# BLOCH VARIETIES AND QUANTUM ERGODICITY FOR PERIODIC GRAPH OPERATORS

*By*

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**Abstract.** For periodic graph operators, we establish criteria to determine the overlaps of spectral band functions based on Bloch varieties. One criterion states that for a large family of periodic graph operators, the irreducibility of Bloch varieties implies no non-trivial periods for spectral band functions. This particularly shows that spectral band functions of discrete periodic Schrödinger operators on $\mathbb{Z}^d$ have no non-trivial periods, answering positively a question asked by Mckenzie and Sabri [Quantum ergodicity for periodic graphs, Comm. Math. Phys. 403 (2023), 1477–1509].

## 1 Introduction and main results

Algebraic and analytic properties of both Bloch and Fermi varieties play a crucial role in the study of spectral theory of periodic Schrödinger operators and related models. We refer readers to a recent review [18] which focuses on techniques arising from Bloch and Fermi varieties. In the continuous setting, Bloch and Fermi varieties are often analytic. For discrete periodic graph operators, both Bloch and Fermi varieties are algebraic in appropriate coordinates.

Recently there have been remarkable developments in using various tools such as algebraic methods, techniques in geometric combinatorics and theory in complex analysis of multi-variables to study the (ir)reducibility of Bloch and Fermi varieties, isospectrality, density of states, and critical points of spectral band functions of periodic graph operators [4, 5, 7, 8, 10, 11, 13, 17, 19, 20, 21, 22, 23, 24, 26, 27].

The main goal of this paper is to develop tools from algebraic geometry arising from Bloch varieties to understand overlaps and periods of spectral band functions of periodic graph operators. One of our motivations comes from a recent arxiv preprint of Mckenzie and Sabri [25], where they proved the quantum ergodicity for a family of periodic graph operators under an assumption on overlaps and periods



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of the spectral band functions.[1] As corollaries, we give criteria to verify for which periodic graphs, the assumption is satisfied.

Our main results are general and independent of periodic graph operators. Assume that $\mathcal{A} = \mathcal{A}(z)$ is a $Q \times Q$ matrix and each entry of $\mathcal{A}(z)$ is a Laurent polynomial of $z = (z_1, z_2, \dots, z_d) \in (\mathbb{C}^\star)^d$, where $\mathbb{C}^\star = \mathbb{C}\backslash\{0\}$. Let $z_j = e^{2\pi i k_j}$, $j = 1, 2, \dots, d$ and $A(k) = \mathcal{A}(z)$ with $k = (k_1, k_2, \dots, k_d)$.

Obviously, $A(k)$ is periodic with respect to $k$. In the following, $\mathcal{A}(z)$ and $A(k)$ are always the same (with respect to different variables). Assume that for any $k \in \mathbb{R}^d$, $A(k)$ is Hermitian. Denote by $\lambda_A^j(k)$, $k \in \mathbb{R}^d, j = 1, 2, \dots, d$, eigenvalues of $A(k)$ in the non-decreasing order:

$$\lambda_A^1(k) \leq \lambda_A^2(k) \leq \cdots \leq \lambda_A^Q(k), \quad k \in \mathbb{R}^d. \tag{1}$$

For any $\eta = (\eta_1, \eta_2, \dots, \eta_d) \in \mathbb{C}^d$ and $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_d) \in \mathbb{C}^d$, let

$$\eta \odot \zeta = (\eta_1\zeta_1, \eta_2\zeta_2, \dots, \eta_d\zeta_d).$$

Denote by $0_d$ and $1_d$ the zero vector and unit element in $\mathbb{C}^d$: $0_d = (0, 0, \dots, 0)$ and $1_d = (1, 1, \dots, 1)$.

**Definition 1.** We say the spectral band functions of $A(k)$ have no non-trivial periods if the following statement holds. If for some $\alpha \in \mathbb{R}^d$ and $s, w \in \{1, 2, \dots, Q\}$, the set

$$\{k \in \mathbb{R}^d : \lambda_A^s(k + \alpha) = \lambda_A^w(k)\} \tag{2}$$

has positive Lebesgue measure, then we must have $\alpha = 0_d \mod \mathbb{Z}^d$ and $s = w$.

