

PERTURBATIONS OF EXPONENTIAL MAPS: NON-RECURRENT DYNAMICS

By

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Abstract. We study perturbations of non-recurrent parameters in the exponential family. It is shown that the set of such parameters has Lebesgue measure zero. This particularly implies that the set of escaping parameters has Lebesgue measure zero, which complements a result of Qiu from 1994. Moreover, we show that non-recurrent parameters can be approximated by hyperbolic ones.

1 Introduction and main results

In this paper we study the exponential family

$$f_\lambda(z) = \lambda e^z \quad \text{for } \lambda \in \mathbb{C} \setminus \{0\}.$$

Regarded as the simplest transcendental functions, they have attracted a lot of attention in transcendental dynamics since 1980s (see [Ber93] for an introduction to the field). Considerable effort has been put to explore this family by Devaney and his coauthors (see [Dev84, DK84] for instance), Baker and Rippon [BR84], Rempe [Rem03], etc. Currently we have a good understanding of these functions both in the dynamical and parameter spaces. Several challenging problems are, however, still open up to now. For instance, it is unknown whether the bifurcation locus of the exponential family has Lebesgue measure zero. Our paper can be viewed as a contribution to this problem.

We will focus on parameters λ for which the only singular value 0 is in the Julia set and the ω -limit set of 0 does not contain itself. Such parameters are called **non-recurrent**. We first show that non-recurrent dynamics are rare in the Lebesgue sense.

Theorem 1. *The set of non-recurrent parameters in the exponential family has zero Lebesgue measure.*

We make several remarks on this result. Let $J(f_\lambda)$ be the Julia set of f_λ .

Remark 1.

- Badeńska considered parameters in the exponential family with bounded post-singular sets (called **post-singularly bounded parameters**) and proved a similar result [Bad11, Theorem 1] for such parameters. This implies directly that such parameters are non-recurrent in the above sense. So our theorem gives a generalization of her result.
- A parameter λ is **escaping** if $f_\lambda^n(0) \rightarrow \infty$ as $n \rightarrow \infty$. Qiu proved in 1994 that the set of escaping parameters has Hausdorff dimension two [Qiu94]. It was, however, unknown since then whether this set has Lebesgue measure zero (which is the motivation of the present paper). Our result confirms this and thus complements his result, since escaping parameters are also non-recurrent. However, escaping parameters may not be rare in the above sense for general families of transcendental functions. In [Qiu94] it is shown that the set of escaping parameters in the sine family $\{\lambda \sin(z) : \lambda \in \mathbb{C} \setminus \{0\}\}$ has positive Lebesgue measure.
- Non-recurrent parameters are considerably more general due to the non-compactness of the phase space. In the above settings, the singular value either has a bounded orbit or tends to ∞ under iterates. Non-recurrent parameters, however, can be post-singularly unbounded without being escaping. In other words, the singular orbit could possibly oscillate between some compact set and ∞ . So in this sense our result is more general.
- For non-recurrent parameters, we note the following difference between escaping and non-escaping ones. For non-escaping non-recurrent maps, the singular value belongs to the radial Julia set (and hence can go from small scales to large scales by univalent iterates); see [RvS11, Section 3] and [Rem09] for a discussion and related results on this set, while for escaping parameters, the singular value does not lie in this set. (We are grateful to Lasse Rempe for making this comment.)

Remark 2. In the study of exponential dynamics, one is often led to a comparison with the (complex) quadratic family $z \mapsto z^2 + c$ which gives rise to the famous Mandelbrot set. For quadratic polynomials, non-recurrence is equivalent to requiring the post-critical set to be a hyperbolic set. However, for exponential maps, that the post-singular set is a hyperbolic set means that the map is post-singularly bounded. Thus, the notion of non-recurrence is weaker in the exponential setting.

A parameter λ is **hyperbolic** if f_λ has an attracting cycle. One of the main problems in the dynamics of exponential maps is whether hyperbolic exponential maps are dense in the parameter space (**density of hyperbolicity**). This is

still widely open. Equivalently, one asks if every non-hyperbolic map can be approximated by hyperbolic maps. As an application of the argument used in the proof of Theorem 1, we show that this is true for all non-recurrent parameters.

Theorem 2. *Every non-recurrent exponential map can be approximated by hyperbolic maps.*

The result was partially known for some non-recurrent parameters; see, for instance, Devaney [Dev85], Zhou and Li [ZL89] and Ye [Ye94]. Indeed, rigidity implies that such parameters can be approximated by hyperbolic ones. For post-singularly finite exponential maps, rigidity was known; see Benini [Ben11]. For escaping parameters this was proved by Rempe in [Rem06].

Remark 3. Dobbs proved a result for post-singularly bounded parameters [Dob15, Main Theorem], saying that such parameters are Lebesgue density points of the set of hyperbolic maps. It is plausible that this is also true for non-recurrent parameters.

We have mainly focused on non-recurrent exponential maps and their relevant properties in this paper. The parameter space of the exponential family has a rich structure and has attracted a lot of attention. Without going into any further details, we refer to [EL92, DFJ02, Qiu94, UZ07, RS09, BBS08, Ben15, Ben11, LPS16, Ber17] and references therein.

2 Preliminaries

In this section we give some preliminary results for non-recurrent exponential maps. These include uniform expansion on the post-singular set and the existence of a holomorphic motion along this orbit.

