THE NUMBER OF ZEROS OF LINEAR COMBINATIONS OF L-FUNCTIONS NEAR THE CRITICAL LINE*

By

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Abstract. In this paper, we investigate the zeros near the critical line of linear combinations of L-functions belonging to a large class, which conjecturally contains all L-functions arising from automorphic representations on $\mathrm{GL}(n)$. More precisely, if $L_1, \ldots, L_J$ are distinct primitive L-functions with $J \geq 2$, and b_j are any non-zero real numbers, we prove that the number of zeros of $F(s) = \sum_{j=1}^{J} b_j L_j(s)$ in the region $\mathrm{Re}(s) \geq 1/2 + 1/G(T)$ and $\mathrm{Im}(s) \in [T, 2T]$ is asymptotic to $K_0 T G(T)/\sqrt{\log G(T)}$ uniformly in the range $\log\log T \leq G(T) \leq (\log T)^{\nu}$, where K_0 is a certain positive constant that depends on J and the L_j. This establishes a generalization of a conjecture of Hejhal in this range. Moreover, the exponent ν verifies $\nu \asymp 1/J$ as J grows.

1 Introduction

The theory of L-functions has become a central part of modern number theory, due to its connection to various arithmetic, geometric and algebraic objects. L-functions are represented by Dirichlet series which are absolutely convergent in half-planes. They satisfy certain conditions, including having a meromorphic continuation, an Euler product over primes and a functional equation. The prototypical example of an L-function is the Riemann zeta function. Other important examples include Dirichlet L-functions attached to primitive Dirichlet characters, and the Hasse–Weil L-functions attached to elliptic curves. The Langlands program predicts that all L-functions arise from automorphic representations over $\mathrm{GL}(n)$.

L-functions are predicted to verify several hypotheses, the most important of which is the generalized Riemann hypothesis (GRH), which asserts that all non-trivial zeros of L-functions lie on the critical line $\mathrm{Re}(s) = 1/2$. On the other hand, there exist various Dirichlet series which have arithmetical significance and satisfy a functional equation, but are not L-functions, since they do not possess an

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Euler product. Most of these functions can be expressed as linear combinations of L-functions. Important examples include Epstein zeta functions associated to quadratic forms, and the zeta function attached to ideal classes in number fields. Unlike L-functions, these zeta functions are not expected to satisfy the GRH, and some of them might even possess zeros inside the region of absolute convergence. The first to have investigated such a phenomenon are Davenport and Heilbronn [3], who proved in 1936 that the Epstein zeta function of a positive definite quadratic form of class number ≥ 2 has infinitely many zeros in the half-plane of absolute convergence $\mathrm{Re}(s) > 1$.

For a complex valued function $f(z)$, we shall denote by $N_f(\sigma_1, \sigma_2, T)$ the number of zeros of f in the rectangle $\sigma_1 \leq \mathrm{Re}(s) \leq \sigma_2$ and $T \leq \mathrm{Im}(s) \leq 2T$. We also let $N_f(\sigma, T)$ be the number of zeros of f in the region $\mathrm{Re}(s) \geq \sigma$ and $T \leq \mathrm{Im}(s) \leq 2T$. Voronin [24] proved that $N_E(\sigma_1, \sigma_2, T) \gg T$ for any $1/2 < \sigma_1 < \sigma_2 < 1$ fixed, where $E(s, Q)$ is the Epstein zeta function attached to a binary quadratic form Q with integral coefficients and with class number at least 2. Lee [14] improved this result to an asymptotic formula $N_E(\sigma_1, \sigma_2, T) \sim c(\sigma_1, \sigma_2)T$ for some positive constant $c(\sigma_1, \sigma_2)$. Gonek and Lee [4] obtained a quantitative bound for the error term in this asymptotic formula, and this was subsequently improved by Lamzouri [11] who showed that one can obtain a saving of a power of $\log T$ in the error term.

Throughout this paper we let $J \geq 2$ be an integer, $b_1, \dots, b_J$ be non-zero real numbers such that $\sum_{j=1}^{J} b_j^2 = 1$, and we define

$$F(s) := F_{L_1,\dots,L_J}(s) = \sum_{j=1}^{J} b_j L_j(s), \tag{1.1}$$

for L-functions $L_1, \dots, L_J$. Lee, Nakamura and Pańkowski [16] generalized Voronin's result to zeros of linear combinations of L-functions in the strip $1/2 < \sigma_1 < \sigma_2 < 1$, where σ_1, σ_2 are fixed. More precisely, they established that

$$N_F(\sigma_1, \sigma_2, T) \gg T,$$

if the L_j belong to the Selberg class of L-functions, and verify a stronger version of the Selberg orthogonality conjecture (see (1.3) below). In the special case where the L_j are Dirichlet L-functions or Hecke L-functions attached to the ideal class characters of a quadratic imaginary field, one obtains an asymptotic formula $N_F(\sigma_1, \sigma_2, T) \sim c_F(\sigma_1, \sigma_2)T$ as $T \to \infty$, for a certain positive constant $c_F(\sigma_1, \sigma_2)$ (See [2], [10] and [14]). Furthermore, by using the methods of the proof of Theorem 1.1 below, one can generalize this result by showing that

$$N_F(\sigma_1, \sigma_2, T) = c_F(\sigma_1, \sigma_2)T + O\Big(\frac{T}{(\log T)^{\delta}}\Big),$$

if the L-functions $L_1, \dots, L_J$ satisfy the assumptions A1–A5 below and $\delta = \delta(J, F, \sigma_1, \sigma_2)$ is a positive constant.

Although linear combinations of L-functions have many zeros off the critical line, it was conjectured by Montgomery that 100% of the zeros of $F(s)$ lie on the critical line, if the L_j are primitive[1] L-functions satisfying assumptions A1 and A2 below. Bombieri and Hejhal [1] established this conjecture if the L_j satisfy the assumptions A1, A2, A3 and A5 below, conditionally on the GRH and a zero-spacing hypothesis for each of the L_j. Unconditionally, Selberg [20] established that a positive proportion of the zeros of $F(s)$ lie on the critical line, in the special case where all of the L_j are Dirichlet L-functions having the same parity and conductor.

In this paper we study the zeros of the linear combination $F(s)$ where the L-functions $L_1, \dots, L_J$ satisfy the following assumptions:

A1: (Euler product) For $j = 1, \dots, J$ and $\mathrm{Re}(s) > 1$ we have

$$L_j(s) = \prod_p \prod_{i=1}^{d} \left(1 - \frac{\alpha_{j,i}(p)}{p^s}\right)^{-1}$$

where $|\alpha_{j,i}(p)| \le p^{\theta}$ for some fixed $0 \le \theta < 1/2$ and for every $i = 1, \dots, d$.

A2: (Functional equation) The functions $L_1, L_2, \dots, L_J$ satisfy the same functional equation

$$\Lambda_j(s) = \omega \overline{\Lambda_j(1 - \bar{s})},$$

where

$$\Lambda_j(s) := L_j(s) Q^s \prod_{\ell=1}^{k} \Gamma(\lambda_\ell s + \mu_\ell),$$

$|\omega| = 1$, $Q > 0$, $\lambda_\ell > 0$ and $\mu_\ell \in \mathbb{C}$ with $\mathrm{Re}(\mu_\ell) \ge 0$.

A3: (Ramanujan hypothesis on average)

$$\sum_{p \le x} \sum_{i=1}^{d} |\alpha_{j,i}(p)|^2 = O(x^{1+\epsilon})$$

holds for every $\epsilon > 0$ and for every $j = 1, \dots, J$ as $x \to \infty$.

A4: (Zero density hypothesis) There exist positive constants c_1, c_2 such that for all $1 \le j \le J$ and all $\sigma \ge 1/2$ we have

$$N_{L_j}(\sigma, T) \ll T^{1 - c_1(\sigma - 1/2)} (\log T)^{c_2}. \tag{1.2}$$

[1] An L-function is primitive if it cannot be written as a product of non-trivial L-functions, where the trivial L-function is the constant 1.

A5: (Selberg orthogonality conjecture) By assumption A1 we can write

$$\log L_j(s) = \sum_p \sum_{k=1}^{\infty} \frac{\beta_{L_j}(p^k)}{p^{ks}},$$

where $\beta_{L_j}(p^k)$ are complex numbers. Then, for all $1 \le j,k \le J$ there exist constants $\xi_j > 0$ and $c_{j,k}$ such that

$$\sum_{p\le x} \frac{\beta_{L_j}(p)\overline{\beta_{L_k}(p)}}{p} = \delta_{j,k}\xi_j \log\log x + c_{j,k} + O\Big(\frac{1}{\log x}\Big), \tag{1.3}$$

where $\delta_{j,k} = 0$ if $j \ne k$ and $\delta_{j,k} = 1$ if $j = k$.

Remark. The assumptions A1–A5 are standard, and are expected to hold for all L-functions arising from automorphic representations on GL(n). In particular, they are verified by GL(1) and GL(2) L-functions, which are the Riemann zeta function and Dirichlet L-functions, and L-functions attached to Hecke holomorphic or Maass cusps forms. For GL(1) L-functions, the Selberg orthogonality conjecture boils down to the fact that $L(s,\chi)$ is regular and non-zero at $s = 1$, if χ is a non-principal Dirichlet character. For GL(2) L-functions, assumptions A3 and A5 are handled using the Ranking–Selberg convolution, while assumption A4 is proved in [17] for L-functions attached to holomorphic cusp forms, and in [19] for L-functions attached to Maass forms. Assumptions A4 and A5 are used to investigate the joint distribution of $\log L_1, \log L_2, \ldots, \log L_J$ near the critical line, which is a key component of the proof of Theorem 1.1 below. To this end, the zero density hypothesis A4 is used to approximate $\log L_j(\sigma + it)$ by short Dirichlet polynomials for "almost all" points t, while the Selberg orthogonality conjecture A5 insures the "statistical independence" of the functions $\log L_j(\sigma + it)$, when σ is very close to $1/2$.

For each $1 \le j \le J$, write L_j as a Dirichlet series $L_j(s) = \sum_{n=1}^{\infty} \frac{\alpha_{L_j}(n)}{n^s}$, which is absolutely convergent for $\mathrm{Re}(s) > 1$ by assumption A3. Then $F(s)$ has a Dirichlet series representation

$$\sum_{n=1}^{\infty} \frac{\alpha_F(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\sum_{j=1}^{J} b_j \alpha_{L_j}(n)}{n^s}.$$

Since the first non-zero term dominates the others, $F(s)$ has no zeros in $\mathrm{Re}(s) > A$ for some constant $A > 0$. Hence, we may define

$$\sigma_F := \sup\{\mathrm{Re}(\rho) : F(\rho) = 0\} \le A.$$

Moreover, it follows from assumption A2 that $F(s)$ satisfies the functional equation

$$F(s)Q^s \prod_{\ell=1}^{k} \Gamma(\lambda_\ell s + \mu_\ell) = \omega\overline{F(1-\bar{s})}Q^{1-s} \prod_{\ell=1}^{k} \Gamma(\lambda_\ell(1-s) + \bar{\mu}_\ell). \tag{1.4}$$

Furthermore, since $F(s)$ has no zeros on $\mathrm{Re}(s) > \sigma_F$, by (1.4) we see that

$$\Big\{ -\frac{\mu_\ell + m}{\lambda_\ell} : -\frac{\mathrm{Re}(\mu_\ell) + m}{\lambda_\ell} < 1 - \sigma_F, \ell = 1, \dots, k \text{ and } m = 0, 1, 2, \dots \Big\}$$

is the set of (trivial) zeros of $F(s)$ on $\mathrm{Re}(s) < 1 - \sigma_F$. All the other zeros are in the strip $1 - \sigma_F \le \mathrm{Re}(s) \le \sigma_F$ and we may call them the non-trivial zeros. The number of nontrivial zeros $\beta + i\gamma$ of $F(s)$ with $0 < \gamma < T$ is denoted by $N_F(T)$ and it is well-known that

$$N_F(T) \sim \frac{d_F}{2\pi} T \log T$$

as $T \to \infty$, where $d_F = 2 \sum_{\ell=1}^{k} \lambda_\ell$.

Bombieri and Hejhal [1] conjectured that the order of magnitude of the number of zeros of $F(s)$ off the critical line and up to height T should be

$$\frac{T \log T}{\sqrt{\log \log T}}.$$

Motivated by this conjecture, Hejhal [5, 6] studied the zeros of linear combinations of L-functions near the critical line. Suppose that L_1 and L_2 satisfy assumptions A1, A2, A4, and A5, as well as the Ramanujan–Petersson conjecture (which asserts that $|\alpha_{j,i}(p)| \le 1$ for all i, j and p) instead of the weaker assumption A3. Let $F(s) = \cos(\alpha) L_1(s) + \sin(\alpha) L_2(s)$, where α is a real number. Then Hejhal [5] proved that for "almost all" α with respect to a certain measure, we have

$$\frac{TG(T)}{\sqrt{\log \log T}} \ll N_F\Big(\frac{1}{2} + \frac{1}{G(T)}, T\Big) \ll \frac{TG(T)}{\sqrt{\log \log T}} \tag{1.5}$$

in the range $(\log T)^\delta \le G(T) \le \frac{\log T}{(\log \log T)^\kappa}$ where $\delta > 0$ and $1 < \kappa < 3$ are fixed. He also conjectured (see [5, Section 6]) that the following asymptotic formula should hold for all $\alpha \notin \frac{\pi}{2}\mathbb{Z}$ in the same range of $G(T)$:

$$N_F\Big(\frac{1}{2} + \frac{1}{G(T)}, T\Big) \sim \frac{\sqrt{\xi_1 + \xi_2}}{8\pi^{3/2}} \frac{TG(T)}{\sqrt{\log G(T)}}. \tag{1.6}$$

In the short note [6], Hejhal discussed a generalization of the bounds (1.5) to linear combinations with three or more L-functions, but did not provide a complete proof of these bounds.

In this paper, we prove a quantitative generalization of the conjectured asymptotic formula (1.6) for any linear combination of L-functions $F(s) = \sum_{j=1}^{J} b_j L_j(s)$ as in (1.1), though in a smaller range of the parameter $G(T)$. More precisely, our main result is the following theorem.

Theorem 1.1. *Let $F(s)$ be defined by* (1.1)*, where the L-functions $L_1, \dots, L_J$ satisfy assumptions A1–A5. Let T be large, $\xi = \max_{j\le J} \xi_j$, and*

$$0 < \nu < 1/(12J + 7 + 16J\sqrt{3\xi})$$

be a fixed real number. Then for $\sigma = 1/2 + 1/G(T)$ with $\log\log T \le G(T) \le (\log T)^\nu$ we have

$$N_F(\sigma, T) = K_0 \frac{TG(T)}{\sqrt{\log G(T)}} + O\Big(\frac{TG(T)}{(\log G(T))^{5/4}}\Big),$$

where

$$K_0 = K_0(J; \xi_1, \xi_2, \dots, \xi_J) := \frac{1}{4\pi^{J/2+1}\prod_{j=1}^J \sqrt{\xi_j}} \sum_{n=1}^J \int_{\mathcal{R}_n} e^{-\sum_{j=1}^J u_j^2/\xi_j} u_n d\mathbf{u}$$

and

$$\mathcal{R}_n := \{\mathbf{u} \in \mathbb{R}^J : u_n = \max\{u_1, \dots, u_J\}\}.$$

Remark. In the case $J = 2$, an easy calculation shows that K_0 matches the constant predicted by Conjecture (1.6) of Hejhal. Indeed, we have in this case

$$K_0 = \frac{1}{4\pi^2\sqrt{\xi_1\xi_2}}\Big(\int_{u_1\ge u_2} e^{-u_1^2/\xi_1 - u_2^2/\xi_2} u_1 du_1 du_2 + \int_{u_2\ge u_1} e^{-u_1^2/\xi_1 - u_2^2/\xi_2} u_2 du_1 du_2\Big).$$

We can compute the first integral as

$$\int_{u_1\ge u_2} e^{-u_1^2/\xi_1 - u_2^2/\xi_2} u_1 du_1 du_2 = \frac{\xi_1}{2}\int_{-\infty}^{\infty} e^{-u_2^2(1/\xi_1 + 1/\xi_2)} du_2 = \frac{\sqrt{\pi}\xi_1}{2}\sqrt{\frac{\xi_1\xi_2}{\xi_1+\xi_2}}.$$

Evaluating the second integral similarly, we thus deduce that

$$K_0 = \frac{1}{4\pi^2\sqrt{\xi_1\xi_2}}\Big(\frac{\sqrt{\pi}\xi_1}{2}\sqrt{\frac{\xi_1\xi_2}{\xi_1+\xi_2}} + \frac{\sqrt{\pi}\xi_2}{2}\sqrt{\frac{\xi_1\xi_2}{\xi_1+\xi_2}}\Big) = \frac{\sqrt{\xi_1+\xi_2}}{8\pi^{3/2}},$$

as desired.

Remark. In [15], Lee obtained an analogue of Theorem 1.1 (in a larger range of $G(T)$) in the case where $F(s) = E(s, Q)$ is the Epstein zeta function attached to a binary quadratic form Q with integral coefficients and class number 2 or 3. In this case $E(s, Q)$ can be expressed as the linear combination of two Hecke L-functions. However, the method of [15] does not seem to generalize to the case of linear combinations of three or more L-functions.

2 Strategy of the proof of Theorem 1.1, key ingredients and detailed results

Let $F(s)$ be defined by (1.1) where the L_j satisfy assumptions A1–A5. In order to count the number of zeros of $F(s)$ in the region $\mathrm{Re}(s) > \sigma$, $T \le \mathrm{Im}(s) \le 2T$ we shall use Littlewood's lemma in a standard way. Let $\sigma_0 > \sigma_F$. Then, $F(s)$ has no zeros in $\mathrm{Re}(s) \ge \sigma_0$ and hence by Littlewood's lemma (see [21, (9.9.1)]), we have

$$\begin{aligned}\int_\sigma^{\sigma_0} N_F(u,T)du = \frac{1}{2\pi}\int_T^{2T} \log|F(\sigma+it)|dt - \frac{1}{2\pi}\int_T^{2T}\log|F(\sigma_0+it)|dt \\ + \frac{T}{2\pi}(\sigma-\sigma_0)\log n_0 + O_F(\log T),\end{aligned} \tag{2.1}$$

where n_0 is the smallest positive integer such that $\alpha_F(n_0) \neq 0$. In order to estimate the integrals on the right hand side of this asymptotic formula, we shall construct a probabilistic random model for $F(\sigma + it)$. Recall that

$$F(\sigma+it) = \sum_{j=1}^{J} b_j L_j(\sigma+it) = \sum_{j=1}^{J} b_j \prod_p \prod_{i=1}^{d}\Big(1 - \frac{\alpha_{j,i}(p)}{p^{\sigma+it}}\Big)^{-1}.$$

Let $\{\mathbb{X}(p)\}_p$ be a sequence of independent random variables, indexed by the prime numbers, and uniformly distributed on the unit circle. For $1 \le j \le J$ we consider the random Euler products

$$L_j(\sigma,\mathbb{X}) := \prod_p \prod_{i=1}^{d}\Big(1 - \frac{\alpha_{j,i}(p)\mathbb{X}(p)}{p^{\sigma}}\Big)^{-1}.$$

These products converge almost surely for $\sigma > 1/2$ by Kolmogorov's three series theorem. We shall prove that the integral $\frac{1}{T}\int_T^{2T}\log|F(\sigma+it)|dt$ is very close to the expectation of $\log|F(\sigma,\mathbb{X})|$, where the probabilistic random model $F(\sigma,\mathbb{X})$ is defined by

$$F(\sigma,\mathbb{X}) := \sum_{j=1}^{J} b_j L_j(\sigma,\mathbb{X}).$$

Theorem 2.1. *Let $J \ge 2$ be an integer, $\xi = \max_{j\le J}\xi_j$, and*

$$0 < \nu < 1/(12J + 7 + 16J\sqrt{3\xi})$$

be a fixed real number. Let T be large. There exists a positive constant $\beta > 0$ such that for $\sigma = 1/2 + 1/G(T)$ with $\log\log T \le G(T) \le (\log T)^\nu$ we have

$$\frac{1}{T}\int_T^{2T}\log|F(\sigma+it)|dt = \mathbb{E}(\log|F(\sigma,\mathbb{X})|) + O\Big(\frac{1}{(\log T)^\beta}\Big),$$

where here and throughout we denote by $\mathbb{E}(\cdot)$ the expectation.

Thus, in order to estimate $N_F(\sigma, T)$ it remains to investigate the function

$$\mathcal{M}(\sigma) := \mathbb{E}(\log |F(\sigma, \mathbb{X})|),$$

and more precisely to estimate the difference $\mathcal{M}(\sigma) - \mathcal{M}(\sigma + h)$ for small h. We shall investigate this quantity in Section 7 and prove the following result.

Theorem 2.2. *Let*

$$G_1(T) := G(T)\frac{\log G(T)}{\log G(T) - 1} \quad \text{and} \quad G_2(T) := G(T)\frac{\log G(T)}{\log G(T) + 1}.$$

Assume that $G(T) \geq 4$. Then for each $i = 1, 2$ we have

$$\mathcal{M}\Big(\frac{1}{2} + \frac{1}{G(T)}\Big) - \mathcal{M}\Big(\frac{1}{2} + \frac{1}{G_i(T)}\Big) = (-1)^i \frac{2\pi K_0}{(\log G(T))^{3/2}} + O\Big(\frac{1}{(\log G(T))^{9/4}}\Big),$$

where the constant K_0 is defined in Theorem 1.1.

We now show how to deduce Theorem 1.1 from Theorems 2.1 and 2.2.

Proof of Theorem 1.1 assuming Theorems 2.1 and 2.2. First, note that

$$\frac{1}{2} + \frac{1}{G_i(T)} = \sigma + (-1)^i/(G(T)\log G(T)).$$

Since $N_F(w, T)$ is a decreasing function of w for each T, we see that

$$\int_{\sigma}^{\frac{1}{2}+\frac{1}{G_2(T)}} N_F(w, T)dw \leq \frac{N_F(\sigma, T)}{G(T)\log G(T)} \leq \int_{\frac{1}{2}+\frac{1}{G_1(T)}}^{\sigma} N_F(w, T)dw. \tag{2.2}$$

By (2.1) and Theorems 2.1 and 2.2, we obtain

$$\begin{aligned}
&\frac{2\pi}{T}\int_{\sigma}^{\frac{1}{2}+\frac{1}{G_i(T)}} N_F(w, T)dw \\
&= \frac{1}{T}\int_T^{2T} \log|F(\sigma + it)| - \log\Big|F\Big(\frac{1}{2} + \frac{1}{G_i(T)} + it\Big)\Big|dt + O\Big(\frac{1}{G(T)\log G(T)}\Big) \\
&= \mathcal{M}(\sigma) - \mathcal{M}\Big(\frac{1}{2} + \frac{1}{G_i(T)}\Big) + O\Big(\frac{1}{G(T)\log G(T)}\Big) \\
&= (-1)^i\frac{2\pi K_0}{(\log G(T))^{3/2}} + O\Big(\frac{1}{(\log G(T))^{9/4}}\Big).
\end{aligned}$$

Inserting these estimates in (2.2) completes the proof. □

We next describe the different ingredients that are used in the proof of Theorem 2.1. The first is a discrepancy bound for the joint distribution of the values of the L-functions $L_j(s)$, which generalizes the results of [13] for the Riemann zeta function. For $\sigma > 1/2$ we let

$$\mathbf{L}(\sigma + it) = (\log|L_1(\sigma + it)|, \ldots, \log|L_J(\sigma + it)|, \arg L_1(\sigma + it), \ldots, \arg L_J(\sigma + it)),$$

and similarly define the random vector

$$\mathbf{L}(\sigma, \mathbb{X}) = (\log|L_1(\sigma, \mathbb{X})|, \ldots, \log|L_J(\sigma, \mathbb{X})|, \arg L_1(\sigma, \mathbb{X}), \ldots, \arg L_J(\sigma, \mathbb{X})).$$

For a Borel set $\mathcal{B}$ in $\mathbb{R}^{2J}$ and for $\sigma = 1/2 + 1/G(T)$, we define

$$\Phi_T(\mathcal{B}) := \frac{1}{T}\text{meas}\{t \in [T, 2T] : \mathbf{L}(\sigma + it) \in \mathcal{B}\} \tag{2.3}$$

and

$$\Phi_T^{\text{rand}}(\mathcal{B}) := \mathbb{P}(\mathbf{L}(\sigma, \mathbb{X}) \in \mathcal{B}), \tag{2.4}$$

where here and throughout, meas will denote the Lebesgue measure on $\mathbb{R}$. We will prove that the measure Φ_T^{rand} is absolutely continuous and investigate its density function $H_T(\mathbf{u}, \mathbf{v})$ in Section 7.

We define the **discrepancy** between these two distributions as

$$\mathbf{D}_T(\mathcal{B}) := \Phi_T(\mathcal{B}) - \Phi_T^{\text{rand}}(\mathcal{B}).$$

Then we prove the following result which generalizes [13, Theorem 1.1], and might be of independent interest.

