\{w(x) - \psi(x) : x \in G\},$$

then $\nabla w(x_1) = \nabla \psi(x_1)$, $w(x_1) > \psi(x_1) > 0$ and $\mathscr{A}_v(w - \psi)(x_1) \ge 0$. Thus,

$$0 \ge \mathcal{L}(w - \psi)(x_1) = \mathscr{A}_v(w - \psi)(x_1) + (w^{\xi} - \psi^{\xi})(x_1) + c_0\Big(\frac{|\nabla \psi|^2}{\psi} - \frac{|\nabla w|^2}{w}\Big).$$

Since the last two terms are positive, we derive a contradiction. Therefore $w \le \psi$ in $B_R(x_0)$. In particular,

$$w(x_0) \le \psi(x_0) = c'_{N,\xi,c_0} R^{-\frac{2}{\xi-1}}. \tag{26}$$

By letting $R \to \operatorname{dist}(x, \partial\Omega)$, we obtain (2.1). □

The next lemma is the extension of formula (2.6) in [7]. The new formula, valid for every $m > 1$, is rather tricky and requires cumbersome calculations since we have to take into account several terms appearing when $m \neq 2$.

Lemma 2.2. *Assume that v is a nonnegative C^3 function in Ω. Let $z = |\nabla v|^2$, then we have*

$$\begin{aligned}\frac{1}{2}\mathscr{A}_v(z) + \frac{1}{N}z^{2-m}(\Delta_m v)^2 + z^{1-\frac{m}{2}}\langle\nabla\Delta_m v, \nabla v\rangle \\ \leq \frac{(N+2)(m-2)}{2N}z^{-\frac{m}{2}}\Delta_m v\langle\nabla z, \nabla v\rangle + \frac{m-2}{4}\frac{|\nabla z|^2}{z} \\ - \frac{(2N+m-2)(m-2)}{4N}\frac{\langle\nabla z, \nabla v\rangle^2}{z^2}, \quad \textit{on } \{z > 0\}.\end{aligned} \tag{27}$$

Proof. Using $z = |\nabla v|^2$, $\nabla z = 2D^2 v\nabla v$, and

$$\Delta_m v = |\nabla v|^{m-2}\Delta v + (m-2)|\nabla v|^{m-4}\langle D^2 v\nabla v, \nabla v\rangle,$$

we obtain

$$\Delta v = z^{1-\frac{m}{2}}\Delta_m v - \frac{m-2}{2}\frac{\langle\nabla z, \nabla v\rangle}{z}, \quad \text{on } \{z > 0\}. \tag{28}$$

A routine computation yields that

$$\begin{aligned}(\Delta v)^2 &= z^{2-m}(\Delta_m v)^2 - (m-2)z^{-\frac{m}{2}}\Delta_m v\langle\nabla z, \nabla v\rangle \\ &\quad + \frac{(m-2)^2}{4}\frac{\langle\nabla z, \nabla v\rangle^2}{z^2}, \\ \nabla\Delta v &= z^{1-\frac{m}{2}}\nabla\Delta_m v - \frac{m-2}{2}z^{-\frac{m}{2}}\Delta_m v\nabla z \\ &\quad + \frac{m-2}{2}\frac{\langle\nabla z, \nabla v\rangle\nabla z}{z^2} - \frac{m-2}{2}\frac{\nabla\langle\nabla z, \nabla v\rangle}{z},\end{aligned} \tag{29}$$

and

$$\begin{aligned}\langle\nabla\Delta v, \nabla v\rangle &= z^{1-\frac{m}{2}}\langle\nabla\Delta_m v, \nabla v\rangle - \frac{m-2}{2}z^{-\frac{m}{2}}\Delta_m v\langle\nabla z, \nabla v\rangle \\ &\quad + \frac{m-2}{2}\frac{\langle\nabla z, \nabla v\rangle^2}{z^2} - \frac{m-2}{2}\frac{\langle\nabla\langle\nabla z, \nabla v\rangle, \nabla v\rangle}{z}.\end{aligned} \tag{30}$$

Noting that

$$\nabla\langle\nabla z, \nabla v\rangle = D^2 z\nabla v + D^2 v\nabla z,$$

and

$$\langle D^2 v\nabla z, \nabla v\rangle = \langle D^2 v\nabla v, \nabla z\rangle = \frac{1}{2}|\nabla z|^2,$$

we get

$$\langle \nabla \langle \nabla z, \nabla v \rangle, \nabla v \rangle = \langle D^2 z \nabla v, \nabla v \rangle + \frac{1}{2}|\nabla z|^2. \tag{31}$$

Combining (31) with (30), we have

$$\begin{aligned} \langle \nabla \Delta v, \nabla v \rangle = z^{1-\frac{m}{2}} \langle \nabla \Delta_m v, \nabla v \rangle - \frac{m-2}{2} z^{-\frac{m}{2}} \Delta_m v \langle \nabla z, \nabla v \rangle \\ + \frac{m-2}{2} \frac{\langle \nabla z, \nabla v \rangle^2}{z^2} - \frac{m-2}{2} \frac{\langle D^2 z \nabla v, \nabla v \rangle}{z} \\ - \frac{m-2}{4} \frac{|\nabla z|^2}{z}. \end{aligned} \tag{32}$$

By the Böchner formula, we have

$$\begin{aligned} \frac{1}{2} \Delta |\nabla v|^2 &= |D^2 v|^2 + \langle \nabla \Delta v, \nabla v \rangle \\ &\geq \frac{1}{N} (\Delta v)^2 + \langle \nabla \Delta v, \nabla v \rangle. \end{aligned} \tag{33}$$

Replacing (29) and (32) into (33), we deduce

$$\begin{aligned} \frac{1}{2} \Delta z \geq & -\frac{m-2}{2} \frac{\langle D^2 z \nabla v, \nabla v \rangle}{z} + \frac{1}{N} z^{2-m} (\Delta_m v)^2 \\ & + z^{1-\frac{m}{2}} \langle \nabla \Delta_m v, \nabla v \rangle - \frac{(N+2)(m-2)}{2N} z^{-\frac{m}{2}} \Delta_m v \langle \nabla z, \nabla v \rangle \\ & + \frac{(2N+m-2)(m-2)}{4N} \frac{\langle \nabla z, \nabla v \rangle^2}{z^2} - \frac{m-2}{4} \frac{|\nabla z|^2}{z}. \end{aligned}$$

The above inequality can be rewritten as

$$\begin{aligned} & \frac{1}{2} \mathscr{A}_v(z) + \frac{1}{N} z^{2-m} (\Delta_m v)^2 + z^{1-\frac{m}{2}} \langle \nabla \Delta_m v, \nabla v \rangle \\ & \quad \leq \frac{(N+2)(m-2)}{2N} z^{-\frac{m}{2}} \Delta_m v \langle \nabla z, \nabla v \rangle + \frac{m-2}{4} \frac{|\nabla z|^2}{z} \\ & \qquad - \frac{(2N+m-2)(m-2)}{4N} \frac{\langle \nabla z, \nabla v \rangle^2}{z^2}, \end{aligned}$$

which yields (27). □

The following Bernstein estimate for solutions of (1) is essential in the proofs of Theorems 1.1, 1.2 and 1.4.

Lemma 2.3. *Assume that u is a C^1 solution of* (1) *in a domain Ω, with $m > 1$ and M, p, q arbitrary real numbers. Let $z = |\nabla u|^2$. Then for any $0 < a < \frac{1}{N}$ and $0 < b < \frac{M^2}{N}$, there exists a positive constant $c_1 = c_1(N, M, m, q, a, b)$ such that*

$$\begin{aligned} \frac{1}{2} \mathscr{A}_u(z) + a u^{2p} z^{2-m} + \frac{2M}{N} |u|^{p-1} u z^{\frac{q}{2}-m+2} + b z^{q-m+2} - p |u|^{p-1} z^{2-\frac{m}{2}} \\ \leq c_1 \frac{|\nabla z|^2}{z}, \quad \text{on } \{z > 0\}. \end{aligned} \tag{34}$$

Remark 2.4. From the proof of Lemma 2.3, it is evident that when $m = 2$, the constant a above can be allowed to satisfy $0 < a \le \frac{1}{N}$, while when $Nq = (N+2)(m-2)$, the constant b above can be chosen such that $0 < b \le \frac{M^2}{N}$.

Proof. As in Theorem 3.4.7 in [37], since u is a C^1 solution of (1) in Ω then $u \in C^2$ in $\{x \in \Omega : \nabla u \neq 0\}$, that is for $\{z > 0\}$. Furthermore, we have

$$z^{2-m}(\Delta_m u)^2 = u^{2p} z^{2-m} + 2M|u|^{p-1} u z^{\frac{q}{2}-m+2} + M^2 z^{q-m+2},$$

$$z^{1-\frac{m}{2}} \langle \nabla \Delta_m u, \nabla u \rangle = -p|u|^{p-1} z^{2-\frac{m}{2}} - \frac{Mq}{2} z^{\frac{q-m}{2}} \langle \nabla z, \nabla u \rangle,$$

and

$$z^{-\frac{m}{2}} \Delta_m u \langle \nabla z, \nabla u \rangle = -|u|^{p-1} u z^{-\frac{m}{2}} \langle \nabla z, \nabla u \rangle - M z^{\frac{q-m}{2}} \langle \nabla z, \nabla u \rangle.$$

Inserting these identities into (27), we arrive at

$$\begin{aligned} &\frac{1}{2}\mathscr{A}_u(z) + \frac{1}{N} u^{2p} z^{2-m} + \frac{2M}{N}|u|^{p-1} u z^{\frac{q}{2}-m+2} + \frac{M^2}{N} z^{q-m+2} - p|u|^{p-1} z^{2-\frac{m}{2}} \\ &\quad \le -\frac{(N+2)(m-2)}{2N}|u|^{p-1} u z^{-\frac{m}{2}} \langle \nabla z, \nabla u \rangle \\ &\qquad + \Big(\frac{Mq}{2} - \frac{M(N+2)(m-2)}{2N}\Big) z^{\frac{q-m}{2}} \langle \nabla z, \nabla u \rangle + \frac{m-2}{4}\frac{|\nabla z|^2}{z} \\ &\qquad - \frac{(2N+m-2)(m-2)}{4N}\frac{\langle \nabla z, \nabla u \rangle^2}{z^2}, \quad \text{on } \{z > 0\}. \end{aligned} \tag{35}$$