First we give some notations which will be used throughout this paper. We will use $D(a, r)$ for a Euclidean disk of radius r centered at a . Let $\text{dist}(z, A)$ be the Euclidean distance of a point z to a set A . The term $Df_\lambda^n(z)$ will always mean the derivative of f_λ^n at the point z (i.e., the phase derivative). Sometimes we also use $(f_\lambda^n)'(z)$ as the same meaning for $Df_\lambda^n(z)$. On the other hand, $\partial_\lambda f_\lambda(z)$ will mean the derivative in λ (i.e., the parameter derivative). The complex plane is always denoted by \mathbb{C} . Moreover, $\text{meas} A$ is the two-dimensional Lebesgue measure of a set $A \subset \mathbb{C}$ and $\text{dens}(A, B)$ is the density of A in B , i.e., $\text{dens}(A, B) = \text{meas}(A \cap B) / \text{meas}(B)$. By $a \sim b$ we mean that there exists a constant $C > 0$ such that $\frac{1}{C}b \leq a \leq Cb$.

Let f_λ be an exponential map. The **post-singular set** of f_λ is defined as

$$\mathcal{P}(f_\lambda) := \overline{\bigcup_{n \geq 0} f_\lambda^n(0)}.$$

Recall that λ is called **non-recurrent** if $0 \in \mathcal{J}(f_\lambda)$ and $0 \notin \omega(0)$. In other words, there exists $\Delta > 0$ such that

$$(2.1) \quad \mathcal{P}(f_\lambda) \cap D(0, \Delta) = \{0\}.$$

Note that non-recurrent exponential maps cannot have attracting or parabolic cycles. That Siegel disks do not exist was proved by Rempe and van Strien [RvS11, Corollary 2.10]. So we see that non-recurrent exponential maps have empty Fatou sets.

Definition 2.1. f_λ is said to be Δ -**non-recurrent** if there exists $\Delta > 0$ such that (2.1) holds. Equivalently, we say that such a parameter λ is Δ -non-recurrent.

One of the ingredients in the proof of our theorem is the expansion along the post-singular set of non-recurrent maps. The following result was proved by Benini [Ben15, Corollary A].

Lemma 2.1. *Let λ be a non-recurrent parameter. Then there exist $\tilde{N} \in \mathbb{N}$ and $\tilde{\gamma} > 1$ such that for any $k \geq \tilde{N}$ and for any $z \in \mathcal{P}(f_\lambda)$ we have*

$$|Df_\lambda^k(z)| > \tilde{\gamma}.$$

Let λ be a Δ -non-recurrent parameter. Let $\tilde{N}, \tilde{\gamma}$ be as in the above lemma. For any integer n there exist k and $j \in \mathbb{N}$ such that $n = k\tilde{N} + j$, where $k \geq 0$ and $0 \leq j < \tilde{N}$. So for any $z \in \mathcal{P}(f_\lambda)$ we see from the above lemma that

$$(2.2) \quad \begin{aligned} |Df_\lambda^n(z)| &= |Df_\lambda^{k\tilde{N}+j}(z)| = |Df_\lambda^{k\tilde{N}}(f_\lambda^j(z))| |Df_\lambda^j(z)| \\ &= \prod_{m=0}^{k-1} |Df_\lambda^{\tilde{N}}(f_\lambda^{m\tilde{N}+j}(z))| |Df_\lambda^j(z)| \\ &\geq \tilde{\gamma}^k |Df_\lambda^j(z)| = \tilde{\gamma}^k \prod_{i=1}^j |f_\lambda^i(z)|. \end{aligned}$$

Since f_λ is Δ -non-recurrent, we have $\prod_{i=1}^j |f_\lambda^i(z)| \geq \Delta^j$. Put

$$\tilde{C} = \min\left\{1, \left(\frac{\Delta}{\tilde{\gamma}}\right)^{\tilde{N}}\right\}.$$

We obtain from (2.2) that

$$(2.3) \quad |Df_\lambda^n(z)| \geq \tilde{C}\tilde{\gamma}^k = \tilde{C}\tilde{\gamma}^{(n-j)/\tilde{N}} \geq \tilde{C}\tilde{\gamma}^{(n-\tilde{N})/\tilde{N}} = \frac{\tilde{C}}{\tilde{\gamma}} \tilde{\gamma}^{n/\tilde{N}} =: C_1 \gamma_1^n,$$

where $\gamma_1 > 1$ and $C_1 > 0$.

More generally, a parameter λ is said to be **summable**, if $0 \in \mathcal{J}(f_\lambda)$ and

$$\sum_{n=0}^{\infty} \frac{1}{|Df_\lambda^n(0)|} < \infty.$$

See [UZ07]. Therefore, for non-recurrent parameter λ , by taking $z = 0$ in (2.3) we see that

$$\sum_{n=0}^{\infty} \frac{1}{|Df_\lambda^n(0)|} \leq \sum_{n=0}^{\infty} \frac{1}{C_1 \gamma_1^n} < \infty.$$

In other words, we have

Lemma 2.2. *Non-recurrent parameters are summable.*

Relationship (2.3) tells that f_λ is uniformly expanding along $\mathcal{P}(f_\lambda)$. (We say that a map f is **uniformly expanding** on a closed set $E \subset \mathbb{C}$ (which could be unbounded), if there exist $C > 0$ and $\gamma > 1$ such that $|Df^n(z)| \geq C\gamma^n$ for all n and for all $z \in E$. If E is also bounded, then such a set is often called a hyperbolic set.)

Let λ_0 be non-recurrent. Then there exists $\eta > 0$ such that $|Df_{\lambda_0}^n(z)| > C_1 \gamma_1^n$ for all $z \in \mathcal{N}_0$, where $\mathcal{N}_0 := D(\mathcal{P}(f_{\lambda_0}), 100\eta)$ is a neighborhood of $\mathcal{P}(f_{\lambda_0})$. By continuity, parameters close to λ_0 also enjoy this property (but with possibly a slightly smaller exponent). So we have the following.