Theorem 2.3. *Let T be large and $\sigma = 1/2 + 1/G(T)$ where*

$$\log\log T \le G(T) \le \sqrt{\log T}/\log\log T.$$

Then we have

$$\sup_{\mathcal{R}} \left|\mathbf{D}_T(\mathcal{R})\right| \ll \frac{\sqrt{G(T)}\log\log T}{\sqrt{\log T}},$$

where $\mathcal{R}$ runs over all rectangular boxes of $\mathbb{R}^{2J}$ (possibly unbounded) with sides parallel to the coordinate axes.

We shall use this result to approximate the integral $\frac{1}{T}\int_T^{2T} \log|F(\sigma+it)|dt$ by the expectation $\mathbb{E}(\log|F(\sigma, \mathbb{X})|)$. However, in doing so we need to control the large values and the logarithmic singularities of both $\log|F(\sigma+it)|$ and $\log|F(\sigma, \mathbb{X})|$. To this end we prove the following propositions.

Proposition 2.4. *Let T be large, and $\sigma = 1/2 + 1/G(T)$ with*

$$2 < G(T) \le c_0\sqrt{\frac{\log T}{\log\log T}},$$

for some small constant $c_0 > 0$. There exist positive constants $C_1, C_2 > 0$ such that for every positive integer $k \le (\log T)/(C_1 G(T)\log\log T)$ we have

$$\frac{1}{T}\int_T^{2T} |\log|F(\sigma+it)||^{2k}dt \ll (C_2 k\log\log T)^k G(T)^{3k+2}(\max\{k, G(T)^{3/2}\log G(T)\})^{2k}.$$

Proposition 2.5. *Let T be large, and $\sigma = 1/2+1/G(T)$ with $2 < G(T) \le \log T$. There exists a constant $C_3 > 0$ such that for every integer $k \ge 1$ we have*

$$\mathbb{E}[|\log|F(\sigma,\mathbb{X})||^{2k}] \ll (\log\log T)^J (C_3 k(k+\log\log T))^k.$$

We also need the following result on the large deviations of $\log L_j(\sigma+it)$ and $\log L_j(\sigma,\mathbb{X})$.

Lemma 2.6. *Let $1 \le j \le J$, and T be large. Let $\sigma = 1/2 + 1/G(T)$ with $\log\log T \le G(T) \le c_0\sqrt{\log T/\log\log T}$ for some small constant $c_0 > 0$. Then, there exists a positive constant c_1 such that for all $\sqrt{\log G(T)} \le \tau \le (\log\log T)^2$ we have*

$$\frac{1}{T}\text{meas}\{t \in [T,2T] : |\log L_j(\sigma+it)| \ge \tau\} \ll \frac{\tau}{\sqrt{\log G(T)}} \exp\Big(-\frac{\tau^2}{\xi_j \log G(T) + c_1 G_0(T)}\Big),$$

where $G_0(T) = \max\{\sqrt{\log G(T)}, \log_3 T\}$. Furthermore, the same bound holds for

$$\mathbb{P}(|\log L_j(\sigma,\mathbb{X})| > \tau),$$

in the same range of τ.

Our last ingredient in the proof of Theorem 2.1 is the following lemma, which provides bounds for the probability of "the concentration" of the random variable $|F(\sigma,\mathbb{X})|$.

Lemma 2.7. *Let $\varepsilon > 0$ be small and $J \ge 1$ be fixed. Then for any real numbers $\sigma > 1/2$, $R > 0$ and $M > 2\pi$ we have*

$$\mathbb{P}(\mathbf{L}(\sigma,\mathbb{X}) \in [-M,M]^{2J}, \text{ and } R < |F(\sigma,\mathbb{X})| < R+\varepsilon) \ll M^{2J-1} e^{2M}(R\varepsilon + \varepsilon^2),$$

where the implicit constant is absolute.

The plan of the remaining part of the paper is as follows. With all the ingredients now in place, we shall first prove Theorem 2.1 in Section 3. In Section 4 we gather together several preliminary results that will be used in subsequent sections, and prove Lemma 2.6. These will be used to bound the discrepancy of the joint distribution of $\log L_j(s)$ and prove Theorem 2.3 in Section 5. Then in Section 6, we shall establish Proposition 2.4. Finally in Section 7 we shall investigate the distribution of the random vector $\mathbf{L}(\sigma,\mathbb{X})$ and establish Theorem 2.2, Proposition 2.5, and Lemma 2.7.

3 Proof of Theorem 2.1

In this section we establish Theorem 2.1 using the ingredients listed in the previous section, namely Theorem 2.3, Propositions 2.4 and 2.5, and Lemmas 2.6 and 2.7.

We let $\sigma = 1/2 + 1/G(T)$ where $\log\log T \leq G(T) \leq (\log T)^{\theta}$, and $0 < \theta \leq 1/2$ is a real number that we shall optimize later. We start by showing how to use Lemma 2.6 and Proposition 2.4 to control the large values and the logarithmic singularities of $\log |F(\sigma + it)|$. Let $\alpha > 0$ be a positive constant to be chosen and define

$$\mathcal{L} := \alpha \log\log T, \quad \text{and} \quad M := (G(T)\log\log T)^3.$$

We define the following sets:

$$S_1(T) := \{t \in [T, 2T] : \mathbf{L}(\sigma + it) \in (-\mathcal{L}, \mathcal{L})^{2J}\},$$
$$S_2(T) := \{t \in [T, 2T] : \log |F(\sigma + it)| > -M\},$$

and

$$S_0(T) := S_1(T) \cap S_2(T).$$

Let

$$\xi := \xi_{\max} := \max_{j \leq J} \xi_j.$$

Then it follows from Lemma 2.6 that

$$\begin{aligned}\text{meas}([T, 2T] \setminus S_1(T)) &\leq \sum_{j=1}^{J} \text{meas}\{t \in [T, 2T] : |\log L_j(\sigma + it)| \geq \mathcal{L}\} \\ &\ll \frac{Te^{O(\sqrt{\log\log T})}}{(\log T)^{\alpha^2/(\xi\theta)}}.\end{aligned}$$

On the other hand, using Proposition 2.4 with $k = \lfloor (\log\log T)^{5/4} \rfloor$ gives

$$\begin{aligned}\text{meas}([T, 2T] \setminus S_2(T)) &\leq \frac{1}{M^{2k}} \int_T^{2T} |\log |F(\sigma + it)||^{2k} dt \\ &\ll T \exp(-(\log\log T)^{5/4}).\end{aligned}$$

Therefore we deduce that

$$\text{meas}([T, 2T] \setminus S_0(T)) \ll \frac{Te^{O(\sqrt{\log\log T})}}{(\log T)^{\alpha^2/(\xi\theta)}}. \tag{3.1}$$

Combining this bound with Proposition 2.4, and using Hölder's inequality with $r = \lfloor \log\log T \rfloor$ we have

$$\begin{aligned}
&\left| \int_{t\in[T,2T]\setminus S_0(T)} \log|F(\sigma+it)|dt \right| \\
&\leq (\text{meas}\{t \in [T,2T] \setminus S_0(T)\})^{1-1/2r} \left(\int_T^{2T} |\log|F(\sigma+it)||^{2r} dt \right)^{1/2r} \\
&\ll \left(\frac{Te^{O(\sqrt{\log\log T})}}{(\log T)^{\alpha^2/(\xi\theta)}} \right)^{1-1/2r} (T(\log T)^{(6r+2)\theta}(\log\log T)^{4r})^{1/2r} \\
&\ll \frac{Te^{O(\sqrt{\log\log T})}}{(\log T)^{\alpha^2/(\xi\theta)-3\theta}}.
\end{aligned} \tag{3.2}$$

We now define for $\tau \in \mathbb{R}$

$$\Psi_T(\tau) := \frac{1}{T}\text{meas}\{t \in S_0(T) : \log|F(\sigma+it)| > \tau\}$$

and similarly

$$\Psi_{\text{rand}}(\tau) := \mathbb{P}(\mathbb{X} \in \mathcal{S}, \text{ and } \log|F(\sigma,\mathbb{X})| > \tau),$$

where $\mathcal{S}$ is the event $\mathbf{L}(\sigma,\mathbb{X}) \in (-\mathcal{L},\mathcal{L})^{2J}$ and $\log|F(\sigma,\mathbb{X})| > -M$. First we observe that for $t \in S_0(T)$ we have

$$\log|F(\sigma+it)| \leq \log\left(\sum_{j=1}^J |b_j L_j(\sigma+it)|\right) \leq \log\left(\sum_{j=1}^J |b_j| e^{\mathcal{L}}\right) = \mathcal{L} + \log\left(\sum_{j=1}^J |b_j|\right),$$

and hence we have

$$\Psi_T(\tau) = \Psi_{\text{rand}}(\tau) = 0, \tag{3.3}$$

for

$$\tau > \widetilde{\mathcal{L}} := \mathcal{L} + \log\left(\sum_{j=1}^J |b_j|\right).$$

Using a geometric covering argument, we prove the following result which shows that $\Psi_T(\tau)$ is very close to $\Psi_{\text{rand}}(\tau)$ uniformly in τ. This will imply Theorem 2.1.

Proposition 3.1. *Let T be large. Then we have*

$$\sup_{\tau\leq\widetilde{\mathcal{L}}} |\Psi_T(\tau) - \Psi_{\text{rand}}(\tau)| \ll \frac{(\log\log T)^{2J+1}}{(\log T)^{(1-\theta-16J\alpha)/(4J+2)}}.$$

Proof. We let $0 < \varepsilon = \varepsilon(T) \leq e^{-2\mathcal{L}}$ be a small parameter to be chosen. We shall consider three cases depending on the size of τ.

Case 1: $\tau \le -M$. In this case, it follows from the definition of the set $S_0(T)$ together with (3.1) that

$$\begin{aligned}\Psi_T(\tau) &= \frac{\text{meas}(S_0(T))}{T}\\ &= \frac{1}{T}\text{meas}\{t \in [T, 2T] : \mathbf{L}(\sigma + it) \in (-\mathcal{L}, \mathcal{L})^{2J}\} + O\Big(\frac{1}{(\log T)^2}\Big).\end{aligned}$$

Similarly, using Proposition 2.5 with $k = \lfloor \log\log T \rfloor$ we have

$$\mathbb{P}(\log |F(\sigma, \mathbb{X})\big| < -M) \le \frac{1}{M^{2k}}\mathbb{E}(|\log |F(\sigma, \mathbb{X})||^{2k}) \ll \frac{1}{(\log T)^2}.$$

Therefore, it follows from the definition of the event $\mathcal{S}$ that

$$\Psi_{\text{rand}}(\tau) = \mathbb{P}\big(\mathbf{L}(\sigma, \mathbb{X}) \in (-\mathcal{L}, \mathcal{L})^{2J}\big) + O\Big(\frac{1}{(\log T)^2}\Big).$$

Hence Theorem 2.3 yields

$$\Psi_T(\tau) = \Psi_{\text{rand}}(\tau) + O\Big(\frac{\log\log T}{(\log T)^{(1-\theta)/2}}\Big). \tag{3.4}$$

Case 2: $-M < \tau \le \log(C_3\varepsilon) + \mathcal{L}$, where C_3 is a suitably large constant. In this case we have

$$\Psi_T(\tau) = \frac{1}{T}\text{meas}\{t \in [T, 2T] : \mathbf{L}(\sigma + it) \in (-\mathcal{L}, \mathcal{L})^{2J} \setminus \mathcal{U}_J(e^{\tau}, \mathcal{L})\}, \tag{3.5}$$

where $\mathcal{U}_J(y, \mathcal{L})$ is the bounded subset of $\mathbb{R}^{2J}$ defined by

$$\mathcal{U}_J(y, \mathcal{L}) := \Big\{(u_1, \dots, u_J, v_1, \dots, v_J) \in \mathbb{R}^{2J} : |u_j|, |v_j| < \mathcal{L} \text{ for all } 1 \le j \le J, \text{ and } \Big|\sum_{j=1}^{J} b_j e^{u_j + iv_j}\Big| \le y\Big\}.$$

We cover $\mathcal{U}_J(e^{\tau}, \mathcal{L})$ with $K(\tau)$ distinct hypercubes $\mathcal{B}_k(\tau)$ of the form $\prod_{j=1}^{2J}[z_j, z_j + \varepsilon)$ with non-empty intersection with $\mathcal{U}_J(e^{\tau}, \mathcal{L})$. Note that

$$K(\tau) \ll \Big(\frac{\mathcal{L}}{\varepsilon}\Big)^{2J}.$$

Now, let $1 \le k \le K(\tau)$ and $(u_1, \dots, u_J, v_1, \dots, v_J) \in \mathcal{B}_k(\tau) \cap \mathcal{U}_J(e^{\tau}, \mathcal{L})$. Recall that this intersection is non-empty by construction. Subsequently, for any $(x_1, \dots, x_J, y_1, \dots, y_J) \in \mathcal{B}_k(\tau)$ we have $|x_j - u_j| \le \varepsilon$ and $|y_j - v_j| \le \varepsilon$ for all $1 \le j \le J$. Hence, we deduce that $|x_j|, |y_j| < \mathcal{L} + \varepsilon$ for all $1 \le j \le J$ and

$$\Big|\sum_{j=1}^{J} b_j e^{x_j + iy_j}\Big| = \Big|\sum_{j=1}^{J} b_j e^{u_j + iv_j}\Big| + O(\varepsilon e^{\mathcal{L}}) \le C_4 \varepsilon e^{\mathcal{L}}$$

for some positive constant C_4 since $e^\tau \le C_3\varepsilon e^{\mathcal{L}}$ by our assumption. Therefore, we have shown that

$$\mathcal{U}_J(e^\tau, \mathcal{L}) \subset \bigcup_{k\le K(\tau)} \mathcal{B}_k(\tau) \subset \mathcal{U}_J(C_4\varepsilon e^{\mathcal{L}}, \mathcal{L}+\varepsilon).$$

Hence, appealing to Theorem 2.3 we obtain

$$\begin{aligned}
&\frac{1}{T}\text{meas}\{t\in[T,2T] : \mathbf{L}(\sigma+it)\in\mathcal{U}_J(e^\tau,\mathcal{L})\}\\
&\quad\le \sum_{k=1}^{K(\tau)}\frac{1}{T}\text{meas}\{t\in[T,2T] : \mathbf{L}(\sigma+it)\in\mathcal{B}_k(\tau)\}\\
&\quad= \sum_{k=1}^{K(\tau)}\mathbb{P}(\mathbf{L}(\sigma,\mathbb{X})\in\mathcal{B}_k(\tau)) + O\Big(\frac{K(\tau)\log\log T}{(\log T)^{(1-\theta)/2}}\Big)\\
&\quad\le \mathbb{P}(\mathbf{L}(\sigma,\mathbb{X})\in\mathcal{U}_J(C_4\varepsilon e^{\mathcal{L}},\mathcal{L}+\varepsilon)) + O\Big(\frac{\mathcal{L}^{2J+1}}{\varepsilon^{2J}(\log T)^{(1-\theta)/2}}\Big).
\end{aligned}\tag{3.6}$$

Moreover, it follows from Lemma 2.7 that

$$\mathbb{P}(\mathbf{L}(\sigma,\mathbb{X})\in\mathcal{U}_J(C_4\varepsilon e^{\mathcal{L}},\mathcal{L}+\varepsilon)) \ll \mathcal{L}^{2J}e^{4\mathcal{L}}\varepsilon^2.\tag{3.7}$$

Combining this bound with (3.5) and (3.6) gives

$$\begin{aligned}
\Psi_T(\tau) &= \frac{1}{T}\text{meas}\{t\in[T,2T] : \mathbf{L}(\sigma+it)\in(-\mathcal{L},\mathcal{L})^{2J}\}\\
&\quad + O\Big(\mathcal{L}^{2J}e^{4\mathcal{L}}\varepsilon^2 + \frac{\mathcal{L}^{2J+1}}{\varepsilon^{2J}(\log T)^{(1-\theta)/2}}\Big).
\end{aligned}$$

Similarly, it follows from (3.7) that

$$\Psi_{\text{rand}}(\tau) = \mathbb{P}(\mathbf{L}(\sigma,\mathbb{X})\in(-\mathcal{L},\mathcal{L})^{2J}) + O(\mathcal{L}^{2J}e^{4\mathcal{L}}\varepsilon^2).$$

Thus, using Theorem 2.3 we deduce that in this case

$$\Psi_T(\tau) = \Psi_{\text{rand}}(\tau) + O\Big(\mathcal{L}^{2J}e^{4\mathcal{L}}\varepsilon^2 + \frac{\mathcal{L}^{2J+1}}{\varepsilon^{2J}(\log T)^{(1-\theta)/2}}\Big).\tag{3.8}$$

Case 3: $\log(C_3\varepsilon)+\mathcal{L} < \tau \le \widetilde{\mathcal{L}}$. In this case we have

$$\Psi_T(\tau) = \frac{1}{T}\text{meas}\{t\in[T,2T] : \mathbf{L}(\sigma+it)\in\mathcal{V}_J(e^\tau,\mathcal{L})\},$$

where $\mathcal{V}_J(y,\mathcal{L})$ is the bounded subset of $\mathbb{R}^{2J}$ defined by

$$\mathcal{V}_J(y,\mathcal{L}) := \Big\{(u_1,\dots,u_J,v_1,\dots,v_J)\in\mathbb{R}^{2J} : |u_j|,|v_j| < \mathcal{L} \text{ for all } 1\le j\le J, \text{ and } \Big|\sum_{j=1}^J b_j e^{u_j+iv_j}\Big| > y\Big\}.$$

Similarly as before, we cover $\mathcal{V}_J(e^\tau, \mathcal{L})$ with $\widetilde{K}(\tau)$ distinct hypercubes $\widetilde{\mathcal{B}}_k(\tau)$, each of which has non-empty intersection with $\mathcal{V}_J(e^\tau, \mathcal{L})$ and sides of length ε. The number of such hypercubes is

$$\widetilde{K}(\tau) \ll \left(\frac{\mathcal{L}}{\varepsilon}\right)^{2J}.$$

Now, let $1 \le k \le \widetilde{K}(\tau)$ and $(u_1, \ldots, u_J, v_1, \ldots, v_J) \in \widetilde{\mathcal{B}}_k(\tau) \cap \mathcal{V}_J(e^\tau, \mathcal{L})$. Then, for any $(x_1, \ldots, x_J, y_1, \ldots, y_J) \in \widetilde{\mathcal{B}}_k(\tau)$ we have $|x_j - u_j| \le \varepsilon$ and $|y_j - v_j| \le \varepsilon$ for all $1 \le j \le J$. Hence, we deduce that $|x_j|, |y_j| < \mathcal{L} + \varepsilon$ for all $1 \le j \le J$ and

$$\left|\sum_{j=1}^{J} b_j e^{x_j + iy_j}\right| = \left|\sum_{j=1}^{J} b_j e^{u_j + iv_j}\right| + O(\varepsilon e^{\mathcal{L}}) > e^\tau - \frac{C_3}{2}\varepsilon e^{\mathcal{L}}$$

if C_3 is suitably large. Therefore, we have shown that

$$\mathcal{V}_J(e^\tau, \mathcal{L}) \subset \bigcup_{k \le \widetilde{K}(\tau)} \widetilde{\mathcal{B}}_k(\tau) \subset \mathcal{V}_J\Big(e^\tau - \frac{C_3}{2}\varepsilon e^{\mathcal{L}}, \mathcal{L} + \varepsilon\Big). \tag{3.9}$$

Now, using Theorem 2.3 we deduce

$$\begin{aligned} \Psi_T(\tau) &\le \sum_{k=1}^{\widetilde{K}(\tau)} \frac{1}{T}\text{meas}\{t \in [T, 2T] : \mathbf{L}(\sigma + it) \in \widetilde{\mathcal{B}}_k(\tau)\} \\ &= \sum_{k=1}^{\widetilde{K}(\tau)} \mathbb{P}(\mathbf{L}(\sigma, \mathbb{X}) \in \widetilde{\mathcal{B}}_k(\tau)) + O\Big(\frac{\widetilde{K}(\tau)\log\log T}{(\log T)^{(1-\theta)/2}}\Big) \\ &\le \mathbb{P}\Big(\mathbf{L}(\sigma, \mathbb{X}) \in \mathcal{V}_J\Big(e^\tau - \frac{C_4}{2}\varepsilon e^{\mathcal{L}}, \mathcal{L} + \varepsilon\Big)\Big) + O\Big(\frac{\mathcal{L}^{2J+1}}{\varepsilon^{2J}(\log T)^{(1-\theta)/2}}\Big). \end{aligned} \tag{3.10}$$

Moreover, it follows from Lemma 2.7 that

$$\mathbb{P}\Big(\mathbf{L}(\sigma, \mathbb{X}) \in (-\mathcal{L} - \varepsilon, \mathcal{L} + \varepsilon)^{2J} \text{ and } e^\tau - \frac{C_3}{2}\varepsilon e^{\mathcal{L}} < \Big|\sum_{j=1}^{J} b_j L_j(\sigma, \mathbb{X})\Big| \le e^\tau\Big) \ll \mathcal{L}^{2J} e^{4\mathcal{L}}\varepsilon,$$

since $\tau \le \widetilde{\mathcal{L}} = \mathcal{L} + O(1)$ by our assumption. Furthermore, since the density $H_T(\mathbf{u}, \mathbf{v})$ of the random vector $\mathbf{L}(\sigma, \mathbb{X})$ is uniformly bounded in $\mathbf{u}, \mathbf{v}$ by Lemma 7.2, we have

$$\mathbb{P}(\mathbf{L}(\sigma, \mathbb{X}) \in (-\mathcal{L} - \varepsilon, \mathcal{L} + \varepsilon)^{2J} \setminus (-\mathcal{L}, \mathcal{L})^{2J}) \ll \mathcal{L}^{2J-1}\varepsilon. \tag{3.11}$$

Combining these bounds we obtain

$$\mathbb{P}\Big(\mathbf{L}(\sigma, \mathbb{X}) \in \mathcal{V}_J\Big(e^\tau - \frac{C_4}{2}\varepsilon e^{\mathcal{L}}, \mathcal{L} + \varepsilon\Big)\Big) = \Psi_{\text{rand}}(\tau) + O(\mathcal{L}^{2J} e^{4\mathcal{L}}\varepsilon).$$

Hence, inserting this estimate in (3.10) we deduce

$$\Psi_T(\tau) \le \Psi_{\text{rand}}(\tau) + O\Big(\mathcal{L}^{2J} e^{4\mathcal{L}} \varepsilon + \frac{\mathcal{L}^{2J+1}}{\varepsilon^{2J} (\log T)^{(1-\theta)/2}}\Big). \tag{3.12}$$

We now proceed to prove the corresponding lower bound. Let τ_1 be such that $e^\tau = e^{\tau_1} - \frac{C_3}{2} \varepsilon e^{\mathcal{L}}$. Then, it follows from (3.9) and Theorem 2.3 that

$$\begin{aligned}
&\frac{1}{T}\text{meas}\{t \in [T, 2T] : \mathbf{L}(\sigma + it) \in \mathcal{V}_J(e^\tau, \mathcal{L} + \varepsilon)\} \\
&\quad \ge \sum_{k=1}^{\widetilde{K}(\tau_1)} \frac{1}{T}\text{meas}\{t \in [T, 2T] : \mathbf{L}(\sigma + it) \in \widetilde{\mathcal{B}}_k(\tau_1)\} \\
&\quad = \sum_{k=1}^{\widetilde{K}(\tau_1)} \mathbb{P}(\mathbf{L}(\sigma, \mathbb{X}) \in \widetilde{\mathcal{B}}_k(\tau_1)) + O\Big(\frac{\widetilde{K}(\tau) \log\log T}{(\log T)^{(1-\theta)/2}}\Big) \\
&\quad \ge \Psi_{\text{rand}}(\tau_1) + O\Big(\frac{\mathcal{L}^{2J+1}}{\varepsilon^{2J} (\log T)^{(1-\theta)/2}}\Big).
\end{aligned} \tag{3.13}$$

Moreover, by Lemma 2.7 we have

$$\begin{aligned}
&\Psi_{\text{rand}}(\tau_1) \\
&= \Psi_{\text{rand}}(\tau) + O\Bigg(\mathbb{P}\Bigg(\mathbf{L}(\sigma, \mathbb{X}) \in (-\mathcal{L}, \mathcal{L})^{2J} : e^\tau < \Bigg|\sum_{j=1}^{J} b_j L_j(\sigma, \mathbb{X})\Bigg| \le e^\tau + \frac{C_3}{2}\varepsilon e^{\mathcal{L}}\Bigg)\Bigg) \\
&= \Psi_{\text{rand}}(\tau) + O(\mathcal{L}^{2J} e^{4\mathcal{L}} \varepsilon).
\end{aligned}$$