Since there are only finitely many choices of $s, w \in \{1, 2, \dots, Q\}$, the spectral band functions of $A(k)$ have no non-trivial periods if and only if for any $\alpha \in \mathbb{R}^d$ with $\alpha \not\equiv 0_d \mod \mathbb{Z}^d$, the set

$$S_1(\alpha) = \{k \in \mathbb{R}^d : \text{there exist } s \text{ and } w \text{ such that } \lambda_A^s(k + \alpha) = \lambda_A^w(k)\} \tag{3}$$

has Lebesgue measure zero, and the set

$$S_2 = \{k \in \mathbb{R}^d : \text{there exist distinct } s \text{ and } w \text{ such that } \lambda_A^s(k) = \lambda_A^w(k)\} \tag{4}$$

has Lebesgue measure zero.

[1] In July 2022, Mostafa Sabri asked me a question (see Question 1 below) whether or not discrete periodic Schrödinger operators on $\mathbb{Z}^d$ satisfy the assumption, which he and Theo McKenzie need to establish quantum ergodicity. I became interested in this problem and finally wrote this paper.

Let

$$\mathcal{P}_{\mathcal{A}}(z,\lambda)=\det(\mathcal{A}(z)-\lambda I_{Q\times Q}). \tag{5}$$

Note that $\mathcal{P}_{\mathcal{A}}(z,\lambda)$ is a Laurent polynomial in $z$ and a polynomial in $\lambda$. Let $\mathcal{P}^l_{\mathcal{A}}(z,\lambda)$, $l=1,2,\dots,K$ be the non-trivial irreducible[2] factors of $\mathcal{P}_{\mathcal{A}}(z,\lambda)$:

$$\mathcal{P}_{\mathcal{A}}(z,\lambda)=(-1)^Q\prod_{l=1}^{K}\mathcal{P}^l_{\mathcal{A}}(z,\lambda). \tag{6}$$

It is easy to see that $\mathcal{P}^l_{\mathcal{A}}(z,\lambda)$ must depend on $\lambda$. Since $\mathcal{P}_{\mathcal{A}}(z,\lambda)$ is a polynomial in $\lambda$ with the highest degree term (in $\lambda$) $(-1)^Q\lambda^Q$, we can normalize $\mathcal{P}^l_{\mathcal{A}}(z,\lambda)$ in the following way: $\mathcal{P}^l_{\mathcal{A}}(z,\lambda)$ is a Laurent polynomial in $z$ and a polynomial in $\lambda$, and the coefficient of highest degree term of $\lambda$ is 1.

We say $\mathcal{P}_{\mathcal{A}}(z,\lambda)$ is square-free if for all distinct $l_1$ and $l_2$ in $\{1,2,\dots,K\}$,

$$\mathcal{P}^{l_1}_{\mathcal{A}}(z,\lambda)\not\equiv\mathcal{P}^{l_2}_{\mathcal{A}}(z,\lambda). \tag{7}$$

Given $\alpha\in\mathbb{R}^d$ with $\alpha\not\equiv 0_d \mod \mathbb{Z}^d$, we say $\mathcal{P}_{\mathcal{A}}(z,\lambda)$ satisfies condition $C_\alpha$ if for all $l_1$ and $l_2$ in $\{1,2,\dots,K\}$,

$$\mathcal{P}^{l_1}_{\mathcal{A}}(z,\lambda)\not\equiv\mathcal{P}^{l_2}_{\mathcal{A}}(\zeta\odot z,\lambda), \tag{8}$$

where $\zeta=(e^{2\pi i\alpha_1},e^{2\pi i\alpha_2},\dots,e^{2\pi i\alpha_d})$.