Lemma 2.3. *Let λ_0 be a non-recurrent parameter. Then there exist r_0 sufficiently small and constants $C > 0$, $\gamma > 1$ such that for all $r \leq r_0$ and for all $\lambda \in D(\lambda_0, r)$, whenever $f_\lambda^j(z) \in \mathcal{N} := D(\mathcal{P}(f_{\lambda_0}), 50\eta)$ for all $j \leq n$ we have*

$$|Df_\lambda^n(z)| \geq C\gamma^n.$$

The uniform expansion also implies the existence of a holomorphic motion over $\mathcal{P}(f_{\lambda_0})$ in a neighborhood of the parameter λ_0 .

Lemma 2.4 (Holomorphic motion). *Let λ_0 be non-recurrent. Then for sufficiently small $r > 0$ there exists a holomorphic motion*

$$h : D(\lambda_0, r) \times \mathcal{P}(f_{\lambda_0}) \rightarrow \mathbb{C}$$

such that h_{λ_0} is identity, h is holomorphic in λ and injective in z . For all $\lambda \in D(\lambda_0, r)$, we have

$$h_\lambda \circ f_{\lambda_0} = f_\lambda \circ h_\lambda$$

for $z \in \mathcal{P}(f_{\lambda_0})$. Moreover, there exists a constant $\alpha > 0$ such that

$$|h_\lambda(z) - z| \leq \alpha |\lambda - \lambda_0|$$

for any $z \in \mathcal{P}(f_{\lambda_0})$ and $\lambda \in D(\lambda_0, r)$.

Proof. Let \mathcal{N}_0 be the neighborhood of $\mathcal{P}(f_{\lambda_0})$ as above. Then by the above Lemma 2.3, we see that f_λ is uniformly expanding in \mathcal{N} , where $\lambda \in D(\lambda_0, r)$. By taking $r_1 > 0$ sufficiently small such that $D(z, r_1) \subset \mathcal{N}$ for all $z \in \mathcal{P}(f_{\lambda_0})$ and using the expansion of f_{λ_0} in \mathcal{N} , we can choose a sufficiently small r such that the following holds: For all $\lambda \in D(\lambda_0, r)$ and all $z \in \mathcal{N}$ we have

$$D(f_{\lambda_0}(z), r_1) \subset f_\lambda(D(z, r_1)).$$

This means that the pullback $f_\lambda^{-n}(D(z, r_1))$ of $D(z, r_1)$ under f_λ^n shrinks exponentially, where the inverse branch of f_λ is taken suitably. Therefore, we can find for each $z \in \mathcal{P}(f_{\lambda_0})$ a unique point z_λ such that $f_\lambda^n(z_\lambda) \in D(f_{\lambda_0}^n(z), r_1)$ for all n . Put $h_\lambda(z) := z_\lambda$. It follows from the construction of h_λ that

$$h_\lambda \circ f_{\lambda_0} = f_\lambda \circ h_\lambda$$

for $z \in \mathcal{P}(f_{\lambda_0})$. Moreover, h_λ is injective in z . That h is holomorphic in λ can be seen by looking at $z_\lambda^n := f_\lambda^{-n}(f_{\lambda_0}^n(z))$. By the above argument z_λ^n converges to z_λ uniformly as $n \rightarrow \infty$. So h is indeed holomorphic in λ . To sum up, $h(\lambda, z)$ thus defined is a holomorphic motion.

The last statement of the lemma follows from the above construction. Observe first that

$$(2.4) \quad |w_\lambda^1 - w| = |\log(\lambda/\lambda_0)|$$

holds for any $w \in \mathcal{P}(f_{\lambda_0})$. Let z and z_λ be as above. So we have

$$(2.5) \quad \begin{aligned} |h_\lambda(z) - z| &= |z_\lambda - z| \leq |z_\lambda^1 - z| + \sum_{n=1}^{\infty} |z_\lambda^{n+1} - z_\lambda^n| \\ &\leq |z_\lambda^1 - z| + \sum_{n=1}^{\infty} \frac{1}{|Df_\lambda^n(z)|} |f_{\lambda_0}^n(z) - f_\lambda^{-1}(f_{\lambda_0}^{n+1}(z))| \\ &\leq \left(1 + \sum_{n=1}^{\infty} \frac{1}{C\gamma^n}\right) \left|\log \frac{\lambda}{\lambda_0}\right| \\ &\leq \alpha |\lambda - \lambda_0|. \end{aligned}$$

In the second inequality of the above estimate, we have taken the inverse branch of f_λ sending $f_{\lambda_0}^n(z)$ to $f_{\lambda_0}^{n+1}(z)$. In the third inequality, we used Lemma 2.3 and (2.4). Since $r > 0$ is taken sufficiently small, so we have the local expansion: $|\log \lambda/\lambda_0| \sim |\lambda - \lambda_0|$. This, together with the fact that $\gamma > 1$, gives the constant α . \square

The holomorphic motion and the expansion together imply that any two points in \mathcal{N} will repel each other for all maps in $D(\lambda_0, r)$. Our analysis later tells that more information can be drawn during this process. For this purpose, we will specify a number $\delta > 0$ such that the set of all points z satisfying $\text{dist}(z, h_\lambda(\mathcal{P}(f_{\lambda_0}))) < 10\delta$ is contained in \mathcal{N} for all $\lambda \in D(\lambda_0, r)$. For instance, one can choose $\delta < 2\eta$ so that Lemmas 2.3 and 2.4 hold.

Using the above lemma, we have the following result.

