DETERMINANTS OF LAPLACIANS ON RANDOM HYPERBOLIC SURFACES

By

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Dedicated to Peter Sarnak on the occasion of his 70th birthday

Abstract. For sequences (X_j) of random closed hyperbolic surfaces with volume $\mathrm{Vol}(X_j)$ tending to infinity, we prove that there exists a universal constant $E > 0$ such that for all $\epsilon > 0$, the regularized determinant of the Laplacian satisfies

$$\frac{\log\det(\Delta_{X_j})}{\mathrm{Vol}(X_j)} \in [E-\epsilon, E+\epsilon]$$

with high probability as $j \to +\infty$. This result holds for various models of random surfaces, including the Weil–Petersson model.

1 Introduction and results

1.1 On determinants. Let $X = \Gamma\backslash\mathbb{H}^2$ be a compact connected hyperbolic surface obtained as a quotient of the hyperbolic plane $\mathbb{H}^2$ by a discrete co-compact torsion-free group of orientation-preserving isometries. The hyperbolic Laplacian Δ_X on $L^2(X)$ has a pure point spectrum which we denote by

$$0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_j \le \cdots .$$

For all $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ large enough, we know by Weyl's law that the spectral zeta function

$$\zeta_X(s) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s}$$

is well defined and holomorphic. The regularized determinant is then usually defined by

$$\log\det(\Delta_X) := -\zeta_X'(0),$$

provided one can prove an analytic extension to $s = 0$ of ζ_X. Practically, one performs a meromorphic continuation by noticing that for large $\mathrm{Re}(s)$ we have

$$\zeta_X(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(\mathrm{Tr}(e^{-t\Delta_X}) - 1)dt,$$

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where $e^{-t\Delta_X}$ is the heat semi-group. The small time asymptotics at $t \sim 0$ of the heat kernel is the main tool which allows to "renormalize" the divergent behavior at $t = 0$ and obtain the meromorphic continuation; see, for example, Chavel [11, p. 156].

In the literature, Polyakov's string theory [35, 14] has emphasized the role of determinants on Riemann surfaces. In particular, the computation of "partition functions" in perturbative string theory involves formal sums over all genera of averages of determinants over the moduli space which have proved since then to be divergent; see Wolpert [41]. Several authors have provided [5, 13, 36] some explicit formulas for regularized determinants for various Laplace-like operators on Riemann surfaces. In small genus, it is possible to compute accurately such determinants by reducing to certain sums over closed geodesics which provide a fast convergence; see [34, 38]. In variable curvature, the behavior of determinants in a conformal class has been studied by Osgood, Phillips and Sarnak [30], in particular the constant curvature metric maximizes the determinant.

In higher dimensions, determinants of Laplacians on differential forms are related to the so-called analytic torsion, which in turn is related to important topological invariants by results of Cheeger and Müller [12, 28]. In particular the work of Bergeron–Venkatesh [3] establishes exponential growth of the analytic torsion for certain families of covers of arithmetic manifolds.

In the case of Riemann surfaces, if Γ is a co-compact arithmetic Fuchsian group derived from a quaternion algebra, one can define congruence covers

$$X_{\mathscr{P}} := \Gamma(\mathscr{P})\backslash\mathbb{H}^2$$

of $X = \Gamma\backslash\mathbb{H}^2$ by looking at prime ideals $\mathscr{P}$ in the ring of integers of the corresponding number field. We denote by $\|\mathscr{P}\|$ the norm of ideals. Using the uniform spectral gap of these surfaces proved by Sarnak and Xue in [37], together with the fact that the injectivity radius goes to infinity as $\|\mathscr{P}\| \to \infty$, see in [19], one can readily show (for example by using the arguments from [2] or as a direct application of Theorem 3.1) that

$$\lim_{\|\mathscr{P}\|\to\infty} \frac{\log\det \Delta_{X_{\mathscr{P}}}}{\mathrm{Vol}(X_{\mathscr{P}})} = E,$$

where $E > 0$ is some universal constant. Arithmetic surfaces being highly non-generic, it is therefore natural to ask if this behavior is typical among larger families of surfaces whose volume (equivalently genus) goes to infinity.

1.2 Models of random surfaces In this paper we will focus on the behaviour of determinants of the Laplacian in the large volume (equivalently large genus) regime, using probabilistic tools. The first historical model of random compact Riemann surfaces in the mathematics literature is perhaps the model of Brooks–Makover [9] which is based on random 3-regular graphs as follows. Consider $\mathcal{G}_n$ a 3-regular graph on $2n$ vertices, endowed with an orientation $\mathcal{O}$.[1] On the (finite) set of all possible pairs $(\mathcal{G}_n, \mathcal{O})$, one can put a probability measure (which is not the uniform measure) introduced first by Bollobás [4]; see [9, Section 5] for a summary on this construction which allows tractable computations in the large n regime. By glueing $2n$ ideal hyperbolic triangles according to $(\mathcal{G}_n, \mathcal{O})$ in such a way that the feet of the altitudes in adjacent triangles match up, one then obtains a random finite area hyperbolic surface $S_n^O := S^O(\mathcal{G}_n, \mathcal{O})$ with $\mathrm{Vol}(S_n) = 2\pi n$. It is possible to show, see [8], that all surfaces in S_n^O are actually (non ramified) covers of the modular surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2$.

One can then conformally compactify S_n^O by cutting cusps and filling them with discs. Provided that S_n^O has genus at least 2, we then denote by S_n^C the unique hyperbolic surface in the conformal class of this compactification; see [9, Section 3]. In §4 of the same paper they also show that there exists a constant $C_0 > 0$ such that with high probability as $n \to +\infty$,

$$\mathrm{Vol}(S_n^C) \geq C_0 n.$$

Most of the geometric properties of S_n^O (and then S_n^C, after a mild loss) can be read off from the combinatorics of $\mathcal{G}_n$.

Another more recent discrete model of random surfaces is the so-called random cover model which has been studied and used recently in [21, 26, 29]. In what follows, we fix a compact surface $X = \Gamma\backslash\mathbb{H}^2$, "the base surface". Let $\phi_n : \Gamma \to \mathcal{S}_n$ be a group homomorphism, where $\mathcal{S}_n$ is the symmetric group of permutations of $[n] := \{1, \dots, n\}$. The discrete group Γ acts on $\mathbb{H}^2 \times [n]$ by

$$\gamma.(z, j) := (\gamma(z), \phi_n(\gamma)(j)).$$

The resulting quotient $X_n := \Gamma\backslash(\mathbb{H}^2 \times [n])$ is then a finite cover of degree n of X, possibly not connected. By considering the (finite) space of all homomorphisms $\phi_n : \Gamma \to \mathcal{S}_n$, endowed with the uniform probability measure, we obtain a notion of **random covering surfaces** of degree n, $X_n \to X$. Let us remark that we can also view (up to isometry) the random cover X_n as

$$X_n = \bigsqcup_{k=1}^{p} \Gamma_k\backslash\mathbb{H}^2,$$

where each Γ_k is the (a priori non-normal) subgroup of Γ given by

$$\Gamma_k = \mathrm{Stab}_\Gamma(i_k) = \{\gamma \in \Gamma : \phi_n(\gamma)(i_k) = i_k\},$$

[1] an orientation on a graph is a function which assigns to each vertex v of the graph a cyclic ordering of the edges emanating from v.

where $i_1, \ldots, i_p \in [n]$ are representatives of the orbits of Γ (acting on $[n]$ via ϕ_n). In general, the cover X_n is not connected, but it follows directly from [20] that the probability that this cover is connected tends to 1 as n goes to infinity. In this model, we have $\mathrm{Vol}(X_n) = n\mathrm{Vol}(X)$.

A smooth model of random hyperbolic surfaces is given by the **moduli space** $\mathscr{M}_g$ of closed hyperbolic surfaces with genus g, up to isometry. It is often defined as the quotient

$$\mathscr{M}_g = \mathscr{T}_g/\mathrm{MCG},$$

where $\mathscr{T}_g$ is the Teichmüller space of hyperbolic metrics on a surface S of genus g and

$$\mathrm{MCG} = \mathrm{Diff}(S)/\mathrm{Diff}_0(S)$$

is the group of isotopy classes of diffeomorphisms on S, aka the mapping class group. We refer the reader, for example, to [10], chapter 6 for more details. A symplectic form ω_{WP} lives naturally on $\mathscr{T}_g$ and descends to the moduli space, endowing it with a natural notion of volume, Weil–Petersson volume. The moduli space is a non-compact finite-dimensional orbifold, but as a consequence of Bers' theorem on pants decomposition, see [10, Theorem 5.1.2], it has a finite volume with respect to this Weil–Petersson volume. We can therefore normalize this measure and obtain a probability measure on $\mathscr{M}_g$. Notice that in this case if $X \in \mathscr{M}_g$, $\mathrm{Vol}(X) = 4\pi(g-1)$ by Gauss–Bonnet. The calculation of Weil–Petersson volumes of the moduli space by Mirzakhani [24] has made possible [25] the large genus asymptotic analysis of various geometric and spectral quantities; see, for example, [27, 43, 42] for recent works in that direction.