**Theorem 1.1.** *The following statements hold:*

(1) *For all $\alpha\not\equiv 0_d \mod \mathbb{Z}^d$, $\mathcal{P}_{\mathcal{A}}(z,\lambda)$ satisfies $C_\alpha$ if and only if $S_1(\alpha)$ has Lebesgue measure zero;*
(2) *$\mathcal{P}_{\mathcal{A}}(z,\lambda)$ is square-free if and only if $S_2$ has Lebesgue measure zero.*

**Remark 1.1.** From the proof of Theorem 1.1, one can see that

- for any $\alpha\in\mathbb{R}^d$ with $\alpha\not\equiv 0_d \mod \mathbb{Z}^d$, either $S_1(\alpha)=\mathbb{R}^d$ or $\mathrm{Leb}(S_1(\alpha))=0$;
- either $S_2=\mathbb{R}^d$ or $\mathrm{Leb}(S_2)=0$.

Theorem 1.1 immediately implies

**Corollary 1.2.** *Assume that $\mathcal{P}_{\mathcal{A}}(z,\lambda)$ is square-free, and for any*

$$\zeta=(\zeta_1,\zeta_2,\dots,\zeta_d)\in\mathbb{C}^d\backslash\{1_d\}$$

*with $|\zeta_j|=1, j=1,2,\dots,d$, and any $l_1$ and $l_2$ in $\{1,2,\dots,K\}$,*

$$\mathcal{P}^{l_1}_{\mathcal{A}}(z,\lambda)\not\equiv\mathcal{P}^{l_2}_{\mathcal{A}}(\zeta\odot z,\lambda). \tag{9}$$

*Then the spectral band functions of $A(k)$ have no non-trivial periods.*

[2]Non-trivial Laurent polynomials mean non-monomials, that is, monomials are units in the Laurent polynomial ring.

Let $L_N^d = \{0, 1, \dots, N-1\}^d$.

**Corollary 1.3.** *Given any $m = (m_1, m_2, \dots, m_d) \in L_N^d \backslash \{0_d\}$, let*

$$\zeta(m, N) = (e^{2\pi i \frac{m_1}{N}}, e^{2\pi i \frac{m_2}{N}}, \dots, e^{2\pi i \frac{m_d}{N}}).$$

*Assume that there exists $N_0$ such that for any $N \geq N_0$, any $m \in L_N^d \backslash \{0_d\}$, and any $l_1$, $l_2$ in $\{1, 2, \dots, K\}$,*

$$\mathcal{P}_{\mathcal{A}}^{l_1}(z, \lambda) \not\equiv \mathcal{P}_{\mathcal{A}}^{l_2}(\zeta(m, N) \odot z, \lambda). \tag{10}$$

*Then for any $s, w \in \{1, 2, \dots, Q\}$,*

$$\lim_{N\to\infty} \sup_{\substack{m\in L_N^d \\ m\neq 0_d}} \frac{\#\{r \in L_N^d : \lambda_A^s(\frac{r_j+m_j}{N}) - \lambda_A^w(\frac{r_j}{N}) = 0\}}{N^d} = 0, \tag{11}$$

*where $r = (r_1, r_2, \dots, r_d)$.*

For the Laurent polynomial, irreducibility implies square-free, so Corollaries 1.2 and 1.3 imply

**Corollary 1.4.** *Assume that $\mathcal{P}_{\mathcal{A}}(z, \lambda)$ is irreducible and for any*

$$\zeta = (\zeta_1, \zeta_2, \dots, \zeta_d) \in \mathbb{C}^d \backslash \{1_d\}$$

*with $|\zeta_j| = 1, j = 1, 2, \dots, d$,*

$$\mathcal{P}_{\mathcal{A}}(z, \lambda) \not\equiv \mathcal{P}_{\mathcal{A}}(\zeta \odot z, \lambda). \tag{12}$$