Lemma 2.5. *Let λ_0 be non-recurrent. Then there exist $r > 0$ sufficiently small, constants C and $\gamma > 1$ such that for any $\lambda \in D(\lambda_0, r)$ we have*

$$|f_\lambda^n(z) - f_\lambda^n(w)| \geq C\gamma^n|z - w|,$$

if $f_\lambda^j(z), f_\lambda^j(w) \in \mathcal{N}$ and $|f_\lambda^j(z) - f_\lambda^j(w)| \leq \delta$ for all $j \leq n$.

3 Phase-parameter relation and distortions

In this section we prove some distortion results by using the expansion property in the previous section. This, together with a transversality result of Urbański and Zdunik [UZ07], gives us sufficiently good control over the distortion in the parameter space.

3.1 Distortions. Define

$$(3.1) \quad \zeta_n(\lambda) := f_\lambda^n(0).$$

Now we assume that λ_0 is a Δ_0 -non-recurrent parameter for some $\Delta_0 > 0$. Let $D(\lambda_0, r)$ be a disk of radius r around λ_0 for some sufficiently small $r > 0$. For simplicity we put $\Lambda_0 := \mathcal{P}(f_{\lambda_0})$. We also define

$$\mu_n(\lambda) = h_\lambda(\zeta_n(\lambda_0)),$$

where h_λ is the holomorphic motion in Lemma 2.4.

We shall use the following simple lemma.

Lemma 3.1. *Let $a_k \in \mathbb{C}$ for $1 \leq k \leq n$. Then*

$$\left| \prod_{k=1}^n (1 + a_k) - 1 \right| \leq \exp\left(\sum_{k=1}^n |a_k|\right) - 1.$$

Lemma 3.2. *Let λ_0 be non-recurrent. For any $\varepsilon > 0$ there exist $\delta > 0$ and $r > 0$ sufficiently small such that for any $\lambda \in D(\lambda_0, r)$, if $\zeta_j(\lambda), \mu_j(\lambda) \in \mathcal{N}$ and $|\zeta_j(\lambda) - \mu_j(\lambda)| \leq \delta$ for all $0 \leq j \leq n$, then*

$$\left| \frac{(f_\lambda^n)'(\mu_0(\lambda))}{(f_\lambda^n)'(\zeta_0(\lambda))} - 1 \right| < \varepsilon.$$

Proof. By the chain rule,

$$(3.2) \quad \left| \frac{(f_\lambda^n)'(\mu_0(\lambda))}{(f_\lambda^n)'(\xi_0(\lambda))} - 1 \right| = \left| \prod_{j=0}^{n-1} \frac{f_\lambda'(f_\lambda^j(\mu_0(\lambda)))}{f_\lambda'(f_\lambda^j(\xi_0(\lambda)))} - 1 \right| \\ = \left| \prod_{j=0}^{n-1} \left(1 + \frac{f_\lambda'(f_\lambda^j(\mu_0(\lambda)))}{f_\lambda'(f_\lambda^j(\xi_0(\lambda)))} - 1 \right) - 1 \right|.$$

We put

$$v_j := \frac{f_\lambda'(f_\lambda^j(\mu_0(\lambda)))}{f_\lambda'(f_\lambda^j(\xi_0(\lambda)))} - 1 = \frac{f_\lambda'(\mu_j(\lambda))}{f_\lambda'(\xi_j(\lambda))} - 1.$$

By Lemma 3.1, to show that (3.2) can be made arbitrarily small, it suffices to prove that $\sum_{j=0}^{n-1} |v_j|$ is sufficiently small. Note first that

$$f_\lambda'(\xi_j(\lambda)) = f_\lambda(\xi_j(\lambda)) = \xi_{j+1}(\lambda) \quad \text{and} \quad f_\lambda'(\mu_j(\lambda)) = f_\lambda(\mu_j(\lambda)) = \mu_{j+1}(\lambda).$$

So,

$$\sum_{j=0}^{n-1} |v_j| = \sum_{j=0}^{n-1} \left| \frac{f_\lambda'(\mu_j(\lambda))}{f_\lambda'(\xi_j(\lambda))} - 1 \right| = \sum_{j=0}^{n-1} \left| \frac{\mu_{j+1}(\lambda)}{\xi_{j+1}(\lambda)} - 1 \right|.$$

By the Δ_0 -non-recurrence of f_{λ_0} and Lemma 2.3, we have

$$\sum_{j=0}^{n-1} |v_j| \leq \frac{2}{\Delta_0} \sum_{j=0}^{n-1} |\mu_{j+1}(\lambda) - \xi_{j+1}(\lambda)| \\ \leq \frac{2}{\Delta_0} \sum_{j=0}^{n-1} \frac{1}{C} \gamma^{j+1-n} |\mu_n(\lambda) - \xi_n(\lambda)| \\ \leq \frac{2}{\Delta_0} C_1(\gamma) \delta.$$

Here $C_1(\gamma) > 0$ is a constant depending on γ . Now since δ can be made arbitrarily small we see that (3.2) is also small. This completes the proof of the lemma. \square

Lemma 3.3. *Let λ_0 be non-recurrent. For any $\varepsilon > 0$ there exist $\delta > 0$ and $r > 0$ sufficiently small such that for any $\lambda_1, \lambda_2 \in D(\lambda_0, r)$, if $\xi_j(\lambda_i), \mu_j(\lambda_i) \in \mathcal{N}$ and $|\xi_j(\lambda_i) - \mu_j(\lambda_i)| \leq \delta$ for $i = 1, 2$ and all $0 \leq j \leq n$, then*

$$\left| \frac{(f_{\lambda_1}^n)'(0)}{(f_{\lambda_2}^n)'(0)} - 1 \right| < \varepsilon.$$