For all of the previous models, we will denote by $\mathbb{P}$ the associated probability measure, which depends either on n or g, which are both proportional to the volume. We say that an event $\mathcal{A}$ is asymptoticaly almost sure (a.a.s.), or holds with high probability, if $\mathbb{P}(\mathcal{A})$ tends to 1 as the volume of surfaces tends to infinity. The expectation of any relevant random variable will also be denoted by $\mathbb{E}$.

1.3 Main result.

Theorem 1.1. *There exists a universal constant $E > 0$ such that for all the above models of random surfaces, for all $\epsilon > 0$ we have*

$$\frac{\log\det(\Delta_X)}{\mathrm{Vol}(X)} \in [E-\epsilon, E+\epsilon],$$

a.a.s. as $\mathrm{Vol}(X) \to +\infty$.

The constant E is actually explicit and is approximately 0.0538; see §2 for an exact description. This result shows that exponential growth of the determinant is typical when the genus goes to infinity. This low dimensional result is consistent, in a much simpler setting, with the conjectures on the exponential growth of the analytic torsion and the torsion homology for higher dimensional hyperbolic manifolds; see, for example, the paper of Bergeron–Venkatesh [3] and references therein.

Note that the above statement says that the random variable $\frac{\log\det(\Delta_X)}{\mathrm{Vol}(X)}$ converges in probability to the constant E. What about other modes of convergence? It is possible to derive from Theorem 1.1 a convergence result for the expectation of $|\log\det(\Delta)|^{\beta}$; see §5, Theorem 5.1: for both models of random covers and Weil–Petersson, we show the existence of exponents $\beta > 0$ such that

$$\lim_{\mathrm{Vol}(X)\to\infty} \mathbb{E}\left(\frac{|\log\det(\Delta_X)|^{\beta}}{\mathrm{Vol}(X)^{\beta}}\right) = E^{\beta}.$$

The paper is organized as follows. In §2 we recall how one establishes an identity for $\log\det(\Delta_X)$ which involves infinite sums over closed geodesics via the Heat trace formula. In §3 we prove an abstract Theorem which guarantees the exponential growth of determinants as long as a certain natural list of assumptions are satisfied. These hypotheses turn out to be valid a.a.s. for the probabilistic models listed above, and this is established in §4. In §5, we derive from Theorem 1.1 a convergence result for the expectation, based on some moments estimates for the systole and the smallest positive eigenvalue.

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2 Heat kernels and determinants

In this section, we recall some standard calculations on regularized determinants mostly taken from [5, Appendix B]. Our goal is to show how the heat trace formula allows to derive an identity for $\log\det\Delta_X$.

On the hyperbolic plane $\mathbb{H}^2$, the heat kernel $p_t(x, y)$ (see, for example, [10, chapter 7]) has an explicit formula given by

$$p_t(x, y) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{d(x,y)}^{\infty} \frac{re^{-r^2/4t}dr}{\sqrt{\cosh r - \cosh d(x, y)}},$$

where $d(x, y)$ denotes the hyperbolic distance in $\mathbb{H}^2$. On the quotient $X = \Gamma\backslash\mathbb{H}^2$, we can recover the heat kernel by summing over the group, i.e.,

$$h_t^X(x, y) = \sum_{\gamma\in\Gamma} p_t(x, \gamma y).$$

Convergence of the above series on any compact subset of $\mathbb{H}^2$ is guaranteed by the lattice counting bound

$$N_\Gamma(x, y, T) := \#\{\gamma \in \Gamma : d(\gamma x, y) \leq T\} = O(e^T), \tag{1}$$

which is standard and follows from a basic volume argument. The semi-group of operators $e^{-t\Delta_X}$ is then of trace class and one has the explicit "heat trace formula"

$$\begin{aligned} \mathrm{Tr}(e^{-t\Delta_X}) = \sum_j e^{-t\lambda_j} &= \mathrm{Vol(X)}\frac{e^{-t/4}}{(4\pi t)^{3/2}} \int_0^{\infty} \frac{re^{-r^2/4t}}{\sinh(r/2)}dr \\ &+ \frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{k\geq 1}\sum_{\gamma\in\mathcal{P}} \frac{\ell(\gamma)}{2\sinh(k\ell(\gamma)/2)}e^{-(k\ell(\gamma))^2/4t}, \end{aligned} \tag{2}$$

where $\mathcal{P}$ stands for the set of primitive conjugacy classes in Γ (i.e., oriented primitive closed geodesics on X) and if $\gamma \in \mathcal{P}$, $\ell(\gamma)$ is the length. For more details on the calculation of this trace and more generally Selberg's formula, see [17, 10]. Setting

$$S_X(t) := \frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{k\geq 1}\sum_{\gamma\in\mathcal{P}} \frac{\ell(\gamma)}{2\sinh(k\ell(\gamma)/2)}e^{-(k\ell(\gamma))^2/4t},$$

it is easy to see from the spectral side of the trace formula that $|S_X(t) - 1|$ is exponentially small as $t \to +\infty$. We also observe that $S_X(t)$ is exponentially small as $t \to 0$. We now write (for $\mathrm{Re}(s)$ large)

$$\zeta_X(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}(\mathrm{Tr}(e^{-t\Delta_X}) - 1)dt = \zeta_X^{(1)}(s) + \zeta_X^{(2)}(s),$$

where we have set

$$\zeta_X^{(1)}(s) =: \mathrm{Vol(X)}\frac{1}{4\pi\Gamma(s)} \int_0^{\infty} t^{s-1}\frac{e^{-t/4}}{\sqrt{4\pi}t^{3/2}} \int_0^{\infty} \frac{re^{-r^2/4t}}{\sinh(r/2)}drdt,$$

and

$$\zeta_X^{(2)}(s) =: \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(S_X(t) - 1)dt.$$

Writing (for $\mathrm{Re}(s)$ large)

$$\begin{aligned}\zeta_X^{(2)}(s) &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1}(S_X(t) - 1)dt + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1}(S_X(t) - 1)dt \\ &= \frac{-1}{\Gamma(s+1)} + \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} S_X(t)dt + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1}(S_X(t) - 1)dt,\end{aligned}$$

we notice that the last two integrals make sense for all $s \in \mathbb{C}$. Therefore $\zeta_X^{(2)}(s)$ has an analytic extension to $\mathbb{C}$ and using elementary facts on the gamma function (in particular that it has a simple pole at $s = 0$ with residue 1), we have that

$$-\frac{d}{ds}\bigg|_{s=0} \zeta_X^{(2)}(s) = -\Gamma'(1) - \int_0^1 \frac{S_X(t)}{t} dt - \int_1^\infty \frac{(S_X(t) - 1)}{t} dt.$$

On the other hand, for large $\mathrm{Re}(s)$ we use that[2]

$$\frac{e^{-t/4}}{\sqrt{4\pi} t^{3/2}} \int_0^\infty \frac{r e^{-r^2/4t}}{\sinh(r/2)} dr = \int_{-\infty}^{+\infty} x \tanh(\pi x) e^{-(x^2+1/4)t} dx,$$

see [5, p. 593], and one can compute the Mellin transform to obtain (again for $\mathrm{Re}(s)$ large)

$$\zeta_X^{(1)}(s) = \frac{2\mathrm{Vol}(X)}{4\pi} \int_0^\infty \frac{u \tanh(\pi u)}{(u^2 + 1/4)^s} du.$$

This function can be analytically continued to $s = 0$, see for example [5, Appendix B], and the value can be actually computed as

$$-\frac{d}{ds}\bigg|_{s=0} \zeta_X^{(1)}(s) = \frac{\mathrm{Vol}(X)}{4\pi}(4\zeta'(-1) - 1/2 + \log(2\pi)) := \mathrm{Vol}(\mathrm{X})E,$$

with $\zeta'(-1) = 1/12 - \log(A)$ and A is the so-called **Glaisher–Kinkelin constant**, which is for example defined by

$$A = \lim_{n\to\infty} \frac{\prod_{k=1}^n k^k}{e^{-n^2/4} n^{n^2/2+n/2+1/12}}.$$

We have $E \approx 0,0538$. Using in addition that $\Gamma'(1) = -\gamma_0$, where

$$\gamma_0 = \lim_{n\to+\infty} \sum_{k=1}^n \frac{1}{k} - \log(n)$$

[2]For example, one can use the identity valid for $x \in \mathbb{R}$

$$\tanh(\pi x) = \frac{1}{\pi} \sum_{k=0}^\infty \frac{2x}{x^2 + (k+1/2)^2} = \frac{1}{\pi} \int_0^\infty \frac{\sin(ux)}{\sinh(u/2)} du.$$

is the **Euler constant**, we have obtained the celebrated identity

$$\log\det \Delta_X = \mathrm{Vol}(X)E + \gamma_0 - \int_0^1 \frac{S_X(t)}{t}dt - \int_1^\infty \frac{(S_X(t)-1)}{t}dt. \tag{3}$$

This formula can be interpreted multiplicatively the via Selberg zeta function at $s = 1$; see [5, 13, 36].