Next we estimate each term in the right-hand side of (35). By the Cauchy–Schwartz inequality and then, thanks to Young's inequality, we have for any $\varepsilon, \varepsilon' > 0$

$$|u|^{p-1} u z^{-\frac{m}{2}} |\langle \nabla z, \nabla u \rangle| \le \varepsilon u^{2p} z^{2-m} + \frac{1}{4\varepsilon}\frac{|\nabla z|^2}{z},$$

and

$$z^{\frac{q-m}{2}} |\langle \nabla z, \nabla u \rangle| \le \varepsilon' z^{q-m+2} + \frac{1}{4\varepsilon'}\frac{|\nabla z|^2}{z}.$$

Note also that

$$\frac{\langle \nabla z, \nabla u \rangle^2}{z^2} \le \frac{|\nabla z|^2}{z}.$$

Let $\varepsilon_1 := \frac{(N+2)|m-2|}{2N}\varepsilon$ and $\varepsilon_2 := |\frac{Mq}{2} - \frac{M(N+2)(m-2)}{2N}|\varepsilon'$. We infer that

$$\begin{aligned} &\frac{1}{2}\mathscr{A}_u(z) + \Big(\frac{1}{N} - \varepsilon_1\Big) u^{2p} z^{2-m} + \frac{2M}{N}|u|^{p-1} u z^{\frac{q}{2}-m+2} \\ &\qquad + \Big(\frac{M^2}{N} - \varepsilon_2\Big) z^{q-m+2} - p|u|^{p-1} z^{2-\frac{m}{2}} \le c_1 \frac{|\nabla z|^2}{z}, \end{aligned}$$

where $c_1 = c_1(N, m, \varepsilon_1, \varepsilon_2) > 0$. Set $a = \frac{1}{N} - \varepsilon_1$ and $b = \frac{M^2}{N} - \varepsilon_2$. Taking ε_1 and ε_2 small enough such that $a, b > 0$, then (34) follows. □

Now we step into the proof of Theorem 1.1.

Proof of Theorem 1.1. Let u be a positive solution of (1). Consider the following change of variables

$$u(x) = \alpha^{\frac{m}{p-m+1}} v(y), \quad y = \alpha x, \quad x \in \Omega, \tag{36}$$

with $\alpha = M^{-\frac{p-m+1}{(p+1)q-mp}}$.

Then $|\nabla v| = |\nabla_y v| = \alpha^{-\frac{p+1}{p-m+1}} |\nabla u|$ and $\Delta_m v = \alpha^{-\frac{mp}{p-m+1}} \Delta_m u$ so that v is a positive C^1 solution of

$$-\Delta_m v = |v|^{p-1} v + |\nabla v|^q \quad \text{in } \Omega_\alpha, \tag{37}$$

where $\Omega_\alpha := \{y \in \mathbb{R}^N : y = \alpha x, \ x \in \Omega\}$.

Let $z = |\nabla v|^2$, so that (34) becomes

$$\frac{1}{2}\mathscr{A}_v(z) + av^{2p} z^{2-m} + \frac{2}{N}|v|^{p-1} v z^{\frac{q}{2}-m+2} + bz^{q-m+2} - p|v|^{p-1} z^{2-\frac{m}{2}} \le c_1 \frac{|\nabla z|^2}{z}, \quad \text{on } \{z > 0\},$$

indeed v is a positive solution of (1) with $M = 1$. In turn

$$\frac{1}{2}\mathscr{A}_v(z) + av^{2p} z^{2-m} + bz^{q-m+2} - p|v|^{p-1} z^{2-\frac{m}{2}} \le c_1 \frac{|\nabla z|^2}{z}, \quad \text{on } \{z > 0\}, \tag{38}$$

with $0 < a, b < \frac{1}{N}$ as in (34) and $c_1 = c_1(N, m, q, a, b)$.

Suppose $q > \frac{mp}{p+1}$. In this case, it immediately follows that $q - m + 2 > 1$ by conditions assumed on p. By the Young's inequality with exponents $2p/(p-1)$ and $2p/(p+1)$, for $\varepsilon_3 > 0$, we have

$$p|v|^{p-1} z^{2-\frac{m}{2}} = p|v|^{p-1} z^{\frac{(2-m)(p-1)}{2p}} z^{1+\frac{2-m}{2p}} \le \varepsilon_3 v^{2p} z^{2-m} + c_2 z^{\frac{2p+2-m}{p+1}}.$$

Since $2p > m-2$ and $q(p+1) > mp$, a further application of the Young's inequality with exponents $(q-m+2)(p+1)/(2p+2-m)$ and its conjugate gives, for $\varepsilon_4 > 0$,

$$c_2 z^{\frac{2p+2-m}{p+1}} \le \varepsilon_4 z^{q-m+2} + c_3,$$

where $c_2 = c_2(p, \varepsilon_3) > 0$ and $c_3 = c_3(m, p, q, c_2, \varepsilon_4) > 0$. Hence by (38),

$$\frac{1}{2}\mathscr{A}_v(z) + A_1 v^{2p} z^{2-m} + A_2 z^{q-m+2} \le c_1 \frac{|\nabla z|^2}{z} + c_3,$$

where $A_1 = a - \varepsilon_3$ and $A_2 = b - \varepsilon_4$. Taking ε_3 and ε_4 small enough such that $A_1, A_2 > 0$, then

$$\frac{1}{2}\mathscr{A}_v(z) + A_2 z^{q-m+2} \le c_1 \frac{|\nabla z|^2}{z} + c_3.$$

Letting $\tilde{z} = (z - (\frac{c_3}{A_2})^{\frac{1}{q-m+2}})_+$, thus $z \geq \tilde{z}$, and with $q - m + 2 > 1$, we obtain

$$\frac{1}{2}\mathscr{A}_v(\tilde{z}) + A_2\tilde{z}^{q-m+2} \leq c_1\frac{|\nabla\tilde{z}|^2}{\tilde{z}}, \quad \text{on } \left\{z > \left(\frac{c_3}{A_2}\right)^{\frac{1}{q-m+2}}\right\}.$$

Using Lemma 2.1, we derive

$$\tilde{z}(y) \leq c_4(\text{dist}(y, \partial\Omega_\alpha))^{-\frac{2}{q-m+1}},$$

where $c_4 = c_4(m, q, c_1, A_2) > 0$, and with $\tilde{z} = |\nabla v(y)|^2 - c$, $c > 0$, using that $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$, it follows that

$$|\nabla v(y)| \leq c_4'(1 + (\text{dist}(y, \partial\Omega_\alpha))^{-\frac{1}{q-m+1}}), \quad y \in \Omega_\alpha. \tag{39}$$

In view of the change of variables (36), we finally obtain (10).

Now consider the case $\Omega = \mathbb{R}^N$ and assume that u is a positive solution of (1) in $\mathbb{R}^N$. Fix $y \in \mathbb{R}^N$ such that $|y| < 2n$. Using (39) with $\Omega_\alpha = B_{2n}(0)$, we see that

$$|\nabla v(y)| \leq c_4'(1 + (2n - |y|)^{-\frac{1}{q-m+1}}), \quad y \in B_{2n}(0).$$

Taking $n \to \infty$ yields

$$|\nabla v(y)| \leq c_4', \quad y \in \mathbb{R}^N,$$

so that (11) follows immediately thanks to the change of variables. □

Proof of Theorem 1.2. Let u be a positive solution of (1) and let v be the function defined in (36) where now $\Omega = \Omega_\alpha = \mathbb{R}^N$. If $z = |\nabla v|^2$, since we have $\max\{m-1, \frac{m}{2}\} < q < \frac{mp}{p+1}$, then for any $\varepsilon_5 > 0$ we have

$$pv^{p-1}z^{2-\frac{m}{2}} = pv^{p-1}z^{\frac{(2-m)(2q-m)}{2q}}z^{\frac{m(q-m+2)}{2q}} \leq \varepsilon_5 z^{q-m+2} + c_5 v^{\frac{2q(p-1)}{2q-m}}z^{2-m},$$

where $c_5 = c_5(m, p, q, \varepsilon_5) > 0$. Inserting this inequality into (38), we obtain

$$\frac{1}{2}\mathscr{A}_v(z) + v^{2p}z^{2-m}(a - c_5 v^{\frac{2mp-2q(p+1)}{2q-m}}) + A_3 z^{q-m+2} \leq c_1\frac{|\nabla z|^2}{z},$$

where $A_3 = b - \varepsilon_5$ with ε_5 small enough such that $A_3 > 0$. If

$$\max v \leq c_{N,m,p,q} := \left(\frac{a}{c_5}\right)^{\frac{2q-m}{2mp-2q(p+1)}},$$

which is equivalent to (12) by virtue of (36), we get

$$\frac{1}{2}\mathscr{A}_v(z) + A_3 z^{q-m+2} \leq c_1\frac{|\nabla z|^2}{z}.$$

From Lemma 2.1, applied with $\xi = q - m + 2 > 1$, we conclude that $z \equiv 0$ in $\mathbb{R}^N$, in turn v is identically constant and thus $v \equiv 0$ in $\mathbb{R}^N$ from the equation (37). □

Proof of Theorem 1.4. Let u be a positive solution of (1) in Ω and let $q = \frac{mp}{p+1}$ by assumption. For $B > 0$ to be chosen and $a \in (0, 1/N)$ if $m \neq 2$ or $a \in (0, 1/N]$ if $m = 2$, consider the auxiliary function Φ defined for $Z > 0$ by

$$\Phi(Z) = u^p Z^{2-m} + BZ^{q-m+2} - \sqrt{\frac{p}{a}} u^{\frac{p-1}{2}} Z^{2-\frac{m}{2}}.$$