*Then the spectral band functions of $A(k)$ have no non-trivial periods and for any $s, w \in \{1, 2, \dots, Q\}$,*

$$\lim_{N\to\infty} \sup_{\substack{m\in L_N^d \\ m\neq 0_d}} \frac{\#\{r \in L_N^d : \lambda_A^s(\frac{r_j+m_j}{N}) - \lambda_A^w(\frac{r_j}{N}) = 0\}}{N^d} = 0.$$

**Theorem 1.5.** *Assume $\mathcal{P}_{\mathcal{A}}(z, \lambda)$ is irreducible. Then for any $a \neq 0$ and any $\alpha \in \mathbb{R}^d$, the set*

$$\{k \in \mathbb{R}^d : \text{there exist } s \text{ and } w \text{ such that } \lambda_A^s(k + \alpha) = \lambda_A^w(k) + a\} \tag{13}$$

*has Lebesgue measure zero.*

In [25], under the assumption (11), Mckenzie and Sabri proved the quantum ergodicity for periodic graph operators. Roughly speaking, quantum ergodicity means that most eigenfunctions on periodic graphs are equidistributed. We refer readers to [25] for the precise description of quantum ergodicity.

Now we want to discuss the applications of our main results to quantum ergodicity. In this paper, we focus on Corollary 1.4. The requirement (12) is easy to verify. So the only restriction of applying Corollary 1.4 is the irreducibility of $\mathcal{P}_{\mathcal{A}}(z,\lambda)$. In applications, starting with a periodic graph operator $H$ and Floquet–Bloch boundary condition (depends on $k\in\mathbb{R}^d$ or $z\in(\mathbb{C}^\star)^d$), we obtain a matrix $A(k)$ ($\mathcal{A}(z)$). The irreducibility of $\mathcal{P}(z,\lambda)$ in Corollary 1.4 essentially (up to multiplicity) means the irreducibility of the Bloch variety of $H$ (modulo periodicity):

$$B=\{(k,\lambda)\in\mathbb{C}^{d+1}: z_j=e^{2\pi i k_j}, \mathcal{P}_{\mathcal{A}}(z,\lambda)=0\}. \tag{14}$$

Recently, the author applied algebraic methods to obtain more general proofs of irreducibility for Laurent polynomials, including proving the irreducibility of Bloch and Fermi varieties (Fermi variety is the level set of the Bloch variety) for discrete periodic Schrödinger operators in arbitrary dimension [20], which previously were only studied in two and three dimensions [1, 14, 2, 16, 3]. The approach in [20] has been developed by Fillman, Matos and the author to prove the irreducibility of Bloch varieties for a large family of periodic graph operators [10]. So Corollary 1.4 may be applicable to many models (see Remark 1.2). In the following, we only discuss one case in detail: discrete periodic Schrödinger operators on $\mathbb{Z}^d$. In [25] Mckenzie and Sabri asked a question:

**Question 1.** Do discrete periodic Schrödinger operators on $\mathbb{Z}^d$ satisfy the assumption (11)?

As an application of Corollary 1.4, we answer Question 1 positively.

Let us give the precise definition of discrete periodic Schrödinger operators on $\mathbb{Z}^d$. Given positive integers $q_j$, $j=1,2,\ldots,d$, let $\Gamma=q_1\mathbb{Z}\oplus q_2\mathbb{Z}\oplus\cdots\oplus q_d\mathbb{Z}$. We say that a function $V:\mathbb{Z}^d\to\mathbb{R}$ is $\Gamma$-periodic (or just periodic) if for any $\gamma\in\Gamma$, $V(n+\gamma)=V(n)$.

Let $\Delta$ be the discrete Laplacian on $\mathbb{Z}^d$, namely

$$(\Delta u)(n)=\sum_{||n'-n||_1=1} u(n'),$$

where $n=(n_1,n_2,\ldots,n_d)\in\mathbb{Z}^d$, $n'=(n'_1,n'_2,\ldots,n'_d)\in\mathbb{Z}^d$ and

$$||n'-n||_1=\sum_{i=1}^d |n_i-n'_i|.$$

Consider the discrete Schrödinger operator on $\mathbb{Z}^d$,

$$H=\Delta+V, \tag{15}$$

where $V$ is $\Gamma$-periodic.