3 An abstract deterministic statement

Theorem 1.1 actually follows from a more general deterministic result for sequences of compact surfaces satisfying certain hypotheses denoted by $\mathscr{H}_1$ and $\mathscr{H}_2$. More precisely, if (X_k) is a sequence of compact connected hyperbolic surfaces with $\mathrm{Vol}(X_k) \to +\infty$, let $\mathcal{P}_k$ denote the set of oriented primitive closed geodesics on X_k, and let Δ_k be the hyperbolic Laplacian on X_k. We also denote by $\ell_0(X_k)$ the length of the shortest closed geodesic on X_k. Let $C > 0$, $\eta > 0$, $L > 0$ and $0 < \alpha < 1/2$ be some constants.

We say that the sequence (X_k) satisfies hypothesis $\mathscr{H}_1(\eta)$ if for all $k \in \mathbb{N}$ we have

$$\lambda_1(\Delta_k) \geq \eta. \tag{4}$$

We say that the sequence (X_k) satisfies hypothesis $\mathscr{H}_2(C, L, \alpha)$ if for all $k \in \mathbb{N}$ we have the following bound on the number of closed geodesics:

$$N_k(L) := N_{X_k}(L) := \#\{(\gamma, m) \in \mathcal{P}_k \times \mathbb{N} : m\ell(\gamma) \leq L\} \leq C\mathrm{Vol}(X_k)^\alpha. \tag{5}$$

In this paper $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural integers starting at 1. We point out that exponential growth of Laplace determinants is established in the literature for families of covers for which a uniform spectral gap holds and the injectivity radius of the manifolds goes to infinity; see, for example, [3] and [2]. Typical examples are congruence covers of arithmetic hyperbolic manifolds and Laplacians twisted by a “strongly acyclic” representation which ensures a uniform spectral gap. In random models of surfaces, having the injectivity radius grow to infinity is atypical and we establish the result under the weaker assumption of small growth of the number of closed geodesics with bounded length.

Theorem 1.1 will follow from the following deterministic result.

Theorem 3.1. *Fix some $\eta > 0$ and $0 < \alpha < 1$. Assume that (X_k) satisfies $\mathscr{H}_1(\eta)$ and $\mathscr{H}_2(C_0, L_0, \alpha)$ with $L_0 = 2\mathrm{arcsinh}(1)$ for some $C_0 > 0$. Then for all $\epsilon > 0$, there exists $L_\epsilon > 0$ such that if in addition (X_k) satisfies $\mathscr{H}_2(C_\epsilon, L_\epsilon, \alpha)$*

for some $C_\epsilon > 0$*, then uniformly for all* $\mathrm{Vol}(X_k)$ *large,*

$$\frac{\log\det(\Delta_{X_k})}{\mathrm{Vol}(X_k)} \in [E-\epsilon, E+\epsilon],$$

where $E > 0$ *is the universal constant from above.*

All the positive constants denoted by $C_1, C_2, \dots, C_j$ below depend only on C_0, L_0. Before we give a proof of Theorem 3.1, we will need a preliminary Lemma which is needed to control uniformly sums over closed geodesics.

Lemma 3.2. *Under hypothesis* $\mathscr{H}_2(C_0, L_0, \alpha)$ *where* $L_0 = 2\mathrm{arcsinh}(1)$, *there exists* $C_1 > 0$ *such that for all k and all* $T \geq 0$*,*

$$N_k(T) \leq C_1 \mathrm{Vol}(X_k) e^T.$$

Proof. A result of Buser ([10] Lemma 6.6.4) says that for any compact connected hyperbolic surface of genus g, the number of oriented closed geodesics with length $\leq T$ which are not iterates of primitive closed geodesics of length $\leq 2\mathrm{arcsinh}(1)$ is bounded from above by

$$(g-1)e^{T+6}.$$

Therefore we have

$$N_k(T) \leq \frac{\mathrm{Vol}(X_k)}{4\pi} e^{T+6} + \#\{(\gamma, m) \in \mathcal{P}_k \times \mathbb{N} : m\ell(\gamma) \leq T \text{ and } \ell(\gamma) \leq 2\mathrm{arcsinh}(1)\}.$$

On the other hand we have

$$\#\{(\gamma, m) \in \mathcal{P}_k \times \mathbb{N} : m\ell(\gamma) \leq T \text{ and } \ell(\gamma) \leq 2\mathrm{arcsinh}(1)\} \leq \sum_{\ell(\gamma) \leq \mathrm{arcsinh}(1)} \frac{T}{\ell(\gamma)}.$$

We can observe that by definition of $N_k(L)$, we have

$$N_k(L) = \sum_{m\ell(\gamma)\leq L} 1 = \sum_{\ell(\gamma)\leq L} \left[\frac{L}{\ell(\gamma)}\right],$$

where [.] is the integer part. Hence we can write

$$\sum_{\ell(\gamma)\leq L} \frac{1}{\ell(\gamma)} \leq \frac{2}{L} N_{X_k}(L). \tag{6}$$

Going back to the estimate of $N_{X_k}(T)$ and using $\mathscr{H}_2(C_0, L_0, \alpha)$ with $L_0 = \mathrm{arcsinh}(1)$, we get

$$\begin{aligned} N_k(T) &\leq \mathrm{Vol}(X_k) e^T \frac{e^6}{4\pi} + \frac{2}{\mathrm{arcsinh}(1)} N_{X_k}(\mathrm{arcsinh}(1)) \\ &\leq C_1 \mathrm{Vol}(X_k) e^T; \end{aligned}$$

the proof is done. □

Lemma 3.3. *Under hypotheses $\mathscr{H}_1(\eta)$ and $\mathscr{H}_2(C_0, L_0, \alpha)$ where L_0 is as above, there exists $C_2 > 0$ such that for all k and all $t \geq 1$,*

$$|S_{X_k}(t) - 1| \leq C_2 \mathrm{Vol}(X_k) e^{-\eta_0 t},$$

where $\eta_0 = \min(\eta, 1/4)$.

Proof. In this proof we will use Vinogradov's notation $A \ll B$, meaning that $A \leq CB$ where $C > 0$ is a universal constant. By formula (2) we have

$$\mathrm{Tr}(e^{-t\Delta_{X_k}}) - 1 = \mathrm{Vol}(\mathrm{X_k}) \frac{e^{-t/4}}{(4\pi t)^{3/2}} \int_0^\infty \frac{re^{-r^2/4t}}{\sinh(r/2)} dr + S_{X_k}(t) - 1,$$

and therefore we get

$$|S_{X_k}(t) - 1| \ll \mathrm{Vol}(X_k) e^{-t/4} + \sum_{j=1}^{\infty} e^{-t\lambda_j(X_k)},$$

for all $t \geq 1$. On the other hand, using the uniform spectral gap we have

$$\sum_{j=1}^{\infty} e^{-t\lambda_j(X_k)} = \sum_{j=1}^{\infty} e^{-(t-1)\lambda_j - \lambda_j} \leq e^{-(t-1)\eta} \mathrm{tr}(e^{-\Delta_{X_k}}).$$

Going back to formula (2) with $t = 1$, we have also

$$\mathrm{tr}(e^{-\Delta_{X_k}}) \ll \mathrm{Vol}(X_k) + S_{X_k}(1)$$

and

$$\begin{aligned} S_{X_k}(1) &\ll \sum_{m \geq 1} \sum_{\gamma \in \mathcal{P}_k} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} e^{-(m\ell(\gamma))^2/4} \\ &\ll \int_0^\infty \frac{u}{2\sinh(u/2)} e^{-u^2/4} dN_k(u), \end{aligned}$$

where we have used the Stieltjes integral notation with the measure dN_k associated to the counting function N_k. We can use Lemma 3.2 to bound $N_k(u)$ as

$$N_k(u) \leq C_1 \mathrm{Vol}(X_k) e^u,$$

and a summation by parts shows that

$$\begin{aligned} \int_0^\infty \frac{u}{2\sinh(u/2)} e^{-u^2/4} dN_k(u) &= -\int_0^\infty N_k(u) \frac{d}{du} \left\{ \frac{u}{2\sinh(u/2)} e^{-u^2/4} \right\} du \\ &\ll C_1 \mathrm{Vol}(X_k), \end{aligned}$$

which ends the proof. □

Proof of Theorem 3.1. First notice that by formula (3), we have

$$\left|\frac{\log\det(\Delta_{X_k})}{\mathrm{Vol}(X_k)} - E\right| \leq O(\mathrm{Vol}(X_k)^{-1}) + \mathcal{D}^{(1)}_{X_k} + \mathcal{D}^{(2)}_{X_k},$$

where

$$\mathcal{D}^{(1)}_{X_k} = \frac{1}{\mathrm{Vol}(X_k)}\int_1^\infty \frac{|S_{X_k}(t)-1|}{t}dt,\ \ \mathcal{D}^{(2)}_{X_k} = \frac{1}{\mathrm{Vol}(X_k)}\int_0^1 \frac{S_{X_k}(t)}{t}dt.$$