In particular, $\Phi(Z) = Z^{2-m}\psi(Z)$ where

$$\psi(Z) = u^p + BZ^{\frac{mp}{p+1}} - \sqrt{\frac{p}{a}} u^{\frac{p-1}{2}} Z^{\frac{m}{2}}$$

with

$$\psi(0) = u^p > 0 \quad \text{and} \quad \psi'(Z) = \frac{mBp}{p+1} Z^{\frac{m}{2}-1}\Big[Z^{\frac{m(p-1)}{2(p+1)}} - \frac{p+1}{2\sqrt{ap}} u^{\frac{p-1}{2}}\Big],$$

so that $\psi(Z)$ achieves its minimum at

$$Z_0 = \Big(\frac{p+1}{2B\sqrt{ap}}\Big)^{\frac{2(p+1)}{m(p-1)}} u^{\frac{p+1}{m}} > 0,$$

and

$$\psi(Z) \geq \psi(Z_0) = \Big[1 - \frac{p-1}{(4ap)^{\frac{p}{p-1}}}\Big(\frac{p+1}{B}\Big)^{\frac{p+1}{p-1}}\Big]u^p.$$

Denoting

$$M_+ = \frac{(p+1)(p-1)^{\frac{p-1}{p+1}}}{(4ap)^{\frac{p}{p+1}}} > 0,$$

we obtain, choosing $B \geq M_+$, that $\psi(Z_0) \geq 0$ yielding $\psi(Z) \geq 0$ for all $Z > 0$ and consequently $\Phi(Z) \geq 0$ for all $Z > 0$.

Observing that condition (13) reads as $M > \big(aN\big)^{\frac{p}{p+1}} M_+$, we consider

$$\begin{aligned} &\frac{1}{2}\mathscr{A}_u(z) + a\Big(u^p z^{1-\frac{m}{2}} + \frac{M_+}{(Na)^{1/(p+1)}} z^{1+\frac{q-m}{2}}\Big)^2 \\ &\quad + \Big[b - a\frac{M_+^2}{(Na)^{\frac{2}{p+1}}}\Big] z^{q-m+2} - pu^{p-1} z^{2-\frac{m}{2}} \\ &= \frac{1}{2}\mathscr{A}_u(z) + au^{2p}z^{2-m} + a\frac{M_+^2}{(Na)^{\frac{2}{p+1}}} z^{q-m+2} + 2a\frac{M_+}{(Na)^{\frac{1}{p+1}}} u^p z^{2-m+\frac{q}{2}} \\ &\quad + \Big(b - a\frac{M_+^2}{(Na)^{\frac{2}{p+1}}}\Big) z^{q-m+2} - pu^{p-1} z^{2-\frac{m}{2}} \\ &\leq \frac{1}{2}\mathscr{A}_u(z) + au^{2p}z^{2-m} + \frac{2M}{N}u^p z^{2-m+\frac{q}{2}} + bz^{q-m+2} - pu^{p-1}z^{2-\frac{m}{2}}, \end{aligned} \tag{40}$$

where in the last inequality we have used $M \geq \big(aN\big)^{\frac{p}{p+1}} M_+$. We have thus obtained the left-hand side of inequality (34) for u positive, so in the set where $|\nabla u| \neq 0$ and for $b < M^2/N$, we get

$$\frac{1}{2}\mathscr{A}_u(z) + a\Big(u^p z^{1-\frac{m}{2}} + \frac{M_+}{(Na)^{1/(p+1)}} z^{1+\frac{q-m}{2}}\Big)^2 + \Big[b - a\frac{M_+^2}{(Na)^{\frac{2}{p+1}}}\Big] z^{q-m+2} - p u^{p-1} z^{2-\frac{m}{2}} \leq c_1 \frac{|\nabla z|^2}{z}.$$

We claim that

$$a\Big(u^p z^{1-\frac{m}{2}} + \frac{M_+}{(Na)^{1/(p+1)}} z^{1+\frac{q-m}{2}}\Big)^2 - p u^{p-1} z^{2-\frac{m}{2}} \geq 0. \tag{41}$$

Indeed, noting that for any $B \geq M_+$, we have

$$\begin{aligned}(u^p|\nabla u|^{2-m} &+ B|\nabla u|^{q-m+2})^2 - \frac{p}{a} u^{p-1}|\nabla u|^{4-m} \\ &= \Big(u^p|\nabla u|^{2-m} + B|\nabla u|^{q-m+2} + \sqrt{\frac{p}{a}}\, u^{\frac{p-1}{2}}|\nabla u|^{2-\frac{m}{2}}\Big)\cdot \Phi(|\nabla u|) \geq 0,\end{aligned}$$

where $0 < a \leq \frac{1}{N}$, then (41) immediately follows choosing $B = M_+/(Na)^{1/(p+1)}$, with $z = |\nabla u|^2$.

Consequently,

$$\frac{1}{2}\mathscr{A}_u(z) + \Big[b - a\frac{M_+^2}{(Na)^{2/(p+1)}}\Big] z^{q-m+2} \leq c_1 \frac{|\nabla z|^2}{z}.$$

Now, it is possible to choose a and b such that

$$a\frac{M_+^2}{(Na)^{\frac{2}{p+1}}} < b < \frac{M^2}{N},$$

indeed the above condition is equivalent to

$$a > N^{\frac{p-1}{p+1}} \Big[\frac{(p+1)(p-1)^{\frac{p-1}{p+1}}}{M(4p)^{\frac{p}{p+1}}}\Big]^2 = \frac{1}{N} - \varepsilon$$

with

$$\varepsilon = \frac{1}{N} - N^{\frac{p-1}{p+1}} \Big[\frac{(p+1)(p-1)^{\frac{p-1}{p+1}}}{M(4p)^{\frac{p}{p+1}}}\Big]^2 > 0$$

by (13) and since $N > N^{\frac{p}{p+1}}$.

By using again Lemma 2.1, with $\xi = q - m + 2 > 1$, we obtain

$$|\nabla u(x)| \leq c_{N,M,m,p,q} (\mathrm{dist}(x, \partial\Omega))^{-\frac{1}{q-m+1}},$$

which is exactly (14) via $q = \frac{mp}{p+1}$.

3 Proof of Theorems 1.5, 1.6 and 1.7

Proposition 3.1. **(A)** *Let $M \geq 0$, $m > 1$, $q \geq 0$ and either $N \leq m$ and $p > 0$ or $N > m$ and $0 < p \leq \frac{N(m-1)}{N-m}$. Then, there exist no positive solutions of* (1) *in $\mathbb{R}^N \setminus \overline{B}_R$ for $R > 0$.*

(B) *Let $M > 0$, $m > 1$, $N > 1$, $p \geq 0$ and $m - 1 < q \leq \frac{N(m-1)}{N-1}$. Then there exist no positive radial solutions of* (1) *in $\mathbb{R}^N \setminus \overline{B}_R$ for $R \geq 0$.*

(C) *Let $N > m$, $m > 1$, $M \geq 0$, $q \geq 0$, $p > \frac{N(m-1)}{N-m}$ and let $u = u(|x|) = u(r)$ be a positive radial solution of* (1) *in $\mathbb{R}^N \setminus \overline{B}_R$. Then, there exists $\rho > R$ such that*

$$u(r) \leq c_0 r^{-\frac{m}{p-m+1}}, \quad r > \rho, \tag{42}$$

with $c_0 = [2N(1 - 2^{-\frac{m}{m-1}})^{-(m-1)}(\frac{m}{p-m+1})^{m-1}]^{\frac{1}{p-m+1}}$ and

$$|u_r(r)| \leq c_0 \frac{N-m}{m-1} r^{-\frac{p+1}{p-m+1}}, \quad r > \rho. \tag{43}$$

(D) *Let $N > 1$, $m > 1$, $M > 0$, $p \geq 0$, $q > \frac{N(m-1)}{N-1}$, and let $u(x) = u(r)$ be a positive radial solution of* (1) *in $\mathbb{R}^N \setminus \overline{B}_R$. There exists $\rho > 2R$ such that*

$$|u_r(r)| \leq c_1 r^{-\frac{1}{q-m+1}}, \quad r > \frac{\rho}{2}, \tag{44}$$

with

$$c_1 = \Big(\frac{q(N-1) - N(m-1)}{M(q-m+1)}\Big)^{\frac{1}{q-m+1}}.$$

Moreover, if $\frac{N(m-1)}{N-1} < q < m$,

$$u(r) \leq c_1 \frac{q-m+1}{m-q} r^{-\frac{m-q}{q-m+1}}, \quad r > \frac{\rho}{2}. \tag{45}$$

Proof. **(A)** When $M \geq 0$, every solution u of (1) satisfies the inequality

$$-\Delta_m u \geq |u|^{p-1} u, \quad \text{in } \mathbb{R}^N \setminus \overline{B}_R.$$

Then, assertion (A) follows by [3, Theorems 3.3 (iii) and 3.4 (ii)] and [35, Theorem I′].

