INSIGHTS ON THE CESÀRO OPERATOR: SHIFT SEMIGROUPS AND INVARIANT SUBSPACES*

By

EVA A. GALLARDO-GUTIÉRREZ AND JONATHAN R. PARTINGTON

Abstract. A closed subspace is invariant under the Cesàro operator $\mathcal{C}$ on the classical Hardy space $H^2(\mathbb{D})$ if and only if its orthogonal complement is invariant under the C_0-semigroup of composition operators induced by the affine maps $\varphi_t(z) = e^{-t}z + 1 - e^{-t}$ for $t \geq 0$ and $z \in \mathbb{D}$. The corresponding result also holds in the Hardy spaces $H^p(\mathbb{D})$ for $1 < p < \infty$. Moreover, in the Hilbert space setting, by linking the invariant subspaces of $\mathcal{C}$ to the lattice of the closed invariant subspaces of the standard right-shift semigroup acting on a particular weighted L^2-space on the line, we exhibit a large class of non-trivial closed invariant subspaces and provide a complete characterization of the finite codimensional ones, establishing, in particular, the limits of such an approach towards describing the lattice of all invariant subspaces of $\mathcal{C}$. Finally, we present a functional calculus argument which allows us to extend a recent result by Mashreghi, Ptak and Ross regarding the square root of $\mathcal{C}$ and discuss its invariant subspaces.

1 Introduction and preliminaries

Despite the fact that one of the most classical transformations of sequences is the Cesàro operator $\mathcal{C}$, there are still many questions about it unsettled. Recall that $\mathcal{C}$ takes a complex sequence $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ to that with n-th entry:

$$(\mathcal{C}\,\mathbf{a})_n = \frac{1}{n+1}\sum_{k=0}^{n} a_k \quad (n \geq 0).$$

Upon identifying sequences with Taylor coefficients of power series, $\mathcal{C}$ acts formally on $f(z) = \sum_{k=0}^{\infty} a_k z^k$ as

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right)z^n. \tag{1.1}$$

*Both authors are partially supported by Plan Nacional I+D grant no. PID2019-105979GB-I00, Spain. The first author is also supported by the Spanish Ministry of Science and Innovation, through the "Severo Ochoa Programme for Centres of Excellence in R&D" (CEX2019-000904-S) and from the Spanish National Research Council, through the "Ayuda extraordinaria a Centros de Excelencia Severo Ochoa" (20205CEX001).

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 152 (2024)
DOI 10.1007/s11854-023-0305-0

Indeed, if f is a holomorphic function on the unit disc $\mathbb{D}$ so is $\mathcal{C}(f)$ and moreover, $\mathcal{C}$ is an isomorphism of the Fréchet space $\mathcal{H}(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$ endowed with the topology of uniform convergence on compacta.

Nevertheless, this is no longer true when $\mathcal{C}$ is restricted to the classical Hardy spaces $H^p(\mathbb{D})$, $1 \leq p < \infty$. A classical result of Hardy concerning trigonometric series along with M. Riesz's theorem yields that $\mathcal{C}$ is bounded on $H^p(\mathbb{D})$ for $1 < p < \infty$. Likewise, Siskakis proved that $\mathcal{C}$ is bounded on $H^1(\mathbb{D})$ (providing even an alternative proof of the boundedness on $H^p(\mathbb{D})$ for $1 < p < \infty$; see [34], [35]). However 0 belongs to the spectrum of $\mathcal{C}$ in $H^p(\mathbb{D})$ and hence, $\mathcal{C}$ is not an isomorphism [34].

Note that (1.1) can be written as

$$\mathcal{C}(f)(z) = \{ \begin{cases} \frac{1}{z}\int_0^z \frac{f(\xi)}{1-\xi}\, d\xi, & z \in \mathbb{D} \setminus \{0\}, \\ f(0), & z = 0, \end{cases}$$

for $z \in \mathbb{D}$. There is an extensive literature on the Cesàro operator, and more general, on integral operators, acting on a large variety of spaces of analytic functions regarding its boundedness, compactness or spectral picture (see [1] or [3], for instance).