Finally, we use Theorem 2.3 together with (3.11) to deduce

$$\begin{aligned}
&\frac{1}{T}\text{meas}\{t \in [T, 2T] : \mathbf{L}(\sigma + it) \in \mathcal{V}_J(e^\tau, \mathcal{L} + \varepsilon)\} - \Psi_T(\tau) \\
&\quad \le \frac{1}{T}\text{meas}\{t \in [T, 2T] : \mathbf{L}(\sigma + it) \in (-\mathcal{L} - \varepsilon, \mathcal{L} + \varepsilon)^{2J} \setminus (-\mathcal{L}, \mathcal{L})^{2J}\} \\
&\quad = \mathbb{P}(\mathbf{L}(\sigma, \mathbb{X}) \in (-\mathcal{L} - \varepsilon, \mathcal{L} + \varepsilon)^{2J} \setminus (-\mathcal{L}, \mathcal{L})^{2J}) + O\Big(\frac{\log\log T}{(\log T)^{(1-\theta)/2}}\Big) \\
&\quad \ll \mathcal{L}^{2J-1}\varepsilon + \frac{\log\log T}{(\log T)^{(1-\theta)/2}}.
\end{aligned}$$

Inserting these estimates in (3.13) yields

$$\Psi_T(\tau) \ge \Psi_{\text{rand}}(\tau) + O\Big(\mathcal{L}^{2J} e^{4\mathcal{L}} \varepsilon + \frac{\mathcal{L}^{2J+1}}{\varepsilon^{2J} (\log T)^{(1-\theta)/2}}\Big). \tag{3.14}$$

Thus we deduce from the estimates (3.4), (3.8) and (3.12) and (3.14) that in all cases we have

$$\Psi_T(\tau) = \Psi_{\text{rand}}(\tau) + O\Big(\mathcal{L}^{2J} e^{4\mathcal{L}} \varepsilon + \frac{\mathcal{L}^{2J+1}}{\varepsilon^{2J} (\log T)^{(1-\theta)/2}}\Big).$$

The desired result follows by choosing

$$\varepsilon = \frac{(\log\log T)^{1/(2J+1)}}{(\log T)^{(1-\theta+8\alpha)/(4J+2)}}. \qquad \square$$

Proof of Theorem 2.1. By (3.3) we have

$$\int_{-M}^{\widetilde{\mathcal{L}}} \Psi_T(\tau)d\tau = \int_{-M}^{\widetilde{\mathcal{L}}} \frac{1}{T} \int_{\substack{t\in S_0(T) \\ \log|F(\sigma+it)|>\tau}} dt d\tau = \frac{1}{T}\int_{t\in S_0(T)} (\log|F(\sigma+it)| + M)dt.$$

Combining this identity with (3.2) and using that $\text{meas}(S_0(T)) = T\Psi_T(-M)$ we obtain

$$\begin{aligned}\frac{1}{T}\int_T^{2T} &\log|F(\sigma+it)|dt \\ &= \int_{-M}^{\widetilde{\mathcal{L}}} \Psi_T(\tau)d\tau - M\Psi_T(-M) + O\Big(\frac{e^{O(\sqrt{\log\log T})}}{(\log T)^{\alpha^2/(\xi\theta)-3\theta}}\Big).\end{aligned} \tag{3.15}$$

We now repeat the exact same approach for the random model $F(\sigma, \mathbb{X})$. Using the same argument leading to (3.2) but with Lemma 2.5 instead of Lemma 2.4, we deduce similarly that

$$\mathbb{E}(\log|F(\sigma,\mathbb{X})|) = \mathbb{E}(\mathbf{1}_{\mathcal{S}}\cdot\log|F(\sigma,\mathbb{X})|) + O\Big(\frac{e^{O(\sqrt{\log\log T})}}{(\log T)^{\alpha^2/(\xi\theta)}}\Big),$$

where $\mathbf{1}_{\mathcal{S}}$ is the indicator function of $\mathcal{S}$. Therefore, reproducing the argument leading to (3.15) we obtain

$$\mathbb{E}(\log|F(\sigma,\mathbb{X})|) = \int_{-M}^{\widetilde{\mathcal{L}}} \Psi_{\text{rand}}(\tau)d\tau - M\Psi_{\text{rand}}(-M) + O\Big(\frac{e^{O(\sqrt{\log\log T})}}{(\log T)^{\alpha^2/(\xi\theta)}}\Big). \tag{3.16}$$

Combining (3.15) and (3.16) together with Proposition 3.1 we deduce that

$$\begin{aligned}\frac{1}{T}\int_T^{2T} &\log|F(\sigma+it)|dt \\ &= \mathbb{E}(\log|F(\sigma,\mathbb{X})|) + O\Big(\frac{(\log\log T)^{2J+4}}{(\log T)^{(1-\theta-16J\alpha)/(4J+2)-3\theta}} + \frac{e^{O(\sqrt{\log\log T})}}{(\log T)^{\alpha^2/(\xi\theta)-3\theta}}\Big).\end{aligned} \tag{3.17}$$

We first require that θ satisfies $\theta < \alpha/\sqrt{3\xi}$, so that the exponent of $\log T$ in the second error term of (3.17) is negative. Therefore, we might choose α be slightly bigger than $\sqrt{3\xi}\theta$. Hence, in order to insure that the exponent of $\log T$ in the error term of (3.17) is negative, we thus require that θ satisfies the inequality

$$12J\theta + 7\theta + 16J\sqrt{3\xi}\theta < 1 \iff \theta < \frac{1}{12J + 7 + 16J\sqrt{3\xi}}.$$

This completes the proof. $\square$

4 Preliminary results

In this section we provide several of the technical lemmas that we shall need later. We first record several useful facts. Since

$$\log L_j(s) = \sum_p \sum_{i=1}^{d} \sum_{k=1}^{\infty} \frac{\alpha_{j,i}(p)^k}{kp^{ks}},$$

we see that

$$\beta_{L_j}(p^k) = \frac{1}{k} \sum_{i=1}^{d} \alpha_{j,i}(p)^k$$

and

$$|\beta_{L_j}(p^k)| \le \frac{d}{k} p^{k\theta} \tag{4.1}$$

for $k = 1, 2, \ldots$ and $j = 1, \ldots, J$. For later use, we remark that

$$\sum_{k=3}^{\infty} \sum_p \frac{|\beta_{L_j}(p^k)|}{p^{k/2}} < \infty, \quad \sum_{k=2}^{\infty} \sum_p \frac{|\beta_{L_j}(p^k)|^2}{p^k} < \infty, \quad \sum_p \frac{|\beta_{L_j}(p)|^4}{p^2} < \infty. \tag{4.2}$$

One can easily show (4.2) by applying assumption A3 and inequalities

$$|\beta_{L_j}(p^k)| \le \frac{1}{k} \sum_{i=1}^{d} |\alpha_{j,i}(p)|^k \le \frac{p^{(k-2)\theta}}{k} \sum_{i=1}^{d} |\alpha_{j,i}(p)|^2 \quad \text{for } k \ge 2 \tag{4.3}$$

and

$$|\beta_{L_j}(p)|^2 \le \left(\sum_{i=1}^{d} |\alpha_{j,i}(p)| \right)^2 \le d \sum_{i=1}^{d} |\alpha_{j,i}(p)|^2. \tag{4.4}$$

Lemma 4.1. *Let $1 \le j, k \le J$. Then uniformly for $1/2 < \sigma \le 1$ we have*

$$\begin{aligned} &\sum_p \frac{\beta_{L_j}(p)\overline{\beta_{L_k}(p)}}{p^{2\sigma}} \\ &\quad = \delta_{j,k}\xi_j \log\left(\frac{1}{\sigma - 1/2}\right) + c'_{j,k} + O\left((\sigma - 1/2)\log\left(\frac{1}{\sigma - 1/2}\right)\right) \end{aligned} \tag{4.5}$$

for some constants $c'_{j,k}$. Moreover, uniformly for $1/2 < \sigma \le 1$ we have

$$\begin{aligned} \sum_{p>Y} \frac{\beta_{L_j}(p)\overline{\beta_{L_k}(p)}}{p^{2\sigma}} &= \delta_{j,k}\xi_j \log\left(\frac{1}{(\sigma - 1/2)\log Y}\right) + O(1) \\ &\qquad \text{if } 2 \le Y \le \exp\left(\frac{1}{2\sigma - 1}\right), \end{aligned} \tag{4.6}$$

and

$$\sum_{p^n>Y}\frac{|\beta_{L_j}(p^n)|^2}{p^{2n\sigma}} \ll \frac{Y^{1-2\sigma}}{(2\sigma-1)\log Y} \quad \text{if } Y \geq \exp\Big(\frac{1}{2\sigma-1}\Big). \tag{4.7}$$

Proof. We start by proving (4.5). First, by partial summation and (1.3) we derive

$$\begin{aligned}\sum_{p}\frac{\beta_{L_j}(p)\overline{\beta_{L_k}(p)}}{p^{2\sigma}} =&\delta_{j,k}\xi_j\int_2^\infty \frac{u^{-2\sigma}}{\log u}du + b_{j,k}\\ &+O\Big(\Big(\sigma-\frac{1}{2}\Big)\Big(1+\int_2^\infty\frac{u^{-2\sigma}}{\log u}du\Big)\Big),\end{aligned} \tag{4.8}$$

for some constants $b_{j,k}$. To evaluate the integral on the right-hand side of this estimate we use the substitution $w = (2\sigma-1)\log u$. This gives

$$\begin{aligned}\int_2^\infty \frac{u^{-2\sigma}}{\log u}du &= \int_{(2\sigma-1)\log 2}^\infty e^{-w}\frac{dw}{w}\\ &= \int_{(2\sigma-1)\log 2}^1 \frac{dw}{w} + \int_{(2\sigma-1)\log 2}^1 (e^{-w}-1)\frac{dw}{w} + \int_1^\infty e^{-w}\frac{dw}{w}\\ &= \log\Big(\frac{1}{\sigma-1/2}\Big) - \log(2\log 2) - \gamma + O(\sigma-1/2),\end{aligned} \tag{4.9}$$

where

$$\gamma = \int_0^1 (1-e^{-w})\frac{dw}{w} - \int_1^\infty e^{-w}\frac{dw}{w}$$

is the Euler–Mascheroni constant. Inserting this estimate in (4.8) implies (4.5).

We next establish (4.6). Similarly to (4.8) one has

$$\sum_{p>Y}\frac{\beta_{L_j}(p)\overline{\beta_{L_k}(p)}}{p^{2\sigma}} = \delta_{j,k}\xi_j\int_Y^\infty\frac{u^{-2\sigma}}{\log u}du + O(1).$$

To estimate this integral we again use the substitution $w = (2\sigma-1)\log u$. Then similarly to (4.9) one obtains

$$\int_Y^\infty \frac{u^{-2\sigma}}{\log u}du = \int_{(2\sigma-1)\log Y}^\infty e^{-w}\frac{dw}{w} = \log\Big(\frac{1}{(\sigma-1/2)\log Y}\Big) + O(1),$$

which implies (4.6).

We finally turn to the proof of (4.7). By partial summation and (1.3) it follows that in the range $Y > \exp(1/(2\sigma-1))$ we have

$$\begin{aligned}\sum_{p>Y}\frac{|\beta_{L_j}(p)|^2}{p^{2\sigma}} &\ll \int_Y^\infty u^{-2\sigma}\frac{du}{\log u} + \frac{Y^{1-2\sigma}}{\log Y} \ll \int_{(2\sigma-1)\log Y}^\infty e^{-w}\frac{dw}{w} + \frac{Y^{1-2\sigma}}{\log Y}\\ &\ll \frac{Y^{1-2\sigma}}{(2\sigma-1)\log Y}.\end{aligned}$$

Now, we will bound the contribution of the prime powers. By (4.1) and (4.3), we have

$$|\beta_{L_j}(p^n)|^2 \leq \frac{dp^{2(n-1)\theta}}{n^2}\sum_{i=1}^{d}|\alpha_{j,i}(p)|^2,$$

and hence by assumption A3 and partial summation, we have

$$\begin{aligned}\sum_{\substack{p^n>Y\\ n\geq 2}}\frac{|\beta_{L_j}(p^n)|^2}{p^{2n\sigma}} &\ll \sum_{p>\sqrt{Y}}\sum_{n=2}^{\infty}\frac{\sum_{i=1}^{d}|\alpha_{j,i}(p)|^2}{n^2p^{2(\sigma-\theta)n+2\theta}}+\sum_{p\leq\sqrt{Y}}\sum_{n\geq\frac{\log Y}{\log p}}\frac{\sum_{i=1}^{d}|\alpha_{j,i}(p)|^2}{n^2p^{2(\sigma-\theta)n+2\theta}}\\ &\ll \sum_{p>\sqrt{Y}}\frac{\sum_{i=1}^{d}|\alpha_{j,i}(p)|^2}{p^{4\sigma-2\theta}}+\sum_{p\leq\sqrt{Y}}\frac{\sum_{i=1}^{d}|\alpha_{j,i}(p)|^2}{Y^{2(\sigma-\theta)}p^{2\theta}}\\ &\ll Y^{(1+\varepsilon-4\sigma+2\theta)/2}+Y^{-2(\sigma-\theta)+(1+\varepsilon-2\theta)/2}\end{aligned}$$

for any $\varepsilon > 0$. By choosing ε sufficiently small, we obtain

$$\sum_{p^n>Y}\frac{|\beta_{L_j}(p^n)|^2}{p^{2n\sigma}} \ll \frac{Y^{1-2\sigma}}{(2\sigma-1)\log Y},$$

which completes the proof. □

Let $L(s)$ be an L-function satisfying assumptions A1–A4. Here and throughout, we define for $Y \geq 2$ and $\sigma, t \in \mathbb{R}$

$$R_{L,Y}(\sigma+it) := \sum_{p^n\leq Y}\frac{\beta_L(p^n)}{p^{n(\sigma+it)}} \quad \text{and} \quad R_{L,Y}(\sigma,\mathbb{X}) := \sum_{p^n\leq Y}\frac{\beta_L(p^n)\mathbb{X}(p)^n}{p^{n\sigma}},$$

where $\{\mathbb{X}(p)\}_p$ is a sequence of independent random variables, uniformly distributed on the unit circle.

Our next result shows that $\log L(\sigma+it)$ can be approximated by $R_{L,Y}(\sigma+it)$ for $1/2+1/G(T) \leq \sigma \leq 1$, and for all $t \in [T, 2T]$ except for an exceptional set with a very small measure. This is accomplished using the zero-density estimates (1.2).

Lemma 4.2. *Let $L(s)$ be an L-function satisfying assumptions A1–A4. Let T be large and $G(T)$ be such that $2 < G(T) \leq c_0\sqrt{\log T/\log\log T}$, for some suitably small constant $c_0 > 0$. Put $Y = e^{AG(T)\log\log T}$ for a constant $A \geq 5$. Then there is a positive constant c_1 such that for all $t \in [T, 2T]$ except for a set of measure $\ll T\exp(-c_1\log T/G(T))$, we have*

$$\log L(\sigma+it) = R_{L,Y}(\sigma+it) + O\Big(\frac{1}{(\log T)^{A/2-2}}\Big)$$

uniformly for $\sigma \geq 1/2+1/G(T)$.

Proof. By Perron's formula we have

$$(4.10)\qquad \begin{aligned} R_{L,Y}(\sigma+it) =& \frac{1}{2\pi i}\int_{c-iY}^{c+iY} \log L(\sigma+it+w)\frac{Y^w}{w}dw \\ &+ O\Big(Y^{-\sigma+1/4}\sum_p\sum_{n=1}^{\infty}\frac{|\beta_L(p^n)|}{p^{5n/4}|\log(Y/p^n)|}\Big) \end{aligned}$$

where $c = 5/4 - \sigma$. To bound the error term of this last estimate, we split the sum over primes into three parts: $p^n \le Y/2$, $Y/2 < p^n < 2Y$ and $p^n \ge 2Y$. The terms in the first and third parts satisfy $|\log(Y/p^n)| \ge \log 2$, and hence their contribution is

$$\ll Y^{-\sigma+1/4}\sum_p\sum_{n=1}^{\infty}\frac{|\beta_L(p^n)|}{p^{5n/4}} = Y^{-\sigma+1/4}\Big(\sum_p\frac{|\beta_L(p)|}{p^{5/4}} + O(1)\Big) \ll Y^{-1/4}$$

by (4.1) and (4.4). To handle the contribution of the terms $Y/2 < p^n < 2Y$, we put $r = Y - p^n$, and use the lower bound $|\log(Y/p^n)| \gg |r|/Y$. Then the contribution of these terms is

$$\ll Y^{-\sigma+\theta-5/4}\sum_{r\le Y}\frac{1}{r} \ll Y^{-1/2+\theta}\log Y.$$

Let $w_0 = -1/(2G(T))$ and assume that $L(\sigma+it+w)$ has no zeros in the half-strip given by $\mathrm{Re}(w) \ge -3/(4G(T))$, $|\mathrm{Im}(w)| \le Y+1$. Then in the slightly smaller half-strip $\{w : \mathrm{Re}(w) \ge w_0, |\mathrm{Im}(w)| \le Y\}$ we have

$$(4.11)\qquad \frac{L'}{L}(\sigma+it+w) \ll G(T)\log T$$

(see [7, Proposition 5.7]). Observe that this holds for all $t \in [T, 2T]$ except for t in a set of measure

$$\begin{aligned} &\ll Y\cdot N\Big(\frac{1}{2}+\frac{1}{4G(T)}, 2T\Big) \ll T^{1-c_2/(4G(T))}(\log T)^{c_3}\exp(AG(T)\log\log T) \\ &\ll T\exp\Big(-c_1\frac{\log T}{G(T)}\Big) \end{aligned}$$

for some constants $c_1, c_2, c_3 > 0$ by (1.2) and our assumption on $G(T)$. Now, integrating both sides of (4.11) along the horizontal segment from w to $w+B$, where B is sufficiently large, we see that for such t we have

$$\log L(\sigma+it+w) \ll G(T)\log T,$$

for all w such that $\mathrm{Re}(w) \ge w_0$, $|\mathrm{Im}(w)| \le Y$. Using this and shifting the contour to the left in (4.10), we obtain

$$\begin{aligned} R_{L,Y}(\sigma+it) - \log L(\sigma+it) &= \frac{1}{2\pi i}\left(\int_{c-iY}^{w_0-iY} + \int_{w_0-iY}^{w_0+iY} + \int_{w_0+iY}^{c+iY}\right) \log L(\sigma+it+w)\frac{Y^w}{w}dw \\ &\quad + O(Y^{-1/2+\theta}\log Y + Y^{-1/4}) \\ &\ll G(T)(\log T)Y^{w_0}\log Y \ll \frac{1}{(\log T)^{A/2-2}}. \end{aligned}$$

This proves the lemma. □

We now establish the analogous result for the random model $\log L(\sigma, \mathbb{X})$.

Lemma 4.3. *Let $\sigma > 1/2$ and $Y \geq \exp(1/(2\sigma-1))$. Then, for all $\varepsilon > 0$ we have*

$$\mathbb{P}(|\log L_j(\sigma, \mathbb{X}) - R_{L_j,Y}(\sigma, \mathbb{X})| \geq \varepsilon) \ll \frac{Y^{1-2\sigma}}{\varepsilon^2(2\sigma-1)\log Y}.$$

Proof. By Chebyshev's inequality we have

$$\mathbb{P}(|\log L_j(\sigma, \mathbb{X}) - R_{L_j,Y}(\sigma, \mathbb{X})| \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\mathbb{E}(|\log L_j(\sigma, \mathbb{X}) - R_{L_j,Y}(\sigma, \mathbb{X})|^2). \tag{4.12}$$

Furthermore, observe that

$$\begin{aligned} \mathbb{E}(|\log L_j(\sigma, \mathbb{X}) - R_{L_j,Y}(\sigma, \mathbb{X})|^2) &= \sum_{p_1^n, p_2^m > Y} \frac{\beta_{L_j}(p_1^n)\overline{\beta_{L_j}(p_2^m)}}{p_1^{n\sigma}p_2^{m\sigma}}\mathbb{E}(\mathbb{X}(p_1)^n\overline{\mathbb{X}(p_2)}^m) \\ &= \sum_{p^n > Y} \frac{|\beta_{L_j}(p^n)|^2}{p^{2n\sigma}}, \end{aligned}$$

since $\mathbb{E}(\mathbb{X}(p_1)^n\overline{\mathbb{X}(p_2)^m}) = 1$ only when $n = m$ and $p_1 = p_2$, and is 0 otherwise. Combining this identity with Lemma 4.1 and (4.12) completes the proof. □

We also need a standard mean value estimate, which follows from [22, Lemma 3.3].

Lemma 4.4. *Let $z \geq 2$ be a real number and k be a positive integer such that $k \leq \log T/\log z$. Let $\{a(p)\}_p$ be a sequence of complex numbers. Let $\{\mathbb{X}(p)\}_p$ be a sequence of independent random variables uniformly distributed on the unit circle. Then we have*

$$\frac{1}{T}\int_T^{2T}\left|\sum_{p\leq z}\frac{a(p)}{p^{it}}\right|^{2k} dt \ll k!\left(\sum_{p\leq z}|a(p)|^2\right)^k,$$

and

$$\mathbb{E}\left(\left|\sum_{p\leq z}a(p)\mathbb{X}(p)\right|^{2k}\right) \leq k!\left(\sum_{p\leq z}|a(p)|^2\right)^k.$$

Using this result we bound the $(2k)$th moments of $R_{L_j,Y}(\sigma+it)$ and $R_{L_j,Y}(\sigma, \mathbb{X})$.

Lemma 4.5. *Let $1 \le j \le J$. Let $\sigma \ge 1/2$ and $Y \ge 2$ be real numbers. Then for all positive integers $k \le \log T/\log Y$, we have*

$$\frac{1}{T}\int_T^{2T} |R_{L_j,Y}(\sigma+it)|^{2k} dt \ll k!(\xi_j \log\log Y + O(\sqrt{\log\log Y}))^k,$$

and

$$\mathbb{E}(|R_{L_j,Y}(\sigma,\mathbb{X})|^{2k}) \ll k!(\xi_j \log\log Y + O(\sqrt{\log\log Y}))^k.$$

Proof. We only prove the first estimate, as the second is similar and simpler. First, by (4.2) we have

$$R_{L_j,Y}(\sigma+it) = \sum_{p\le Y} \frac{\beta_{L_j}(p)}{p^{\sigma+it}} + \sum_{p\le\sqrt{Y}} \frac{\beta_{L_j}(p^2)}{p^{2\sigma+2it}} + O(1)$$

uniformly in Y and t. Hence it follows from Minkowski's inequality that

$$\begin{aligned}(4.13)\qquad &\left(\int_T^{2T} |R_{L_j,Y}(\sigma+it)|^{2k} dt\right)^{1/2k} \\ &\le \left(\int_T^{2T} \left|\sum_{p\le Y}\frac{\beta_{L_j}(p)}{p^{\sigma+it}}\right|^{2k} dt\right)^{1/2k} \\ &\quad + \left(\int_T^{2T} \left|\sum_{p\le\sqrt{Y}}\frac{\beta_{L_j}(p^2)}{p^{2\sigma+2it}}\right|^{2k} dt\right)^{1/2k} + C_1 T^{1/2k},\end{aligned}$$

for some constant $C_1 > 0$. Now, it follows from Lemma 4.4 together with Stirling's formula and assumption A5 that

$$\begin{aligned}(4.14)\qquad \frac{1}{T}\int_T^{2T} \left|\sum_{p\le Y}\frac{\beta_{L_j}(p)}{p^{\sigma+it}}\right|^{2k} dt &\ll k!\left(\sum_{p\le Y}\frac{|\beta_{L_j}(p)|^2}{p^{2\sigma}}\right)^k \\ &\le k!(\xi_j\log\log Y + O(1))^k,\end{aligned}$$

since $\sigma \ge 1/2$. Similarly, by Lemma 4.4 and a simple change of variable we have

$$\begin{aligned}(4.15)\qquad \int_T^{2T} \left|\sum_{p\le\sqrt{Y}}\frac{\beta_{L_j}(p^2)}{p^{2\sigma+2it}}\right|^{2k} dt &= \frac{1}{2}\int_{2T}^{4T} \left|\sum_{p\le\sqrt{Y}}\frac{\beta_{L_j}(p^2)}{p^{2\sigma+it}}\right|^{2k} dt \\ &\ll Tk!\left(\sum_{p\le Y}\frac{|\beta_{L_j}(p^2)|^2}{p^2}\right)^k.\end{aligned}$$

Now, by (4.1), (4.3), assumption A3 and partial summation we obtain

$$\sum_{p\le Y}\frac{|\beta_{L_j}(p^2)|^2}{p^2} \ll \sum_{p\le Y}\sum_{i=1}^{d}\frac{|\alpha_{j,i}(p)|^2}{p^{2-2\theta}} \ll 1.$$

Combining this estimate with (4.13), (4.14) and (4.15) completes the proof. □

As a consequence of this result and Lemma 4.2, we establish Lemma 2.6.