Proof. The proof is similar to the proof of the above lemma. First we note that by Lemma 3.2 it suffices to show the following holds:

$$(3.3) \quad \left| \frac{Df_{\lambda_1}^n(\mu_0(\lambda_1))}{Df_{\lambda_2}^n(\mu_0(\lambda_2))} - 1 \right| = \left| \prod_{j=0}^{n-1} \frac{f_{\lambda_1}'(\mu_j(\lambda_1))}{f_{\lambda_2}'(\mu_j(\lambda_2))} - 1 \right| = \left| \prod_{j=0}^{n-1} \frac{\mu_{j+1}(\lambda_1)}{\mu_{j+1}(\lambda_2)} - 1 \right| < \varepsilon.$$

With

$$w_j = \frac{\mu_{j+1}(\lambda_1)}{\mu_{j+1}(\lambda_2)} - 1,$$

it is enough to show, by Lemma 3.1, that $\sum_{j=0}^{n-1} |w_j|$ is sufficiently small. By the Δ_0 -non-recurrence of f_{λ_0} and Lemma 2.4, we have

$$\begin{aligned} |w_j| &\leq \frac{2}{\Delta_0} |\mu_{j+1}(\lambda_1) - \mu_{j+1}(\lambda_2)| \\ &= \frac{2}{\Delta_0} |h_{\lambda_1}(\zeta_{j+1}(\lambda_0)) - h_{\lambda_2}(\zeta_{j+1}(\lambda_0))| \\ &\leq \frac{2}{\Delta_0} (|h_{\lambda_1}(\zeta_{j+1}(\lambda_0)) - h_{\lambda_0}(\zeta_{j+1}(\lambda_0))| + |h_{\lambda_2}(\zeta_{j+1}(\lambda_0)) - h_{\lambda_0}(\zeta_{j+1}(\lambda_0))|) \\ &\leq \frac{2\alpha}{\Delta_0} (|\lambda_1 - \lambda_0| + |\lambda_2 - \lambda_0|). \end{aligned}$$

So we have

$$(3.4) \quad \sum_{j=0}^{n-1} |w_j| \leq \frac{2\alpha}{\Delta_0} n (|\lambda_1 - \lambda_0| + |\lambda_2 - \lambda_0|).$$

Now it follows from Lemma 2.5, that for $\lambda \in D(\lambda_0, r)$ we also have

$$(3.5) \quad \begin{aligned} \delta &\geq |\zeta_n(\lambda) - \mu_n(\lambda)| = |f_\lambda^n(\zeta_0(\lambda)) - f_\lambda^n(\mu_0(\lambda))| \\ &\geq C\gamma^n |\zeta_0(\lambda) - \mu_0(\lambda)| \\ &= C\gamma^n |h_\lambda(0)|, \end{aligned}$$

where h_λ is the holomorphic motion and thus has the local expansion

$$(3.6) \quad h_\lambda(0) = a_K (\lambda - \lambda_0)^K + \mathcal{O}((\lambda - \lambda_0)^{K+1})$$

for some constant a_K and $K \in \mathbb{N}$. Combining (3.5) with (3.6) we get an estimate for the time n : For some constant $C' > 0$ depending only on δ but not on n ,

$$(3.7) \quad n \log \gamma \leq -C' \log |\lambda - \lambda_0|.$$

Putting (3.7) into (3.4), we see that

$$\sum_{j=0}^{n-1} |w_j| = \mathcal{O}(|\lambda_1 - \lambda_0| \log |\lambda_1 - \lambda_0| + |\lambda_2 - \lambda_0| \log |\lambda_2 - \lambda_0|),$$

which can be made sufficiently close to 0 since r can be taken sufficiently small and $|\lambda_i - \lambda_0| < r$ for $i = 1, 2$. Thus (3.3) holds. This finishes the proof of the lemma. \square

3.2 Phase-parameter relations: transversality. In the following we need to compare the phase and parameter derivatives. We start with some calculations. Recall from (3.1) that

$$\xi_n(\lambda) = f_\lambda^n(0).$$

Now we write

$$\begin{aligned}\xi_n(\lambda) &= f(\lambda, \xi_{n-1}(\lambda)), \\ \alpha_j(\lambda) &= f'_\lambda(\xi_j(\lambda))\end{aligned}$$

and

$$\rho_j(\lambda) = \partial_\lambda f(\lambda, \xi_j(\lambda)).$$

Then for all $n \geq 0$, we have, by induction,

$$\begin{aligned}\xi'_{n+1}(\lambda) &= f'_\lambda(\xi_n(\lambda)) \cdot \xi'_n(\lambda) + \partial_\lambda f(\lambda, \xi_n(\lambda)) \\ &= \alpha_n(\lambda) \cdot \xi'_n(\lambda) + \rho_n(\lambda) \\ &= \alpha_n(\lambda)(\alpha_{n-1}(\lambda) \cdot \xi'_{n-1}(\lambda) + \rho_{n-1}(\lambda)) + \rho_n(\lambda) \\ &= \prod_{j=1}^n \alpha_j(\lambda) \cdot \xi'_1(\lambda) + \sum_{j=1}^{n-1} \left(\prod_{k=j+1}^n \alpha_k(\lambda) \right) \rho_j(\lambda) + \rho_n(\lambda) \\ &= \prod_{j=1}^n \alpha_j(\lambda) \left(\xi'_1(\lambda) + \sum_{j=1}^n \frac{1}{\lambda^j \prod_{k=0}^{j-1} e^{\xi_k(\lambda)}} \right) \\ &= \prod_{j=1}^n \alpha_j(\lambda) \left(\xi'_1(\lambda) + \sum_{j=1}^n \prod_{k=1}^j \frac{1}{\xi_k(\lambda)} \right).\end{aligned}$$