Let us fix $\epsilon > 0$. We first investigate $\mathcal{D}^{(1)}_{X_k}$. Using $\mathscr{H}_1(\eta)$ and $\mathscr{H}_2(C_0, \alpha, L_0)$, we can use Lemma 3.3 and write

$$\int_1^\infty \frac{|S_{X_k}(t)-1|}{t}dt \leq \int_1^R \frac{S_{X_k}(t)}{t}dt + \log(R) + C_2\mathrm{Vol}(X_k)\int_R^\infty \frac{e^{-\eta_0 t}}{t}dt,$$

for any $R > 1$. Fixing $R = R(\epsilon)$ so large that

$$C_2\int_R^\infty \frac{e^{-\eta_0 t}}{t}dt \leq \epsilon,$$

we have

$$\mathcal{D}^{(1)}_{X_k} \leq \frac{1}{\mathrm{Vol}(X_k)}\int_1^R \frac{S_{X_k}(t)}{t}dt + \frac{\log(R)}{\mathrm{Vol}(X_k)} + \epsilon.$$

We now pick $L_1 > 1$ (to be adjusted later on) and write

$$\int_1^R \frac{S_{X_k}(t)}{t}dt = \int_1^R \frac{S^{L_1,-}_{X_k}(t)}{t}dt + \int_1^R \frac{S^{L_1,+}_{X_k}(t)}{t}dt,$$

where

$$S^{L_1,-}_{X_k}(t) = \frac{e^{-t/4}}{(4\pi t)^{1/2}}\sum_{m\ell(\gamma)\leq L_1}\frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)}e^{-(m\ell(\gamma))^2/4t},$$

$$S^{L_1,+}_{X_k}(t) = \frac{e^{-t/4}}{(4\pi t)^{1/2}}\sum_{m\ell(\gamma)> L_1}\frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)}e^{-(m\ell(\gamma))^2/4t}.$$

Clearly we have

$$\int_1^R \frac{S^{L_1,+}_{X_k}(t)}{t}dt \leq C_3\sum_{m\ell(\gamma)>L_1}\frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)}e^{-(m\ell(\gamma))^2/4R},$$

for some universal constant $C_3 > 0$. Using Lemma 3.2 and a summation by parts, we deduce that

$$\int_1^R \frac{S^{L_1,+}_{X_k}(t)}{t}dt \leq C_4\mathrm{Vol}(\mathrm{X_k})\int_{L_1}^\infty \left|\frac{d}{du}\left(\frac{u}{\sinh(u/2)}e^{-u^2/(4R)}\right)\right|e^u du.$$

We now take $L_1 = L_1(\epsilon)$ so large that

$$C_4 \int_{L_1}^{\infty} \Big| \frac{d}{du} \Big(\frac{u}{\sinh(u/2)} e^{-u^2/(4R)} \Big) \Big| e^u du < \epsilon.$$

We now observe that if $\mathscr{H}_2(C, L_1, \alpha)$ holds, we have

$$\int_1^R \frac{S_{X_k}^{L,-}(t)}{t} dt = \sum_{m\ell(\gamma) \le L_1} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} \int_1^R e^{-(m\ell(\gamma))^2/4t} \frac{e^{-t/4}}{\sqrt{4\pi} t^{3/2}} dt$$
$$\le C_5 L_1 N_k(L_1) \le C_5 C L_1 \mathrm{Vol}(X_k)^\alpha \le C_6(L_1, C) \mathrm{Vol}(X_k)^\alpha,$$

for some possibly large constant $C_6(L_1, C) > 0$. In a nutshell, we have obtained, provided that $\mathscr{H}_2(C, L_1, \alpha)$ is satisfied with $L_1 = L_1(\epsilon)$ taken large enough,

$$\limsup_{\mathrm{Vol}(X_k) \to +\infty} \mathcal{D}_{X_k}^{(1)} \le 2\epsilon.$$

We now turn our attention to $\mathcal{D}_{X_k}^{(2)}$, and this is where a good control of sums over short geodesics is required. We first use the same idea as above by writing

$$\mathcal{D}_{X_k}^{(2)} = \frac{1}{\mathrm{Vol}(X_k)} \int_0^1 \frac{S_{X_k}(t)}{t} dt = \frac{1}{\mathrm{Vol}(X_k)} \int_0^1 \frac{S_{X_k}^{L_2,-}(t)}{t} dt + \frac{1}{\mathrm{Vol}(X_k)} \int_0^1 \frac{S_{X_k}^{L_2,+}(t)}{t} dt.$$

Writing for $t > 0$,

$$S_{X_k}^{L_2,+}(t) \le C_7 t^{-1/2} \sum_{m\ell(\gamma) > L_2} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} e^{-(m\ell(\gamma))^2/4t},$$

where $C_7 > 0$ is universal, we have by Fubini

$$\int_0^1 \frac{S_{X_k}^{L_2,+}(t)}{t} dt \le C_7 \sum_{m\ell(\gamma) > L_2} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} G(m\ell(\gamma)),$$

where for $u > 0$,

$$G(u) = \int_0^1 t^{-3/2} e^{-u^2/4t} dt.$$

Notice that $u \mapsto G(u)$ is a decreasing function and by a change of variable we have actually for all $u > 0$,

$$G(u) = \frac{4}{u} \int_{u/2}^{\infty} e^{-x^2} dx.$$

We have therefore the bound

$$G(u) = \frac{4}{u} \int_{u/2}^{\infty} e^{-x^2/2 - x^2/2} dx \le \frac{4}{u} e^{-u^2/8} \int_0^{\infty} e^{-x^2/2} dx = \frac{2\sqrt{2\pi}}{u} e^{-u^2/8}.$$

As a consequence we get for $L_2 > 1$

$$\int_0^1 \frac{S_{X_k}^{L_2,+}(t)}{t}dt \leq C_8 \sum_{m\ell(\gamma)>L_2} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} e^{-(m\ell(\gamma))^2/8},$$

and by using Lemma 3.2 and a summation by parts, we can definitely fix $L_2 = L_2(\epsilon)$ large enough so that

$$\int_0^1 \frac{S_{X_k}^{L_2,+}(t)}{t}dt \leq C_8 \sum_{m\ell(\gamma)>L_2} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} e^{-(m\ell(\gamma))^2/8} \leq \mathrm{Vol}(X_k)\epsilon.$$

From the above bound on $G(u)$ we also deduce

$$\int_0^1 \frac{S_{X_k}^{L_2,-}(t)}{t}dt \leq C_9 \sum_{m\ell(\gamma)\leq L_2} \frac{\ell(\gamma)}{\sinh(m\ell(\gamma)/2)} \frac{1}{m\ell(\gamma)}$$
$$\leq C_9' \sum_{m\ell(\gamma)\leq L_2} \frac{1}{m^2\ell(\gamma)}.$$

By writing

$$\sum_{m\ell(\gamma)\leq L_2} \frac{1}{m^2\ell(\gamma)} = \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{\ell(\gamma)\leq L_2/m} \frac{1}{\ell(\gamma)} \leq \frac{\pi^2}{6} \sum_{\ell(\gamma)\leq L_2} \frac{1}{\ell(\gamma)},$$

we can use estimate (6) and $\mathscr{H}_2(C, L_2(\epsilon), \alpha)$ and we have again as above

$$\int_0^1 \frac{S_{X_k}^{L_2,-}(t)}{t}dt \leq C_{10}(L_2, C)\mathrm{Vol}(X_k)^{\alpha},$$

where $C_{10}(L_2, C) > 0$ is some (possibly very large) constant depending on L_2, C and $0 < \alpha < 1$. We have therefore shown that whenever (X_k) satisfies $\mathscr{H}_2(C, L(\epsilon), \alpha)$ for some $C > 0$ and with $L(\epsilon) = \max\{L_1, L_2\}$, we have

$$\limsup_{\mathrm{Vol}(X_k)\to+\infty} \left| \frac{\log\det(\Delta_{X_k})}{\mathrm{Vol}(X_k)} - E \right| \leq 3\epsilon.$$

Theorem 3.1 is proved. □

4 Hypotheses $\mathscr{H}_1$ and $\mathscr{H}_2$ hold with high probability

Theorem 1.1 follows immediately from Theorem 3.1 if one can establish for the three models of random hyperbolic surfaces considered here that there exists $\eta > 0$ and such that $\mathscr{H}_1(\eta)$ holds a.a.s. and also that there exists $0 < \alpha < 1$ such that for all L large, one can find $C > 0$ such that $\mathscr{H}_2(C, L, \alpha)$ also holds a.a.s.