Proof of Lemma 2.6. Let $Y = e^{10G(T)\log\log T}$. By Lemma 4.2 for all $t \in [T, 2T]$ except for a set of measure $\ll T\exp(-c_1 \log T/G(T))$ (for some constant $c_1 > 0$), we have

$$\log L_j(\sigma + it) = R_{L_j,Y}(\sigma + it) + O\Big(\frac{1}{(\log T)^3}\Big).$$

Therefore, it follows from Lemma 4.5 that

$$\begin{aligned}
&\frac{1}{T}\text{meas}\{t \in [T, 2T] : |\log L_j(\sigma + it)| > \tau\} \\
&\quad \le \frac{1}{T}\text{meas}\Big\{t \in [T, 2T] : |R_{L_j,Y}(\sigma + it)| > \tau - \frac{1}{\log T}\Big\} + O(e^{-c_1\frac{\log T}{G(T)}}) \\
&\quad \le \frac{1}{(\tau - 1/\log T)^{2k}}\frac{1}{T}\int_T^{2T} |R_{L_j,Y}(\sigma + it)|^{2k}dt + O(e^{-c_1\frac{\log T}{G(T)}}) \\
&\quad \le k!\Big(\frac{\xi_j \log G(T) + O(\sqrt{\log G(T)} + \log_3 T)}{(\tau - 1/\log T)^2}\Big)^k + O(e^{-c_1\frac{\log T}{G(T)}}).
\end{aligned} \tag{4.16}$$

Using Stirling's formula, and choosing

$$k = \lfloor(\tau - 1/\log T)^2/(\xi_j \log G(T) + C\max\{\sqrt{\log G(T)}, \log_3 T\})\rfloor$$

for some suitably large constant C implies the result.

We now establish the analogous bound for $\log L_j(\sigma, \mathbb{X})$. Let $\varepsilon = 1/(\log T)^2$ and $Y = e^{G(T)(\log\log T)^5}$. Then, it follows from Lemma 4.3 that

$$\begin{aligned}
\mathbb{P}\Big(|\log L_j(\sigma, \mathbb{X}) - R_{L_j,Y}(\sigma, \mathbb{X})| > \frac{1}{(\log T)^2}\Big) &\ll \frac{(\log T)^4}{(\log\log T)^5}e^{-2(\log\log T)^5} \\
&\ll e^{-(\log\log T)^5}.
\end{aligned}$$

Now, using the same argument leading to (4.16) together with Lemma 4.5 we obtain

$$\begin{aligned}
&\mathbb{P}(|\log L_j(\sigma, \mathbb{X})| > \tau) \\
&\quad \le \mathbb{P}\Big(|R_{L_j,Y}(\sigma, \mathbb{X})| > \tau - \frac{1}{\log T}\Big) + O(e^{-(\log\log T)^5}) \\
&\quad \le k!\Big(\frac{\xi_j \log G(T) + O(\sqrt{\log G(T)} + \log_3 T)}{(\tau - 1/\log T)^2}\Big)^k + O(e^{-(\log\log T)^5}).
\end{aligned}$$

Making the same choice of k and using Stirling's formula completes the proof. □

We extend the $\mathbb{X}(p)$ multiplicatively by defining for $n = \prod_p p^\alpha$, $\mathbb{X}(n) = \prod_p \mathbb{X}(p)^\alpha$. We finish this section with the following standard lemma, which we shall use in Section 5 to prove that the characteristic function of the joint distribution of $\log L_j(\sigma + it)$ is very close to that of the joint distribution of $\log L_j(\sigma, \mathbb{X})$.

Lemma 4.6. *Let $b_j(n)$ be complex numbers, such that $|b_j(n)| \le C$ for all $1 \le j \le J$ and $n \ge 1$ and for some constant $C > 0$. Let k_j, k_j' be positive integers for $j \le J$ and write $k = \sum_{j\le J} k_j$ and $k' = \sum_{j \le J} k_j'$. Then uniformly for $Y, T \ge 2$ we have*

$$\frac{1}{T}\int_T^{2T} \prod_{j=1}^{J} \Big(\sum_{n\le Y} \frac{b_j(n)}{n^{it}}\Big)^{k_j} \Big(\overline{\sum_{n\le Y} \frac{b_j(n)}{n^{it}}}\Big)^{k_j'} dt$$
$$= \mathbb{E}\Big(\prod_{j=1}^{J}\Big(\sum_{n\le Y} b_j(n)\mathbb{X}(n)\Big)^{k_j}\Big(\overline{\sum_{n\le Y} b_j(n)\mathbb{X}(n)}\Big)^{k_j'}\Big) + O\Big(\frac{(CY^2)^{k+k'}}{T}\Big).$$

Proof. We have

$$\frac{1}{T}\int_T^{2T} \prod_{j=1}^{J} \Big(\sum_{n\le Y} \frac{b_j(n)}{n^{it}}\Big)^{k_j} \Big(\overline{\sum_{n\le Y} \frac{b_j(n)}{n^{it}}}\Big)^{k_j'} dt$$
$$= \frac{1}{T}\int_T^{2T} \Big(\sum_{n_{i,j}\le Y} \frac{b_1(n_{1,1})\cdots b_1(n_{k_1,1})}{(n_{1,1}\cdots n_{k_1,1})^{it}} \cdots \frac{b_J(n_{1,J})\cdots b_J(n_{k_J,J})}{(n_{1,J}\cdots n_{k_J,J})^{it}}\Big)$$
$$\times \Big(\sum_{m_{i,j}\le Y} \frac{\overline{b_1(m_{1,1})}\cdots \overline{b_1(m_{k_1',1})}}{(m_{1,1}\cdots m_{k_1',1})^{-it}} \cdots \frac{\overline{b_J(m_{1,J})}\cdots \overline{b_J(m_{k_J',J})}}{(m_{1,J}\cdots m_{k_J',J})^{-it}}\Big) dt.$$

The contribution of the diagonal terms is

$$\Sigma_1 = \sum_{\substack{n_{i,j}, m_{i,j}\le Y \\ \prod n_{i,j} = \prod m_{i,j}}} \prod_{j=1}^{J} \Big(\prod_{i=1}^{k_j} b_j(n_{i,j}) \prod_{i=1}^{k_j'} \overline{b_j(m_{i,j})}\Big)$$
$$= \mathbb{E}\Big(\prod_{j=1}^{J}\Big(\sum_{n\le Y} b_j(n)\mathbb{X}(n)\Big)^{k_j}\Big(\overline{\sum_{n\le Y} b_j(n)\mathbb{X}(n)}\Big)^{k_j'}\Big).$$

The off-diagonal contribution is

$$\Sigma_2 = \sum_{\substack{n_{i,j}, m_{i,j}\le Y \\ \prod n_{i,j} \ne \prod m_{i,j}}} \prod_{j=1}^{J} \Big(\prod_{i=1}^{k_j} b_j(n_{i,j}) \prod_{i=1}^{k_j'} \overline{b_j(m_{i,j})}\Big)\Big(\frac{(m/n)^{2iT} - (m/n)^{iT}}{iT\log(m/n)}\Big),$$

where $n = \prod n_{i,j}$ and $m = \prod m_{i,j}$. Since $n, m \le Y^{k+k'}$ and $n \ne m$,

$$\frac{1}{|\log(m/n)|} \ll Y^{k+k'}.$$

Hence, we derive

$$\Sigma_2 \ll \frac{(CY)^{k+k'}}{T} \sum_{\substack{n_{i,j}, m_{i,j}\le Y \\ \prod n_{i,j} \ne \prod m_{i,j}}} 1 \ll \frac{(CY^2)^{k+k'}}{T}.$$

This completes the proof. □

5 Bounding the discrepancy: Proof of Theorem 2.3

Let $\mathbf{u} = (u_1, \ldots, u_J)$ and similarly $\mathbf{v}$, $\mathbf{x}$, and $\mathbf{y}$ be vectors in $\mathbb{R}^J$, and define

$$\widehat{\Phi}_T(\mathbf{x}, \mathbf{y}) := \int_{\mathbb{R}^{2J}} e^{2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})} d\Phi_T(\mathbf{u}, \mathbf{v}),$$

and

$$\widehat{\Phi}_T^{\text{rand}}(\mathbf{x}, \mathbf{y}) := \int_{\mathbb{R}^{2J}} e^{2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})} d\Phi_T^{\text{rand}}(\mathbf{u}, \mathbf{v}),$$

where $\mathbf{x} \cdot \mathbf{u} = \sum_{j=1}^J x_j u_j$ is the dot product. Then by the definitions of Φ_T and Φ_T^{rand} in (2.3) and (2.4), we may write

$$\widehat{\Phi}_T(\mathbf{x}, \mathbf{y}) = \frac{1}{T}\int_T^{2T} \exp\left[2\pi i \sum_{j=1}^J (x_j \log |L_j(\sigma + it)| + y_j \arg L_j(\sigma + it))\right] dt$$

and

$$\widehat{\Phi}_T^{\text{rand}}(\mathbf{x}, \mathbf{y}) = \mathbb{E}\left(\exp\left[2\pi i \sum_{j=1}^J (x_j \log |L_j(\sigma, \mathbb{X})| + y_j \arg L_j(\sigma, \mathbb{X}))\right]\right).$$

Proposition 5.1. *Let T be large and $\sigma = 1/2+1/G(T)$ where $2 < G(T) \leq \frac{\sqrt{\log T}}{\log\log T}$. Let $||\mathbf{x}||_\infty := \sup_{1\leq j\leq J} |x_j|$. Then, for any constant $A > 0$ there exists a constant $c_1 > 0$ such that for all $\mathbf{x}$ and $\mathbf{y}$ with $||\mathbf{x}||_\infty, ||\mathbf{y}||_\infty \leq c_1 \frac{\sqrt{\log T}}{\sqrt{G(T)}\log\log T}$, we have*

$$\widehat{\Phi}_T(\mathbf{x}, \mathbf{y}) = \widehat{\Phi}_T^{\text{rand}}(\mathbf{x}, \mathbf{y}) + O\left(\frac{1}{(\log T)^A}\right).$$

Proof. Let $Y = \exp(BG(T)\log\log T)$, where $B = 2A+6$. Then for every $j \leq J$, it follows from Lemma 4.2 that

$$\log L_j(\sigma + it) = R_{L_j,Y}(\sigma + it) + O\left(\frac{1}{(\log T)^{A+1}}\right) \tag{5.1}$$

for all $t \in [T, 2T]$ except for a set of measure $\ll T\exp(-c_1 \log T/G(T))$ for some constant $c_1 > 0$. Let $\mathcal{A}(T)$ be the set of points $t \in [T, 2T]$ for which (5.1) holds for all $j \leq J$. Then

$$\text{meas}(\mathcal{A}(T)) \ll T\exp(-c_1 \log T/G(T)) \ll T\exp(-\sqrt{\log T}).$$

Hence by (5.1), $\widehat{\Phi}_T(\mathbf{x},\mathbf{y})$ equals

$$\frac{1}{T}\int_{\mathcal{A}(T)} \exp\left(2\pi i\Big(\sum_{j=1}^{J}(x_j\mathrm{Re}\log L_j(\sigma+it)+y_j\mathrm{Im}\log L_j(\sigma+it))\Big)\right)dt$$
$$+O(e^{-\sqrt{\log T}})$$
$$=\frac{1}{T}\int_{T}^{2T} \exp\left(2\pi i\Big(\sum_{j=1}^{J}(x_j\mathrm{Re}R_{L_j,Y}(\sigma+it)+y_j\mathrm{Im}R_{L_j,Y}(\sigma+it))\Big)\right)dt$$
$$+O\Big(\frac{1}{(\log T)^A}\Big).$$

Let $N=[\log T/(10BG(T)\log\log T)]$. Then, it follows from the previous estimate that $\widehat{\Phi}_T(\mathbf{x},\mathbf{y})$ equals

$$\sum_{n=0}^{2N-1}\frac{(2\pi i)^n}{n!}\frac{1}{T}\int_T^{2T}\left(\sum_{j=1}^{J}(x_j\mathrm{Re}R_{L_j,Y}(\sigma+it)+y_j\mathrm{Im}R_{L_j,Y}(\sigma+it))\right)^n dt+E_1, \tag{5.2}$$

where

$$\begin{aligned}E_1 &\ll \frac{1}{(\log T)^A}\\ &\quad+\frac{(2\pi)^{2N}(||\mathbf{x}||_\infty+||\mathbf{y}||_\infty)^{2N}}{(2N)!}\frac{1}{T}\int_T^{2T}\left(\sum_{j=1}^{J}|R_{L_j,Y}(\sigma+it)|\right)^{2N}dt \\ &\ll \frac{1}{(\log T)^A}+\frac{N!}{(2N)!}(C_2\log\log Y)^N\frac{(c_1^2\log T)^N}{G(T)^N(\log\log T)^{2N}}\ll\frac{1}{(\log T)^A},\end{aligned} \tag{5.3}$$

by Lemma 4.5, Minkowski's inequality and Stirling's formula, where c_1 and C_2 are positive constants.

Next, we handle the main term of (5.2). To this end, we use Lemma 4.6, which implies that for all non-negative integers $k_1,k_2,\dots,k_{2J}$ such that $k_1+\cdots+k_{2J}\le 2N$ we have

$$\frac{1}{T}\int_T^{2T}\prod_{j=1}^{J}(R_{L_j,Y}(\sigma+it))^{k_j}\prod_{\ell=1}^{J}(\overline{R_{L_j,Y}(\sigma+it)})^{k_{J+\ell}}dt$$
$$=\mathbb{E}\left(\prod_{j=1}^{J}(R_{L_j,Y}(\sigma,\mathbb{X}))^{k_j}\prod_{\ell=1}^{J}(\overline{R_{L_j,Y}(\sigma,\mathbb{X})})^{k_{J+\ell}}\right)+O(T^{-1/2}).$$

Let $z_j=x_j+iy_j$ and $\overline{z_j}$ be its complex conjugate. Then it follows from this estimate that for all $0 \le n \le 2N$ we have

$$\frac{1}{T}\int_T^{2T}\left(\sum_{j=1}^J (x_j \mathrm{Re} R_{L_j,Y}(\sigma+it) + y_j \mathrm{Im} R_{L_j,Y}(\sigma+it))\right)^n dt$$
$$= \frac{1}{2^n}\frac{1}{T}\int_T^{2T}\left(\sum_{j=1}^J (\overline{z_j} R_{L_j,Y}(\sigma+it) + z_j \overline{R_{L_j,Y}(\sigma+it)})\right)^n dt$$
$$= \frac{1}{2^n}\sum_{\substack{k_1,\dots,k_{2J}\ge 0\\ k_1+\cdots+k_{2J}=n}}\binom{n}{k_1,\dots,k_{2J}}$$
$$\times \frac{1}{T}\int_T^{2T}\prod_{j=1}^J(\overline{z_j} R_{L_j,Y}(\sigma+it))^{k_j}\prod_{\ell=1}^J (z_\ell \overline{R_{L_\ell,Y}(\sigma+it)})^{k_{J+\ell}} dt$$
$$= \frac{1}{2^n}\sum_{\substack{k_1,\dots,k_{2J}\ge 0\\ k_1+\cdots+k_{2J}=n}}\binom{n}{k_1,k_2,\dots,k_{2J}}\mathbb{E}\left(\prod_{j=1}^J(\overline{z_j} R_{L_j,Y}(\sigma,\mathbb{X}))^{k_j}\prod_{\ell=1}^J(z_\ell\overline{R_{L_j,Y}(\sigma,\mathbb{X})})^{k_{J+\ell}}\right)$$
$$+ O\left(T^{-1/2}\left(\sum_{j=1}^J |z_j|\right)^n\right)$$
$$= \mathbb{E}\left[\left(\sum_{j=1}^J (x_j \mathrm{Re} R_{L_j,Y}(\sigma,\mathbb{X}) + y_j \mathrm{Im} R_{L_j,Y}(\sigma,\mathbb{X}))\right)^n\right]$$
$$+ O(T^{-1/2} J^n (||\mathbf{x}||_\infty + ||\mathbf{y}||_\infty)^n).$$

Inserting this estimate in (5.2), we derive that $\widehat{\Phi}_T(\mathbf{x},\mathbf{y})$ equals

$$\begin{aligned}&\sum_{n=0}^{2N-1}\frac{(2\pi i)^n}{n!}\mathbb{E}\left(\left(\sum_{j=1}^J (x_j \mathrm{Re} R_{L_j,Y}(\sigma,\mathbb{X}) + y_j \mathrm{Im} R_{L_j,Y}(\sigma,\mathbb{X}))\right)^n\right)\\ &+ O((\log T)^{-A})\\ &\quad = \mathbb{E}\left(\exp\left(2\pi i\sum_{j=1}^J (x_j \mathrm{Re} R_{L_j,Y}(\sigma,\mathbb{X}) + y_j \mathrm{Im} R_{L_j,Y}(\sigma,\mathbb{X}))\right)\right)\\ &\qquad + O((\log T)^{-A}),\end{aligned} \tag{5.4}$$

where the last estimate follows by Lemma 4.5 and the same argument as in (5.3).

Let $\varepsilon > 0$ be a parameter to be chosen, and define $\mathcal{B}_\varepsilon$ to be the event

$$|\log L_j(\sigma,\mathbb{X}) - R_{L_j,Y}(\sigma,\mathbb{X})| < \varepsilon,$$

for all $j \le J$. Let $\mathcal{B}_\varepsilon^c$ be the complement of $\mathcal{B}_\varepsilon$. Then it follows from Lemma 4.3 that

$$\mathbb{P}(\mathcal{B}_\varepsilon^c) \ll \frac{1}{\varepsilon^2(\log T)^{2B}}.$$

Let $\mathbf{1}_{\mathcal{B}_\varepsilon}$ be the indicator function of the event $\mathcal{B}_\varepsilon$. Then it follows from this estimate that $\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x},\mathbf{y})$ equals

$$\mathbb{E}\Big(\mathbf{1}_{\mathcal{B}_\varepsilon}\cdot\exp\Big(2\pi i\sum_{j=1}^{J}(x_j\mathrm{Re}\log L_j(\sigma,\mathbb{X})+y_j\mathrm{Im}\log L_j(\sigma,\mathbb{X}))\Big)\Big)+O\Big(\frac{1}{\varepsilon^2(\log T)^{2B}}\Big)$$
$$=\mathbb{E}\Big(\mathbf{1}_{\mathcal{B}_\varepsilon}\cdot\exp\Big(2\pi i\sum_{j=1}^{J}(x_j\mathrm{Re}R_{L_j,Y}(\sigma,\mathbb{X})+y_j\mathrm{Im}R_{L_j,Y}(\sigma,\mathbb{X}))\Big)\Big)$$
$$+O\Big(\varepsilon+\frac{1}{\varepsilon^2(\log T)^{2B}}\Big)$$
$$=\mathbb{E}\Big(\exp\Big(2\pi i\sum_{j=1}^{J}(x_j\mathrm{Re}R_{L_j,Y}(\sigma,\mathbb{X})+y_j\mathrm{Im}R_{L_j,Y}(\sigma,\mathbb{X}))\Big)\Big)+O\Big(\varepsilon+\frac{1}{\varepsilon^2(\log T)^{2B}}\Big).$$

Choosing $\varepsilon=(\log T)^{-2B/3}$ and inserting this estimate in (5.4) completes the proof. □

The deduction of Theorem 2.3 from Proposition 5.1 uses Beurling–Selberg functions. For $z\in\mathbb{C}$ let

$$H(z)=\Big(\frac{\sin\pi z}{\pi}\Big)^2\Big(\sum_{n=-\infty}^{\infty}\frac{\mathrm{sgn}(n)}{(z-n)^2}+\frac{2}{z}\Big)\quad\text{and}\quad K(z)=\Big(\frac{\sin\pi z}{\pi z}\Big)^2.$$

Beurling proved that the function $B^+(x)=H(x)+K(x)$ majorizes $\mathrm{sgn}(x)$ and its Fourier transform has restricted support in $(-1,1)$. Similarly, the function $B^-(x)=H(x)-K(x)$ minorizes $\mathrm{sgn}(x)$ and its Fourier transform has the same property (see [23, Lemma 5]).

Let $\Delta>0$ and a,b be real numbers with $a<b$. Take $\mathcal{I}=[a,b]$ and define

$$F_{\mathcal{I},\Delta}(z)=\frac{1}{2}(B^-(\Delta(z-a))+B^-(\Delta(b-z))).$$

Then we have the following lemma, which is proved in [12] (see Lemma 7.1 therein and the discussion above it).

Lemma 5.2. *The function $F_{\mathcal{I},\Delta}$ satisfies the following properties*

(1) *For all $x\in\mathbb{R}$ we have $|F_{\mathcal{I},\Delta}(x)|\le 1$ and*

$$0\le\mathbf{1}_{\mathcal{I}}(x)-F_{\mathcal{I},\Delta}(x)\le K(\Delta(x-a))+K(\Delta(b-x)).\tag{5.5}$$

(2) *The Fourier transform of $F_{\mathcal{I},\Delta}$ is*

$$\widehat{F}_{\mathcal{I},\Delta}(y)=\begin{cases}\widehat{\mathbf{1}}_{\mathcal{I}}(y)+O(\frac{1}{\Delta}) & \text{if } |y|<\Delta,\\ 0 & \text{if } |y|\ge\Delta.\end{cases}\tag{5.6}$$

Proof of Theorem 2.3. First, it follows from Lemma 2.6 that with $\tau = (\log\log T)^2$ we have

$$\frac{1}{T}\text{meas}\{t \in [T, 2T] : \mathbf{L}(\sigma + it) \notin [-(\log\log T)^2, (\log\log T)^2]^{2J}\} \ll \frac{1}{(\log T)^{10}},$$

and

$$\mathbb{P}(\mathbf{L}(\sigma, \mathbb{X}) \notin [-(\log\log T)^2, (\log\log T)^2]^{2J}) \ll \frac{1}{(\log T)^{10}}.$$

Therefore, it suffices to consider rectangular regions

$$R \subset [-(\log\log T)^2, (\log\log T)^2]^{2J}.$$

Let $A = J + 3$ and c_1 be the corresponding constant in Proposition 5.1. Let

$$\Delta := c_1 \frac{\sqrt{\log T}}{\sqrt{G(T)}\log\log T},$$

and

$$R = \prod_{j=1}^{J}[a_j, b_j] \times \prod_{j=1}^{J}[c_j, d_j]$$

for $j = 1, \ldots, J$, with $0 < b_j - a_j, d_j - c_j \le 2(\log\log T)^2$. We also write $\mathcal{I}_j = [a_j, b_j]$ and $\mathcal{J}_j = [c_j, d_j]$. By Fourier inversion, (5.6), and Proposition 5.1 we have

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T}\prod_{j=1}^{J} F_{\mathcal{I}_j,\Delta}(\log|L_j(\sigma + it)|)F_{\mathcal{J}_j,\Delta}(\arg L_j(\sigma + it))\,dt \\
&= \int_{\mathbb{R}^{2J}}\left(\prod_{j=1}^{J}\widehat{F}_{\mathcal{I}_j,\Delta}(x_j)\widehat{F}_{\mathcal{J}_j,\Delta}(y_j)\right)\widehat{\Phi}_T(-\mathbf{x}, -\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} \\
&= \int_{\substack{|x_j|,|y_j|\le\Delta \\ j=1,2,\ldots,J}}\left(\prod_{j=1}^{J}\widehat{F}_{\mathcal{I}_j,\Delta}(x_j)\widehat{F}_{\mathcal{J}_j,\Delta}(y_j)\right)\widehat{\Phi}_T^{\text{rand}}(-\mathbf{x}, -\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} + E_2 \qquad (5.7)\\
&= \mathbb{E}\left(\prod_{j=1}^{J} F_{\mathcal{I}_j,\Delta}(\log|L_j(\sigma, \mathbb{X})|)F_{\mathcal{J}_j,\Delta}(\arg L_j(\sigma, \mathbb{X}))\right) + O\left(\frac{1}{(\log T)^2}\right),
\end{aligned}$$

where $E_2 = O(\Delta^{2J}\frac{(\log\log T)^{4J}}{(\log T)^A})$.