(B) Let u be a radial positive solution of (1) in $\mathbb{R}^N \setminus \overline{B}_R$, $R \geq 0$. Thus, $u = u(r) = u(|x|)$ satisfies (1) in the radial form, that is

$$-r^{1-N}(r^{N-1}|u_r|^{m-2}u_r)_r = u^p + M|u_r|^q, \quad r > R. \tag{46}$$

It follows that

$$r \mapsto w(r) := -r^{N-1}|u_r|^{m-2}u_r$$

is strictly increasing on (R, ∞), thus it admits a limit $l \in (-\infty, \infty]$. If $l \leq 0$, then $u_r(r) > 0$ on (R, ∞). Hence $u(r) \geq u(s_0) := c > 0$ for some $s_0 > R$ and for all $r \geq s_0$, so that

$$(r^{N-1} u_r^{m-1})_r \leq -c^p r^{N-1}, \quad r \geq s_0,$$

in turn, by integration from s to r, with $s_0 < s < r$, we arrive at

$$(u_r(r))^{m-1} \leq \frac{s^{N-1}}{r^{N-1}}(u_r(s))^{m-1} - \frac{c^p}{N}\left(r - \frac{s^N}{r^{N-1}}\right),$$

which implies $u_r(r) \to -\infty$, thus $u(r) \to -\infty$ as $r \to \infty$, a contradiction. Therefore, $w(r) \to l \in (0, \infty]$ as $r \to \infty$ and there exists $r_l > R$ such that $u_r(r) < 0$ on (r_l, ∞), so that $w = r^{N-1}|u_r|^{m-1} > 0$ on (r_l, ∞). By (46), we have for $M > 0$

$$w_r \geq M r^{-\frac{(N-1)(q-m+1)}{m-1}} w^{\frac{q}{m-1}},$$

yielding

$$(w^{-\frac{q-m+1}{m-1}})_r \leq -\frac{q-m+1}{m-1} M r^{-\frac{(N-1)(q-m+1)}{m-1}}. \tag{47}$$

Integrating (47) on (s, r) with $s > r_l$, if $q = \frac{N(m-1)}{N-1}$, we obtain

$$w^{-\frac{1}{N-1}}(r) - w^{-\frac{1}{N-1}}(s) \leq -\frac{M}{N-1} \ln \frac{r}{s}, \tag{48}$$

while if $q < \frac{N(m-1)}{N-1}$, we have

$$\begin{aligned} & w^{-\frac{q-m+1}{m-1}}(r) - w^{-\frac{q-m+1}{m-1}}(s) \\ & \quad \leq -\frac{M(q-m+1)}{N(m-1) - q(N-1)}\left(r^{\frac{N(m-1)-q(N-1)}{m-1}} - s^{\frac{N(m-1)-q(N-1)}{m-1}}\right). \end{aligned} \tag{49}$$

Letting $r \to \infty$, we obtain that both right-hand sides of (48) and (49) tend to $-\infty$, with $N(m-1) - q(N-1) > 0$, namely

$$w^{-\frac{1}{N-1}}(r),\ w^{-\frac{q-m+1}{m-1}}(r) \to \infty \quad \text{as } r \to \infty.$$

This contradicts $\lim_{r\to\infty} w(r) = l > 0$, concluding the proof of (B).

(C) Let $u(x) = u(r)$ be a positive radial solution of (1) in $\mathbb{R}^N \setminus \overline{B}_R$. Arguing as in **(B)**, there exists $r_l > R$ such that $u_r(r) < 0$ on (r_l, ∞). By (46), with $M \geq 0$, we have for $r > \rho_1 := 2r_l$

$$r^{N-1}|u_r(r)|^{m-1} \geq \int_{\frac{r}{2}}^{r} \tau^{N-1}u^p(\tau)d\tau \geq \frac{r^N u^p(r)}{N}\Big(1 - \frac{1}{2^N}\Big) \geq \frac{r^N u^p(r)}{2N},$$

yielding

$$(u^{-\frac{p-m+1}{m-1}})_r \geq \frac{p-m+1}{m-1}\Big(\frac{r}{2N}\Big)^{\frac{1}{m-1}}. \tag{50}$$

Integrating (50) on $(\frac{r}{2}, r)$ we obtain

$$u(r) \leq c_0 r^{-\frac{m}{p-m+1}}, \quad r > 2\rho_1, \tag{51}$$

with

$$c_0 = (2N)^{\frac{1}{p-m+1}}(1 - 2^{-\frac{m}{m-1}})^{-\frac{m-1}{p-m+1}}\Big(\frac{m}{p-m+1}\Big)^{\frac{m-1}{p-m+1}},$$

which yields (42).

To prove (43), we set

$$v(t) = u(t^{-\frac{m-1}{N-m}})$$

with $t \in (0, \rho_1^{-\frac{N-m}{m-1}})$. By (51) we see that $v(t) \to 0$ as $t \to 0^+$. By (46), using that

$$v_t(t) = -\frac{m-1}{N-m}u_r(r)r^{\frac{N-1}{m-1}},$$

and $r = t^{-\frac{m-1}{N-m}}$, we obtain

$$\begin{aligned} v_{tt}(t) &= \frac{m-1}{(N-m)^2}r^{\frac{2(N-1)}{m-1}}\Big[(m-1)u_{rr} + \frac{N-1}{r}u_r\Big] \\ &= \frac{m-1}{(N-m)^2}r^{\frac{(3-m)(N-1)}{m-1}}|u_r|^{2-m}(r^{N-1}|u_r|^{m-2}u_r)_r \leq 0. \end{aligned}$$

Using the mean value theorem in $(0, t)$, we derive, with v_t increasing since $u_r < 0$,

$$v_t(t) \leq \frac{v(t)}{t},$$

so that, replacing the expression of v_t, we obtain the following

$$\begin{aligned} |u_r(r)| &\leq \frac{N-m}{m-1}t^{\frac{N-1}{N-m}}\frac{v(t)}{t} \\ &= \frac{N-m}{m-1}\frac{v(t)}{t^{-\frac{m-1}{N-m}}} = \frac{N-m}{m-1}\frac{u(r)}{r}, \quad r > 2\rho_1, \end{aligned}$$

so that, using (42) with $\rho = 2\rho_1$, then (43) follows immediately.

(D) Let u be a radial positive solution of (1) in $\mathbb{R}^N \setminus \overline{B}_R$, $R > 0$. Arguing as in the first part of **(B)**, but now assuming $q > \frac{N(m-1)}{N-1}$, inequality (49) is still valid, so that letting $r \to \infty$ on both sides of (49), we obtain that there exists ρ such that for all $s > \frac{\rho}{2}$

$$l^{-\frac{q-m+1}{m-1}} - w^{-\frac{q-m+1}{m-1}}(s) \le -\frac{M(q-m+1)}{q(N-1)-N(m-1)} s^{-\frac{q(N-1)-N(m-1)}{m-1}},$$

hence

$$w(s) \le \Big(\frac{q(N-1)-N(m-1)}{M(q-m+1)}\Big)^{\frac{m-1}{q-m+1}} s^{\frac{q(N-1)-N(m-1)}{q-m+1}}, \quad s > \frac{\rho}{2},$$

thus, from $w(r) = r^{N-1}|w_r(r)|^{m-1}$, we get

$$|u_r(r)| \le \Big(\frac{q(N-1)-N(m-1)}{M(q-m+1)}\Big)^{\frac{1}{q-m+1}} r^{-\frac{1}{q-m+1}}, \quad r > \frac{\rho}{2},$$

which yields (44). Then (45) follows by integrating (44) from r to ∞. □

Proof of Theorem 1.5. Let u be a positive supersolution of (1) in $\overline{B}_R^c$ for some $R > 0$. By Proposition 3.1 **(A)**, we know that when $M \ge 0$, the result is valid, even in a larger range for p. Thus, let us deal with the remaining case $M < 0$ and $N = m$ with $p > m-1$ or $N > m$ with $m-1 < p < \frac{N(m-1)}{N-m}$.

Setting $u = v^\sigma$ with $\sigma > 1$, we obtain

$$-\Delta_m v \ge (\sigma-1)(m-1)\frac{|\nabla v|^m}{v} + \sigma^{1-m} v^{m+\sigma(p-m+1)-1}$$
$$+ M\sigma^{q-m+1} v^{(\sigma-1)(q-m+1)}|\nabla v|^q,$$

and then setting $z = |\nabla v|^m$ yields

$$-\Delta_m v \ge \sigma^{1-m}\frac{\Psi(z)}{v}, \tag{52}$$

where

$$\Psi(z) = \sigma^{m-1}(\sigma-1)(m-1)z + M\sigma^q v^{(\sigma-1)(q-m+1)+1} z^{\frac{q}{m}} + v^{m+\sigma(p-m+1)}.$$

Since $q = \frac{mp}{p+1}$, it is easy to see that $\Psi(z)$ achieves its minimum at

$$z_0 = \Big(\frac{|M|p\sigma^{1-\frac{m}{p+1}}}{(\sigma-1)(m-1)(p+1)}\Big)^{p+1} v^{m+\sigma(p-m+1)},$$

and

$$\Psi(z_0) = \Big[1 - \Big(\frac{|M|}{p+1}\Big)^{p+1}\Big(\frac{\sigma p}{(\sigma-1)(m-1)}\Big)^p\Big] v^{m+\sigma(p-m+1)}. \tag{53}$$

For the case of $N > m$, we choose σ such that

$$m + \sigma(p - m + 1) - 1 = \frac{N(m-1)}{N-m},$$

namely

$$\sigma = \frac{m(m-1)}{(N-m)(p-m+1)},$$

in turn $\sigma > 1$ by $p < \frac{N(m-1)}{N-m}$ and

$$\Psi(z) \geq \Psi(z_0) = \Big[1 - \Big(\frac{|M|}{p+1}\Big)^{p+1}\Big(\frac{mp}{N(m-1) - p(N-m)}\Big)^p\Big] v^{\frac{N(m-1)}{N-m}+1}.$$

We derive that if $|M| < \mu^*(N)$, where $\mu^*(N)$ is given in (15), then inequality (52) gives

$$-\Delta_m v \geq \delta v^{\frac{N(m-1)}{N-m}} \quad \text{in } \mathbb{R}^N \setminus \overline{B}_R, \tag{54}$$

for some $\delta > 0$. Hence, Proposition 3.1 **(A)** yields the required contradiction, since no positive solutions of (54) can exist in exterior domains of $\mathbb{R}^N$.