Indeed we then have for all $\epsilon > 0$, and all $\mathrm{Vol}(X)$ large enough,

$$\mathbb{P}\Big(\frac{\log\det(\Delta_X)}{\mathrm{Vol}(X)} \in [E-\epsilon, E+\epsilon]\Big)$$
$$\geq \mathbb{P}(X \in \mathscr{H}_1(\eta) \cap \mathscr{H}_2(C_0, L_0, \alpha) \cap \mathscr{H}_2(C, L(\epsilon), \alpha)),$$

with

$$\lim_{\mathrm{Vol}(X)\to\infty} \mathbb{P}(X \in \mathscr{H}_1(\eta) \cap \mathscr{H}_2(C_0, L_0, \alpha) \cap \mathscr{H}_2(C, L(\epsilon), \alpha)) = 1.$$

4.1 Random covers. First we recall that the cover $X_n \to X$ may not be connected but we know from [20] that

$$\lim_{n\to+\infty} \mathbb{P}(X_n \text{ connected}) = 1.$$

We can therefore either restrict ourselves to connected surfaces X_n or modify the definition of the regularized determinant by setting

$$\zeta_{X_n}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(\mathrm{Tr}(e^{-t\Delta_X}) - d_n)dt,$$

where d_n is the number of connected components of X_n and use the fact that $d_n = 1$ with high probability.

It was recently shown in [21] that there exists a uniform spectral gap for random covers X_n a.a.s. as $n \to +\infty$. More precisely we have for all $0 < \eta < \min\{3/16, \lambda_1(X)\}$,

$$\lim_{n\to+\infty} \mathbb{P}(\lambda_1(X_n) \geq \eta) = 1.$$

Therefore $\mathscr{H}_1(\eta)$ holds a.a.s. provided η is taken small enough. On the other hand, property $\mathscr{H}_2$ is less obvious from the existing litterature and will require some explanations. We recall that given a random homomorphism $\phi_n : \Gamma \to \mathcal{S}_n$, one can define a unitary representation ρ_n of Γ by setting

$$\rho_n(\gamma)(f) := f \circ \phi_n(\gamma)^{-1},$$

where $f \in L^2([n])$, and the representation space is $L^2([n]) \simeq \mathbb{C}^n$. The main interest of this representation is the following fact, often called the "Venkov–Zograf induction formula". For all $\mathrm{Re}(s) > 1$, one can define the Selberg zeta function of X_n by

$$Z_{X_n}(s) := \prod_{m\geq 0} \prod_{\gamma\in\mathcal{P}_{X_n}} (1 - e^{-(s+m)\ell(\gamma)}).$$

One can also look at the twisted Selberg zeta function of the base $X = \Gamma\backslash\mathbb{H}^2$ defined for $\mathrm{Re}(s) > 1$ by

$$Z_{X,\rho_n}(s) := \prod_{m\geq 0} \prod_{\gamma\in\mathcal{P}_X} \det(I - \rho_n(\gamma)e^{-(s+m)\ell(\gamma)}).$$

It turns out that we have for all s, $Z_{X_n}(s) = Z_{X,\rho_n}(s)$; see [40, p. 51]. By computing logarithmic derivatives we have, for all $\mathrm{Re}(s) > 1$,

$$\frac{Z'_{X_n}(s)}{Z_{X_n}(s)} = \sum_{\gamma\in\mathcal{P}_{X_n}}\sum_{q\geq 1}\frac{\ell(\gamma)e^{-sq\ell(\gamma)}}{1-e^{-q\ell(\gamma)}} = \sum_{\gamma\in\mathcal{P}_X}\sum_{q\geq 1}\frac{\ell(\gamma)\mathrm{tr}(\rho_n(\gamma^q))e^{-sq\ell(\gamma)}}{1-e^{-q\ell(\gamma)}}.$$

Let $\phi \in C_0^\infty(\mathbb{R}^+)$ be a compactly supported smooth test function, and set

$$\psi(s) := \int_0^\infty e^{sx}\phi(x)dx.$$

One can check that $\psi(s)$ is actually analytic on $\mathbb{C}$ and by the Fourier inversion formula we have, for all $A > 1$,

$$\begin{aligned}\frac{1}{2i\pi}\int_{A-i\infty}^{A+i\infty}\frac{Z'_{X_n}(s)}{Z_{X_n}(s)}\psi(s)ds &= \sum_{\gamma\in\mathcal{P}_{X_n}}\sum_{q\geq 1}\frac{\ell(\gamma)}{1-e^{-q\ell(\gamma)}}\phi(q\ell(\gamma))\\ &= \sum_{\gamma\in\mathcal{P}_X}\sum_{q\geq 1}\frac{\ell(\gamma)\mathrm{tr}(\rho_n(\gamma^q))}{1-e^{-q\ell(\gamma)}}\phi(q\ell(\gamma)).\end{aligned}$$

See, for example, [18] §3 for more details on the derivation of this formula. By carefully choosing the test function ϕ we deduce that for all $\mathcal{L} \in \mathbb{R}^+$, we recover the identity

$$\sum_{\gamma\in\mathcal{P}_{X_n}}\sum_{\substack{q\geq 1\\ q\ell(\gamma)=\mathcal{L}}}\ell(\gamma) = \sum_{\gamma\in\mathcal{P}_X}\sum_{\substack{q\geq 1\\ q\ell(\gamma)=\mathcal{L}}}\ell(\gamma)\mathrm{tr}(\rho_n(\gamma^q)).$$

Notice that this formula can be proved directly by group theoretic arguments; see, for example, in [16], in the proof of theorem 7.1. From this identity we deduce that for all $L > 0$, we have

$$\sum_{\gamma\in\mathcal{P}_{X_n}}\sum_{\substack{q\geq 1\\ q\ell(\gamma)\leq L}}\ell(\gamma) = \sum_{\gamma\in\mathcal{P}_X}\sum_{\substack{q\geq 1\\ q\ell(\gamma)\leq L}}\ell(\gamma)\mathrm{tr}(\rho_n(\gamma^q)).$$

In particular we have

$$\ell_0(X_n)N_{X_n}(L) \leq L\sum_{\gamma\in\mathcal{P}_X}\sum_{\substack{q\geq 1\\ q\ell(\gamma)\leq L}}\mathrm{tr}(\rho_n(\gamma^q)),$$

where $\ell_0(X_n)$ denotes the shortest closed geodesic length on X_n. We point out that we have actually $\mathrm{tr}(\rho_n(\gamma^q)) = \mathrm{Fix}(\phi_n(\gamma^q))$, where $\mathrm{Fix}(\sigma)$ denotes the number of fixed points of the permutation σ acting on $[n]$. From the combinatorial analysis of Magee–Puder [22, 21], we know that for all primitive $\gamma \in \Gamma$ and $q \geq 1$, we have

$$\lim_{n\to\infty}\mathbb{E}(\mathrm{Fix}(\phi_n(\gamma^q))) = d(q),$$

where $d(q)$ stands for the number of divisors of q. Noticing that in the random cover model we have always $\ell_0(X_n) \geq \ell_0(X)$, this is enough to conclude that for all L, we have

$$\lim_{n\to\infty} \mathbb{E}(N_{X_n}(L)) \leq C(\Gamma, L),$$

where $C(\Gamma, L) > 0$ is some (possibly large) constant. Applying Markov's inequality, we get that for all $\varepsilon > 0$ and L fixed,

$$\lim_{n\to\infty} \mathbb{P}(N_{X_n}(L) \leq \mathrm{Vol}(X_n)^{\varepsilon}) = 1.$$

As a conclusion, in the random cover model, $\mathscr{H}_2(C, L, \alpha)$ is satisfied a.a.s. for all L large and all $\alpha > 0$.

4.2 Brooks–Makover model. In the paper [9, Theorem 2.2], they show that there exists a constant $C_1 > 0$ such that as $n \to \infty$,

$$\mathbb{P}(\lambda_1(S_n^C) \geq C_1) \to 1.$$

In other words, property $\mathscr{H}_1(\eta)$ is satisfied a.a.s. for some $\eta > 0$.

We point out that contrary to the model of random covers, the systole of S_n^C can be arbitrarily small, but we actually know that there exists $C_2 > 0$ such that as $n \to +\infty$,

$$\mathbb{P}(\ell_0(S_n^C) \geq C_2) \to 1.$$

Counting results for closed geodesics follow from the later work of Petri [32]. More precisely, one can derive from [32] the following fact.