Let $\{\mathbf{e}_j\}$, $j = 1, 2, \dots, d$, be the standard basis in $\mathbb{Z}^d$:

$$\mathbf{e}_1 = (1, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \quad \mathbf{e}_d = (0, 0, \dots, 0, 1).$$

Let us consider the equation

$$(\Delta u)(n) + V(n)u(n) = \lambda u(n), \quad n \in \mathbb{Z}^d, \tag{16}$$

with the so-called Floquet–Bloch boundary condition

$$u(n + q_j \mathbf{e}_j) = z_j u(n) = e^{2\pi i k_j} u(n), \quad j = 1, 2, \dots, d, \text{ and } n \in \mathbb{Z}^d. \tag{17}$$

Let $D_V(k)$ ($\mathcal{D}_V(z)$) be the periodic operator $\Delta + V$ with the Floquet–Bloch boundary condition (17) with respect to variables $k$ (variables $z$). $D_V(k)$ can be realized as a $Q \times Q$ matrix, where $Q = q_1 q_2 \cdots q_d$. Let $\lambda_V^j(k)$, $j = 1, 2, \dots, Q$ be the standard spectral band functions of $\Delta + V$ (applying (1) with $A(k) = D_V(k)$).

**Corollary 1.6.** *For any discrete periodic Schrödinger operators $\Delta + V$, we have that for any $s, w \in \{1, 2, \dots, Q\}$,*

$$\lim_{N\to\infty} \sup_{\substack{m \in L_N^d \\ m \neq 0_d}} \frac{\#\{r \in L_N^d : \lambda_V^s(\frac{r_j + m_j}{N}) - \lambda_V^w(\frac{r_j}{N}) = 0\}}{N^d} = 0. \tag{18}$$

**Remark 1.2.**
- Corollary 1.6 answers Question 1 positively.
- Following the proof of Corollary 1.6 step by step, one can show that Schrödinger operators with periodic potentials on the triangular lattice (see [10] for the precise definition) satisfy (18).

Finally, we remark that in this paper, we discuss (see Theorems 1.1 and 1.5) overlaps between two spectral band functions by shifting the quasi-momenta and locations of bands, namely $\lambda_A^s(k + \alpha)$ and $\lambda_A^w(k) + a$. Overlaps between two spectral bands $[\min_k \lambda_A^s(k), \max_k \lambda_A^s(k)]$ and $[\min_k \lambda_A^w(k), \max_k \lambda_A^w(k)]$ (related to the discrete Bethe–Sommerfeld conjecture) have been studied in [12, 15, 6, 9].

## 2 Proof of Theorem 1.1, Corollary 1.3 and Theorem 1.5

**Proof of Theorem 1.1.** Recall that $\alpha \neq 0_d \mod \mathbb{Z}^d$, $\zeta_j = e^{2\pi i \alpha_j}$ and $z_j = e^{2\pi i k_j}$, $j = 1, 2, \dots, d$. Let

$$P_A(k, \lambda) = \mathcal{P}_{\mathcal{A}}(z, \lambda) = \det(A(k) - \lambda I).$$

Note that $P_A(k, \lambda)$ is analytic.

If $\mathcal{P}_{\mathcal{A}}$ does not satisfy $C_\alpha$, then there exist $l_1$ and $l_2$ in $\{1, 2, \dots, K\}$ ($l_1$ may equal $l_2$) such that

$$\mathcal{P}_{\mathcal{A}}^{l_1}(z, \lambda) \equiv \mathcal{P}_{\mathcal{A}}^{l_2}(\zeta \odot z, \lambda). \tag{19}$$

This implies that for any $k$, $A(k)$ and $A(k+\alpha)$ have at least one common eigenvalue and hence there exist $s, w$ such that $\lambda_A^s(k+\alpha) = \lambda_A^w(k)$. Therefore, $S_1(\alpha) = \mathbb{R}^d$.