If $N = m$, for a fixed $\sigma > 1$, if

$$|M| < (p+1)\Big(\frac{(\sigma-1)(m-1)}{\sigma p}\Big)^{\frac{p}{p+1}} := \mu_m^*, \tag{55}$$

then, from (52) and (53), we have

$$-\Delta_m v \geq \delta v^{m+\sigma(p-m+1)-1} \quad \text{in } \mathbb{R}^N \setminus \overline{B}_R$$

for some $\delta > 0$. Since $m+\sigma(p-m+1)-1 > 0$, then the result follows immediately from Proposition 3.1 **(A)**. In particular,

$$\mu_m^* \to \mu^*(m) = (p+1)\Big(\frac{m-1}{p}\Big)^{\frac{p}{p+1}} \quad \text{as } \sigma \to \infty,$$

thus, choosing σ large enough, condition $M > -\mu^*(N)$ holds also for $N = m$. □

Proof of Theorem 1.6. We perform the proof by a contradiction argument. Let us assume that there exists a positive supersolution u of (1) satisfying (16). Without loss of generality, let us assume that $u > 1$ in Ω, otherwise, we could replace Ω with the set $\{u > 1\}$. Take $v = \log u$, so that v is positive with $u > 1$. By $q = \frac{mp}{p+1}$, we obtain

$$-\Delta_m v \geq F(|\nabla v|^m), \tag{56}$$

where

$$F(X) = (m-1)X + e^{(p-m+1)v} + Me^{(q-m+1)v} X^{\frac{p}{p+1}}.$$

Obviously $F(X) > 0$ for any $X \geq 0$ when $M \geq 0$. On the other hand, in the case $M < 0$, it is not hard to see that $F(X)$ achieves its minimum at

$$X_0 = \Big(\frac{|M|p}{(m-1)(p+1)}\Big)^{p+1} e^{(p-m+1)v},$$

and

$$F(X) \geq F(X_0) = \Big[1 - \Big(\frac{p}{m-1}\Big)^p \Big(\frac{|M|}{p+1}\Big)^{p+1}\Big] e^{(p-m+1)v}$$

for all $X \geq 0$. Therefore, if

$$|M| \leq (p+1)\Big(\frac{m-1}{p}\Big)^{\frac{p}{p+1}} = \mu^*(m), \tag{57}$$

where μ^* is as in (15), then $F(X_0) \geq 0$, so that we see that v solves

$$\begin{cases} -\Delta_m v \geq 0, & \text{in } \Omega, \\ \lim_{\operatorname{dist}(x,\partial\Omega)\to 0} v(x) = \infty. \end{cases} \tag{58}$$

Clearly, when Ω is bounded, v is larger than the m-harmonic function with any boundary value $k > 0$. Letting $k \to \infty$ we derive a contradiction.

When Ω is an exterior domain, namely $\Omega = \mathbb{R}^N \setminus \overline{B}_R$, so that $\Omega^c = B_R$, we may assume $B_{R_1} \subset \Omega^c \subset B_{R_2}$ for some $R_2 > R_1 > 0$. Define

$$d = \frac{(N-1)(R_2 - R_1)}{(m-1)R_1} + 1 > 0$$

and

$$w(x) = (R_2 - |x|)^d, \quad \text{in } B_{R_2}\backslash\Omega^c.$$

We have

$$-\Delta_m w = d^{m-1}(R_2 - |x|)^{d(m-1)-m}\Big[(N-1)\frac{R_2 - |x|}{|x|} - (d-1)(m-1)\Big],$$

thus the choice of d and the decreasing monotonicity of $(R_2 - y)/y$ in (R_1, R_2) give that w is a solution of

$$\begin{cases} -\Delta_m w \leq 0, & \text{in } B_{R_2}\backslash\Omega^c, \\ w \leq (R_2 - R_1)^d, & \text{on } \partial\Omega, \\ w = 0, & \text{on } \partial B_{R_2}. \end{cases} \tag{59}$$

Hence by the weak comparison principle in [35, Lemma 2.2], we get $v \geq kw$ in $B_{R_2}\backslash\Omega^c$ for any $k > 0$. Letting $k \to \infty$ we derive a contradiction once again. □

Proof of Theorem 1.7. Let $u \in C^2(\Omega \setminus \{0\})$ be a positive solution of (1) in $\Omega \setminus \{0\}$. Let $\overline{B_1} \subset \Omega$. By [1, Theorem 1.1],

$$u^{m-1} \in \mathcal{M}^{\frac{N}{N-m}}(B_1), \quad |\nabla u|^{m-1} \in \mathcal{M}^{\frac{N}{N-1}}(B_1),$$

where $\mathcal{M}^r = L^{r,\infty}$ denotes the Marcinkiewicz space or Lorentz space of index (r, ∞). In order to fit with Serrin's formalism, we write (1) as

$$-\Delta_m u = Du^{m-1} + E|\nabla u|^{m-1},$$

where $D = u^{p-m+1}$ and $E = M|\nabla u|^{q-m+1}$. Then

$$D \in \mathcal{M}^{\frac{N(m-1)}{(N-m)(p-m+1)}}(B_1), \quad E \in \mathcal{M}^{\frac{N(m-1)}{(N-1)(q-m+1)}}(B_1).$$

Since $m-1 < p < \frac{N(m-1)}{N-m}$ and $m-1 < q < \frac{N(m-1)}{N-1}$, we have

$$\frac{N(m-1)}{(N-m)(p-m+1)} > \frac{N}{m}, \quad \frac{N(m-1)}{(N-1)(q-m+1)} > N. \tag{60}$$

Since $\mathcal{M}^r(B_1) \hookrightarrow L^{r-\delta}(B_1)$ for any $r > \delta > 0$, we infer that

$$D \in L^{\frac{N}{m}+\delta}(B_1), \quad E \in L^{N+\delta}(B_1).$$

Thus u verifies the Harnack inequality in $B_1 \setminus \{0\}$ by [34, Theorem 5]. This implies that

$$\max_{|x|=r} u(x) \le K \min_{|x|=r} u(x), \quad \forall r \in (0, 1/2], \tag{61}$$

where $K > 0$ depends on the norms of D and E.

Moreover, since $u(x) = u(r)$ on $\{x : |x| = r\}$ is m-superharmonic when $M \ge 0$, i.e., $-(r^{N-1}|u_r|^{m-2}u_r)_r \ge 0$, there exists some $k > 0$ such that

$$u(r) \le kr^{\frac{m-N}{m-1}}. \tag{62}$$

Indeed, by monotonicity decreasing of $r^{N-1}|u_r|^{m-2}u_r$, there exists $k_0 > 0$ such that

$$r^{N-1}|u_r|^{m-2}u_r \ge -k_0,$$

which yields

$$u_r \ge -k_0^{\frac{1}{m-1}} r^{\frac{1-N}{m-1}}, \quad \text{for } r \in (0, 1]. \tag{63}$$

Integrating (63) on $(r, 1)$, we obtain

$$u(1) - u(r) \ge k(1 - r^{\frac{m-N}{m-1}}),$$

where $k = k_0^{\frac{1}{m-1}} \frac{m-1}{N-m}$. It follows that

$$u(r) \le u(1) - k + kr^{\frac{m-N}{m-1}} \le k' r^{\frac{m-N}{m-1}}, \tag{64}$$

in a suitable right neighborhood of 0, with $m < N$, so that (62) holds.

Combining with (61), we arrive

$$u(x) \le Kk|x|^{\frac{m-N}{m-1}}.$$

According to (60) and (3), we see that the function $g := |u|^{p-1}u + M|\nabla u|^q$ satisfies the (ϕ, m)-scaling-growth property defined by [37, Definition 3.1], thus the estimate on the gradient is standard and follows [37, Lemma 3.3.2]. □

4 Proof of Theorem 1.9

Proof of Theorem 1.9. Let u be a positive solution of (1) . Set $v = u^{-\frac{1}{\beta}}$, with $\beta \neq 0$ to be determined later and let $z = |\nabla v|^2$. Then

$$\Delta_m v = (\beta+1)(m-1)\frac{z^{\frac{m}{2}}}{v} + \frac{|\beta|^{2-m}}{\beta} v^{\sigma} + M|\beta|^{q-m}\beta v^s z^{\frac{q}{2}}, \tag{65}$$

where

$$\begin{cases} \sigma = m - \beta(p-m+1) - 1, \\ s = (\beta+1)(m-q-1). \end{cases} \tag{66}$$

By (65), we obtain

$$\begin{aligned} z^{2-m}(\Delta_m v)^2 &= (\beta+1)^2(m-1)^2\frac{z^2}{v^2} + \beta^{2(1-m)} v^{2\sigma} z^{2-m} \\ &\quad + M^2\beta^{2(q-m+1)} v^{2s} z^{q-m+2} + 2M|\beta|^{q-2m+2} v^{\sigma+s} z^{\frac{q}{2}-m+2} \\ &\quad + 2M|\beta|^{q-m}\beta(\beta+1)(m-1) v^{s-1} z^{\frac{q-m}{2}+2} \\ &\quad + \frac{2|\beta|^{2-m}}{\beta}(\beta+1)(m-1) v^{\sigma-1} z^{2-\frac{m}{2}}, \end{aligned} \tag{67}$$

$$\begin{aligned} & z^{1-\frac{m}{2}}\langle \nabla \Delta_m v, \nabla v\rangle \\ &= -(\beta+1)(m-1)\frac{z^2}{v^2} + \frac{\sigma|\beta|^{2-m}}{\beta} v^{\sigma-1} z^{2-\frac{m}{2}} + sM|\beta|^{q-m}\beta v^{s-1} z^{\frac{q-m}{2}+2} \\ &\quad + \frac{q}{2}M|\beta|^{q-m}\beta v^s z^{\frac{q-m}{2}}\langle \nabla z, \nabla v\rangle + \frac{m}{2}(\beta+1)(m-1)\frac{\langle \nabla z, \nabla v\rangle}{v}, \end{aligned} \tag{68}$$

and

$$
\begin{aligned}
&z^{-\frac{m}{2}}\Delta_m v\langle\nabla z,\nabla v\rangle\\
&\quad=(\beta+1)(m-1)\frac{\langle\nabla z,\nabla v\rangle}{v}\\
&\qquad+\frac{|\beta|^{2-m}}{\beta}v^{\sigma}z^{-\frac{m}{2}}\langle\nabla z,\nabla v\rangle\\
&\qquad+M|\beta|^{q-m}\beta v^{s}z^{\frac{q-m}{2}}\langle\nabla z,\nabla v\rangle.
\end{aligned}
\tag{69}
$$