Next note that $\widehat{K}(\xi) = \max(0, 1 - |\xi|)$. Applying Fourier inversion, Proposition 5.1 with $J = 1$, and Lemma 7.1 we obtain

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T} K(\Delta(\log|L_1(\sigma + it)| - \alpha))\,dt \\
&\qquad = \frac{1}{\Delta}\int_{-\Delta}^{\Delta}\left(1 - \frac{|\xi|}{\Delta}\right)e^{-2\pi i\alpha\xi}\widehat{\Phi}_T(\xi, 0, \ldots, 0)\,d\xi \ll \frac{1}{\Delta},
\end{aligned}$$

where α is an arbitrary real number. By this and (5.5) we have that

$$\begin{aligned}\frac{1}{T}\int_T^{2T} & F_{\mathcal{I}_1,\Delta}(\operatorname{Re}\log L_1(\sigma+it))\,dt \\ &= \frac{1}{T}\int_T^{2T} \mathbf{1}_{\mathcal{I}_1}\Big(\operatorname{Re}\log L_1(\sigma+it))dt + O(1/\Delta).\end{aligned} \tag{5.8}$$

Lemma 5.2 implies that $|F_{\mathcal{I}_j,\Delta}(x)|, |F_{\mathcal{J}_j,\Delta}(x)| \le 1$ for $j = 1, \ldots, J$. Hence, by this and (5.8) we have

$$\begin{aligned}&\frac{1}{T}\int_T^{2T}\prod_{j=1}^{J} F_{\mathcal{I}_j,\Delta}(\operatorname{Re}\log L_j(\sigma+it))F_{\mathcal{J}_j,\Delta}(\arg L_j(\sigma+it))\,dt \\ &= \frac{1}{T}\int_T^{2T}\mathbf{1}_{\mathcal{I}_1}(\operatorname{Re}\log L_1(\sigma+it))F_{\mathcal{J}_1,\Delta}(\arg L_1(\sigma+it)) \\ &\qquad\times\prod_{j=2}^{J} F_{\mathcal{I}_j,\Delta}(\operatorname{Re}\log L_j(\sigma+it))F_{\mathcal{J}_j,\Delta}(\arg L_j(\sigma+it))\,dt + O(1/\Delta).\end{aligned}$$

By using the same argument, one can prove analogs of (5.8) for $\operatorname{Re}\log L_j(\sigma+it)$ with $2 \le j \le J$ and $\arg L_j(\sigma+it)$ with $1 \le j \le J$. We then derive

$$\begin{aligned}&\frac{1}{T}\int_T^{2T}\prod_{j=1}^{J} F_{\mathcal{I}_j,\Delta}(\operatorname{Re}\log L_j(\sigma+it))F_{\mathcal{J}_j,\Delta}(\arg L_j(\sigma+it))\,dt \\ &=\frac{1}{T}\int_T^{2T}\prod_{j=1}^{J}\mathbf{1}_{\mathcal{I}_j,\Delta}(\operatorname{Re}\log L_j(\sigma+it))\mathbf{1}_{\mathcal{J}_j,\Delta}(\arg L_j(\sigma+it))\,dt + O_J\Big(\frac{1}{\Delta}\Big) \\ &=\Phi_T(R) + O_J\Big(\frac{1}{\Delta}\Big).\end{aligned} \tag{5.9}$$

A similar argument shows that

$$\mathbb{E}\bigg(\prod_{j=1}^{J} F_{\mathcal{I}_j,\Delta}(\operatorname{Re}\log L_j(\sigma,\mathbb{X}))F_{\mathcal{J}_j,\Delta}(\arg L_j(\sigma,\mathbb{X}))\bigg) = \Phi_T^{\mathrm{rand}}(R) + O_J\Big(\frac{1}{\Delta}\Big). \tag{5.10}$$

Inserting the estimates (5.9) and (5.10) in (5.7) completes the proof. □

6 L^{2k} norm of $\log|\sum_{j=1}^{J} b_jL_j(\sigma+it)|$: Proof of Proposition 2.4

To prove Proposition 2.4 we follow the same strategy as in the proof of [13, Proposition 2.5] for the Riemann zeta function, but we encounter new technical difficulties which we shall describe later.

We first start with the following classical lemma, which is a generalization of a lemma of Landau (see Lemma α in [21, Section 3.9]).

Lemma 6.1 ([13, Lemma 5.1]). *Let* $0 < r \ll 1$. *Also, let* $s_0 = \sigma_0 + it$ *and suppose* $f(z)$ *is analytic in* $|z - s_0| \leq r$. *Define*

$$M_r(s_0) = \max_{|z-s_0|\leq r} \left|\frac{f(z)}{f(s_0)}\right| + 3 \quad \text{and} \quad N_r(s_0) = \sum_{|\varrho - s_0|\leq r} 1,$$

where the last sum runs over the zeros ϱ *of* $f(z)$ *in the closed disk of radius* r *centered at* s_0. *Then for* $0 < \delta < r/2$ *and* $|z - s_0| \leq r - 2\delta$ *we have*

$$\frac{f'}{f}(z) = \sum_{|\rho - s_0|\leq r-\delta} \frac{1}{s-\rho} + O\Big(\frac{1}{\delta^2}(\log M_r(s_0) + N_{r-\delta}(s_0)(\log 1/\delta + 1))\Big).$$

Recall that $L_j(s)$ has a Dirichlet series representation

$$L_j(s) = \sum_{n=1}^{\infty} \frac{\alpha_{L_j}(n)}{n^s},$$

for $\mathrm{Re}(s) > 1$. We shall apply Lemma 6.1 to the following function

$$f(z) = \frac{n_0^z}{\sum_{j\leq J} b_j \alpha_{L_j}(n_0)} \sum_{j=1}^{J} b_j L_j(z), \tag{6.1}$$

where n_0 is the smallest positive integer n such that $\sum_{j=1}^{J} b_j \alpha_{L_j}(n) \neq 0$. We let ρ run over the zeros of f. We recall that $\sigma = 1/2 + 1/G(T)$, and choose

$$\delta := \frac{1}{5G(T)}, \quad r := \sigma_0 - \frac{1}{2} - \frac{1}{2G(T)}, \quad \text{and } R := r + \delta.$$

where σ_0 is taken to be large (but fixed) so that $|f(\sigma_0 + it)| \geq 1/10$ and $\min_\rho |s_0 - \rho| \geq 1/10$ uniformly in t. A straightforward generalization of [13, Lemmas 5.2 and 5.3] leads to the following result. To be precise, we only include the major steps of the proof.

Lemma 6.2. *Let* σ, δ, r, R, *and* $s_0 = \sigma_0 + it$ *be as above. Then there exists an absolute constant* $c > 0$ *such that for every positive integer* k *we have*

$$\int_T^{2T} \left|\log\left|\sum_{j=1}^{J} b_j L_j(\sigma + it)\right|\right|^{2k} dt$$

$$\ll c^k (G(T)^{3/2}k + G(T)^3 \log G(T))^{2k} \sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor} \left(\log\left(\left|\sum_{j=1}^{J} b_j L_j(s_n)\right| + 3\right)\right)^{2k},$$

where $s_n = \sigma_n + it_n$ *for* $n > 0$ *is a point at which* $|\sum_{j=1}^{J} b_j L_j(s)|$ *achieves its maximum value on the set* $\bigcup_{n\leq t\leq n+1} D_R(\sigma_0 + it)$, *and* $D_R(z)$ *is the disc of radius* R *centered at* z.

Proof. First, applying Lemma 6.1 to $f(z)$ defined in (6.1) we obtain for $|z - s_0| \le r - 2\delta$ that

$$\frac{f'(z)}{f(z)} = \sum_{|\rho - s_0| \le r-\delta} \frac{1}{z-\rho} + O\Big(\frac{1}{\delta^2}(\log M_r(s_0) + N_{r-\delta}(s_0)(\log 1/\delta + 1))\Big)$$

where

$$M_r(s_0) = \max_{|z-s_0|\le r} \left|\frac{f(z)}{f(s_0)}\right| + 3 \quad \text{and} \quad N_r(s_0) = \sum_{|\varrho - s_0| \le r} 1.$$

Now, a standard application of Jensen's formula shows that (see [13, (5.4)])

$$N_{r-\delta}(s_0) \le \frac{r}{\delta}(\log M_r(s_0) + \log 10).$$

Hence we derive

$$\frac{f'(z)}{f(z)} = \sum_{|\rho - s_0| \le r-\delta} \frac{1}{z-\rho} + O(G(T)^3 \log G(T) \log M_r(s_0))$$

for $|z - s_0| \le r - 2\delta$. We integrate both sides from $s_0 = \sigma_0 + it$ to $s = \sigma + it$ and take the real parts, to obtain

$$\begin{aligned} \log|f(s)| &- \log|f(s_0)| \\ &= \sum_{|\rho - s_0| \le r-\delta} \log|s-\rho| + O(N_{r-\delta}(s_0) + G(T)^3 \log G(T) \log M_r(s_0)) \\ &= \sum_{|\rho - s_0| \le r-\delta} \log|s-\rho| + O(G(T)^3 \log G(T) \log M_r(s_0)), \end{aligned}$$

since $\log|s_0 - \rho| = O(1)$ for all zeros ρ with $|\rho - s_0| \le r - \delta$ by our choice of σ_0. Furthermore, since $\log|f(s_0)| = O(1)$ and $\log|f(s)| = \log|\sum_{j=1}^J b_j L_j(s)| + O(1)$, we deduce that

$$\log\left|\sum_{j=1}^J b_j L_j(\sigma+it)\right| \le \sum_{|\rho - s_0| \le r-\delta} \log|\sigma + it - \rho| + c_1 G(T)^3 \log G(T) \log M_r(s_0),$$

for some positive constant c_1. We now use the simple inequality

$$(x+y)^{2k} \le 2^{2k} \max(|x|,|y|)^{2k} \le 2^{2k}(|x|^{2k} + |y|^{2k})$$

for all real numbers x, y, to deduce that

$$\begin{aligned} \int_T^{2T} \left|\log\left|\sum_{j=1}^J b_j L_j(\sigma+it)\right|\right|^{2k} dt \\ \le 4^k \int_T^{2T} \Big(\sum_{|\rho - s_0| \le r-\delta} |\log|\sigma + it - \rho||\Big)^{2k} dt \\ + (2c_1 G(T)^3 \log G(T))^{2k} \int_T^{2T} (\log M_r(s_0))^{2k} dt. \end{aligned} \tag{6.2}$$

For $n > 0$, let $s_n = \sigma_n + it_n$ be a point at which $|\sum_{j=1}^J b_jL_j(s)|$ achieves its maximum value on the set $\bigcup_{n\le t\le n+1} D_R(\sigma_0 + it)$. Then, we note that

$$\begin{aligned}\int_T^{2T} (\log M_r(s_0))^{2k}dt &\le \sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor} \int_n^{n+1} (\log M_r(s_0))^{2k}dt \\ &\ll c_2^k \sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor} \left(\log \left(\left| \sum_{j=1}^J b_jL_j(s_n)\right| + 3\right)\right)^{2k}\end{aligned} \tag{6.3}$$

for some absolute constant $c_2 > 0$. Furthermore, a straightforward generalization of the proof of [13, Lemma 5.3] implies that

$$\begin{aligned}&\int_T^{2T} \left(\sum_{|\rho-s_0|\le r-\delta} |\log|\sigma + it - \rho||\right)^{2k} dt \\ &\ll (c_3k)^{2k} \sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor} \left(\frac{1}{\delta} \sum_{\ell\le 1/\sqrt{\delta}} \log M_R(\sigma_0 + i(n + \ell\sqrt{\delta}))\right)^{2k} \\ &\ll \frac{(c_3k)^{2k}}{\delta^{3k}} \sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor} \left(\log \left(\left| \sum_{j=1}^J b_jL_j(s_n)\right| + 3\right)\right)^{2k},\end{aligned} \tag{6.4}$$

where $c_3 > 0$ is an absolute constant. Inserting the estimates (6.3) and (6.4) in (6.2) completes the proof. □

In the case of the Riemann zeta function, in order to bound

$$\sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor} (\log|\zeta(s_n)| + 3)^{2k},$$

the authors of [13] use Jensen's inequality together with standard estimates for the second moment of $\zeta(s)$. However, estimates for the second moment are not known in general for the L-functions in our class. Using a different approach, we were able to overcome this problem and establish the following result.

Lemma 6.3. *Let δ, r, R, and $s_0 = \sigma_0 + it$ be as above. Let $D_a(z)$ be the disc of radius a centered at z. For $n > 0$, let $s_n = \sigma_n + it_n$ be a point at which $|\sum_{j=1}^J b_jL_j(s)|$ achieves its maximum value on the set $\bigcup_{n\le t\le n+1} D_R(\sigma_0+it)$. Then there exist positive constants c_1 and c_2 such that for all positive integers $k \le \log T/(c_1G(T)\log\log T)$ we have*

$$\sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor} \left(\log \left(\left| \sum_{j=1}^J b_jL_j(s_n)\right| + 3\right)\right)^{2k} \ll TG(T)^2(c_2k\log\log T)^k.$$

Proof. We first observe that

$$\left(\log\left(\left|\sum_{j=1}^{J} b_j L_j(s_n)\right| + 3\right)\right)^{2k} \leq C_1^k \max_{j\leq J}\{(\log(|L_j(s_n)|+3))^{2k}\}$$
$$\leq C_1^k \sum_{j=1}^{J}(\log(|L_j(s_n)|+3))^{2k}$$

for some constant $C_1 > 0$ that depends on J and the b_j. Thus, we have

$$\sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor}\left(\log\left(\left|\sum_{j=1}^{J} b_j L_j(s_n)\right| + 3\right)\right)^{2k} \leq C_1^k \sum_{j=1}^{J}\sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor}(\log(|L_j(s_n)|+3))^{2k}.$$

To prove the lemma, it is enough to show that

$$\sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor}(\log(|L_j(s_n)|+3))^{2k} \ll TG(T)^2(C_2 k \log\log T)^k$$

for every $j \leq J$ and for some constant $C_2 > 0$.

Without loss of generality, we only consider the case $j = 1$. Let

$$\mathcal{A}_1(T) := \{\lfloor T\rfloor \leq n \leq \lfloor 2T\rfloor : |L_1(s_n)| \leq 5\},$$
$$\mathcal{A}_2(T) := \{\lfloor T\rfloor \leq n \leq \lfloor 2T\rfloor : |L_1(s_n)| > 5, L_1(s) = 0 \text{ for some } |s - s_n| \leq \delta\},$$
$$\mathcal{A}_3(T) := \{\lfloor T\rfloor \leq n \leq \lfloor 2T\rfloor : |L_1(s_n)| > 5, L_1(s) \neq 0 \text{ for all } |s - s_n| \leq \delta\}.$$

Then we see that

$$\sum_{n\in\mathcal{A}_1(T)}(\log(|L_1(s_n)|+3))^{2k} \leq T(\log 8)^{2k}. \tag{6.5}$$

To bound the sum over $\mathcal{A}_2(T)$, we use the classical Phragmen–Lindelöf principle which implies that there exists $\kappa > 0$ (which might depend on x and d in assumption A1) such that

$$|L_1(x+iy)| \ll (1+|y|)^{\kappa}. \tag{6.6}$$

If $|\rho - s_n| \leq \delta$ and $L_1(\rho) = 0$, then $\mathrm{Re}(\rho) \geq 1/2 + 1/(10G(T))$. By assumption A4 we have

$$\begin{aligned}\sum_{n\in\mathcal{A}_2(T)}(\log(|L_1(s_n)|+3))^{2k} &\ll C_3^k(\log T)^{2k} N_{L_1}(1/2+1/(10G(T)), T)\\ &\ll C_3^k T e^{-C_4\frac{\log T}{G(T)}+2k\log\log T}\\ &\ll C_3^k T e^{-\frac{C_4}{2}\frac{\log T}{G(T)}}\end{aligned} \tag{6.7}$$

for some constants $C_3, C_4 > 0$ and for $k \le \log T/(c_1 G(T)\log\log T)$ by choosing c_1 sufficiently large. Lastly, for each $n \in \mathcal{A}_3(T)$ we have

$$0 \le \log(|L_1(s_n)|+3) \le 2\log|L_1(s_n)| \ll \frac{1}{\pi\delta^2}\iint_{D_\delta(s_n)} \log|L_1(x+iy)|dxdy,$$

since $\log|L_1(s)|$ is subharmonic by [18, Theorem 17.3]. By Jensen's inequality applied to the convex function $\varphi(x) = x^{2k}$, we have

$$\begin{aligned}(\log(|L_1(s_n)|+3))^{2k} &\ll C_5^k\left(\frac{1}{\pi\delta^2}\iint_{D_\delta(s_n)}\log|L_1(x+iy)|dxdy\right)^{2k}\\ &\ll C_5^k\frac{1}{\pi\delta^2}\iint_{D_\delta(s_n)}(\log|L_1(x+iy)|)^{2k}dxdy\\ &\ll C_5^k\frac{1}{\delta^2}\iint_{D_{R'}(\sigma_0+it_n)}(\log|L_1(x+iy)|)^{2k}dxdy\end{aligned}$$

for some $C_5 > 0$ and $R' = R + \delta$. Thus,

$$\begin{aligned}&\sum_{n\in\mathcal{A}_3(T)}(\log(|L_1(s_n)|+3))^{2k}\\ &\qquad\ll C_5^k\sum_{n\in\mathcal{A}_3(T)}\frac{1}{\delta^2}\iint_{D_{R'}(\sigma_0+it_n)}(\log|L_1(x+iy)|)^{2k}dxdy.\end{aligned} \tag{6.8}$$

Let $\mathcal{S}_\ell = \{n \in \mathcal{A}_3(T) : n \equiv \ell \pmod{4\lceil R'\rceil + 2}\}$. If $m, n \in \mathcal{S}_\ell$ and $m \ne n$ then $|m-n| \ge 4\lceil R'\rceil + 2$; so that $|t_m - t_n| \ge 2R' + 1$. This implies that

$$D_{R'}(\sigma_0+it_n)\cap D_{R'}(\sigma_0+it_m) = \emptyset.$$

Thus, since the disks are disjoint we see that

$$\begin{aligned}&\sum_{n\in\mathcal{S}_\ell}\frac{1}{\delta^2}\iint_{D_{R'}(\sigma_0+it_n)}(\log|L_1(x+iy)|)^{2k}dxdy\\ &\qquad\ll G(T)^2\int_{\sigma_0-R'}^{\sigma_0+R'}\int_{T-2R'-1}^{2T+2R'+1}(\log|L_1(x+iy)|)^{2k}dydx.\end{aligned} \tag{6.9}$$

By adding (6.9) for all $\ell \pmod{4\lceil R'\rceil + 2}$ and using (6.8), we see that

$$\begin{aligned}&\sum_{n\in\mathcal{A}_3(T)}(\log(|L_1(s_n)|+3))^{2k}\\ &\qquad\ll C_5^k G(T)^2\int_{\sigma_0-R'}^{\sigma_0+R'}\int_{T-2R'-1}^{2T+2R'+1}(\log|L_1(x+iy)|)^{2k}dydx.\end{aligned} \tag{6.10}$$

Note that

$$\sigma_0 - R' = \sigma_0 - r - 2\delta = 1/2 + 1/(10G(T)).$$

Let $Y = \exp(100G(T)\log\log T)$. Then it follows from Lemma 4.2 that uniformly for $x \geq \sigma_0 - R'$ we have

$$\log L_1(x+iy) = R_{L_1,Y}(x+iy) + O\Big(\frac{1}{(\log T)^3}\Big), \tag{6.11}$$

for all $y \in [T, 2T]$ except for a set of measure $\ll T\exp(-C_6 \log T/G(T))$, for some constant $C_6 > 0$. Let $\mathcal{A}(T)$ be the set of points $y \in [T, 2T]$ for which (6.11) holds and let $\mathcal{A}^c(T)$ be its complement in $[T - 2R' - 1, 2T + 2R' + 1]$. Then we have

$$\text{meas}(\mathcal{A}^c(T)) \ll T\exp\Big(-C_6\frac{\log T}{G(T)}\Big).$$

We now split the inner integral on the right-hand side of (6.10) in two parts, the first over $\mathcal{A}(T)$ and the second over $\mathcal{A}^c(T)$. By (6.6) we obtain

$$\begin{aligned}\int_{\sigma_0-R'}^{\sigma_0+R'}\int_{\mathcal{A}^c(T)} (\log|L_1(x+iy)|)^{2k} dydx &\ll \text{meas}(\mathcal{A}^c(T))(C_7\log T)^{2k} \\ &\ll T\exp\Big(-\frac{C_6\log T}{2G(T)}\Big)\end{aligned} \tag{6.12}$$

for some positive constant C_7, where the last estimate follows from our assumption on k.

Furthermore, if $y \in \mathcal{A}(T)$, then for $x \geq \sigma_0 - R'$ we have

$$(\log|L_1(x+iy)|)^{2k} \ll 4^k(|R_{L_1,Y}(x+iy)|^{2k} + 1)$$

by (6.11). Thus, by Lemma 4.5 and Stirling's formula we obtain

$$\begin{aligned}&\int_{\sigma_0-R'}^{\sigma_0+R'}\int_{\mathcal{A}(T)} (\log|L_1(x+iy)|)^{2k}\, dydx \\ &\ll 4^k\Big(\int_{\sigma_0-R'}^{\sigma_0+R'}\int_T^{2T} |R_{L_1,Y}(x+iy)|^{2k} dydx + T\Big) \ll T(C_8 k\log\log T)^k\end{aligned} \tag{6.13}$$

for some positive constant C_8. Inserting the estimates (6.12) and (6.13) in (6.10) gives

$$\sum_{n\in\mathcal{A}_3(T)} (\log(|L_1(s_n)| + 3))^{2k} \ll TG(T)^2(C_9 k\log\log T)^k$$

for some constant $C_9 > 0$. This with (6.5) and (6.7) proves the lemma. □

Proposition 2.4 follows from Lemmas 6.2 and 6.3.

7 Analysis of the random model: Proofs of Theorem 2.2, Proposition 2.5 and Lemma 2.7

Recall that

$$\mathbf{L}(\sigma, \mathbb{X}) = (\log|L_1(\sigma, \mathbb{X})|, \ldots, \log|L_J(\sigma, \mathbb{X})|, \arg L_1(\sigma, \mathbb{X}), \ldots, \arg L_J(\sigma, \mathbb{X})).$$

We define its partial sum

$$\mathbf{L}_q(\sigma, \mathbb{X}) = (\log|L_{1,q}(\sigma, \mathbb{X})|, \dots, \log|L_{J,q}(\sigma, \mathbb{X})|, \arg L_{1,q}(\sigma, \mathbb{X}), \dots, \arg L_{J,q}(\sigma, \mathbb{X}))$$

for a positive integer q, where

$$\log L_{j,q}(\sigma, \mathbb{X}) := \sum_{p \le q} \sum_{k=1}^{\infty} \frac{\beta_{L_j}(p^k)\mathbb{X}(p)^k}{p^{k\sigma}}.$$

We also define

$$\mathbf{L}_{>q}(\sigma, \mathbb{X}) := \mathbf{L}(\sigma, \mathbb{X}) - \mathbf{L}_q(\sigma, \mathbb{X}). \tag{7.1}$$

For a Borel set $\mathcal{B}$ in $\mathbb{R}^{2J}$ and for $\sigma = 1/2 + 1/G(T)$, we define

$$\begin{aligned} \Phi_{q,T}^{\text{rand}}(\mathcal{B}) &= \mathbb{P}(\mathbf{L}_q(\sigma, \mathbb{X}) \in \mathcal{B}), \\ \Phi_{>q,T}^{\text{rand}}(\mathcal{B}) &:= \mathbb{P}(\mathbf{L}_{>q}(\sigma, \mathbb{X}) \in \mathcal{B}) \end{aligned} \tag{7.2}$$

and their Fourier transforms

$$\begin{aligned} \widehat{\Phi}_{q,T}^{\text{rand}}(\mathbf{x}, \mathbf{y}) &:= \int_{\mathbb{R}^{2J}} e^{2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})} d\Phi_{q,T}^{\text{rand}}(\mathbf{u}, \mathbf{v}), \\ \widehat{\Phi}_{>q,T}^{\text{rand}}(\mathbf{x}, \mathbf{y}) &:= \int_{\mathbb{R}^{2J}} e^{2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})} d\Phi_{>q,T}^{\text{rand}}(\mathbf{u}, \mathbf{v}) \end{aligned} \tag{7.3}$$

for $\mathbf{x} = (x_1, \dots, x_J) \in \mathbb{R}^J$ and $\mathbf{y} = (y_1, \dots, y_J) \in \mathbb{R}^J$.

7.1 Upper bounds for the density functions and the Fourier transforms of $\mathbf{L}(\sigma, \mathbb{X})$, $\mathbf{L}_q(\sigma, \mathbb{X})$, and $\mathbf{L}_{>q}(\sigma, \mathbb{X})$.

In this subsection, we prove that the distribution functions of $\mathbf{L}(\sigma, \mathbb{X})$, $\mathbf{L}_q(\sigma, \mathbb{X})$, and $\mathbf{L}_{>q}(\sigma, \mathbb{X})$ are absolutely continuous, and provide bounds for their density functions and Fourier transforms. These will be used to prove Proposition 2.5 and Lemma 2.7. We start with the following lemma.

Lemma 7.1. *Let $A > 0$ be a given real number. Then, there exists a positive integer $q(A)$ such that*

$$\widehat{\Phi}_{q,T}^{\text{rand}}(\mathbf{x}, \mathbf{y}) \ll_{q,A} (1 + ||\mathbf{x}||_2 + ||\mathbf{y}||_2)^{-A}$$

for every $q \ge q(A)$, where

$$||\mathbf{x}||_2 := \sqrt{\sum_{j \le J} |x_j|^2}.$$

Furthermore, for any positive integer q we have

$$\widehat{\Phi}_{>q,T}^{\text{rand}}(\mathbf{x}, \mathbf{y}) \ll_{q,A} (1 + ||\mathbf{x}||_2 + ||\mathbf{y}||_2)^{-A}.$$

Thus, $\Phi_{q,T}^{\text{rand}}$ is absolutely continuous for sufficiently large $q > 0$ and $\Phi_{>q,T}^{\text{rand}}$ is absolutely continuous for any $q > 0$.

Proof. The absolute continuity of $\Phi^{\text{rand}}_{q,T}$ and $\Phi^{\text{rand}}_{>q,T}$ follows from the inequalities in the lemma (see [9, Section 3]). Thus, it is enough to prove these inequalities.