Simple calculations imply that

$$\begin{aligned} S_1(\alpha) &= \{k \in \mathbb{R}^d : \text{there exist } s, w \text{ such that } \lambda_A^s(k+\alpha) = \lambda_A^w(k)\} \\ &= \{k \in \mathbb{R}^d : \text{and there exists } \lambda \text{ such that } P_A(k,\lambda) = P_A(\alpha+k,\lambda) = 0\} \\ &= \operatorname{Proj}_k\{(k,\lambda) \in \mathbb{R}^{d+1} : P_A(k,\lambda) = P_A(\alpha+k,\lambda) = 0\}, \end{aligned} \tag{20}$$

where $\operatorname{Proj}_k$ is the projection to $k$ variables.

If $\mathcal{P}_{\mathcal{A}}$ satisfies $C_\alpha$, we have that for any $l_1$ and $l_2$ in $\{1,2,\ldots,K\}$, $\{(z,\lambda) \in (\mathbb{C}^\star)^d \times \mathbb{C} : \mathcal{P}_{\mathcal{A}}^{l_1}(z,\lambda) = 0\}$ and $\{(z,\lambda) \in (\mathbb{C}^\star)^d \times \mathbb{C} : \mathcal{P}_{\mathcal{A}}^{l_2}(\zeta \odot z,\lambda) = 0\}$ are not identical, where $\zeta = (e^{2\pi i\alpha_1}, e^{2\pi i\alpha_2}, \ldots, e^{2\pi i\alpha_d})$, and hence the algebraic variety $\{(z,\lambda) : \mathcal{P}_{\mathcal{A}}^{l_1}(z,\lambda) = \mathcal{P}_{\mathcal{A}}^{l_2}(\zeta \odot z,\lambda) = 0\}$ has algebraic dimension $d-1$. We conclude that the algebraic variety $\{(z,\lambda) : \mathcal{P}_{\mathcal{A}}(z,\lambda) = \mathcal{P}_{\mathcal{A}}(\zeta \odot z,\lambda) = 0\}$ has algebraic dimension $d-1$. This implies that $\{(k,\lambda) \in \mathbb{R}^{d+1} : P_A(k,\lambda) = P_A(\alpha+k,\lambda) = 0\}$ has real (analytic) dimension at most $d-1$. By (20), one has that $S_1(\alpha)$ (as a subset in $\mathbb{R}^d$) has Lebesgue measure zero. We finish the proof of part 1.

If $\mathcal{P}_{\mathcal{A}}$ is not square-free, then there exist distinct $l_1$ and $l_2$ in $\{1,2,\ldots,K\}$ such that

$$\mathcal{P}_{\mathcal{A}}^{l_1}(z,\lambda) \equiv \mathcal{P}_{\mathcal{A}}^{l_2}(z,\lambda). \tag{21}$$

This implies that for any $k$, $A(k)$ has repeated eigenvalues and hence there exist distinct $s, w \in \{1,2,\ldots,Q\}$ such that $\lambda_A^s(k) = \lambda_A^w(k)$. Therefore, $S_2 = \mathbb{R}^d$.

It is clear that

$$\begin{aligned} S_2 &= \{k \in \mathbb{R}^d : \text{ there exist distinct } s, w \text{ such that } \lambda_A^s(k) = \lambda_A^w(k)\} \\ &= \{k \in \mathbb{R}^d : A(k) \text{ has repeated eigenvalues}\} \\ &= \{k \in \mathbb{R}^d : \text{there exists } \lambda \text{ such that } P_A(k,\lambda) = \partial_\lambda P_A(k,\lambda) = 0\} \\ &= \operatorname{Proj}_k\{(k,\lambda) \in \mathbb{R}^{d+1} : P_A(k,\lambda) = \partial_\lambda P_A(k,\lambda) = 0\}. \end{aligned} \tag{22}$$