Proposition 4.1. *For all $L > 0$ fixed, we can find an integer N_L and a finite set of words*

$$\mathcal{W}_L \subset \{l, r\}^{N_L},$$

such that with high probability as $n \to \infty$,

$$N_n(L) := \#\{(\gamma, m) \in \mathcal{P}_{S_n^C} \times \mathbb{N}^* : m\ell(\gamma) \leq L\} \leq \sum_{w\in\mathcal{W}_L} Z_{n,w},$$

where $Z_{n,w}$ are integer-valued random variables. In addition, each $Z_{n,w}$ converges in the sense of moments (and hence in distribution) as $n \to \infty$ to a Poisson variable with expectation $\lambda_w > 0$.

By applying Markov's inequality, one deduces readily that for all $\varepsilon > 0$ we have a.a.s.

$$\sum_{w\in\mathcal{W}_L} Z_{n,w} \leq n^{\epsilon} \leq C_L \mathrm{Vol}(S_n^C)^{\varepsilon}.$$

This is enough to conclude that for all $\varepsilon > 0$, with high probability as $n \to \infty$, we have

$$N_n(L) \leq C\mathrm{Vol}(S_n^C)^\varepsilon,$$

and therefore property $\mathscr{H}_2(C, \alpha, L)$ holds for any choice of $\alpha > 0$, just like in the previous model.

Let us now give some details on the proof of Proposition 4.1. The first step is to reduce the problem to a counting bound for S_n^O. In [9, Section 3], they introduce the notion of a "large cusps" condition for S_n^O. This condition is satisfied a.a.s. for S_n^O as $n \to \infty$; see [9, Theorem 2.1]. The main interest of this condition is [9, Theorem 3.2], see also [32, Lemma 2.5], which allows to show that provided this "large cusps" condition is satisfied, one can bound

$$N_n(L) \leq \#\{(\gamma, m) \in \mathcal{P}_{S_n^0} \times \mathbb{N}^* : m\ell(\gamma) \leq 2L\} =: N_n^O(2L).$$

As explained in [9], §4, closed geodesics and their length in S_n^O can be described via the combinatorial data of $(\mathcal{G}_n, \mathcal{O})$: any closed geodesic in S_n^O corresponds to a word $w \in \{l, r\}^N$, for some $N > 0$. To this word one can associate a matrix M_w in $SL_2(\mathbb{N})$ via the rule

$$M_w = W_1 \cdots W_N,$$

where $W_j = \mathcal{L}$ if $w_j = l$ and $W_j = \mathcal{R}$ if $w_j = r$, where

$$\mathcal{L} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The length ℓ_w of the corresponding geodesic on S_n^O is then given by

$$\mathrm{tr}(M_w) = 2\cosh(\ell_w/2).$$

All we need to check is that fixing L implies finiteness of the corresponding set of words (by bounding their word length). This is done in [33, Lemma 3.1]. One obtains therefore that L being fixed, there exists a finite subset $\mathcal{W}_L \subset \{l, r\}^{N_L}$ for some large $N_L > 0$, such that for all n, all closed geodesics with length $\leq 2L$ on S_n^O are given by words $w \in W_L$. Proposition 4.1 now follows directly from [32, Theorem B].

4.3 Weil–Petersson model. If f is a measurable non-negative function on moduli space $\mathscr{M}_g$, we will denote by $\int_{\mathscr{M}_g} f(X)dX$ the corresponding integral with respect to Weil–Petersson volume. In the latter V_g will denote the Weil–Petersson volume of $\mathscr{M}_g$ so that the expectation of f is given by

$$\mathbb{E}_g(f) := \frac{1}{V_g}\int_{\mathscr{M}_g} f(X)dX.$$

In [25], the following fact was proved. There exists $\eta > 0$ such that as $g \to +\infty$, we have

$$\mathbb{P}(\lambda_1(X) \geq \eta) \to 1.$$

The constant η given by Mirzakhani follows from Cheeger's inequality and an estimate a.a.s. of Cheeger's isoperimetric constant. It was shown independently in [42, 43] that one can actually take $\eta = 3/16 - \epsilon$, and more recently $\eta = 2/9 - \epsilon$ by Anantharaman and Monk [1]. This shows that $\mathscr{H}_1(\eta)$ holds with high probability as $g \to +\infty$ for some universal $\eta > 0$. In [25, Theorem 4.2] and the remark after, Mirzakhani proved that there exists a universal $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$, one has for all g large

$$\mathbb{P}(\ell_0(X) \leq \epsilon) \leq C\epsilon^2,$$

where C is uniform in g. In particular for all $\varepsilon > 0$, we get for all g large

$$\mathbb{P}(\ell_0(X) \geq \mathrm{Vol}(X)^{-\varepsilon}) \geq 1 - O(\mathrm{Vol}(X)^{-2\varepsilon}).$$

On the other hand, in the paper [26], Mirzakhani and Petri showed that for all $L > 0$ fixed, the random variable

$$N_g^0(L) := \#\{\gamma \in \mathcal{P}_X : \ell(\gamma) \leq L\}$$

converges in distribution as $g \to +\infty$ to a Poisson variable Z_{λ_L} with parameter

$$\lambda_L := \int_0^L \frac{e^t + e^{-t} - 2}{2t} dt.$$

Moreover, we have also convergence of all moments with $p \in \mathbb{N}$

$$\lim_{g \to \infty} \mathbb{E}((N_g^0(L))^p) = \mathbb{E}(Z_{\lambda_L}^p).$$

An application of Markov's inequality then shows that for all $\varepsilon > 0$, for all g large enough

$$\mathbb{P}(N_g^0(L) \leq \mathrm{Vol}(X)^{\varepsilon}) \geq 1 - C_L \mathrm{Vol}(X)^{-\varepsilon}.$$

To control the counting function

$$N_g(L) := \#\{(\gamma, m) \in \mathcal{P}_X \times \mathbb{N}^* : m\ell(\gamma) \leq L\},$$

we write

$$N_g(L) \leq \sum_{m=1}^{[L/\ell_0(X)]+1} N_g^0(L/m) \leq N_g^0(L)\Big(1 + \frac{L}{\ell_0(X)}\Big).$$

For all $\varepsilon > 0$, we have with high probability as $g \to +\infty$

$$N_g(L) \leq \mathrm{Vol}(X)^{\varepsilon} + L\mathrm{Vol}(X)^{2\varepsilon} = O_L(\mathrm{Vol}(X)^{2\varepsilon}),$$

and thus for all L large, there exists $C_L > 0$ such that $\mathscr{H}_2(C_L, L, \alpha)$ holds with high probability for all $\alpha > 0$ on $\mathscr{M}_g$ as $g \to +\infty$.

5 Convergence results for moments of $\log\det(\Delta_X)$.

In this last section, we give the proof of the following fact.

Theorem 5.1. *In the Weil–Petersson model, for all* $0 < \beta < 1$ *we have*

$$\lim_{g\to\infty} \frac{1}{V_g(4\pi(g-1))^\beta} \int_{\mathcal{M}_g} |\log\det(\Delta_X)|^\beta dX = E^\beta.$$

In the random cover model, let $\chi_0 = \mathbf{1}_{\{X_n \text{ connected}\}}$*. Then as* $n \to +\infty$*, for all* $\beta > 0$*, we have*

$$\lim_{n\to\infty} \mathbb{E}\Big(\frac{|\log\det(\Delta_{X_n})|^\beta}{\mathrm{Vol}(X_n)^\beta}\chi_0\Big) = E^\beta.$$

Proof for the Weil–Petterson model. The proof for the Weil–Petersson model is a rather direct consequence of Theorem 1.1 and some estimates of Mirzakhani [24]. We first need an a priori estimate for $|\log\det(\Delta_X)|$ which follows from similar ideas as in Theorem 3.1, without the probabilistic input. We use Vinogradov's notation $\ll$ where the implied constant is universal.