We first define for any prime p

$$\varphi_{p,\sigma}(\mathbf{x},\mathbf{y}) := \mathbb{E}\left[\exp\left(2\pi i\sum_{j=1}^{J}(x_j\mathrm{Re}(g_j(\mathbb{X}(p)p^{-\sigma})) + y_j\mathrm{Im}(g_j(\mathbb{X}(p)p^{-\sigma})))\right)\right], \tag{7.4}$$

where

$$g_j(u) = \sum_{k=1}^{\infty}\beta_{L_j}(p^k)u^k.$$

Then we find that

$$\begin{aligned}\widehat{\Phi}^{\text{rand}}_{q,T}(\mathbf{x},\mathbf{y}) &= \mathbb{E}[\exp(2\pi i(x_1,\dots,x_J,y_1,\dots,y_J)\cdot\mathbf{L}_q(\sigma,\mathbb{X}))]\\ &= \prod_{p\le q}\varphi_{p,\sigma}(\mathbf{x},\mathbf{y}).\end{aligned} \tag{7.5}$$

By [14, Lemma 2.5] there is a constant $C>0$ such that

$$|\varphi_{p,\sigma}(\mathbf{x},\mathbf{y})| \le \frac{Cp^{\sigma/2}}{(\sum_{j=1}^{J}(x_j^2+y_j^2))^{1/4}} \tag{7.6}$$

if

$$\left|\sum_{j=1}^{J}\beta_{L_j}(p)(x_j-iy_j)\right|^2 \ge \delta\sum_{j=1}^{J}(x_j^2+y_j^2)$$

for some constant $\delta>0$. Note that [14, Lemma 2.5] holds even for complex coefficients a_j with minor modification. In that case the condition in the last line of the lemma should be

$$\left|\sum_{j=1}^{J}a_j(y_j-iy_j')\right| \ge \delta||\mathbf{y}||_2.$$

Let $q_1>0$ be a large positive integer to be chosen later and define a sequence q_n of integers inductively by $q_{n+1}=2^{q_n}$. We shall prove that given $\mathbf{u}$ and $\mathbf{v}$, there exists a prime p in the interval $(q_{n-1},q_n]$ such that

$$\left|\sum_{j=1}^{J}\beta_{L_j}(p)(x_j-iy_j)\right|^2 \ge \frac{1}{2}(\min_{j\le J}\xi_j)\sum_{j=1}^{J}(x_j^2+y_j^2)$$

holds. Suppose not. Then multiplying both sides by $1/p$ and summing over all primes p in $(q_{n-1},q_n]$ we have

$$\begin{aligned}&\sum_{q_{n-1}<p<q_n}\frac{|\sum_{j=1}^{J}\beta_{L_j}(p)(x_j-iy_j)|^2}{p}\\ &\qquad\le \frac{1}{2}(\min_{j\le J}\xi_j)\sum_{j=1}^{J}(x_j^2+y_j^2)(\log q_{n-1}+O(\log\log q_{n-1})).\end{aligned}$$

On the other hand by (1.3) we see that

$$\begin{aligned}\sum_{q_{n-1}<p<q_n} \frac{|\sum_{j=1}^J \beta_{L_j}(p)(x_j - iy_j)|^2}{p} &= \sum_{j=1}^J (x_j^2 + y_j^2)\Bigg(\sum_{q_{n-1}<p\le q_n} \frac{|\beta_{L_j}(p)|^2}{p} + O(1)\Bigg) \\ &= \sum_{j=1}^J (x_j^2 + y_j^2)(\xi_j \log q_{n-1} + O(\log\log q_{n-1})) \\ &\geq (\min_{j\le J} \xi_j) \sum_{j=1}^J (x_j^2 + y_j^2)(\log q_{n-1} + O(\log\log q_{n-1})).\end{aligned}$$

This is a contradiction if q_{n-1} is sufficiently large.

Now, take $q = q_{m+1}$ where $m = \lfloor 4A \rfloor + 1$. Then using (7.6) together with the trivial bound $|\varphi_{p,\sigma}(\mathbf{x}, \mathbf{y})| \leq 1$ we obtain

$$|\widehat{\Phi}^{\text{rand}}_{q,T}(\mathbf{x}, \mathbf{y})| \leq \prod_{n=1}^m \prod_{q_n<p\le q_{n+1}} |\varphi_{p,\sigma}(\mathbf{x}, \mathbf{y})| \leq \prod_{n=1}^m \frac{C q_{n+1}^{\sigma/2}}{(\sum_{j=1}^J x_j^2 + y_j^2)^{1/4}} \leq \frac{C_{q,m}}{(\sum_{j=1}^J x_j^2 + y_j^2)^A}$$

for some constant $C_{q,m} > 0$. This completes the proof.

To prove the second inequality, we choose ℓ such that $q_\ell > q$. Then for $m = \lfloor 4A \rfloor + 1$ we obtain similarly that

$$\begin{aligned}|\widehat{\Phi}^{\text{rand}}_{>q,T}(\mathbf{x}, \mathbf{y})| &\leq \prod_{n=\ell}^{\ell+m-1} \prod_{q_n<p\le q_{n+1}} |\varphi_{p,\sigma}(\mathbf{x}, \mathbf{y})| \leq \prod_{n=1}^m \frac{C q_{n+1}^{\sigma/2}}{(\sum_{j=1}^J x_j^2 + y_j^2)^{1/4}} \\ &\leq \frac{C'_{q,m,\ell}}{(\sum_{j=1}^J x_j^2 + y_j^2)^A}\end{aligned}$$

for some constant $C'_{q,m,\ell} > 0$. □

By Lemma 7.1 and [9, Section 3], there is an integer $q > 0$ such that both $\Phi^{\text{rand}}_{q,T}$ and $\Phi^{\text{rand}}_{>q,T}$ have continuous density functions, say $H_{q,T}(\mathbf{u}, \mathbf{v})$ and $H_{>q,T}(\mathbf{u}, \mathbf{v})$, respectively. One can also see that $H_{>q,T}(\mathbf{u}, \mathbf{v})$ has partial derivatives of any order. Since $\Phi^{\text{rand}}_T = \Phi^{\text{rand}}_{>1,T}$, it follows that Φ^{rand}_T has a continuous density function which we shall denote throughout by $H_T(\mathbf{u}, \mathbf{v})$. These density functions are real valued and nonnegative.

Lemma 7.2. *Let* $0 < \lambda < (24J \max_{j\le J} \xi_j)^{-1}$ *be a fixed real number. For all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^J$ *we have*

$$H_T(\mathbf{u}, \mathbf{v}) \ll_\lambda e^{-\frac{\lambda}{\log G(T)} \sum_{j=1}^J (u_j^2 + v_j{}^2)}.$$

Proof. Let q be a positive integer. By a standard convolution argument we have

$$\begin{aligned} H_T(\mathbf{u},\mathbf{v}) &= \int_{\mathbb{R}^{2J}} H_{q,T}(\mathbf{u}-\mathbf{x},\mathbf{v}-\mathbf{y})d\Phi^{\mathrm{rand}}_{>q,T}(\mathbf{x},\mathbf{y}) \\ &= \int_{\mathbb{R}^{2J}} H_{q,T}(\mathbf{u}-\mathbf{x},\mathbf{v}-\mathbf{y})H_{>q,T}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} \end{aligned}$$

for $\mathbf{u},\mathbf{v}\in\mathbb{R}^J$. Since

$$|\mathbf{L}_q(\sigma,\mathbb{X})|^2 = \sum_{j=1}^{J} |\log L_{j,q}(\sigma,\mathbb{X})|^2 \le \sum_{j=1}^{J}\sum_{p\le q}\sum_{k=1}^{\infty} \frac{|\beta_{L_j}(p^k)|}{p^{k/2}} := R_q^2,$$

we have $H_{q,T}(\mathbf{x},\mathbf{y}) = 0$ for $\sum_{j=1}^{J}(x_j^2+y_j^2) > R_q^2$. Let

$$B_q(\mathbf{u},\mathbf{v}) := \left\{(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2J} : \sum_{j=1}^{J}(x_j-u_j)^2+(y_j-v_j)^2 \le R_q^2\right\}$$

be the $2J$ dimensional ball of radius R_q centered at $(\mathbf{u},\mathbf{v})$, then we see that

$$\begin{aligned} H_T(\mathbf{u},\mathbf{v}) &= \int_{B_q(\mathbf{u},\mathbf{v})} H_{q,T}(\mathbf{u}-\mathbf{x},\mathbf{v}-\mathbf{y})H_{>q,T}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} \\ &\le \Big(\sup_{(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2J}} H_{q,T}(\mathbf{x},\mathbf{y})\Big)\Phi^{\mathrm{rand}}_{>q,T}(B_q(\mathbf{u},\mathbf{v})). \end{aligned}$$

Since the measure $\tilde{\Phi}^{\mathrm{rand}}_{q,\sigma}(\mathcal{B}) := \mathbb{P}[\mathbf{L}_q(\sigma,\mathbb{X})\in\mathcal{B}]$ and its density function $\tilde{H}_{q,\sigma}(\mathbf{x},\mathbf{y})$ depend continuously on $\sigma\ge 1/2$, we see that

$$\sup_{T\ge T_0}\sup_{(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2J}} H_{q,T}(\mathbf{x},\mathbf{y}) \le M_q := \sup_{1/2\le\sigma\le 2/3}\sup_{\mathbf{x},\mathbf{y}\in\mathbb{R}^J} \tilde{H}_{q,\sigma}(\mathbf{x},\mathbf{y}) < \infty$$

for a sufficiently large constant $T_0 > 0$. Hence, we deduce that

$$H_T(\mathbf{u},\mathbf{v}) \le M_q\Phi^{\mathrm{rand}}_{>q,T}(B_q(\mathbf{u},\mathbf{v})). \tag{7.7}$$

Thus, it remains to find an upper bound for $\Phi^{\mathrm{rand}}_{>q,T}(B_q(\mathbf{u},\mathbf{v}))$.

First, we remark that if $(\mathbf{x},\mathbf{y})\in B_q(\mathbf{u},\mathbf{v})$ and $||(\mathbf{u},\mathbf{v})||_2 \ge 2R_q$ then

$$||(\mathbf{x},\mathbf{y})||_2 \ge \frac{1}{2}||(\mathbf{u},\mathbf{v})||_2, \tag{7.8}$$

since otherwise $\frac{1}{2}||(\mathbf{u},\mathbf{v})||_2 < ||(\mathbf{u},\mathbf{v})||_2 - ||(\mathbf{x},\mathbf{y})||_2 \le ||(\mathbf{u},\mathbf{v})-(\mathbf{x},\mathbf{y})||_2 \le R_q$ which contradicts our assumption. Let $\lambda < (24J\max_{j\le J}\xi_j)^{-1}$ be a positive real number. Then it follows from (7.8) that for $(\mathbf{u},\mathbf{v})\in\mathbb{R}^{2J}$ such that $||(\mathbf{u},\mathbf{v})||_2 \ge 2R_q$

we have

$$
\begin{aligned}
\Phi^{\mathrm{rand}}_{>q,T}(B_q(\mathbf{u},\mathbf{v}))
&= \int_{B_q(\mathbf{u},\mathbf{v})} H_{>q,T}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}\\
&\leq e^{-\frac{\lambda}{\log G(T)}\sum_{j=1}^J(u_j^2+v_j{}^2)}\int_{B_q(\mathbf{u},\mathbf{v})} e^{\frac{4\lambda}{\log G(T)}\sum_{j=1}^J(x_j^2+y_j^2)}H_{>q,T}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}\\
&\leq e^{-\frac{\lambda}{\log G(T)}\sum_{j=1}^J(u_j^2+v_j{}^2)}\int_{\mathbb{R}^{2J}} e^{\frac{4\lambda}{\log G(T)}\sum_{j=1}^J(x_j^2+y_j^2)}H_{>q,T}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}.
\end{aligned}
\tag{7.9}
$$

To complete the proof we establish that for any real number $0<\lambda'<(6J\max_{j\leq J}\xi_j)^{-1}$ we have

$$
\int_{\mathbb{R}^{2J}} e^{\frac{\lambda'}{\log G(T)}\sum_{j=1}^J(x_j^2+y_j^2)}H_{>q,T}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} = O_{q,\lambda'}(1)
\tag{7.10}
$$

as $T\to\infty$. Indeed, assuming (7.10) we obtain by (7.7) and (7.9) that

$$
H_T(\mathbf{u},\mathbf{v}) \ll_q e^{-\frac{\lambda}{\log G(T)}\sum_{j=1}^J(u_j^2+v_j{}^2)},
$$

for $||(\mathbf{u},\mathbf{v})||_2\geq 2R_q$. Therefore, choosing q to be large but fixed we deduce that for all $(\mathbf{u},\mathbf{v})\in\mathbb{R}^{2J}$ we have

$$
H_T(\mathbf{u},\mathbf{v}) \ll e^{-\frac{\lambda}{\log G(T)}\sum_{j=1}^J(u_j^2+v_j{}^2)}
$$

where the implicit constant is absolute.

We now proceed to establish (7.10). Our proof is basically the same as the second part of the proof of Proposition 2.2 in [15]. First, note that

$$
\int_{\mathbb{R}^{2J}} e^{\frac{\lambda'}{\log G(T)}\sum_{j=1}^J(x_j^2+y_j^2)}H_{>q,T}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}
= \mathbb{E}\left[\exp\left(\frac{\lambda'}{\log G(T)}\sum_{j=1}^J\left|\sum_{p>q}\sum_{k=1}^\infty\frac{\beta_{L_j}(p^k)\mathbb{X}(p)^k}{p^{k\sigma}}\right|^2\right)\right].
$$

Since by (4.2) we have

$$
\begin{aligned}
\sum_{p>q}\sum_{k=1}^\infty\frac{\beta_{L_j}(p^k)\mathbb{X}(p)^k}{p^{k\sigma}}
&= \sum_{p}\sum_{k=1}^\infty\frac{\beta_{L_j}(p^k)\mathbb{X}(p)^k}{p^{k\sigma}} + O_q(1)\\
&= \sum_p\frac{\beta_{L_j}(p)\mathbb{X}(p)}{p^\sigma} + \sum_p\frac{\beta_{L_j}(p^2)\mathbb{X}(p)^2}{p^{2\sigma}} + O_q(1),
\end{aligned}
$$

we see that

$$\mathbb{E}\left[\exp\left(\frac{\lambda'}{\log G(T)}\sum_{j=1}^{J}\left|\sum_{p>q}\sum_{k=1}^{\infty}\frac{\beta_{L_j}(p^k)\mathbb{X}(p)^k}{p^{k\sigma}}\right|^2\right)\right]$$

$$\ll_q \mathbb{E}\left[\exp\left(\frac{3\lambda'}{\log G(T)}\sum_{j=1}^{J}\left|\sum_{p}\frac{\beta_{L_j}(p)\mathbb{X}(p)}{p^{\sigma}}\right|^2 + \frac{3\lambda'}{\log G(T)}\sum_{j=1}^{J}\left|\sum_{p}\frac{\beta_{L_j}(p^2)\mathbb{X}(p)^2}{p^{2\sigma}}\right|^2\right)\right].$$

By Hölder's inequality, the above is

$$\leq \prod_{j=1}^{J}\mathbb{E}\left[\exp\left(\frac{6J\lambda'}{\log G(T)}\left|\sum_{p}\frac{\beta_{L_j}(p)\mathbb{X}(p)}{p^{\sigma}}\right|^2\right)\right]^{\frac{1}{2J}} \times \mathbb{E}\left[\exp\left(\frac{6J\lambda'}{\log G(T)}\left|\sum_{p}\frac{\beta_{L_j}(p^2)\mathbb{X}(p)^2}{p^{2\sigma}}\right|^2\right)\right]^{\frac{1}{2J}}.$$

By inequality [8, (18.8)] (which is an easy application of Parseval's identity), the above is

$$\leq \prod_{j=1}^{J}\left(1-\frac{6J\lambda'}{\log G(T)}\sum_{p}\frac{|\beta_{L_j}(p)|^2}{p^{2\sigma}}\right)^{-\frac{1}{2J}}\left(1-\frac{6J\lambda'}{\log G(T)}\sum_{p}\frac{|\beta_{L_j}(p^2)|^2}{p^{4\sigma}}\right)^{-\frac{1}{2J}}.$$

By (4.2), we see that

$$\sum_{p}\frac{|\beta_{L_j}(p^2)|^2}{p^{4\sigma}} \leq \sum_{p}\frac{|\beta_{L_j}(p^2)|^2}{p^{2}} < \infty.$$

Furthermore, it follows from (4.5) that

$$\sum_{p}\frac{|\beta_{L_j}(p)|^2}{p^{2\sigma}} = \xi_j \log G(T) + O(1).$$

Hence, we obtain

$$\int_{\mathbb{R}^{2J}} e^{\frac{\lambda'}{\log G(T)}\sum_{j\leq J}(x_j^2+y_j^2)}H_{>q,T}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} \ll_q \prod_{j=1}^{J}\left(1-\frac{6J\lambda'(\xi_j\log G(T)+O(1))}{\log G(T)}\right)^{-\frac{1}{2J}} \ll_{q,\lambda'} 1$$

since $0 < \lambda' < (6J\max_{j\leq J}\xi_j)^{-1}$. This completes the proof of (7.10) and hence the result. □

From the above lemma, we deduce the following proposition and Lemma 2.7.

Proof of Proposition 2.5. By Lemma 7.2 we see that

$$\mathbb{E}\left(\left|\log\left|\sum_{j=1}^{J} b_j L_j(\sigma,\mathbb{X})\right|\right|^{2k}\right) = \int_{\mathbb{R}^{2J}} \left|\log\left|\sum_{j=1}^{J} b_j e^{u_j+iv_j}\right|\right|^{2k} H_T(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v}.$$

Furthermore, it follows from [15, Lemma 2.3] that there exists a constant $C > 0$ such that for any $M > 0$ we have

$$\int_{\mathbb{R}^{2J}} \left|\log\left|\sum_{j=1}^{J} b_j e^{u_j+iv_j}\right|\right|^{2k} e^{-\frac{1}{M}\sum_{j=1}^{J}(u_j^2+{v_j}^2)} d\mathbf{u}d\mathbf{v} \ll M^J (Ck)^k (M+k)^k. \tag{7.11}$$

Applying this result with $M = \log G(T)/\lambda$ completes the proof. □

Proof of Lemma 2.7. First, using that $H_T(\mathbf{u},\mathbf{v})$ is uniformly bounded in $\mathbf{u},\mathbf{v}$ we obtain

$$\begin{aligned} &\mathbb{P}\left(\mathbf{L}(\sigma,\mathbb{X}) \in [-M,M]^{2J} \text{ and } R < \left|\sum_{j=1}^{J} b_j L_j(\sigma,\mathbb{X})\right| < R+\varepsilon\right) \\ &\ll \int_{\substack{\mathbf{u}\in[-M,M]^J, \mathbf{v}\in[-M,M]^J \\ R<|\sum_{j=1}^J b_j e^{u_j+iv_j}|<R+\varepsilon}} d\mathbf{u}d\mathbf{v} \ll M^J \int_{\substack{\mathbf{u}\in[-M,M]^J, \mathbf{v}\in[0,2\pi]^J \\ R<|\sum_{j=1}^J b_j e^{u_j+iv_j}|<R+\varepsilon}} d\mathbf{u}d\mathbf{v}, \end{aligned} \tag{7.12}$$

where the last estimate is obtained by splitting the range of each v_j into intervals of the form $[2k\pi,(2k+1)\pi]$ and using that e^{iv_j} is periodic of period 2π. By the change of variables $r_1 = e^{u_1}$, the last integral in (7.12) equals

$$\int_{[0,2\pi]^{J-1}} \int_{[-M,M]^{J-1}} \left(\int_{\mathcal{R}_0} \frac{dr_1}{r_1} dv_1\right) du_2 \ldots du_J dv_2 \ldots dv_J, \tag{7.13}$$

where

$$\mathcal{R}_0 := \{(r_1,v_1) \in [e^{-M},e^M] \times [0,2\pi] : R < |b_1 r_1 e^{iv_1} + \sum_{j=2}^{J} b_j e^{u_j+iv_j}| < R+\varepsilon\}.$$

We shall now bound the inner integral by changing the polar coordinates (r_1,v_1) to cartesian coordinates x, y, defined by $x = r_1\cos(v_1)$ and $y = r_1\sin(v_1)$. Let $Z = \sum_{j=2}^{J} \frac{b_j}{b_1} e^{u_j+iv_j}$. The set $\{(x,y) \in \mathbb{R}^2 : R_1 < |x+iy+Z| < R_2\}$ corresponds to the annulus of radii R_1, R_2 centered at $-Z$ with volume $\pi(R_2^2 - R_1^2)$. Thus, we have

$$\begin{aligned} \int_{\mathcal{R}_0} \frac{dr_1}{r_1} dv_1 &= \int_{\substack{e^{-M}\le\sqrt{x^2+y^2}\le e^M \\ R/|b_1|<|x+iy+Z|<(R+\varepsilon)/|b_1|}} \frac{dxdy}{x^2+y^2} \\ &\le e^{2M} \int_{R/|b_1|<|x+iy+Z|<(R+\varepsilon)/|b_1|} dxdy = \pi e^{2M} \frac{(R+\varepsilon)^2 - R^2}{|b_1|^2} \\ &\ll e^{2M}(R\varepsilon + \varepsilon^2). \end{aligned}$$

Inserting this estimate in (7.13) and combining it with (7.12) we deduce

$$\mathbb{P}\left(\mathbf{L}(\sigma, \mathbb{X}) \in [-M, M]^{2J} \text{ and } R < \left|\sum_{j=1}^{J} b_j L_j(\sigma, \mathbb{X})\right| < R + \varepsilon\right) \ll M^{2J-1} e^{2M}(R\varepsilon + \varepsilon^2). \quad \square$$

7.2 Asymptotic formulas for $\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x}, \mathbf{y})$ and $H_T(\mathbf{u}, \mathbf{v})$. In order to prove Theorem 2.2 we need an asymptotic formula for the density function $H_T(\mathbf{u}, \mathbf{v})$ that is valid for a certain set of $(\mathbf{u}, \mathbf{v})$. To this end we prove the following result.

Lemma 7.3. *Let $\xi_{\min} = \min_{j \le J} \xi_j$. Then we have*

$$|\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x}, \mathbf{y})| \le e^{-\pi^2 \xi_{\min}(||\mathbf{x}||_2^2 + ||\mathbf{y}||_2^2)(\frac{1}{2}\log G(T) + O(1))} \tag{7.14}$$

for $||\mathbf{x}||_2^2 + ||\mathbf{y}||_2^2 \le e^{2\sqrt{G(T)}}$. Moreover, there exists a constant $c_4 > 0$ such that

$$\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x}, \mathbf{y}) = e^{\mathcal{B}_{2,\sigma}(\mathbf{z})}\left(1 + \sum_{m=3}^{5} \mathcal{B}_{m,\sigma}(\mathbf{z}) + O(||\mathbf{z}||_2^6)\right) \tag{7.15}$$

holds for

$$\mathbf{z} := \mathbf{x} + i\mathbf{y} = (x_1 + iy_1, \ldots, x_J + iy_J) \in \mathbb{C}^J$$

and $||\mathbf{z}||_2 \le c_4$, where each $\mathcal{B}_{m,\sigma}(\mathbf{z})$ is a homogeneous polynomial in $\mathbf{z}$ and $\overline{\mathbf{z}}$ of degree m,

$$\begin{aligned} \mathcal{B}_{2,\sigma}(\mathbf{z}) = & -\pi^2 \log G(T) \sum_{j=1}^{J} \xi_j (x_j^2 + y_j^2) \\ & + \sum_{j_1, j_2 \le J} \left(C_{j_1,j_2} + O\Big(\frac{\log G(T)}{G(T)}\Big)\right)(x_{j_1} - iy_{j_1})(x_{j_2} + iy_{j_2}), \end{aligned} \tag{7.16}$$

for some constants C_{j_1,j_2} and

$$\mathcal{B}_{m,\sigma}(\mathbf{z}) = \mathcal{B}_{m,1/2}(\mathbf{z}) + O\Big(\frac{||\mathbf{z}||_2^m}{G(T)}\Big) \tag{7.17}$$

for $m = 3, 4, 5$.