If $\mathcal{P}_{\mathcal{A}}$ is square-free, then the algebraic variety

$$\{(z,\lambda) : \mathcal{P}_{\mathcal{A}}(z,\lambda) = \partial_\lambda \mathcal{P}_{\mathcal{A}}(z,\lambda) = 0\}$$

has algebraic dimension $d-1$. This implies that

$$\{(k,\lambda) \in \mathbb{R}^{d+1} : P_A(k,\lambda) = \partial_\lambda P_A(k,\lambda) = 0\}$$

has real (analytic) dimension at most $d-1$. By (22), one has that $S_2$ (as a subset in $\mathbb{R}^d$) has Lebesgue measure zero. We finish the proof of part 2. $\square$

**Proof of Corollary 1.3.** Let

$$U = \bigcup_{N\geq N_0} \bigcup_{\substack{m\in L_N^d \\ m\neq 0_d}} S_1\Big(\frac{m}{N}\Big).$$

Applying Part 1 of Theorem 1.1 with all $\alpha = \frac{m}{N}$ with $m \in L_N^d \backslash \{0_d\}$ and $N \geq N_0$, $U$ has Lebesgue measure zero. Therefore, for any $s, w \in \{1, 2, \ldots, Q\}$ one has that

$$\begin{aligned} \lim_{N\to\infty} \sup_{\substack{m\in L_N^d \\ m\neq 0_d}} \frac{\#\{r \in L_N^d : \lambda_A^s(\frac{r_j+m_j}{N}) - \lambda_A^w(\frac{r_j}{N}) = 0\}}{N^d} &\leq \lim_{N\to\infty} \frac{1}{N^d} \sum_{r\in L_N^d} \chi_U\Big(\frac{r_j}{N}\Big) \\ &= 0, \end{aligned} \tag{23}$$

where $\chi_U$ is the characteristic function. $\square$

**Proof of Theorem 1.5.** From the proof of Part 1 of Theorem 1.1, it suffices to show that for any $a \neq 0$ and any $\zeta \in \mathbb{C}^d$ with $|\zeta_j| = 1, j = 1, 2, \ldots, d$,

$$\mathcal{P}_{\mathcal{A}}(z, \lambda) \not\equiv \mathcal{P}_{\mathcal{A}}(\zeta \odot z, \lambda + a). \tag{24}$$

Simple calculations imply that

$$\mathcal{P}_{\mathcal{A}}(z, \lambda) = (-\lambda)^Q + \mathrm{Tr}\mathcal{A}(z)(-\lambda)^{Q-1} + \text{l.o.t} \tag{25}$$

and

$$\mathcal{P}_{\mathcal{A}}(\zeta \odot z, \lambda + a) = (-\lambda)^Q + (-Qa + \mathrm{Tr}\mathcal{A}(\zeta \odot z))(-\lambda)^{Q-1} + \text{l.o.t}, \tag{26}$$

where l.o.t contains terms of $\lambda$ with degree less than or equal to $Q - 2$. Obviously, constant terms in both $\mathrm{Tr}\mathcal{A}(z)$ and $\mathrm{Tr}\mathcal{A}(\zeta \odot z)$ are the same. Then $\mathrm{Tr}\mathcal{A}(z)$ and $-Qa + \mathrm{Tr}\mathcal{A}(\zeta \odot z)$ are different functions. Now (24) follows from (25) and (26).$\square$

## 3 Proof of Corollary 1.6

In this section, we first recall some basics. We refer readers to [20] for details. For $n = (n_1, n_2, \ldots, n_d)$, let $z^n = z_1^{n_1} \cdots z_d^{n_d}$. By abusing the notation, denote $q = (q_1, q_2, \ldots, q_d)$. Let $\hat{V}(n)$, $n \in \mathbb{Z}^d$ be the discrete Fourier transform of $\{V(n)\}$.