Since $\xi'_1(\lambda) = 1$ and

$$\prod_{j=1}^n \alpha_j(\lambda) = \prod_{j=1}^n f'_\lambda(\xi_j(\lambda)) = (f_\lambda^n)'(\lambda),$$

we obtain immediately that

$$\begin{aligned}\xi'_{n+1}(\lambda) &= (f_\lambda^n)'(\lambda) \left(1 + \sum_{j=1}^n \prod_{k=1}^j \frac{1}{\xi_k(\lambda)} \right) \\ (3.8) \quad &= \frac{1}{\lambda} (f_\lambda^{n+1})'(\lambda) \left(1 + \sum_{j=1}^n \frac{1}{(f_\lambda^j)'(\lambda)} \right) \\ &= \frac{1}{\lambda} (f_\lambda^{n+1})'(\lambda) \sum_{j=0}^n \frac{1}{(f_\lambda^j)'(\lambda)}.\end{aligned}$$

We used that $(f_\lambda^j)'(0) = \prod_{k=1}^j f_\lambda^k(0) = \prod_{k=1}^j \xi_k(\lambda)$ and that f_λ^0 is the identity map. The sum appearing in the last line is actually a truncated term (after taking absolute value) occurring in the summability condition. Therefore, for summable λ , the limit

$$L_\lambda := \lim_{n \rightarrow \infty} \sum_{j=0}^n \frac{1}{|(f_\lambda^j)'(0)|} < \infty$$

exists. Actually, one can say more for non-recurrent parameters.

Lemma 3.4. *Let λ be a non-recurrent parameter. Then L_λ exists and is not equal to 0 or ∞ .*

Proof. By Lemma 2.2, λ is summable. It follows then from the work of Urbański and Zdunik [UZ07, Sections 4 and 5] that $L_\lambda \neq 0, \infty$ for the non-recurrent parameter λ . \square

By using Lemma 3.4 and (3.8), we have a relation between phase and parameter derivatives in the following sense: For non-recurrent λ_0 , there exists $L_{\lambda_0} (\neq 0, \infty)$ such that

$$\lim_{n \rightarrow \infty} \frac{\xi'_n(\lambda_0)}{(f_{\lambda_0}^n)'(0)} = \frac{L_{\lambda_0}}{\lambda_0}.$$

This relation is persistent for parameters λ which are sufficiently close to λ_0 as long as expansion on $\mathcal{P}(f_{\lambda_0})$ is ensured and the singular orbit of f_λ stays close to $\mathcal{P}(f_{\lambda_0})$. More precisely, we have:

Lemma 3.5. *Let λ_0 be non-recurrent. Then for any $q \in (0, 1)$ there exist $N_1 > 0$ and $r > 0$ sufficiently small such that the following holds. For any $\lambda \in D(\lambda_0, r)$, if $|\xi_j(\lambda) - \xi_j(\lambda_0)| \leq \delta$ for some $\delta = \delta(r)$ and for all $j \leq n$ with $n \geq N_1$, we have*

$$\left| \frac{\xi'_n(\lambda)}{(f_\lambda^n)'(0)} - \frac{L_{\lambda_0}}{\lambda_0} \right| \leq q \left| \frac{L_{\lambda_0}}{\lambda_0} \right|.$$

Proof. By (3.8) we have

$$\frac{\xi'_n(\lambda)}{(f_\lambda^n)'(0)} = \frac{1}{\lambda} \sum_{j=0}^{n-1} \frac{1}{|(f_\lambda^j)'(0)|}.$$

By taking N_1 sufficiently large so that $N_1 > \tilde{N}$ and using (2.3) and Lemma 2.3 we can have

$$\frac{1}{|\lambda|} \sum_{j=N_1+1}^{\infty} \frac{1}{|(f_\lambda^j)'(0)|} \leq \frac{q}{2} \left| \frac{L_{\lambda_0}}{\lambda_0} \right|.$$

On the other hand, by continuity one can find some $r > 0$ such that if $\lambda \in D(\lambda_0, r)$,

$$\left| \frac{1}{\lambda} \sum_{j=0}^{N_1} \frac{1}{(f_\lambda^j)'(0)} - \frac{L_{\lambda_0}}{\lambda_0} \right| \leq \frac{q}{2} \left| \frac{L_{\lambda_0}}{\lambda_0} \right|.$$

Combining all the above we have

$$\begin{aligned} \left| \frac{\xi_n'(\lambda)}{(f_\lambda^n)'(0)} - \frac{L_{\lambda_0}}{\lambda_0} \right| &\leq \left| \frac{1}{\lambda} \sum_{j=0}^{N_1} \frac{1}{(f_\lambda^j)'(0)} - \frac{L_{\lambda_0}}{\lambda_0} \right| + \frac{1}{|\lambda|} \sum_{j=N_1+1}^{\infty} \frac{1}{|(f_\lambda^j)'(0)|} \\ &\leq q \left| \frac{L_{\lambda_0}}{\lambda_0} \right|. \quad \square \end{aligned}$$

Lemma 3.5 and Lemma 3.3 together imply that the function ξ_n is an almost affine map.