Proof. Recall from (7.5) that

$$\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x}, \mathbf{y}) = \prod_p \varphi_{p,\sigma}(\mathbf{x}, \mathbf{y}), \tag{7.18}$$

where $\varphi_{p,\sigma}(\mathbf{x},\mathbf{y})$ is defined in (7.4). Now, using (7.4) and expanding the exponential we obtain

$$\varphi_{p,\sigma}(\mathbf{x},\mathbf{y}) = \mathbb{E}\left[\exp\left(2\pi i\sum_{j=1}^{J}(x_j\mathrm{Re}(g_j(\mathbb{X}(p)p^{-\sigma})) + y_j\mathrm{Im}(g_j(\mathbb{X}(p)p^{-\sigma})))\right)\right]$$
$$= \sum_{\mathbf{k},\mathbf{l}\in(\mathbb{Z}_{\geq 0})^J}\frac{(\pi i)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}\overline{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\mathbf{l}}}{\mathbf{k}!\mathbf{l}!}\mathbb{E}\left[\prod_{j=1}^{J} g_j\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big)^{k_j}\overline{g_j\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big)}^{\ell_j}\right],$$

where for $\mathbf{k}=(k_1,\dots,k_J)\in(\mathbb{Z}_{\geq 0})^J$, we define $\mathcal{K}(\mathbf{k}) := k_1+\cdots+k_J$, $\mathbf{k}! := k_1!\cdots k_J!$, $\mathbf{z}=\mathbf{x}+i\mathbf{y}$, $\overline{\mathbf{z}}=\mathbf{x}-i\mathbf{y}$ and $\mathbf{z}^{\mathbf{k}} := z_1^{k_1}\cdots z_J^{k_J}$. Let

$$A_{p,\sigma}(\mathbf{k},\mathbf{l}) := \mathbb{E}\left[\prod_{j=1}^{J} g_j\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big)^{k_j}\overline{g_j\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big)}^{\ell_j}\right].$$

Since $A_{p,\sigma}(0,0)=1$, and $A_{p,\sigma}(0,\mathbf{k}) = A_{p,\sigma}(\mathbf{k},0)=0$ for $\mathbf{k}\neq 0$, we deduce that

$$\varphi_{p,\sigma}(\mathbf{x},\mathbf{y}) = 1 + R_{p,\sigma}(\mathbf{z}), \tag{7.19}$$

where

$$R_{p,\sigma}(\mathbf{z}) := \sum_{\mathbf{k}\neq 0}\sum_{\mathbf{l}\neq 0}\frac{(\pi i)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}\overline{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\mathbf{l}}}{\mathbf{k}!\mathbf{l}!}A_{p,\sigma}(\mathbf{k},\mathbf{l}).$$

We now proceed to bound the sum $R_{p,\sigma}(\mathbf{z})$ in a certain range of $\mathbf{z}$ and p. By (4.3) and (4.4), we see that

$$g_j\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big) = \frac{\beta_{L_j}(p)\mathbb{X}(p)}{p^\sigma} + O\bigg(\frac{\sum_{i=1}^d|\alpha_{j,i}(p)|^2}{p^{2\sigma}}\bigg)$$
$$= O\bigg(\frac{1}{p^\sigma}\sqrt{\sum_{i=1}^{d}|\alpha_{j,i}(p)|^2}\bigg) = O\Big(\frac{1}{p^{1/2-\theta}}\Big).$$

Hence, there exists a constant $c_0>0$ such that both

$$\begin{aligned}\Big|g_j\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big)\Big| &\leq c_0\frac{1}{p^\sigma}\sqrt{\sum_{i=1}^{d}|\alpha_{j,i}(p)|^2},\\ \Big|g_j\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big)\Big| &\leq c_0\frac{1}{p^{1/2-\theta}}\end{aligned} \tag{7.20}$$

hold for every prime p and every $j\leq J$. Thus, we obtain

$$\begin{aligned}|R_{p,\sigma}(\mathbf{z})| &\ll \sum_{\mathbf{k}\neq 0}\sum_{\mathbf{l}\neq 0}\frac{1}{\mathbf{k}!\mathbf{l}!}\prod_{j=1}^{J}\bigg(\frac{c_0\pi||\mathbf{z}||_2}{p^{1/2}}\sqrt{\sum_{i=1}^{d}|\alpha_{j,i}(p)|^2}\bigg)^{k_j+\ell_j}\\ &\ll \frac{||\mathbf{z}||_2^2}{p}\sum_{j=1}^{J}\sum_{i=1}^{d}|\alpha_{j,i}(p)|^2 \ll \frac{||\mathbf{z}||_2^2}{p^{1-2\theta}}\end{aligned} \tag{7.21}$$

provided that $||\mathbf{z}||_2 \ll p^{1/2-\theta}$. This implies that in the range $||\mathbf{z}||_2 \le Y$ with $Y := \exp(\sqrt{G(T)})$, there is a constant $c_1 > 0$ such that $|R_{p,\sigma}(\mathbf{z})| \le \frac{1}{2}$ holds for all primes $p \ge c_1 Y^{1/(1/2-\theta)}$. Therefore, using that $|\varphi_{p,\sigma}(\mathbf{x},\mathbf{y})| \le 1$ for all primes p, together with (7.18) and (7.19) we have

$$\begin{aligned} |\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x},\mathbf{y})| &\le \prod_{p \ge c_1 Y^{c(\epsilon)}} |1 + R_{p,\sigma}(\mathbf{z})| \\ &\le \left| \exp\Big(\sum_{p \ge c_1 Y^{c(\epsilon)}} R_{p,\sigma}(\mathbf{z}) + O\Big(\sum_{p \ge c_1 Y^{c(\epsilon)}} |R_{p,\sigma}(\mathbf{z})|^2 \Big) \Big) \right|, \end{aligned} \tag{7.22}$$

for $||\mathbf{z}||_2 \le Y$ and any $\epsilon > 0$ fixed, where

$$c(\epsilon) := (1+\epsilon)/(1/2-\theta).$$

The second p-sum in (7.22) is

$$\begin{aligned} \sum_{p \ge c_1 Y^{c(\epsilon)}} |R_{p,\sigma}(\mathbf{z})|^2 &\ll ||\mathbf{z}||_2^4 \sum_{p \ge c_1 Y^{c(\epsilon)}} \frac{1}{p^{2-2\theta}} \sum_{j=1}^{J} \sum_{i=1}^{d} |\alpha_{j,i}(p)|^2 \\ &\ll ||\mathbf{z}||_2^4 Y^{-2-\epsilon} \ll ||\mathbf{z}||_2^2 Y^{-\epsilon} \end{aligned}$$

for $||\mathbf{z}||_2 \le Y$ and any $\epsilon > 0$ fixed by (7.21), assumption A3 and partial summation. The first p-sum in (7.22) is

$$\begin{aligned} \sum_{p \ge c_1 Y^{c(\epsilon)}} R_{p,\sigma}(\mathbf{z}) &= \sum_{p \ge c_1 Y^{c(\epsilon)}} \sum_{\mathbf{k} \ne 0} \sum_{\mathbf{l} \ne 0} \frac{(\pi i)^{\mathcal{K}(\mathbf{k}+\mathbf{l})} \overline{\mathbf{z}}^{\mathbf{k}} \mathbf{z}^{\mathbf{l}}}{\mathbf{k}!\mathbf{l}!} A_{p,\sigma}(\mathbf{k},\mathbf{l}) \\ &= -\pi^2 \sum_{j_1,j_2 \le J} \overline{z_{j_1}} z_{j_2} \sum_{p \ge c_1 Y^{c(\epsilon)}} \mathbb{E}\Big[g_{j_1}\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big) \overline{g_{j_2}\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big)} \Big] \\ &\quad + O\Big(\sum_{p \ge c_1 Y^{c(\epsilon)}} \sideset{}{^*}\sum_{\mathbf{k},\mathbf{l}} \frac{(\pi ||z||_2)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}!\mathbf{l}!} |A_{p,\sigma}(\mathbf{k},\mathbf{l})| \Big), \end{aligned}$$

where the $*$-sum is over $\mathbf{k},\mathbf{l} \in (\mathbb{Z}_{\ge 0})^J$ with $\mathbf{k} \ne 0, \mathbf{l} \ne 0$ and $\mathcal{K}(\mathbf{k}+\mathbf{l}) \ge 3$. By (7.20), assumption A3 and partial summation, the above O-term is

$$\begin{aligned} &\ll \sideset{}{^*}\sum_{\mathbf{k},\mathbf{l}} \frac{(c_0 \pi ||\mathbf{z}||_2)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}!\mathbf{l}!} \sum_{p \ge c_1 Y^{c(\epsilon)}} \frac{1}{p^{1+(\mathcal{K}(\mathbf{k}+\mathbf{l})-2)(1/2-\theta)}} \sum_{j=1}^{J} \sum_{i=1}^{d} |\alpha_{j,i}(p)|^2 \\ &\ll \sideset{}{^*}\sum_{\mathbf{k},\mathbf{l}} \frac{(c_0 \pi ||\mathbf{z}||_2)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}!\mathbf{l}!} Y^{2-\mathcal{K}(\mathbf{k}+\mathbf{l})-\epsilon/2} \ll Y^{-1-\epsilon/2} ||\mathbf{z}||_2^3 \le Y^{-\epsilon/2} ||\mathbf{z}||_2^2 \end{aligned}$$

for $||\mathbf{z}||_2 \le Y$. Thus we derive

$$\begin{aligned} &|\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x},\mathbf{y})| \\ &\le \left| \exp\Big(-\pi^2 \sum_{j_1,j_2 \le J} \overline{z_{j_1}} z_{j_2} \sum_{p \ge c_1 Y^{c(\epsilon)}} \mathbb{E}\Big[g_{j_1}\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big) \overline{g_{j_2}\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big)} \Big] + O(Y^{-\epsilon/2} ||\mathbf{z}||_2^2) \Big) \right| \end{aligned}$$

for $||\mathbf{z}||_2 \le Y$ and for any $\epsilon > 0$ fixed. Moreover, by (4.2) and (4.6), the sum above equals

$$\begin{aligned}\sum_{j_1,j_2\le J}\overline{z_{j_1}}z_{j_2}\sum_{p\ge c_1 Y^{c(\epsilon)}}\sum_{k=1}^{\infty}\frac{\beta_{L_{j_1}}(p^k)\overline{\beta_{L_{j_2}}(p^k)}}{p^{2k\sigma}}&\\ =\sum_{j_1,j_2\le J}\overline{z_{j_1}}z_{j_2}\sum_{p\ge c_1 Y^{c(\epsilon)}}\frac{\beta_{L_{j_1}}(p)\overline{\beta_{L_{j_2}}(p)}}{p^{2\sigma}}+O(||\mathbf{z}||_2^2)&\\ =\frac{1}{2}\log G(T)\sum_{j=1}^{J}\xi_j|z_j|^2+O(||\mathbf{z}||_2^2).&\end{aligned}$$

Therefore, we deduce that

$$|\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x},\mathbf{y})|\le e^{-(\frac{\pi^2}{2}\log G(T)+O(1))\sum_{j=1}^{J}\xi_j|z_j|^2}\le e^{-\pi^2\xi_{\min}||\mathbf{z}||_2^2(\frac{1}{2}\log G(T)+O(1))} \tag{7.23}$$

for $||\mathbf{z}||_2 \le Y$ where $Y = e^{\sqrt{G(T)}}$. This proves (7.14).

Next we find an asymptotic formula for $\widehat{\Phi}_T^{\mathrm{rand}}$. By (7.21), there is a constant $c_4 > 0$ such that $|R_{p,\sigma}(\mathbf{z})| \le \frac{1}{2}$ for $||\mathbf{z}||_2 \le c_4$ and for every prime p. Hence, it follows from (7.18) and (7.19) that

$$\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x},\mathbf{y})=\exp\left(\sum_p R_{p,\sigma}(\mathbf{z})-\frac{1}{2}\sum_p R_{p,\sigma}(\mathbf{z})^2+O\left(\sum_p |R_{p,\sigma}(\mathbf{z})|^3\right)\right).$$

The O-term above is

$$\ll ||\mathbf{z}||_2^6\sum_p\frac{1}{p^{1+2(1-2\theta)}}\sum_{j=1}^{J}\sum_{i=1}^{d}|\alpha_{j,i}(p)|^2\ll ||\mathbf{z}||_2^6$$

for $||\mathbf{z}||_2 \le c_4$ by (7.21), assumption A3 and partial summation. We observe that the sum $\sum_p R_{p,\sigma}(\mathbf{z}) - \frac{1}{2}\sum_p R_{p,\sigma}(\mathbf{z})^2$ has a power series representation in $z_1, \ldots, z_J, \overline{z_1}, \ldots, \overline{z_J}$ without a constant term. Let $B_\sigma(\mathbf{k},\mathbf{l})$ be the coefficient of $\overline{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\mathbf{l}}$ in this sum. Then we have

$$\begin{aligned}&\sum_{\mathbf{k}\ne 0}\sum_{\mathbf{l}\ne 0}B_\sigma(\mathbf{k},\mathbf{l})\overline{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\mathbf{l}}\\ &\quad=\sum_{\mathbf{k}\ne 0}\sum_{\mathbf{l}\ne 0}\frac{(\pi i)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}\overline{\mathbf{z}}^{\mathbf{k}}\mathbf{z}^{\mathbf{l}}}{\mathbf{k}!\mathbf{l}!}\sum_p A_{p,\sigma}(\mathbf{k},\mathbf{l})\\ &\qquad-\frac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2\ne 0}\sum_{\mathbf{l}_1,\mathbf{l}_2\ne 0}\frac{(\pi i)^{\mathcal{K}(\mathbf{k}_1+\mathbf{l}_1+\mathbf{k}_2+\mathbf{l}_2)}\overline{\mathbf{z}}^{\mathbf{k}_1+\mathbf{k}_2}\mathbf{z}^{\mathbf{l}_1+\mathbf{l}_2}}{\mathbf{k}_1!\mathbf{k}_2!\mathbf{l}_1!\mathbf{l}_2!}\sum_p A_{p,\sigma}(\mathbf{k}_1,\mathbf{l}_1)A_{p,\sigma}(\mathbf{k}_2,\mathbf{l}_2).\end{aligned}$$

Therefore, if $\mathcal{K}(\mathbf{k}+\mathbf{l}) = 2$ or 3, then

$$B_\sigma(\mathbf{k},\mathbf{l})=\frac{(\pi i)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}!\mathbf{l}!}\sum_p A_{p,\sigma}(\mathbf{k},\mathbf{l}),$$

while in the case $\mathcal{K}(\mathbf{k}+\mathbf{l}) \geq 4$, we have

$$B_\sigma(\mathbf{k},\mathbf{l}) = \frac{(\pi i)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}!\mathbf{l}!} \sum_p A_{p,\sigma}(\mathbf{k},\mathbf{l}) - \frac{1}{2} \sum_{\substack{\mathbf{k}_1,\mathbf{k}_2 \neq 0 \\ \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}} \sum_{\substack{\mathbf{l}_1,\mathbf{l}_2 \neq 0 \\ \mathbf{l}_1+\mathbf{l}_2=\mathbf{l}}} \frac{(\pi i)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}_1!\mathbf{k}_2!\mathbf{l}_1!\mathbf{l}_2!} \sum_p A_{p,\sigma}(\mathbf{k}_1,\mathbf{l}_1) A_{p,\sigma}(\mathbf{k}_2,\mathbf{l}_2)$$

For $\mathcal{K}(\mathbf{k}+\mathbf{l}) \geq 4$, we have

$$\begin{aligned} B_\sigma(\mathbf{k},\mathbf{l}) &\ll \frac{\pi^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}!\mathbf{l}!} \sum_p |A_{p,\sigma}(\mathbf{k},\mathbf{l})| \\ &\quad + \sum_{\substack{\mathbf{k}_1,\mathbf{k}_2 \neq 0 \\ \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}} \sum_{\substack{\mathbf{l}_1,\mathbf{l}_2 \neq 0 \\ \mathbf{l}_1+\mathbf{l}_2=\mathbf{l}}} \frac{\pi^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}_1!\mathbf{k}_2!\mathbf{l}_1!\mathbf{l}_2!} \sum_p |A_{p,\sigma}(\mathbf{k}_1,\mathbf{l}_2)||A_{p,\sigma}(\mathbf{k}_2,\mathbf{l}_2)| \\ &\ll \frac{(c_0\pi)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}!\mathbf{l}!} \sum_p \frac{\max_j \sum_i |\alpha_{j,i}(p)|^2}{p^{1+2(1/2-\theta)}} \\ &\quad + \sum_{\substack{\mathbf{k}_1,\mathbf{k}_2 \neq 0 \\ \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}} \sum_{\substack{\mathbf{l}_1,\mathbf{l}_2 \neq 0 \\ \mathbf{l}_1+\mathbf{l}_2=\mathbf{l}}} \frac{(c_0\pi)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}_1!\mathbf{k}_2!\mathbf{l}_1!\mathbf{l}_2!} \sum_p \frac{\max_j \sum_i |\alpha_{j,i}(p)|^2}{p^{1+2(1/2-\theta)}} \\ &\ll \frac{c_5^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}!\mathbf{l}!}, \end{aligned}$$

for some constant $c_5 > 0$, where the implicit constant is independent of $\mathbf{k}$ and $\mathbf{l}$. Hence, we deduce that

$$\sum_{\substack{\mathbf{k},\mathbf{l} \neq 0 \\ \mathcal{K}(\mathbf{k}+\mathbf{l}) \geq 6}} B_\sigma(\mathbf{k},\mathbf{l}) \overline{\mathbf{z}}^{\mathbf{k}} \mathbf{z}^{\mathbf{l}} \ll \sum_{\mathcal{K}(\mathbf{k}+\mathbf{l}) \geq 6} \frac{(c_5||\mathbf{z}||_2)^{\mathcal{K}(\mathbf{k}+\mathbf{l})}}{\mathbf{k}!\mathbf{l}!} \ll ||\mathbf{z}||_2^6$$

for $||\mathbf{z}||_2 \leq c_4$. Therefore, we obtain

$$\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x},\mathbf{y}) = \exp\left(\sum_{m=2}^{5} \mathcal{B}_{m,\sigma}(\mathbf{z}) + O(||\mathbf{z}||_2^6) \right) \tag{7.24}$$

for $||\mathbf{z}||_2 \leq c_4$, where

$$\mathcal{B}_{m,\sigma}(\mathbf{z}) := \sum_{\substack{\mathbf{k},\mathbf{l} \neq 0 \\ \mathcal{K}(\mathbf{k}+\mathbf{l})=m}} B_\sigma(\mathbf{k},\mathbf{l}) \overline{\mathbf{z}}^{\mathbf{k}} \mathbf{z}^{\mathbf{l}}$$

is a homogeneous polynomial in $z_1, \ldots, z_J, \overline{z_1}, \ldots, \overline{z_J}$ of degree m.

For each $\mathbf{k},\mathbf{l} \neq 0$ satisfying $\mathcal{K}(\mathbf{k}+\mathbf{l}) = 3, 4, 5$, $B_\sigma(\mathbf{k},\mathbf{l})$ is a Dirichlet series absolutely convergent for $\mathrm{Re}(s) > 1/2 - \epsilon$ for some $\epsilon > 0$. Thus, each coefficient $B_\sigma(\mathbf{k},\mathbf{l})$ in such case satisfies

$$B_\sigma(\mathbf{k},\mathbf{l}) = B_{1/2}(\mathbf{k},\mathbf{l}) + O(1/G(T)),$$

which proves (7.17). This also implies that

$$\mathcal{B}_{m,\sigma}(\mathbf{z}) = O(||\mathbf{z}||_2^m) \tag{7.25}$$

for $m = 3, 4, 5$ and for $||\mathbf{z}||_2 \leq c_4$. On the other hand, when $m = 2$, we see that

$$\mathcal{B}_{2,\sigma}(\mathbf{z}) = \sum_{j_1,j_2 \leq J} g_{j_1,j_2}(\sigma)\overline{z_{j_1}}z_{j_2}, \tag{7.26}$$

where

$$\begin{aligned} g_{j_1,j_2}(\sigma) :=& -\pi^2 \sum_p \mathbb{E}\Big[g_{j_1}\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big)\overline{g_{j_2}\Big(\frac{\mathbb{X}(p)}{p^\sigma}\Big)}\Big] \\ =& -\pi^2 \sum_p \sum_{k=1}^{\infty} \frac{\beta_{L_{j_1}}(p^k)\overline{\beta_{L_{j_2}}(p^k)}}{p^{2k\sigma}} \\ =& -\pi^2 \sum_p \frac{\beta_{L_{j_1}}(p)\overline{\beta_{L_{j_2}}(p)}}{p^{2\sigma}} - \pi^2 \sum_p \sum_{k=2}^{\infty} \frac{\beta_{L_{j_1}}(p^k)\overline{\beta_{L_{j_2}}(p^k)}}{p^k} + O\Big(\frac{1}{G(T)}\Big). \end{aligned}$$

The second p-sum on the last line is convergent by (4.2). Therefore, it follows from (4.5) that

$$g_{j_1,j_2}(\sigma) = -\pi^2\delta_{j_1,j_2}\xi_{j_1}\log G(T) + C_{j_1,j_2} + O\Big(\frac{\log G(T)}{G(T)}\Big) \tag{7.27}$$

for some constant C_{j_1,j_2}. This proves (7.16). By (7.24) and (7.25) we see that

$$\widehat{\Phi}_T^{\text{rand}}(\mathbf{x},\mathbf{y}) = e^{\mathcal{B}_{2,\sigma}(\mathbf{z})}\left(1 + \sum_{m=3}^{5}\mathcal{B}_{m,\sigma}(\mathbf{z}) + O(||\mathbf{z}||_2^6)\right)$$

holds for $||\mathbf{z}||_2 \leq c_4$. This proves (7.15) and hence completes the proof. □

Using Lemma 7.3 we establish the following result, which is the key ingredient in the proof of Theorem 2.2.

Lemma 7.4. *For all* $\mathbf{u},\mathbf{v} \in \mathbb{R}^J$ *we have*

$$H_T(\mathbf{u},\mathbf{v}) = \frac{1}{(\log G(T))^J} e^{-\frac{1}{\log G(T)}\sum_{j=1}^J \xi_j^{-1}(u_j^2+v_j{}^2)} P_T(\mathbf{u},\mathbf{v}) + O\Big(\frac{1}{(\log G(T))^{J+3}}\Big),$$

where

$$P_T(\mathbf{u},\mathbf{v}) = \frac{1}{\pi^J\prod_{j=1}^J \xi_j} + \sum_{m=2}^{5}\sum_{r=0}^{m}\frac{Q_{r,m}(\mathbf{u},\mathbf{v})}{(\log G(T))^{(m+r)/2}}$$

is a polynomial in $\mathbf{u},\mathbf{v}$ *of degree* ≤ 5 *and* $Q_{r,m}(\mathbf{u},\mathbf{v})$ *is a homogeneous polynomial in* $\mathbf{u},\mathbf{v}$ *of degree* r *for* $r \leq m \leq 5$.