Define

$$\tilde{\mathcal{D}}_V(z) = \mathcal{D}_V(z^q), \tag{27}$$

and

$$\tilde{\mathcal{P}}_V(z, \lambda) = \det(\tilde{\mathcal{D}}_V(z, \lambda) - \lambda I) = \mathcal{P}_V(z^q, \lambda). \tag{28}$$

Let

$$\rho^j_{n_j} = e^{2\pi i \frac{n_j}{q_j}},$$

where $0 \leq n_j \leq q_j - 1, j = 1, 2, \dots, d$.

By the standard discrete Floquet transform (e.g., [20, 18]), one has

**Lemma 3.1.** *$\tilde{\mathcal{D}}_V(z)$ is unitarily equivalent to $B_0 + B_V$, where $B_0$ is a diagonal matrix with entries*

$$B_0(n; n') = \left(\sum_{j=1}^{d}\left(\rho^j_{n_j} z_j + \frac{1}{\rho^j_{n_j} z_j}\right)\right)\delta_{n,n'}, \tag{29}$$

$$B_V(n; n') = \hat{V}(n_1 - n'_1, n_2 - n'_2, \dots, n_d - n'_d), \tag{30}$$

*and*

$$0 \leq n_j \leq q_j - 1, \quad 0 \leq n'_j \leq q_j - 1, \quad j = 1, 2, \dots, d.$$

*In particular,*

$$\tilde{\mathcal{P}}_V(z, \lambda) = \det(B_0 + B_V - \lambda I).$$

Let

$$h(z, \lambda) = \prod_{\substack{0 \leq n_j \leq q_j - 1 \\ 1 \leq j \leq d}} \left(\left(\sum_{j=1}^{d} \rho^j_{n_j} z_j\right) - \lambda\right). \tag{31}$$

**Proof of Corollary 1.6.** Recall that $\mathcal{P}_V(z, \lambda)$ is irreducible [20, 10]. By Corollary 1.4, it suffices to verify that for any $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_d) \in \mathbb{C}^d \backslash \{1_d\}$ with $|\zeta_j| = 1, j = 1, 2, \dots, d$,

$$\mathcal{P}_V(z, \lambda) \not\equiv \mathcal{P}_V(\zeta \odot z, \lambda). \tag{32}$$

It suffices to prove that for any $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_d) \in \mathbb{C}^d$ with $|\zeta_j| = 1, j = 1, 2, \dots, d$ and $\zeta^q \neq 1_d$, one has that

$$\tilde{\mathcal{P}}_V(z, \lambda) \not\equiv \tilde{\mathcal{P}}_V(\zeta \odot z, \lambda). \tag{33}$$

By (31) and Lemma 3.1, $h(z, \lambda)$ is the highest degree component of $\tilde{\mathcal{P}}_V(z, \lambda)$. Therefore, to prove (33), it suffices to show that for any $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_d) \in \mathbb{C}^d$ with $|\zeta_j| = 1, j = 1, 2, \dots, d$ and $\zeta^q \neq 1_d$, one has that

$$h(z, \lambda) \not\equiv h(\zeta \odot z, \lambda). \tag{34}$$

Assume that

$$h(z, \lambda) \equiv h(\zeta \odot z, \lambda). \tag{35}$$

Substituting $\lambda = \sum_{j=1}^{d} z_j$ in (35), one has that $h(z, z_1 + z_2 + \cdots + z_d) \equiv 0$ and hence

$$h(\zeta \odot z, z_1 + z_2 + \cdots + z_d) \equiv 0.$$

Then there exist $n_j, j = 1, 2, \ldots, d$ with $0 \leq n_j \leq q_j - 1$ such that

$$\sum_{j=1}^{d} z_j \equiv \sum_{j=1}^{d} \rho^j_{n_j} \zeta_j z_j.$$

This implies $\zeta_j = e^{-2\pi i \frac{n_j}{q_j}}$, $j = 1, 2, \ldots, d$ and hence $\zeta^q = 1_d$. We reach a contradiction. □

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