Lemma 3.6 (Strong distortion). *Let λ_0 be non-recurrent. For any $\varepsilon > 0$ there exist $\delta > 0$ and $r > 0$ sufficiently small such that for any $\lambda_1, \lambda_2 \in D(\lambda_0, r)$, if $|\xi_j(\lambda_i) - \mu_j(\lambda_i)| \leq \delta$ for $i = 1, 2$ and all $0 \leq j \leq n$ with $n \geq N_1$, then*

$$\left| \frac{\xi_j'(\lambda_1)}{\xi_j'(\lambda_2)} - 1 \right| \leq \varepsilon.$$

With this lemma we get that

$$(3.9) \quad |\xi_n(\lambda_1) - \xi_n(\lambda_2)| \sim |Df_{\lambda_0}^{n-1}(0)| |\lambda_1 - \lambda_2|$$

as long as $\xi_j(\lambda_1)$ and $\xi_j(\lambda_2)$ stay close in a neighborhood of the singular orbit of f_{λ_0} for all $j \leq n$. As $|Df_{\lambda_0}^n(0)|$ grows (at least) exponentially we obtain immediately the following result.

Lemma 3.7 (Large scale). *Let λ_0 be non-recurrent. There exists a number $S > 0$ such that for all $r > 0$ sufficiently small, there is an integer n so that $D(\xi_n(\lambda_0), S/4) \subset \xi_n(D(\lambda_0, r)) \subset \mathcal{N}$ and has diameter at least S .*

Proof. Put $S = \delta/2$ with δ in the above lemma. If $\xi_n(D(\lambda_0, r))$ never reached the large scale, i.e.,

$$\text{diam } \xi_n(D(\lambda_0, r)) < S$$

for all $n \geq 0$, then we have (3.9) satisfied for all n . So we see that, for some constant $C > 0$,

$$S > \text{diam } \xi_n(D(\lambda_0, r)) \geq C\gamma^{n-1}r.$$

But this is impossible since $\gamma > 1$ and n can be taken sufficiently large. Hence, $\xi_n(D(\lambda_0, r)) \supset D(\xi_n(\lambda_0), S/4)$ follows directly from Lemma 3.6. \square

4 Non-recurrent dynamics are rare

It follows from Lemma 3.7 that a small disk around a starting non-recurrent parameter will finally grow to a definite size under the function ζ_n with bounded distortion. As long as this happens, any compact set will finally be covered within a few more iterates. For our purpose, we shall show that a definite portion of parameters in the disk $D(\lambda_0, r)$ are not Δ -non-recurrent (i.e., those λ for which $f_\lambda^n(0)$ belongs to $D(0, \Delta)$).

Let \mathcal{NR} and \mathcal{NR}_Δ be the set of non-recurrent and Δ -non-recurrent parameters in the exponential family, respectively; see Definition 2.1. We prove the following result, which implies directly Theorem 1, since $\mathcal{NR} = \bigcup_{\Delta > 0} \mathcal{NR}_\Delta$.

Proposition 4.1. *Let λ_0 be non-recurrent. For any $\Delta > 0$ and for all sufficiently small $r > 0$, we have*

$$\text{dens}(\mathcal{NR}_\Delta, D(\lambda_0, r)) = \frac{\text{meas}(D(\lambda_0, r) \cap \mathcal{NR}_\Delta)}{\text{meas } D(\lambda_0, r)} < 1.$$

We also need the following general result of Baker [Bak84, Lemma 2.2].

Lemma 4.1. *Let f be a transcendental entire function. Let U be a neighbourhood of $z \in \mathcal{J}(f)$. Then for any compact set K not containing an exceptional point of f there exists $n(K) \in \mathbb{N}$ such that $f^n(U) \supset K$ for all $n \geq n(K)$.*

An exceptional point is a point with finite backward orbit. Note that a transcendental entire function has at most one exceptional point. In our case, exponential functions have 0 as an exceptional point.

To begin with the proof, let us put $D := D(\lambda_0, r)$. Then by Lemma 3.7 for all $r > 0$ sufficiently small one can find n such that $\text{diam } \zeta_n(D) \geq S$ and

$$\zeta_n(D) \supset D(\zeta_n(\lambda_0), S/4) =: V.$$

Now we consider the annulus $A := A(\Delta/4, \Delta)$, centered at the origin with inner and outer radii $\Delta/4$ and Δ respectively. The closure of A is denoted by \bar{A} . Then by the above Lemma 4.1, there exists N such that $f_{\lambda_0}^N(V) \supset \bar{A}$. Put $A' = A(\Delta/2, 3\Delta/4)$. Since N is a finite number depending only on f_{λ_0} and V , we have, by decreasing r if necessary, that for all $\lambda \in \zeta_n^{-1}(V)$,

$$f_\lambda^N(V) \supset A'.$$

With

$$V' = \{z \in V : f_\lambda^N(z) \in A'\},$$

one can find a positive number $c > 0$ such that

$$\text{meas } V' \geq c \text{ meas } V.$$

Let \widehat{D} be the set of parameters in D which are mapped into V' by the map ξ_n . Then by Lemma 3.6, there exists a constant $c' > 0$ depending on c such that

$$\text{meas } \widehat{D} \geq c' \text{ meas } D.$$

Therefore, there is a definite portion of parameters λ in D for which $\xi_{n+N}(\lambda) \in A'$. In other words, such λ 's are not Δ -non-recurrent. This ends the proof of Proposition 4.1.

5 Non-recurrence and hyperbolicity

In the section we prove approximation of non-recurrent parameters by hyperbolic ones. One of the key ingredients is to control derivatives along the singular orbit up to at least time n , which is the time reaching the large scale.