Substituting (67), (68) and (69) into (27), we derive

$$
\begin{aligned}
&\frac{1}{2}\mathscr{A}_v(z)+\Big(\frac{(\beta+1)(m-1)}{N}-1\Big)(\beta+1)(m-1)\frac{z^2}{v^2}\\
&\quad+\Big(\sigma+\frac{2(\beta+1)(m-1)}{N}\Big)\frac{|\beta|^{2-m}}{\beta}v^{\sigma-1}z^{2-\frac{m}{2}}\\
&\quad+\Big(s+\frac{2(\beta+1)(m-1)}{N}\Big)M|\beta|^{q-m}\beta v^{s-1}z^{\frac{q-m}{2}+2}\\
&\quad+\frac{1}{N\beta^{2(m-1)}}v^{2\sigma}z^{2-m}+\frac{M^2\beta^{2(q-m+1)}}{N}v^{2s}z^{q-m+2}\\
&\quad+\frac{2M|\beta|^{q-2m+2}}{N}v^{\sigma+s}z^{\frac{q}{2}-m+2}\\
&\quad-\frac{(N+2)(m-2)}{2N}\frac{|\beta|^{2-m}}{\beta}v^{\sigma}z^{-\frac{m}{2}}\langle\nabla z,\nabla v\rangle\\
&\quad+\Big(\frac{q}{2}-\frac{(N+2)(m-2)}{2N}\Big)M|\beta|^{q-m}\beta v^{s}z^{\frac{q-m}{2}}\langle\nabla z,\nabla v\rangle\\
&\quad+\Big(\frac{m}{2}-\frac{(N+2)(m-2)}{2N}\Big)(\beta+1)(m-1)\frac{\langle\nabla z,\nabla v\rangle}{v}\\
&\quad+\frac{(2N+m-2)(m-2)}{4N}\frac{\langle\nabla z,\nabla v\rangle^2}{z^2}\\
&\quad-\frac{m-2}{4}\frac{|\nabla z|^2}{z}\le 0,\quad\text{ on }\{z>0\}.
\end{aligned}
\tag{70}
$$

Afterwards, set $Y=v^{\lambda}z$ on $\{z>0\}$ for some parameter λ to be determined later. In order to replace $\mathscr{A}_v(z)$ by $\mathscr{A}_v(Y)$, we first calculate

$$
\begin{aligned}
-\Delta z&=\lambda v^{-\lambda-1}Y\Delta v-\lambda(\lambda+1)v^{-2\lambda-2}Y^2\\
&\quad+2\lambda v^{-\lambda-1}\langle\nabla v,\nabla Y\rangle-v^{-\lambda}\Delta Y,
\end{aligned}
\tag{71}
$$

where we have used that

$$v^{-2\lambda-2}Y|\nabla v|^2=v^{-\lambda-2}Y^2.$$

Furthermore, reading the m-Laplacian as $\Delta_m v = \operatorname{div}(z^{\frac{m}{2}-1}\nabla v)$, we get

$$\Delta v = z^{1-\frac{m}{2}}\Delta_m v - \frac{m-2}{2}\frac{\langle \nabla z, \nabla v\rangle}{z}.$$

Then using

$$\langle \nabla z, \nabla v\rangle = -\lambda v^{-2\lambda-1}Y^2 + v^{-\lambda}\langle \nabla v, \nabla Y\rangle, \tag{72}$$

and (65), we obtain

$$\begin{aligned}\Delta v = &\Big[\frac{\lambda(m-2)}{2} + (\beta+1)(m-1)\Big]v^{-\lambda-1}Y + \frac{|\beta|^{2-m}}{\beta}v^{\sigma-\lambda(1-\frac{m}{2})}Y^{1-\frac{m}{2}}\\ &+ M|\beta|^{q-m}\beta v^{s-\lambda(\frac{q-m}{2}+1)}Y^{\frac{q-m}{2}+1} - \frac{m-2}{2}\frac{\langle \nabla v, \nabla Y\rangle}{Y}.\end{aligned} \tag{73}$$

Replacing (73) into (71), we obtain

$$\begin{aligned}-\Delta z = &\lambda\Big[\lambda\Big(\frac{m}{2}-2\Big) + \beta(m-1) + m - 2\Big]v^{-2\lambda-2}Y^2\\ &+ \frac{\lambda|\beta|^{2-m}}{\beta}v^{\sigma-\lambda(2-\frac{m}{2})-1}Y^{2-\frac{m}{2}}\\ &+ \lambda M|\beta|^{q-m}\beta v^{s-\lambda(\frac{q-m}{2}+2)-1}Y^{\frac{q-m}{2}+2}\\ &+ \lambda\Big(3-\frac{m}{2}\Big)v^{-\lambda-1}\langle \nabla v, \nabla Y\rangle - v^{-\lambda}\Delta Y.\end{aligned} \tag{74}$$

Next, we focus on $\frac{\langle D^2 z\nabla v, \nabla v\rangle}{z}$. In view of (31), we have

$$\langle D^2 z\nabla v, \nabla v\rangle = \langle \nabla\langle \nabla z, \nabla v\rangle, \nabla v\rangle - \frac{1}{2}|\nabla z|^2, \tag{75}$$

and using (72) we get

$$\begin{aligned}\langle \nabla\langle \nabla z, \nabla v\rangle, \nabla v\rangle = &\lambda(2\lambda+1)v^{-3\lambda-2}Y^3 - 3\lambda v^{-2\lambda-1}Y\langle \nabla v, \nabla Y\rangle\\ &+ v^{-\lambda}\langle \nabla\langle \nabla v, \nabla Y\rangle, \nabla v\rangle,\end{aligned}$$

and, as in (32) and using $\nabla z = 2D^2 v\nabla v$, we arrive at

$$\begin{cases}\langle \nabla\langle \nabla v, \nabla Y\rangle, \nabla v\rangle = \langle D^2 Y\nabla v, \nabla v\rangle + \frac{1}{2}\langle \nabla z, \nabla Y\rangle,\\ \langle \nabla z, \nabla Y\rangle = -\lambda v^{-\lambda-1}Y\langle \nabla v, \nabla Y\rangle + v^{-\lambda}|\nabla Y|^2.\end{cases}$$

Then, by (75),

$$\langle D^2 z\nabla v, \nabla v\rangle = \lambda(2\lambda+1)v^{-3\lambda-2}Y^3 - \frac{7\lambda}{2}v^{-2\lambda-1}Y\langle\nabla v, \nabla Y\rangle + v^{-\lambda}\langle D^2 Y\nabla v, \nabla v\rangle + \frac{1}{2}v^{-2\lambda}|\nabla Y|^2 - \frac{1}{2}|\nabla z|^2.$$

Thus

$$\frac{\langle D^2 z\nabla v, \nabla v\rangle}{z} = \lambda(2\lambda+1)v^{-2\lambda-2}Y^2 - \frac{7\lambda}{2}v^{-\lambda-1}\langle\nabla v, \nabla Y\rangle + \frac{\langle D^2 Y\nabla v, \nabla v\rangle}{Y} + \frac{1}{2}v^{-\lambda}\frac{|\nabla Y|^2}{Y} - \frac{1}{2}\frac{|\nabla z|^2}{z}. \tag{76}$$

Combining (74) and (76), we derive

$$\begin{aligned} \mathscr{A}_v(z) &= -\Delta z - (m-2)\frac{\langle D^2 z\nabla v, \nabla v\rangle}{z} \\ &= v^{-\lambda}\mathscr{A}_v(Y) + \lambda\left[\lambda\left(2-\frac{3m}{2}\right) + \beta(m-1)\right]v^{-2\lambda-2}Y^2 \\ &\quad + \frac{\lambda|\beta|^{2-m}}{\beta}v^{\sigma-\lambda(2-\frac{m}{2})-1}Y^{2-\frac{m}{2}} + \lambda M|\beta|^{q-m}\beta v^{s-\lambda(\frac{q-m}{2}+2)-1}Y^{\frac{q-m}{2}+2} \\ &\quad + \lambda(3m-4)v^{-\lambda-1}\langle\nabla v, \nabla Y\rangle - \frac{m-2}{2}v^{-\lambda}\frac{|\nabla Y|^2}{Y} + \frac{m-2}{2}\frac{|\nabla z|^2}{z}. \end{aligned} \tag{77}$$