By using Buser's counting bound [10], as in the proof of Lemma 3.2, we have the following universal bound for the number of closed geodesics of a surface with genus g:

$$\begin{aligned} N_X(L) &\leq (g-1)e^{L+6} + \frac{L}{\ell_0(X)}(3g-3) \\ &\ll \mathrm{Vol}(X)e^L\Big(1+\frac{1}{\ell_0(X)}\Big), \end{aligned} \tag{7}$$

for some universal constant $A_1 > 0$. We now prove an a priori upper bound for $|\log\det(\Delta_X)|$. Following ideas of Wolpert [41], it is convenient to write for all $\mathrm{Re}(s)$ large

$$\zeta_X(s) = \sum_{0<\lambda_j<1/4} \lambda_j^{-s} + \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\Big(\mathrm{tr}(e^{-t\Delta_X}) - \sum_{0\leq\lambda_j<1/4} e^{-t\lambda_j}\Big)dt,$$

which can be rewritten as

$$\begin{aligned} \zeta_X(s) = &\sum_{0<\lambda_j<1/4} \lambda_j^{-s} - \sum_{0\leq\lambda_j<1/4} H(s,\lambda_j) + \zeta_X^{(1)}(s) \\ &+ \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\Big(S_X(t) - \sum_{0\leq\lambda_j<1/4} e^{-t\lambda_j}\Big)dt + \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}S_X(t)dt, \end{aligned}$$

where we have set for $\mathrm{Re}(s)$ large

$$H(s,\lambda) := \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}e^{-\lambda t}dt.$$

By integration by parts and elementary properties of the Euler gamma function, we then observe that for all $\lambda \in [0, 1/4]$, $s \mapsto H(s, \lambda)$ has an analytic extension to $s = 0$. Morevover, if we set

$$C(\lambda) := \frac{d}{ds}\Big|_{s=0} H(\lambda, s),$$

then there exists a universal constant $A_1 > 0$ such that for all $\lambda \in [0, 1/4]$,

$$|C(\lambda)| \leq A_1.$$

This formula leads to the identity

$$\log\det(\Delta_X) = \sum_{0<\lambda_j<1/4} \log(\lambda_j) + \sum_{0<\lambda_j<1/4} C(\lambda_j) - \frac{d}{ds}\Big|_{s=0} \zeta_X^{(1)}(s)$$
$$- \int_0^1 \frac{S_X(t)}{t} dt - \int_1^\infty \frac{(S_X(t) - \sum_{0\leq\lambda_j<1/4} e^{-\lambda_j t})}{t} dt.$$

By mimicking the proof of Lemma 3.3 and using the above counting bound, we deduce that for all $t \geq 1$,

$$\Big| S_X(t) - \sum_{0\leq\lambda_j<1/4} e^{-t\lambda_j} \Big| \ll \mathrm{Vol}(X)\Big(1 + \frac{1}{\ell_0(X)}\Big) e^{-t/4}.$$

By Fubini and the estimate on $u \mapsto G(u)$ we have also

$$\int_0^1 \frac{S_X(t)}{t} dt \ll \sum_{m,\gamma} \frac{e^{-(m\ell(\gamma))^2/8}}{m\ell(\gamma)} \ll \sum_{m\ell(\gamma)\leq 1} \frac{1}{m\ell(\gamma)} + \sum_{m\ell(\gamma)>1} e^{-(m\ell(\gamma))^2/8}$$
$$\ll \sum_{m\ell(\gamma)\leq 1} \frac{1}{m\ell(\gamma)} + \mathrm{Vol}(X)\Big(1 + \frac{1}{\ell_0(X)}\Big).$$

By noticing that we have

$$\sum_{m\ell(\gamma)\leq 1} \frac{1}{m\ell(\gamma)} \ll \frac{\log^+ \ell_0^{-1}(X)}{\ell_0(X)} N_X^0(1),$$

where $\log^+(x) = \max\{0, \log(x)\}$ and $N_X^0(L)$ is the counting function for primitive closed geodesics, we have obtained the estimate

$$|\log\det(\Delta_X)| \ll \mathrm{Vol}(X)\Big(1 + |\log(\lambda_*(X))| + \frac{1}{\ell_0(X)} +\Big) + \frac{\log^+ \ell_0^{-1}(X)}{\ell_0(X)} N_X^0(1),$$

where we have set $\lambda_*(X) = \min\{\lambda_1(X), 1/4\}$, and we have used the rough bound of Buser [10]:

$$\#\{\lambda_j < 1/4\} \leq 4g - 3 = 1 + \frac{\mathrm{Vol}(X)}{\pi}.$$

Notice that the optimal bound of Otal–Rosas [31],

$$\#\{\lambda_j < 1/4\} \leq 2g - 2,$$

won't make any difference here.

This estimate of $|\log \det(\Delta_X)|$ is consistent with the fact that $\det(\Delta_X)$ has exponential growth when X approaches certain boundary points of the (compactified) moduli space, a fact that was rigorously established by Wolpert in [41]. In particular, the so-called "bosonic Polyakov integral" involving the determinant over the moduli space is indeed infinite; see [41].

Using the inequality for all $\beta > 0$ and all $a_j \geq 0$

$$\left(\sum_{j=1}^{4} a_j\right)^{\beta} \leq 4^{\beta}(\max_j a_j)^{\beta} \leq 4^{\beta}\left(\sum_{j=1}^{4} a_j^{\beta}\right),$$

we end up with the estimate

$$\frac{|\log \det(\Delta_X)|^{\beta}}{\mathrm{Vol}(X)^{\beta}} \ll \left(1 + |\log(\lambda_*(X))|^{\beta} + \frac{1}{\ell_0(X)^{\beta}} + \mathrm{Vol}(X)^{-\beta}\Big(\frac{\log^+ \ell_0^{-1}(X)}{\ell_0(X)} N_X^0(1)\Big)^{\beta}\right).$$

From Mirzakhani [24, Corollary 4.3], we know that

$$\int_{\mathscr{M}_g} \frac{1}{\ell_0(X)} dX \leq C V_g,$$

where $C > 0$ is independent of g, from which we can deduce easily that for all $0 < \alpha < 1$, we have

$$\int_{\mathscr{M}_g} \Big(\frac{\log^+ \ell_0^{-1}(X)}{\ell_0(X)}\Big)^{\alpha} dX \leq C V_g,$$

for some $C > 0$ uniform in g. Indeed, for all $0 < \alpha < 1$, there exists a universal constant $r_0 > 0$ such that $0 < x \leq r_0$ implies

$$\Big(\frac{|\log x|}{x}\Big)^{\alpha} \leq \frac{1}{x}.$$

By integrating over $\{X \in \mathscr{M}_g : \ell_0(X) \leq r_0\}$ and using Mirzakhani's bound we get the desired bound, while on the complementary set

$$X \mapsto \Big(\frac{\log^+ \ell_0^{-1}(X)}{\ell_0(X)}\Big)^{\alpha}$$

is uniformly bounded.

On the other hand, Cheeger's inequality says that

$$\lambda_1(X) \geq \frac{h_X^2}{4},$$

where h_X is the so-called Cheeger constant of X, which is defined by an isoperimetric quantity; see, for example, in [10], chapter 8. Again by Mirzakhani [24, Theorem 4.8], we know that for all $0 \leq \alpha < 2$, we have

$$\int_{\mathscr{M}_g} \frac{1}{(h_X)^\alpha} dX \leq CV_g,$$

where $C > 0$ is again uniform with respect to g. In particular, we deduce that for all $\beta > 0$

$$\int_{\mathscr{M}_g} |\log(\lambda_*(X))|^\beta dX \leq CV_g.$$

Assuming that $\beta < 1$, we choose $p > 1$ such that $\beta < p\beta < 1$ and let q be such that $1/p + 1/q = 1$. By Hölder's inequality we get

$$\mathbb{E}_g\Big(\Big(\frac{\log^+ \ell_0^{-1}(X)}{\ell_0(X)} N_X^0(1)\Big)^\beta\Big) \leq \Big[\mathbb{E}_g\Big(\Big(\frac{\log^+ \ell_0^{-1}(X)}{\ell_0(X)}\Big)^{\beta p}\Big)\Big]^{1/p} [\mathbb{E}_g(N_X^0(1)^{\beta q})]^{1/q},$$

which by the convergence of moments in Mirzakhani–Petri [26] and the above remarks is uniformly bounded as $g \to +\infty$. We therefore have shown that for all $\beta < 1$, there exists $C > 0$ independent of g such that

$$\int_{\mathscr{M}_g} \frac{|\log\det(\Delta_X)|^\beta}{\mathrm{Vol}(X)^\beta} dX \leq CV_g. \tag{8}$$

We now fix $\epsilon > 0$, and $0 < \beta < 1$. By Theorem 1.1, there exists a subset $\mathcal{A}_g(\epsilon) \subset \mathscr{M}_g$, with $\mathbb{P}(\mathcal{A}_g(\epsilon)) \to 1$ as $g \to +\infty$, such that for all $X \in \mathcal{A}_g(\epsilon)$,

$$(E - \epsilon)^\beta \leq \frac{|\log\det(\Delta_X)|^\beta}{\mathrm{Vol}(X)^\beta} \leq (E + \epsilon)^\beta.$$