We use some notations from the previous section. That is, we use $D := D(\lambda_0, r)$ and $V = D(\xi_n(\lambda_0), S/4)$. Let $D' \subset D$ be such that $\xi_n(D') = V$. Let n be as in Lemma 3.7. Then by this lemma, $\xi_n(D)$ has diameter at least S and contains V . Moreover, it is almost round due to the strong distortion ensured by Lemma 3.6. Now we consider the following disk

$$D_1 := D(z_1, 1)$$

such that

$$\text{Im}(z_1) = -i \arg(\lambda_0) \quad \text{and} \quad \text{Re}(z_1) \geq M,$$

where $\arg(\lambda_0) \in [0, 2\pi)$ and $M > 0$ is some sufficiently large number. Then similarly as above, Lemma 4.1 ensures the existence of a number N such that $f_{\lambda_0}^N(V) \supset D_1$.

Now we see that $f_{\lambda_0}(D_1)$ is a large set which intersects with the horizontal line $\mathcal{L} := \{z : \text{Im}(z) = -i \arg(\lambda_0)\}$ knowing that M is chosen to be large at the beginning. So we can choose a disk

$$D_2 := D(z_2, 1) \subset f_{\lambda_0}(D_1)$$

such that $z_2 \in \mathcal{L}$ and $\text{Re } z_2 \geq e^{M/2} > \text{Re } z_1$. Continuing in a similar way, we obtain a sequence of points z_n such that $z_n \in \mathcal{L}$ and $\text{Re } z_n > \exp(\text{Re } z_{n-1}/2)$. Moreover, with

$$D_n := D(z_n, 1)$$

we have $f_{\lambda_0}(D_n) \supset D_{n+1}$. By choosing z_n in this way, we can actually make sure that $\text{Re } z_n$ grows exponentially.

With

$$x_0 := \max_{j \leq n} \{\text{Re}(z) : z \in \zeta_j(D)\} + 2S,$$

there exists a smallest integer p such that

$$\operatorname{Re}(z_p) \geq x_0.$$

Then we see that $f_{\lambda_0}(D_p)$ will contain a disk $D(z_{p+1}, 3/2)$ satisfying

$$(5.1) \quad z_{p+1} \in \mathcal{L} + \pi i \quad \text{and} \quad \operatorname{Re}(z_{p+1}) > \exp(\operatorname{Re}(z_p))/2.$$

Here $\mathcal{L} + \pi i$ is the translation of \mathcal{L} by πi .

The above construction tells that $f_{\lambda_0}^{N+p}(V) \supset D(z_{p+1}, 3/2)$. By decreasing r (if necessary) we can make sure that for all $\lambda \in D'$,

$$f_{\lambda}^{N+p}(V) \supset D_{p+1} = D(z_{p+1}, 1).$$

Let \widehat{V} be the component of $f_{\lambda}^{-(N+p)}(D_{p+1})$ that is contained in V and let \widehat{D} be the set of points in D' which are mapped into \widehat{V} under the map ζ_n . In other words, $\widehat{D} = \zeta_n^{-1}(\widehat{V}) \subset D$.

Now we claim that parameters in \widehat{D} are hyperbolic parameters. This gives the desired approximation of non-recurrent parameters by hyperbolic ones, since $\widehat{D} \subset D' \subset D(\lambda_0, r)$ and r can be chosen arbitrarily small.

For any $\lambda \in \widehat{D}$, we see that $f_{\lambda}^{n+N+p}(0) \in D_{p+1}$. Now we consider a disk $B := D(f_{\lambda}^{n+N+p}(0), 1)$ centered at $f_{\lambda}^{n+N+p}(0)$ of radius 1. By Koebe's one-quarter theorem we have

$$f_{\lambda}^{-(n+N+p)}(B) \supset D\left(0, \frac{1}{8|Df_{\lambda}^{n+N+p}(0)|}\right),$$

where we choose the inverse branch sending $f_{\lambda}^{n+N+p}(0)$ to 0. On the other hand, the image $f_{\lambda}(B)$ lies in some left half-plane by the choice of z_{p+1} ; see (5.1). More precisely,

$$f_{\lambda}(B) \subset \{z : \operatorname{Re}(z) \leq -|\lambda|e^{\operatorname{Re}(z_{p+1})-1}\}.$$

With

$$r_1 := \frac{1}{8|Df_{\lambda}^{n+N+p}(0)|} \quad \text{and} \quad r_2 := \exp\{-|\lambda|e^{\operatorname{Re}(z_{p+1})-1}\},$$

we want to show that $r_1 > r_2$, which means that a small disk around the singular value 0 is mapped by $f_{\lambda}^{n+N+p+2}$ into itself. In other words, f_{λ} has an attracting cycle and thus is a hyperbolic map. Noting that

$$\begin{aligned} |Df_{\lambda}^{n+N+p}(0)| &= \prod_{j=0}^{n+N+p-1} |f'_{\lambda}(f_{\lambda}^j(0))| = \prod_{j=0}^{n+N+p-1} |\lambda e^{\operatorname{Re}f_{\lambda}^j(0)}| \\ &\leq |\lambda|^{n+N+p} e^{(n+N+p)\operatorname{Re}(z_p)}, \end{aligned}$$

one can obtain the following estimate for r_1 :

$$r_1 \geq \frac{1}{8|\lambda|^{n+N+p} e^{(n+N+p)\operatorname{Re}(z_p)}}.$$

On the other hand, since $\operatorname{Re}(z_{p+1}) = c''|z_{p+1}| = c''|\lambda|e^{\operatorname{Re}(z_p)}$ for a small constant c'' , we get that

$$r_2 \leq \frac{1}{e^{|\lambda| \exp(c''|\lambda| \exp(\operatorname{Re}(z_p)-1))}}.$$

This means that r_2 is much smaller than r_1 . So we see that there is a neighborhood U of 0 which is mapped by $f_\lambda^{n+N+p+2}$ into itself. Therefore, f_λ is a hyperbolic map.

This completes the proof of Theorem 2.

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