Replacing into (70) the following expressions,

$$\begin{aligned} \frac{z^2}{v^2} &= v^{-2\lambda-2}Y^2, \\ v^{\sigma-1}z^{2-\frac{m}{2}} &= v^{\sigma-\lambda(2-\frac{m}{2})-1}Y^{2-\frac{m}{2}}, \\ v^{s-1}z^{\frac{q-m}{2}+2} &= v^{s-\lambda(\frac{q-m}{2}+2)-1}Y^{\frac{q-m}{2}+2}, \\ v^{2\sigma}z^{2-m} &= v^{2\sigma-\lambda(2-m)}Y^{2-m}, \\ v^{2s}z^{q-m+2} &= v^{2s-\lambda(q-m+2)}Y^{q-m+2}, \\ v^{\sigma+s}z^{\frac{q}{2}-m+2} &= v^{\sigma+s-\lambda(\frac{q}{2}-m+2)}Y^{\frac{q}{2}-m+2}, \\ v^{\sigma}z^{-\frac{m}{2}}\langle\nabla z, \nabla v\rangle &= -\lambda v^{\sigma-\lambda(2-\frac{m}{2})-1}Y^{2-\frac{m}{2}} + v^{\sigma-\lambda(1-\frac{m}{2})}Y^{-\frac{m}{2}}\langle\nabla v, \nabla Y\rangle, \\ v^{s}z^{\frac{q-m}{2}}\langle\nabla z, \nabla v\rangle &= -\lambda v^{s-\lambda(\frac{q-m}{2}+2)-1}Y^{\frac{q-m}{2}+2} + v^{s-\lambda(\frac{q-m}{2}+1)}Y^{\frac{q-m}{2}}\langle\nabla v, \nabla Y\rangle, \\ \frac{\langle\nabla z, \nabla v\rangle}{v} &= -\lambda v^{-2\lambda-2}Y^2 + v^{-\lambda-1}\langle\nabla v, \nabla Y\rangle, \\ \frac{\langle\nabla z, \nabla v\rangle^2}{z^2} &= \lambda^2 v^{-2\lambda-2}Y^2 - 2\lambda v^{-\lambda-1}\langle\nabla v, \nabla Y\rangle + \frac{\langle\nabla v, \nabla Y\rangle^2}{Y^2}, \end{aligned}$$

we get an estimate from above for $\mathscr{A}_v(z)$, precisely

$$
\begin{aligned}
\mathscr{A}_v(z) \le \Big\{ & 2(\beta+1)(m-1)\Big[\lambda\Big(\frac{m}{2} - \frac{(N+2)(m-2)}{2N}\Big) - \Big(\frac{(\beta+1)(m-1)}{N} - 1\Big)\Big] \\
& - \lambda^2\Big(m-2+\frac{(m-2)^2}{2N}\Big)\Big\} v^{-2\lambda-2}Y^2 \\
& - \Big[2\sigma + \frac{4(\beta+1)(m-1)}{N} + \lambda\frac{(N+2)(m-2)}{N}\Big]\frac{|\beta|^{2-m}}{\beta} v^{\sigma-\lambda(2-\frac{m}{2})-1}Y^{2-\frac{m}{2}} \\
& - \Big[2s + \frac{4(\beta+1)(m-1)}{N} - \lambda\Big(q - \frac{(N+2)(m-2)}{N}\Big)\Big] \\
& \quad \times M|\beta|^{q-m}\beta v^{s-\lambda(\frac{q-m}{2}+2)-1}Y^{\frac{q-m}{2}+2} \\
& - \frac{2}{N\beta^{2(m-1)}} v^{2\sigma-\lambda(2-m)}Y^{2-m} - 2\frac{M^2\beta^{2(q-m+1)}}{N} v^{2s-\lambda(q-m+2)}Y^{q-m+2} \\
& - \frac{4M}{N}|\beta|^{q-2m+2} v^{\sigma+s-\lambda(\frac{q}{2}-m+2)}Y^{\frac{q}{2}-m+2} \\
& - \Big[2(\beta+1)(m-1)\Big(1-\frac{m-2}{N}\Big) - \lambda(m-2)\Big(2+\frac{m-2}{N}\Big)\Big] \\
& \quad \times v^{-\lambda-1}\langle\nabla v, \nabla Y\rangle \\
& + \frac{(N+2)(m-2)}{N}\frac{|\beta|^{2-m}}{\beta} v^{\sigma-\lambda(1-\frac{m}{2})}Y^{-\frac{m}{2}}\langle\nabla v, \nabla Y\rangle \\
& - \Big(q - \frac{(N+2)(m-2)}{N}\Big)M|\beta|^{q-m}\beta v^{s-\lambda(\frac{q-m}{2}+1)}Y^{\frac{q-m}{2}}\langle\nabla v, \nabla Y\rangle \\
& - \frac{(2N+m-2)(m-2)}{2N}\frac{\langle\nabla v, \nabla Y\rangle^2}{Y^2} \\
& + \frac{m-2}{2}\frac{|\nabla z|^2}{z}, \quad \text{for } z > 0.
\end{aligned} \tag{78}
$$

Replacing (77) in (78) we deduce that for some positive constant

$$c_6 = c_6(N, m, q, \beta, \lambda),$$

the following holds:

$$
\begin{aligned}
& v^{-\lambda}\mathscr{A}_v(Y) + L_1 v^{-2\lambda-2}Y^2 + L_2 v^{\sigma-\lambda(2-\frac{m}{2})-1}Y^{2-\frac{m}{2}} \\
& \quad + L_3 v^{s-\lambda(\frac{q-m}{2}+2)-1}Y^{\frac{q-m}{2}+2} + L_4 v^{2\sigma-\lambda(2-m)}Y^{2-m} \\
& \quad + L_5 v^{2s-\lambda(q-m+2)}Y^{q-m+2} + L_6 v^{\sigma+s-\lambda(\frac{q}{2}-m+2)}Y^{\frac{q}{2}-m+2} \\
& \le c_6\Big\{(v^{-\lambda-1} + v^{\sigma-\lambda(1-\frac{m}{2})}Y^{-\frac{m}{2}})|\langle\nabla v, \nabla Y\rangle| + \frac{\langle\nabla v, \nabla Y\rangle^2}{Y^2} + v^{-\lambda}\frac{|\nabla Y|^2}{Y}\Big\} \\
& \quad + L_7 v^{s-\lambda(\frac{q-m}{2}+1)}Y^{\frac{q-m}{2}}|\langle\nabla v, \nabla Y\rangle|,
\end{aligned} \tag{79}
$$

where

$$
\begin{aligned}
L_1 &= \lambda^2\Big(\frac{(m-2)^2}{2N} - \frac{m}{2}\Big) - \lambda(m-1)\Big(\beta + 2 - \frac{2(\beta+1)(m-2)}{N}\Big)\\
&\quad + 2(\beta+1)(m-1)\Big(\frac{(\beta+1)(m-1)}{N} - 1\Big),\\
L_2 &= \frac{|\beta|^{2-m}}{\beta}\Big\{\lambda\Big(m - 1 + \frac{2(m-2)}{N}\Big) + \frac{4(\beta+1)(m-1)}{N} + 2\sigma\Big\},\\
L_3 &= M|\beta|^{q-m}\beta\Big\{\lambda\Big(m - q - 1 + \frac{2(m-2)}{N}\Big) + \frac{4(\beta+1)(m-1)}{N} + 2s\Big\},\\
L_4 &= \frac{2}{N\beta^{2(m-1)}}, \quad L_5 = \frac{2M^2\beta^{2(q-m+1)}}{N}, \quad L_6 = \frac{4M|\beta|^{q-2m+2}}{N},\\
L_7 &= \Big(q + \frac{(N+2)|m-2|}{N}\Big)M|\beta|^{q-m+1}.
\end{aligned}
$$

In particular, it results that $L_4,\ L_5,\ L_6,\ L_7 > 0$.

Multiplying (79) by υ^{λ} yields

$$
\begin{aligned}
&\mathscr{A}_\upsilon(Y) + L_1\upsilon^{-\lambda-2}Y^2 + L_2\upsilon^{\sigma-\lambda(1-\frac{m}{2})-1}Y^{2-\frac{m}{2}}\\
&\quad + L_3\upsilon^{s-\lambda(\frac{q-m}{2}+1)-1}Y^{\frac{q-m}{2}+2} + L_4\upsilon^{2\sigma+\lambda(m-1)}Y^{2-m}\\
&\quad + L_5\upsilon^{2s-\lambda(q-m+1)}Y^{q-m+2} + L_6\upsilon^{\sigma+s-\lambda(\frac{q}{2}-m+1)}Y^{\frac{q}{2}-m+2}\\
&\le c_6\Big\{(\upsilon^{-1} + \upsilon^{\sigma+\frac{m\lambda}{2}}Y^{-\frac{m}{2}})|\langle\nabla\upsilon, \nabla Y\rangle| + \upsilon^{\lambda}\frac{\langle\nabla\upsilon, \nabla Y\rangle^2}{Y^2} + \frac{|\nabla Y|^2}{Y}\Big\}\\
&\quad + L_7\upsilon^{s-\lambda\frac{q-m}{2}}Y^{\frac{q-m}{2}}|\langle\nabla\upsilon, \nabla Y\rangle|.
\end{aligned}
\tag{80}
$$

Now we estimate each term in the right-hand side of (80). For any $\varepsilon > 0$, using that $|\nabla\upsilon|^2 = \upsilon^{-\lambda}Y$, we have

$$
c_6\frac{|\langle\nabla\upsilon, \nabla Y\rangle|}{\upsilon} \le \upsilon^{-\frac{\lambda}{2}-1}\sqrt{Y}|\nabla Y| \le \varepsilon\upsilon^{-\lambda-2}Y^2 + \frac{c_6^2}{4\varepsilon}\frac{|\nabla Y|^2}{Y}
$$

and

$$
\begin{aligned}
c_6\upsilon^{\sigma+\frac{m\lambda}{2}}Y^{-\frac{m}{2}}|\langle\nabla\upsilon, \nabla Y\rangle| &\le c_6\upsilon^{\sigma+\frac{\lambda}{2}(m-1)}Y^{\frac{2-m}{2}-\frac{1}{2}}|\nabla Y|\\
&\le \varepsilon\upsilon^{2\sigma+\lambda(m-1)}Y^{2-m} + \frac{c_6^2}{4\varepsilon}\frac{|\nabla Y|^2}{Y}.
\end{aligned}
$$

Similarly, L_5 being positive, we get

$$
L_7\upsilon^{s-\lambda\frac{q-m}{2}}Y^{\frac{q-m}{2}}|\langle\nabla\upsilon, \nabla Y\rangle| \le \frac{L_5}{2}\upsilon^{2s-\lambda(q-m+1)}Y^{q-m+2} + \frac{L_7^2}{2L_5}\frac{|\nabla Y|^2}{Y}.
$$

Note also that

$$
\upsilon^{|!\lambda}\frac{\langle\nabla\upsilon, \nabla Y\rangle^2}{Y^2} \le \frac{|\nabla Y|^2}{Y}.
$$