Therefore we have

$$\frac{1}{V_g} \int_{\mathscr{M}_g} \frac{|\log\det(\Delta_X)|^\beta}{\mathrm{Vol}(X)^\beta} dX \leq (E+\epsilon)^\beta + \frac{1}{V_g} \int_{\mathcal{A}_g(\epsilon)^c} \frac{|\log\det(\Delta_X)|^\beta}{\mathrm{Vol}(X)^\beta} dX,$$

while

$$(E - \epsilon)^\beta \mathbb{P}(\mathcal{A}_g(\epsilon)) \leq \frac{1}{V_g} \int_{\mathscr{M}_g} \frac{|\log\det(\Delta_X)|^\beta}{\mathrm{Vol}(X)^\beta} dX.$$

We now apply Hölder's inequality and estimate (8). Since $0 < \beta < 1$, let $q > 1$ be chosen such that $0 < q\beta < 1$ and let $0 < p < \infty$ be such that $1/p + 1/q = 1$, we have

$$\begin{aligned} \frac{1}{V_g} \int_{\mathcal{A}_g(\epsilon)^c} \frac{|\log\det(\Delta_X)|^\beta}{\mathrm{Vol}(X)^\beta} dX &\leq (\mathbb{P}(\mathcal{A}_g(\epsilon)^c))^{1/p} \Big(\frac{1}{V_g} \int_{\mathscr{M}_g} \frac{|\log\det(\Delta_X)|^{q\beta}}{\mathrm{Vol}(X)^{q\beta}} dX\Big)^{1/q} \\ &\leq C(\mathbb{P}(\mathcal{A}_g(\epsilon)^c)^{1/p}. \end{aligned}$$

Since $\lim_{g\to+\infty} \mathbb{P}(\mathcal{A}_g(\epsilon)^c) = 0$, we definitely have for all g large enough

$$(E-\epsilon)^\beta - \epsilon \le \frac{1}{V_g}\int_{\mathcal{M}_g} \frac{|\log\det(\Delta_X)|^\beta}{\mathrm{Vol}(X)^\beta} dX \le (E+\epsilon)^\beta + \epsilon,$$

and the proof is done for the Weil–Petterson case. □

We would like to mention that in this smooth Weil–Petterson model, it is very likely that by using finer estimates one can improve the result to the following statement: for any $\beta \in (0,2)$ we have

$$\lim_{g\to\infty} \frac{1}{V_g(4\pi(g-1))^\beta}\int_{\mathcal{M}_g} |\log\det(\Delta_X)|^\beta dX = E^\beta,$$

while for all $\beta \ge 2$,

$$\int_{\mathcal{M}_g} |\log\det(\Delta_X)|^\beta dX = +\infty.$$

This fact was pointed out to the author by Yunhui Wu and Yuxin He in a private communication and will be published in a separate paper.

Proof for the random covers model. We recall that we have $\chi_0 = \mathbf{1}_{\{X_n \text{ connected}\}}$. In this model, the systole is bounded uniformly from below, so the a priori bound for $\log\det(\Delta_{X_n})$, whenever X_n is connected, is actually

$$\frac{|\log\det(\Delta_{X_n})|^\beta}{\mathrm{Vol}(X_n)^\beta} \le A(1 + |\log(\lambda_*(X))|^\beta),$$

for some constant $A > 0$ independent of n. Using the same arguments as above based on Hölder's inequality, the result follows directly from the next fact.

Proposition 5.2. *Assuming that X_n is connected, then we have for all n large,*

$$\lambda_1(X_n) \ge \frac{C_\Gamma}{n^{3/2}},$$

where C_Γ depends only on the base surface $X = \Gamma\backslash\mathbb{H}^2$. Consequently for every exponent β with $0 < \beta$,

$$\mathbb{E}(|\log(\lambda_*(X))|^\beta \chi_0) \le C,$$

where $C > 0$ is uniform with respect to n.

We first need to prove a deterministic lower bound on $\lambda_1(X_n)$, provided that X_n is connected. We know that the spectrum of Δ_{X_n} coincides, with multiplicity, with the spectrum of Δ_{ρ_n}, which is the Laplacian on the base surface twisted by the unitary representation ρ_n of Γ defined previously. See, for example, [40, p. 51]. If X_n is connected, then

$$\lambda_1(X_n) = \min\{\lambda_1(X), \lambda_0(\Delta_{\rho_n^0})\},$$

where ρ_n^0 is the representation given by

$$\rho_n^0(\gamma)U := U_{\phi_n^{-1}(\gamma)},$$

where

$$U \in V_n^0 := \left\{ U \in \mathbb{C}^n : \sum_{j=1}^n U(j) = 0 \right\}$$

and

$$U_{\phi_n^{-1}(\gamma)}(j) = U(\phi_n^{-1}(\gamma)(j)).$$

Notice that ρ_n^0 is just the orthogonal complement to the trivial representation in ρ_n. Let us fix a system of generators of Γ, denoted by S. A result of Sunada [39] then says that there exists C_S depending only on X and S such that

$$\lambda_0(\Delta_{\rho_n^0}) \geq C_S \inf_{\substack{U \in V_n^0 \\ \|U\|=1}} \max_{\gamma \in S} \|\rho_n^0(\gamma)U - U\|.$$

Let us take $U \in V_n^0$ such that $\|U\| = 1$. We therefore have

$$1 = \|U\|^2 \leq n \max_j (\mathrm{Re}(U(j))^2 + \mathrm{Im}(U(j))^2),$$

and we can assume without loss of generality that we have $\mathrm{Re}(U(j_0)) \geq \frac{1}{\sqrt{2n}}$ for some $j_0 \in [n]$. Because we have in addition

$$\sum_j \mathrm{Re}(U(j)) = 0,$$

there exists also $j_1 \in [n]$ such that $\mathrm{Re}(U(j_1)) \leq 0$. If X_n is connected, then Γ acts transitively on $[n]$ via ϕ_n and there exists $\gamma_0 \in \Gamma$ such that $\phi_n(\gamma_0)^{-1}(j_0) = j_1$. To bound the word length of γ_0, consider the graph with set of vertices $[n]$ and define edges by connecting i to j if there exists $g \in S$ such that $\phi_n(g)(i) = j$. By transitivity of the action, this graph is connected and thus has diameter less than $n - 1$. Therefore we can choose γ_0 with word length (with respect to S) less than $n - 1$. We now have

$$\frac{1}{\sqrt{2n}} \leq |\mathrm{Re}(U(j_0)) - \mathrm{Re}(U(j_1))| \leq \|\rho_n^0(\gamma_0)U - U\|.$$

Writing

$$\gamma_0 = g_1 g_2 \cdots g_m,$$

with $g_j \in S$ and $m \leq n - 1$, we have therefore

$$\|\rho_n^0(\gamma_0)U - U\| \leq \sum_{j=1}^m \|\rho_n^0(g_j)U - U\| \leq (n-1) \max_{g \in S} \|\rho_n^0(g)U - U\|,$$

which yields

$$\max_{g\in S} \|\rho_n^0(g)U - U\| \geq \frac{1}{(n-1)\sqrt{2n}}.$$

The first claim of the proposition is proved. Alternatively, one can use directly a result of Brooks [7] which relates $\lambda_1(X_n)$ to the Cheeger constants of Schreier graphs of the covers (with a choice of generators of Γ) to obtain a similar lower bound $\lambda_1(X_n) \geq C_\Gamma n^{-2}$ which is slightly worse but good enough for our purpose.

From the proof of the uniform spectral gap in [21] one can directly derive that for all $1/4 > r > 0$, for all $\epsilon > 0$, as $n \to \infty$,

$$\mathbb{P}(\lambda_1(X_n) \leq r) \leq \frac{C_\epsilon}{n^{4\sqrt{(1/4-r)}-1-\epsilon}}.$$

From that we deduce that for all $\alpha < 1$ we can find $r_\alpha > 0$ such that

$$\mathbb{P}(\lambda_1(X_n) \leq r_\alpha) \leq \frac{C_\alpha}{n^\alpha}.$$

We now fix any $\beta > 0$ and fix some $0 < \alpha < 1$. We can use the fact that we have (by the lower bound on $\lambda_*(X)$)

$$|\log(\lambda_*(X))|^\beta = O((\log(n))^\beta),$$

where the implied constant is uniform with respect to n. We therefore get, as $n \to \infty$,

$$\mathbb{E}(|\log(\lambda_*(X))|^\beta \chi_0) \leq |\log(r_\alpha)|^\beta (1 + O(n^{-\alpha})) + O(n^{-\alpha}|\log(n)|^\beta) = O(1)$$

and the proof is done. □

We conclude with some comments.

- It would be interesting to know if a similar type of result can be proved for the Brooks–Makover model, in particular can one show that

$$\mathbb{E}\Big(\frac{1}{\ell_0(S_n^C)}\Big)$$

  is finite and uniformly bounded with respect to n ? This will require an effective version of the compactification procedure; see the paper of Mangoubi [23].
- It is likely that our results can be extended to finite area surfaces (see Efrat [15] for the definition and properties of determinants in this context), or even geometrically finite surfaces where a notion of determinant also holds; see [6].

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