Proof. Recall that the density function $H_T(\mathbf{u},\mathbf{v})$ is the inverse Fourier transform of $\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x},\mathbf{y})$, so that

$$H_T(\mathbf{u},\mathbf{v}) = \int_{\mathbb{R}^{2J}} e^{-2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})}\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}.$$

By Lemma 7.1, (7.14) and by changing the variables to polar coordinates, with $\mathbf{z}=\mathbf{x}+i\mathbf{y}$ as in the previous lemma, we find that

$$\begin{aligned}
&H_T(\mathbf{u},\mathbf{v})\\
&= \int_{||\mathbf{z}||_2\le c_4} e^{-2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})}\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}\\
&\quad + O\bigg(\int_{c_4<||\mathbf{z}||_2\le e^{\sqrt{G(T)}}} e^{-\xi_{\min}\log G(T)||\mathbf{z}||_2^2}d\mathbf{x}d\mathbf{y} + \int_{||\mathbf{z}||_2>e^{\sqrt{G(T)}}} \frac{d\mathbf{x}d\mathbf{y}}{(1+||\mathbf{z}||_2)^{2J+1}}\bigg)\\
&= \int_{||\mathbf{z}||_2\le c_4} e^{-2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})}\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}\\
&\quad + O\bigg(\int_{c_4}^{e^{\sqrt{G(T)}}} e^{-\xi_{\min}\log G(T)w^2}w^{2J-1}dw + \int_{e^{\sqrt{G(T)}}}^{\infty} \frac{1}{(1+w)^{2J+1}}w^{2J-1}dw\bigg)\\
&= \int_{||\mathbf{z}||_2\le c_4} e^{-2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})}\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} + O\bigg(\frac{1}{G(T)^{c_4^2\xi_{\min}}\log G(T)}\bigg).
\end{aligned}$$

Therefore, it follows from (7.15) and (7.16) that

$$\begin{aligned}
H_T(\mathbf{u},\mathbf{v}) &= \int_{||\mathbf{z}||_2\le c_4} e^{\mathcal{B}_{2,\sigma}(\mathbf{z})-2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})}\bigg(1+\sum_{m=3}^{5}\mathcal{B}_{m,\sigma}(\mathbf{z}) + O(||\mathbf{z}||_2^6)\bigg)d\mathbf{x}d\mathbf{y}\\
&\quad + O\bigg(\frac{1}{G(T)^{c_4^2\xi_{\min}}\log G(T)}\bigg)\\
&= \int_{||\mathbf{z}||_2\le c_4} e^{\mathcal{B}_{2,\sigma}(\mathbf{z})-2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})}\bigg(1+\sum_{m=3}^{5}\mathcal{B}_{m,\sigma}(\mathbf{z})\bigg)d\mathbf{x}d\mathbf{y}\\
&\quad + O\bigg(\frac{1}{(\log G(T))^{J+3}}\bigg).
\end{aligned}$$

Let

$$\mathcal{B}'_{2,\sigma}(\mathbf{z}) := \mathcal{B}_{2,\sigma}(\mathbf{z}) + \pi^2\log G(T)\sum_{j=1}^{J}\xi_j|z_j|^2,$$

then by (7.26) and (7.27), we have

$$\mathcal{B}'_{2,\sigma}(\mathbf{z}) = \sum_{j_1,j_2\le J}\bigg(C_{j_1,j_2} + O\bigg(\frac{\log G(T)}{G(T)}\bigg)\bigg)\overline{z_{j_1}}z_{j_2}.$$

Thus, by (7.16) we have

$$H_T(\mathbf{u},\mathbf{v}) = \int_{||\mathbf{z}||_2\le c_4} e^{-\pi^2 \log G(T)\sum_{j=1}^J \xi_j(x_j^2+y_j^2)-2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})}\Big(1+\sum_{m=2}^{5}\tilde{\mathcal{B}}_{m,\sigma}(\mathbf{z})\Big)d\mathbf{x}d\mathbf{y} + O\Big(\frac{1}{(\log G(T))^{J+3}}\Big),$$

where each $\tilde{\mathcal{B}}_{m,\sigma}(\mathbf{z})$ for $m=2,3,4,5$ is a homogeneous polynomial of degree m defined to satisfy the identity

$$\Big(1+\mathcal{B}'_{2,\sigma}(\mathbf{z})+\frac{1}{2}\mathcal{B}'_{2,\sigma}(\mathbf{z})^2\Big)\Big(1+\sum_{m=3}^{5}\mathcal{B}_{m,\sigma}(\mathbf{z})\Big) = 1+\sum_{m=2}^{5}\tilde{\mathcal{B}}_{m,\sigma}(\mathbf{z})+O(||\mathbf{z}||_2^6).$$

Furthermore, one has

$$\tilde{\mathcal{B}}_{m,\sigma}(\mathbf{z}) = \sum_{k=0}^{m}\Big(D_{k,m}+O\Big(\frac{\log G(T)}{G(T)}\Big)\Big)\overline{z}^k z^{m-k},$$

for some constants $D_{k,m}$. Thus, the polynomials

$$\tilde{\mathcal{B}}_{m}(\mathbf{z}) := \sum_{k=0}^{m} D_{k,m}\overline{z}^k z^{m-k}$$

are independent of σ and we see that

$$H_T(\mathbf{u},\mathbf{v}) = \int_{||\mathbf{z}||_2\le c_4} e^{-\pi^2 \log G(T)\sum_{j=1}^J \xi_j(x_j^2+y_j^2)-2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})}\Big(1+\sum_{m=2}^{5}\tilde{\mathcal{B}}_{m}(\mathbf{z})\Big)d\mathbf{x}d\mathbf{y} + O\Big(\frac{1}{(\log G(T))^{J+3}}\Big).$$

Extending the range of integration to all of $\mathbb{R}^{2J}$ and changing the x_j and y_j to $x_j/\sqrt{\pi^2\log G(T)}$ and $y_j/\sqrt{\pi^2\log G(T)}$ respectively, we obtain

$$\begin{aligned} &H_T(\mathbf{u},\mathbf{v})\\ &= \int_{\mathbb{R}^{2J}} e^{-\pi^2 \log G(T)\sum_{j=1}^J \xi_j(x_j^2+y_j^2)-2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})}\Big(1+\sum_{m=2}^{5}\tilde{\mathcal{B}}_{m}(\mathbf{z})\Big)d\mathbf{x}d\mathbf{y}\\ &\quad + O((\log G(T))^{-J-3})\\ &= \int_{\mathbb{R}^{2J}} e^{-\sum_{j=1}^J \xi_j(x_j^2+y_j^2)-2\frac{i}{\sqrt{\log G(T)}}(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})}\Big(1+\sum_{m=2}^{5}\frac{\tilde{\mathcal{B}}_{m}(\mathbf{z})}{\pi^m(\log G(T))^{m/2}}\Big)\frac{d\mathbf{x}d\mathbf{y}}{\pi^{2J}(\log G(T))^J}\\ &\quad + O((\log G(T))^{-J-3}). \end{aligned}$$

The above exponent is

$$-\frac{1}{\log G(T)}\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+v_j{}^2)-\sum_{j=1}^{J}\xi_j\Big(\Big(x_j+\frac{iu_j}{\xi_j\sqrt{\log G(T)}}\Big)^2+\Big(y_j+\frac{iv_j}{\xi_j\sqrt{\log G(T)}}\Big)^2\Big).$$

Hence, making the change of variables

$$\tilde{\mathbf{z}}=(\tilde{x}_1+i\tilde{y}_1,\ldots,\tilde{x}_J+i\tilde{y}_J),$$

where

$$\tilde{x}_j=x_j-\frac{iu_j}{\xi_j\sqrt{\log G(T)}},\quad \tilde{y}_j=y_j-\frac{iv_j}{\xi_j\sqrt{\log G(T)}}$$

for every $j=1,\ldots,J$, we see that

$$H_T(\mathbf{u},\mathbf{v})=\frac{1}{(\log G(T))^J}e^{-\frac{1}{\log G(T)}\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+v_j{}^2)}P_T(\mathbf{u},\mathbf{v})+O\Big(\frac{1}{(\log G(T))^{J+3}}\Big),$$

where $P_T(\mathbf{u},\mathbf{v})$ is a polynomial in $\mathbf{u}$ and $\mathbf{v}$ defined by

$$P_T(\mathbf{u},\mathbf{v})=\frac{1}{\pi^{2J}}\int_{\mathbb{R}^{2J}}e^{-\sum_{j=1}^{J}\xi_j(x_j^2+y_j^2)}\Big(1+\sum_{m=2}^{5}\frac{\tilde{\mathcal{B}}_m(\tilde{\mathbf{z}})}{\pi^m(\log G(T))^{m/2}}\Big)d\mathbf{x}d\mathbf{y}.$$

By expanding the polynomial $\pi^{-m}\tilde{\mathcal{B}}_m(\tilde{\mathbf{z}})$, we have

$$\pi^{-m-2J}\tilde{\mathcal{B}}_m(\tilde{\mathbf{z}})=\sum_{k=0}^{m}\frac{Q_{k,m}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v})}{(\log G(T))^{k/2}},$$

where $Q_{k,m}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v})$ is a homogeneous polynomial in $\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}$ of degree m and each term has k factors in $\mathbf{u},\mathbf{v}$. Therefore, we have

$$P_T(\mathbf{u},\mathbf{v})=\frac{1}{\pi^J\prod_{j=1}^{J}\xi_j}+\sum_{m=2}^{5}\sum_{k=0}^{m}\frac{\mathcal{Q}_{k,m}(\mathbf{u},\mathbf{v})}{(\log G(T))^{(m+k)/2}},$$

where

$$\mathcal{Q}_{k,m}(\mathbf{u},\mathbf{v}):=\int_{\mathbb{R}^{2J}}e^{-\sum_{j=1}^{J}\xi_j(x_j^2+y_j^2)}Q_{k,m}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v})d\mathbf{x}d\mathbf{y}$$

is a homogeneous polynomial in $\mathbf{u},\mathbf{v}$ of degree k for $k\le m\le 5$. □

7.3 Proof of Theorem 2.2. Let $\sigma=1/2+G(T)$ and $\sigma_i=1/2+G_i(T)$ for $i=1,2$. Then, recall that

$$\mathcal{M}(\sigma)=\mathbb{E}\Big(\log\Big|\sum_{j=1}^{J}b_jL_j(\sigma,\mathbb{X})\Big|\Big)=\int_{\mathbb{R}^{2J}}\log\Big|\sum_{j=1}^{J}b_je^{u_j+iv_j}\Big|H_T(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v}.$$

Let $M_1 := \sqrt{\eta \log G(T) \log\log G(T)}$, where $\eta > 0$ is a suitably large constant that depends on J and the ξ_j. By the Cauchy–Schwarz inequality, Lemma 7.2 with $\lambda = (30J \max_{j\le J} \xi_j)^{-1}$and equation (7.11) we see that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{2J}\setminus[-M_1,M_1]^{2J}} \log\left|\sum_{j=1}^J b_j e^{u_j+iv_j}\right| H_T(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v}\right| \\
&\qquad\le \left(\int_{\mathbb{R}^{2J}\setminus[-M_1,M_1]^{2J}} \left|\log\left|\sum_{j=1}^J b_j e^{u_j+iv_j}\right|\right|^2 H_T(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v}\right)^{\frac12} \\
&\qquad\quad\times \left(\int_{\mathbb{R}^{2J}\setminus[-M_1,M_1]^{2J}} H_T(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v}\right)^{\frac12} \\
&\qquad\ll (\log G(T))^{\frac{J+1}{2}}\left(\frac{(\log G(T))^{J+\frac12}}{M_1} e^{-\frac{\lambda M_1^2}{\log G(T)}}\right)^{\frac12} \ll \frac{1}{(\log G(T))^3}.
\end{aligned}$$

This implies

$$\mathcal{M}(\sigma) = \int_{[-M_1,M_1]^{2J}} \log\left|\sum_{j=1}^J b_j e^{u_j+iv_j}\right| H_T(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v} + O\Big(\frac{1}{(\log G(T))^3}\Big).$$

We now use the asymptotic formula for $H_T(\mathbf{u},\mathbf{v})$ in Lemma 7.4 to obtain

$$\begin{aligned}
\mathcal{M}(\sigma) &= \frac{1}{(\log G(T))^J} \\
&\quad\times \int_{[-M_1,M_1]^{2J}} \log\left|\sum_{j=1}^J b_j e^{u_j+iv_j}\right| e^{-\frac{1}{\log G(T)}\sum_{j=1}^J \xi_j^{-1}(u_j^2+v_j^{\,2})} P_T(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v} \\
&\quad+ O\Bigg(\frac{1}{(\log G(T))^{J+3}} \int_{[-M_1,M_1]^{2J}} \left|\log\left|\sum_{j=1}^J b_j e^{u_j+iv_j}\right|\right| d\mathbf{u}d\mathbf{v} + \frac{1}{(\log G(T))^3}\Bigg).
\end{aligned}$$

By the Cauchy–Schwarz inequality and (7.11) we have

$$\begin{aligned}
&\left(\int_{[-M_1,M_1]^{2J}} \left|\log\left|\sum_{j=1}^J b_j e^{u_j+iv_j}\right|\right| d\mathbf{u}d\mathbf{v}\right)^2 \\
&\le (2M_1)^{2J} \int_{[-M_1,M_1]^{2J}} \left|\log\left|\sum_{j=1}^J b_j e^{u_j+iv_j}\right|\right|^2 d\mathbf{u}d\mathbf{v} \\
&\ll M_1^{2J} \int_{\mathbb{R}^{2J}} \left|\log\left|\sum_{j=1}^J b_j e^{u_j+iv_j}\right|\right|^2 e^{-\frac{1}{M_1^2}\sum_{j=1}^J (u_j^2+v_j^{\,2})} d\mathbf{u}d\mathbf{v} \ll M_1^{2(2J+1)}.
\end{aligned}$$

Thus we have

$$\mathcal{M}(\sigma) = \frac{1}{(\log G(T))^J} \times \int_{[-M_1,M_1]^{2J}} \log\Big|\sum_{j=1}^{J} b_j e^{u_j+iv_j}\Big| e^{-\frac{1}{\log G(T)}\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+v_j{}^2)} P_T(\mathbf{u},\mathbf{v})\,d\mathbf{u}d\mathbf{v} + O\Big(\frac{(\log\log G(T))^{J+(1/2)}}{(\log G(T))^{5/2}}\Big).$$

We now use Lemma 7.4, which gives

$$\begin{aligned}\mathcal{M}(\sigma) &= \frac{1}{\prod_{j\le J}\xi_j(\pi\log G(T))^J}\int_{[-M_1,M_1]^{2J}} \log\Big|\sum_{j=1}^{J} b_j e^{u_j+iv_j}\Big| e^{-\frac{1}{\log G(T)}\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+v_j{}^2)}\,d\mathbf{u}d\mathbf{v}\\ &\quad+\sum_{m=2}^{5}\sum_{r=0}^{m}\int_{[-M_1,M_1]^{2J}} \log\Big|\sum_{j=1}^{J} b_j e^{u_j+iv_j}\Big| e^{-\frac{1}{\log G(T)}\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+v_j{}^2)} \frac{Q_{r,m}(\mathbf{u},\mathbf{v})\,d\mathbf{u}d\mathbf{v}}{(\log G(T))^{J+(m+r)/2}}\\ &\quad+O\Big(\frac{(\log\log G(T))^{J+(1/2)}}{(\log G(T))^{5/2}}\Big).\end{aligned}$$

By the Cauchy–Schwarz inequality and (7.11), we can replace $[-M_1, M_1]^{2J}$ in both integrals by $\mathbb{R}^{2J}$, at the cost of an error term of size $\ll 1/(\log G(T))^3$ if η is suitably large. Let

$$Q_{r,m}(\mathbf{u},\mathbf{v}) = \sum_{\substack{\mathbf{k},\mathbf{l}\\ \mathcal{K}(\mathbf{k}+\mathbf{l})=r}} q_{r,m,\mathbf{k},\mathbf{l}}\mathbf{u}^{\mathbf{k}}\mathbf{v}^{\mathbf{l}}.$$

Since $Q_{r,m}(\mathbf{u},\mathbf{v})$ is a homogeneous polynomial of degree r, by changing the variables $\mathbf{u}, \mathbf{v}$ to $\sqrt{\log G(T)}\mathbf{u}$, $\sqrt{\log G(T)}\mathbf{v}$, we obtain

$$\begin{aligned}\mathcal{M}(\sigma) &= \frac{1}{\pi^J\prod_{j=1}^{J}\xi_j}\int_{\mathbb{R}^{2J}} \log\Big|\sum_{j=1}^{J} b_j e^{(u_j+iv_j)\sqrt{\log G(T)}}\Big| e^{-\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+v_j{}^2)}\,d\mathbf{u}d\mathbf{v}\\ &\quad+\sum_{m=2}^{5}\sum_{r=0}^{m}\int_{\mathbb{R}^{2J}} \log\Big|\sum_{j=1}^{J} b_j e^{(u_j+iv_j)\sqrt{\log G(T)}}\Big| e^{-\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+v_j{}^2)} \frac{Q_{r,m}(\mathbf{u},\mathbf{v})\,d\mathbf{u}d\mathbf{v}}{(\log G(T))^{m/2}}\\ &\quad+O\Big(\frac{(\log\log G(T))^{J+(1/2)}}{(\log G(T))^{5/2}}\Big)\\ &= \frac{1}{\pi^J\prod_{j=1}^{J}\xi_j}I(0,0,\sigma) + \sum_{m=2}^{5}\sum_{r=0}^{m}\frac{1}{(\log G(T))^{m/2}}\sum_{\substack{\mathbf{k},\mathbf{l}\\ \mathcal{K}(\mathbf{k}+\mathbf{l})=r}} q_{r,m,\mathbf{k},\mathbf{l}}I(\mathbf{k},\mathbf{l},\sigma)\\ &\quad+O\Big(\frac{(\log\log G(T))^{J+(1/2)}}{(\log G(T))^{5/2}}\Big),\end{aligned}$$

where

$$I(\mathbf{k},\mathbf{l},\sigma) := \int_{\mathbb{R}^{2J}} \log\left|\sum_{j=1}^{J} b_j e^{(u_j+iv_j)\sqrt{\log G(T)}}\right| e^{-\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+v_j{}^2)}\mathbf{u}^{\mathbf{k}}\mathbf{v}^{\mathbf{l}}d\mathbf{u}d\mathbf{v}.$$

The above estimation also holds for $\mathcal{M}(\sigma_1)$ and $\mathcal{M}(\sigma_2)$. Therefore, we deduce that

$$\begin{aligned}
&\mathcal{M}(\sigma)-\mathcal{M}(\sigma_i)\\
&=\frac{1}{\pi^J\prod_{j=1}^{J}\xi_j}(I(0,0,\sigma)-I(0,0,\sigma_i))\\
&\quad+\sum_{m=2}^{5}\sum_{r=0}^{m}\frac{1}{(\log G(T))^{m/2}}\sum_{\substack{\mathbf{k},\mathbf{l}\\ \mathcal{K}(\mathbf{k}+\mathbf{l})=r}} q_{r,m,\mathbf{k},\mathbf{l}}(I(\mathbf{k},\mathbf{l},\sigma)-I(\mathbf{k},\mathbf{l},\sigma_i))\\
&\quad+O\Big(\frac{(\log\log G(T))^{J+(1/2)}}{(\log G(T))^{5/2}}\Big).
\end{aligned} \tag{7.28}$$

This integral $I(\mathbf{k},\mathbf{l},\sigma)$ was estimated in [15] when $\mathbf{k},\mathbf{l}$ are fixed and $G(T)$ is a power of $\log T$. Let $\mathcal{R}_n := \{\mathbf{u}\in\mathbb{R}^J : u_n = \max\{u_1,\dots,u_J\}\}$, then $I(\mathbf{k},\mathbf{l},\sigma)$ equals

$$\begin{aligned}
&\sum_{n=1}^{J}\int_{\mathbb{R}^J}\int_{\mathcal{R}_n}\log\left|\sum_{j=1}^{J} b_j e^{(u_j+iv_j)\sqrt{\log G(T)}}\right| e^{-\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+v_j{}^2)}\mathbf{u}^{\mathbf{k}}\mathbf{v}^{\mathbf{l}}d\mathbf{u}d\mathbf{v}\\
&=\sum_{n=1}^{J}\int_{\mathbb{R}^J}\int_{\mathcal{R}_n}\log\left|b_n e^{(u_n+iv_n)\sqrt{\log G(T)}}\right| e^{-\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+v_j{}^2)}\mathbf{u}^{\mathbf{k}}\mathbf{v}^{\mathbf{l}}d\mathbf{u}d\mathbf{v}+\sum_{n=1}^{J}\mathcal{E}_n(\mathbf{k},\mathbf{l},\sigma),
\end{aligned}$$

where $\mathcal{E}_n(\mathbf{k},\mathbf{l},\sigma)$ is defined by

$$\int_{\mathbb{R}^J}\int_{\mathcal{R}_n}\log\left|1+\sum_{j\neq n}\frac{b_j}{b_n}e^{((u_j-u_n)+i(v_j-v_n))\sqrt{\log G(T)}}\right| e^{-\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+v_j{}^2)}\mathbf{u}^{\mathbf{k}}\mathbf{v}^{\mathbf{l}}d\mathbf{u}d\mathbf{v}. \tag{7.29}$$

Moreover, define

$$\begin{aligned}
d_{\mathbf{l}} &:= \int_{\mathbb{R}^J} e^{-\sum_{j=1}^{J} v_j{}^2/\xi_j}\mathbf{v}^{\mathbf{l}}d\mathbf{v},\\
D_1(\mathbf{k},\mathbf{l}) &:= d_{\mathbf{l}}\sum_{n=1}^{J}\int_{\mathcal{R}_n} e^{-\sum_{j=1}^{J} u_j^2/\xi_j}u_n\mathbf{u}^{\mathbf{k}}d\mathbf{u},\\
D_2(\mathbf{k},\mathbf{l}) &:= d_{\mathbf{l}}\sum_{n=1}^{J}\log|b_n|\int_{\mathcal{R}_n} e^{-\sum_{j=1}^{J} u_j^2/\xi_j}\mathbf{u}^{\mathbf{k}}d\mathbf{u},
\end{aligned}$$

then we find that

$$I(\mathbf{k},\mathbf{l},\sigma) = \sqrt{\log G(T)}\cdot D_1(\mathbf{k},\mathbf{l}) + D_2(\mathbf{k},\mathbf{l}) + \sum_{n=1}^{J}\mathcal{E}_n(\mathbf{k},\mathbf{l},\sigma). \tag{7.30}$$

Note that

$$d_{\mathbf{l}} = \begin{cases} 0, & \text{if } \ell_j \text{ is odd for some } j, \\ \prod_{j=1}^{J} \left(\xi_j^{(\ell_j+1)/2}\Gamma((\ell_j+1)/2)\right), & \text{if } \ell_j \text{ is even for all } j. \end{cases}$$

By changing $(\log T)^\theta$ to $G(T)$ in the proof of [15, Proposition 2.4], it follows that

$$\mathcal{E}_n(\mathbf{k}, \mathbf{l}, \sigma) = O\Big(\frac{1}{(\log G(T))^{1/4}}\Big)$$

if $G(T) \gg 1$, but this bound is not sufficient for our purpose. Instead, by (7.30), we estimate the difference

$$\begin{aligned} &I(\mathbf{k}, \mathbf{l}, \sigma) - I(\mathbf{k}, \mathbf{l}, \sigma_i) \\ &\quad=(\sqrt{\log G(T)} - \sqrt{\log G_i(T)})D_1(\mathbf{k}, \mathbf{l}) + \sum_{n=1}^{J}(\mathcal{E}_n(\mathbf{k}, \mathbf{l}, \sigma) - \mathcal{E}_n(\mathbf{k}, \mathbf{l}, \sigma_i)) \\ &\quad=\frac{(-1)^i D_1(\mathbf{k}, \mathbf{l})}{2(\log G(T))^{3/2}} + \sum_{n=1}^{J}(\mathcal{E}_n(\mathbf{k}, \mathbf{l}, \sigma) - \mathcal{E}_n(\mathbf{k}, \mathbf{l}, \sigma_i)) + O\Big(\frac{1}{(\log G(T))^{5/2}}\Big). \end{aligned}$$

By symmetry, we only estimate $\mathcal{E}_1(\mathbf{k}, \mathbf{l}, \sigma) - \mathcal{E}_1(\mathbf{k}, \mathbf{l}, \sigma_i)$. By (7.29) and a simple substitution, we see that

$$\begin{aligned} &\mathcal{E}_1(\mathbf{k}, \mathbf{l}, \sigma) - \mathcal{E}_1(\mathbf{k}, \mathbf{l}, \sigma_i) \\ &=\int_{\mathbb{R}^J}\int_{\mathcal{R}_1} \log\Big|1 + \sum_{j\neq 1}\frac{b_j}{b_1}e^{((u_j-u_1)+i(v_j-v_1))\sqrt{\log G(T)}}\Big| e^{-\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+{v_j}^2)}\mathbf{u}^{\mathbf{k}}\mathbf{v}^{\mathbf{l}}d\mathbf{u}d\mathbf{v} \\ &\quad-\int_{\mathbb{R}^J}\int_{\mathcal{R}_1} \log\Big|1 + \sum_{j\neq 1}\frac{b_j}{b_1}e^{((u_j-u_1)+i(v_j-v_1))\sqrt{\log G_i(T)}}\Big| e^{-\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+{v_j}^2)}\mathbf{u}^{\mathbf{k}}\mathbf{v}^{\mathbf{l}}d\mathbf{u}d\mathbf{v} \\ &=\int_{\mathbb{R}^J}\int_{\mathcal{R}_1} \log\Big|1 + \sum_{j\neq 1}\frac{b_j}{b_1}e^{((u_j-u_1)+i(v_j-v_1))\sqrt{\log G(T)}}\Big| \\ &\quad\times\Big(e^{-\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+{v_j}^2)} - e^{-\frac{\log G(T)}{\log G_i(T)}\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+{v_j}^2)}\Big(\frac{\log G(T)}{\log G_i(T)}\Big)^{J+\mathcal{K}(\mathbf{k}+\mathbf{l})/2}\Big)\mathbf{u}^{\mathbf{k}}\mathbf{v}^{\mathbf{l}}d\mathbf{u}d\mathbf{v}. \end{aligned}$$

Since $\frac{\log G(T)}{\log G_i(T)} = 1 + O(\frac{1}{(\log G(T))^2})$, by adapting the proof of [15, Lemma 2.5], we find that the above is

$$\begin{aligned} &\ll\frac{1}{(\log G(T))^2}\int_{\mathbb{R}^J}\int_{\mathcal{R}_1}\Big|\log\Big|1 + \sum_{j\neq 1}\frac{b_j}{b_1}e^{((u_j-u_1)+i(v_j-v_1))\sqrt{\log G(T)}}\Big|\Big| \\ &\qquad\times e^{-(1+O(\frac{1}{(\log G(T))^2}))\sum_{j=1}^{J}\xi_j^{-1}(u_j^2+{v_j}^2)}\Big(\sum_{j=1}^{J}(u_j^2+{v_j}^2)+1\Big)\mathbf{u}^{\mathbf{k}}\mathbf{v}^{\mathbf{l}}d\mathbf{u}d\mathbf{v} \\ &\ll\frac{1}{(\log G(T))^{9/4}}. \end{aligned}$$

Thus, we deduce that

$$I(\mathbf{k},\mathbf{l},\sigma)-I(\mathbf{k},\mathbf{l},\sigma_i)=\frac{(-1)^iD_1(\mathbf{k},\mathbf{l})}{2(\log G(T))^{3/2}}+O\Big(\frac{1}{(\log G(T))^{9/4}}\Big).$$

Inserting this estimate in (7.28) gives

$$\mathcal{M}(\sigma)-\mathcal{M}(\sigma_i)=\frac{(-1)^iD_1(0,0)}{2\pi^J(\prod_{j=1}^J\xi_j)(\log G(T))^{3/2}}+O\Big(\frac{1}{(\log G(T))^{9/4}}\Big),$$

where

$$D_1(0,0)=\pi^{J/2}\prod_{j=1}^J\sqrt{\xi_j}\sum_{n=1}^J\int_{\mathcal{R}_n}e^{-\sum_{j=1}^J u_j^2/\xi_j}u_n d\mathbf{u}$$

by $\Gamma(1/2)=\sqrt{\pi}$. This completes the proof.

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