Hence, we obtain from (80) that

$$\mathscr{A}_v(Y) + H_1 + H_2 \le c_7 \frac{|\nabla Y|^2}{Y}, \tag{81}$$

where $c_7 = c_7(N, m, q, \beta, \lambda) > 0$, and

$$\begin{aligned} H_1 &:= (L_1 - \varepsilon) v^{-\lambda-2} Y^2 + L_2 v^{\sigma - \lambda(1-\frac{m}{2})-1} Y^{2-\frac{m}{2}} \\ &\quad + (L_4 - \varepsilon) v^{2\sigma + \lambda(m-1)} Y^{2-m} \\ &= v^{-\lambda-2} Y^2 [L_1 - \varepsilon + L_2 v^{\sigma + \lambda\frac{m}{2}+1} Y^{-\frac{m}{2}} + (L_4-\varepsilon) v^{2\sigma+\lambda m+2} Y^{-m}], \end{aligned} \tag{82}$$

and

$$\begin{aligned} H_2 &:= L_3 v^{s-\lambda(\frac{q-m}{2}+1)-1} Y^{\frac{q-m}{2}+2} + \frac{L_5}{2} v^{2s-\lambda(q-m+1)} Y^{q-m+2} \\ &\quad + L_6 v^{\sigma+s-\lambda(\frac{q}{2}-m+1)} Y^{\frac{q}{2}-m+2}. \end{aligned} \tag{83}$$

Now fix

$$\lambda < -2, \quad \beta > 0, \quad 2(\beta+1) + \lambda > 0.$$

By this choice, we immediately see that the positivity of H_2 is ensured, indeed the second and the third terms of H_2 are positive, with $L_5,\ L_6 > 0$; it remains to prove that $L_3 > 0$. This latter follows by the positivity of

$$L_3' = \lambda\Big(m - q - 1 + \frac{2(m-2)}{N}\Big) + \frac{4(\beta+1)(m-1)}{N} + 2s.$$

Since $s = (\beta+1)(m-q-1)$, by (66), and $m - 1 < q < \frac{(N+2)(m-1)}{N}$, by assumption, then we obtain

$$L_3' = \frac{2\lambda(m-2)}{N} + \frac{4(\beta+1)(m-1)}{N} - (q-m+1)[2(\beta+1)+\lambda] > -\frac{2\lambda}{N} > 0.$$

To estimate the term H_1, we consider the following trinomial

$$T_\varepsilon(t) = (L_4 - \varepsilon)t^2 + L_2 t + L_1 - \varepsilon.$$

If its discriminant is strictly negative, then it is possible to find γ small enough so that the discriminant of $(L_4 - \varepsilon - \gamma)t^2 + L_2 t + L_1 - \varepsilon - \gamma$ still remains strictly negative; in turn we can conclude that there exists $\gamma = \gamma(N, m, p, q, \beta, \lambda, \varepsilon) > 0$ such that $T_\varepsilon(t) \ge \gamma(t^2+1)$, and hence

$$\begin{aligned} H_1 &= v^{-\lambda-2} Y^2 T_\varepsilon(v^{\sigma+\frac{m\lambda}{2}+1} Y^{-\frac{m}{2}}) \\ &\ge \gamma(v^{-\lambda-2} Y^2 + v^{2\sigma+\lambda(m-1)} Y^{2-m}). \end{aligned}$$

Since $\lambda < -2$, we can define

$$S = \frac{2\sigma + \lambda(m-1)}{\lambda+2} = m - 1 + \frac{p-m+1}{d}.$$

where $d := -\frac{\lambda+2}{2\beta} > 0$, by the choice of λ and β, so that $S > m-1$.

Since $\frac{2S-m+2}{S+1} > 1$, we have

$$\begin{aligned} Y^{\frac{2S-m+2}{S+1}} &= (\upsilon^{-\lambda-2}Y^2)^{\frac{S}{S+1}}(\upsilon^{(\lambda+2)S}Y^{2-m})^{\frac{1}{S+1}} \\ &\le \upsilon^{-\lambda-2}Y^2 + \upsilon^{(\lambda+2)S}Y^{2-m} \\ &= \upsilon^{-\lambda-2}Y^2 + \upsilon^{2\sigma+\lambda(m-1)}Y^{2-m}. \end{aligned}$$

Therefore,

$$H_1 \ge \gamma Y^{\frac{2S-m+2}{S+1}}. \tag{84}$$

Combining with (81), (82), (83) and (84), we arrive at

$$\mathscr{A}_\upsilon(Y) + \gamma Y^{\frac{2S-m+2}{S+1}} \le c_7 \frac{|\nabla Y|^2}{Y}. \tag{85}$$

By Lemma 2.1, we obtain

$$Y(x) \le c_8(\operatorname{dist}(x,\partial\Omega))^{-\frac{2(S+1)}{S-m+1}} = c_8(\operatorname{dist}(x,\partial\Omega))^{-\frac{2\sigma+m\lambda+2}{\sigma-m+1}},$$

where $c_8 = c_8(S, m, \gamma, c_7) > 0$. It follows that

$$|\nabla u^d(x)| \le c_8'(\operatorname{dist}(x,\partial\Omega))^{-\frac{2\sigma+m\lambda+2}{2(\sigma-m+1)}} = c_8'(\operatorname{dist}(x,\partial\Omega))^{-1-\frac{md}{p-m+1}}, \tag{86}$$

where $c_8' = c_8'(m, \lambda, \beta) > 0$, which is exactly (20). The nonexistence of any positive solution of (1) in $\mathbb{R}^N$ follows consequently.

It remains to prove that the discriminant of the trinomial $T_\varepsilon(t)$ is negative. The discriminant is a polynomial of its coefficients. Hence it suffices to prove that the discriminant of $T_0(t)$ is strictly negative to deduce the same property holds for $T_\varepsilon(t)$ for small enough ε. Note that

$$T_0(t) = L_4 t^2 + L_2 t + L_1,$$

and its discriminant $D = L_2^2 - 4L_1L_4$ satisfies

$$\begin{aligned} D = |\beta|^{2(1-m)} \Big\{ & \Big[\lambda\Big(m-1+\frac{2(m-2)}{N}\Big) + \frac{4(\beta+1)(m-1)}{N} + 2\sigma\Big]^2 \\ & - \frac{8\lambda^2}{N}\Big(\frac{(m-2)^2}{2N} - \frac{m}{2}\Big) + \frac{8\lambda(m-1)}{N}\Big(\beta+2-\frac{2(\beta+1)(m-2)}{N}\Big) \\ & - \frac{16}{N}(\beta+1)(m-1)\Big(\frac{(\beta+1)(m-1)}{N} - 1\Big)\Big\}. \end{aligned}$$

Using $\beta+1 = \frac{2p+\lambda(m-1)-(\lambda+2)S}{2(p-m+1)}$ and $\sigma = \frac{(\lambda+2)S-\lambda(m-1)}{2}$, we further compute

$$\begin{aligned} & \lambda\Big(m-1+\frac{2(m-2)}{N}\Big) + \frac{4(\beta+1)(m-1)}{N} + 2\sigma \\ & = \frac{1}{N(p-m+1)} \\ & \quad \times \{4p(m-1) + 2\lambda[m-1+p(m-2)] \\ & \qquad + (\lambda+2)S[N(p-m+1) - 2(m-1)]\}, \end{aligned}$$

$$\beta+2-\frac{2(\beta+1)(m-2)}{N}=\frac{2p+\lambda(m-1)-(\lambda+2)S}{2N(p-m+1)}[N-2(m-2)]+1,$$

and

$$(\beta+1)(m-1)\Big(\frac{(\beta+1)(m-1)}{N}-1\Big)=\frac{(m-1)[2p+\lambda(m-1)-(\lambda+2)S]}{4N(p-m+1)^2}$$
$$\times\{(m-1)[2p+\lambda(m-1)-(\lambda+2)S]-2N(p-m+1)\}.$$

Thus

$$D=\frac{\beta^{2(1-m)}}{N(p-m+1)}\{(\lambda+2)^2[N(p-m+1)-4(m-1)]S^2$$
$$+4(\lambda+2)[\lambda p(m-2)+2(m-1)(p-1)]S$$
$$+4\lambda^2(p-m+1)+4(\lambda+2)^2p(m-1)\}.$$

Since $\lambda+2\neq 0$, we set $\ell=\frac{\lambda}{\lambda+2}$. By the choice of λ it follows that $\ell>1$. In turn, using also that $1/(\lambda+2)=(1-\ell)/2<0$, we arrive at

$$D=\frac{(\lambda+2)^2\beta^{2(1-m)}}{N(p-m+1)}\{[N(p-m+1)-4(m-1)]S^2-4(p-m+1)\ell S$$
$$+4(m-1)(p-1)S+4(p-m+1)\ell^2+4p(m-1)\},$$

which is equivalent to

$$D=\frac{(\lambda+2)^2\beta^{2(1-m)}}{N(p-m+1)}\Big\{4(p-m+1)\Big(\ell-\frac{S}{2}\Big)^2+D_1(S)\Big\},$$

where

$$D_1(S):=[(N-1)(p-m+1)-4(m-1)]S^2+4(m-1)(p-1)S+4p(m-1).$$

Fix $\ell=\frac{S}{2}$, hence $\beta=\frac{\lambda(m-3)+2(m-1)}{2(p-m+1)}$. As the coefficient of S^2 in $D_1(S)$ is negative if $p<\frac{(N+3)(m-1)}{N-1}$, we can choose S large enough, namely $\lambda<-2$ such that $|\lambda+2|$ is small enough, to reach $D_1(S)<0$. In particular, condition $2(\beta+1)+\lambda>0$ holds true for $\lambda\to-2^-$ being equivalent to $(\lambda+2)p-2\lambda>0$. Consequently $D<0$, concluding the proof of the positivity of T_ε.

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