If we restrict ourselves to the Hilbert space case $H^2(\mathbb{D})$, Kriete and Trutt proved the striking result that the Cesàro operator is **subnormal**, namely, $\mathcal{C}$ on $H^2(\mathbb{D})$ has a normal extension. More precisely, if I denotes the identity operator on $H^2(\mathbb{D})$, they proved that $I - \mathcal{C}$ is unitarily equivalent to the operator of multiplication by the identity function acting on the closure of analytic polynomials on the space $L^2(\mu, \mathbb{D})$ for a particular measure μ (see [24]). An alternative proof of the Kriete and Trutt theorem, based on the connection between $\mathcal{C}$ and composition operator semigroups, was later established by Cowen [9].

For $H^p(\mathbb{D})$, $1 < p < \infty$, Miller, Miller and Smith [29] showed that $\mathcal{C}$ is **subdecomposable**, namely, it has a decomposable extension (the $H^1(\mathbb{D})$ case was proved by Persson [32] ten years later). Decomposable operators were introduced by Foiaş [16] in the sixties as a generalization of spectral operators in the sense of Dunford, and many spectral operators in Hilbert spaces as unitary operators, self-adjoint operators or more generally, normal operators are decomposable (see the monograph [26] for more on the subject).

Normal operators on Hilbert spaces or more generally, decomposable operators on Banach spaces have a rich lattice of non-trivial closed invariant subspaces with a significant description of them. But, very little is known about this description even for concrete examples of subnormal operators as the Cesàro operator, and this will be the main motivation of the present manuscript.

In this context, we discuss invariant subspaces of the Cesàro operator $\mathcal{C}$ on the Hardy space $H^2(\mathbb{D})$. Broadly speaking, we prove a **Beurling–Lax Theorem** for the Cesàro operator and provide a complete characterization of the finite codimensional invariant subspaces of $\mathcal{C}$. The composition semigroup method has turned out to be a powerful tool to study the Cesàro operator and we will make use of such technique in Section 2 to link the invariant subspaces of $\mathcal{C}$ to those of the right-shift semigroup $\{S_\tau\}_{\tau\geq 0}$ acting on a particular weighted $L^2(\mathbb{R}, w(y)dy)$. In particular, we will establish the limits of our approach towards describing completely the lattice of the invariant subspaces of $\mathcal{C}$.

In Section 3, we discuss Phillips functional calculus (as in Haase's book [22]) which will allow us, in particular, to generalize the recent work by Mashreghi, Ptak and Ross [28] regarding the square roots of $\mathcal{C}$. In particular, we will discuss their invariant subspaces.

In order to close this introductory section we collect some preliminaries for the sake of completeness.

1.1 Semigroups of composition operators.

The study of semigroups of composition operators on various function spaces of analytic functions has its origins in the work of Berkson and Porta [5], where they characterize their generators on $H^p(\mathbb{D})$, proving, indeed, that these semigroups are always strongly continuous.

Recall that a one-parameter family $\Phi = \{\varphi_t\}_{t\geq 0}$ of analytic self-maps of $\mathbb{D}$ is called a **holomorphic flow** (or **holomorphic semiflow** by some authors) if it is a continuous family that has a semigroup property with respect to composition, namely

(1) $\varphi_0(z) = z$, for $z \in \mathbb{D}$;
(2) $\varphi_{t+s}(z) = \varphi_t \circ \varphi_s(z)$, for $t, s \geq 0$, and $z \in \mathbb{D}$;
(3) for any $s \geq 0$ and any $z \in \mathbb{D}$, $\lim_{t\to s} \varphi_t(z) = \varphi_s(z)$.

The holomorphic flow Φ is trivial if $\varphi_t(z) = z$ for all $t \geq 0$. Otherwise, we say that Φ is non-trivial. We refer to the recent monograph [6] for more on the subject.

Associated to the holomorphic flow $\Phi = \{\varphi_t\}_{t\geq 0}$ is the family of composition operators $\{C_{\varphi_t}\}_{t\geq 0}$, defined on the space of analytic functions on $\mathbb{D}$ by

$$C_{\varphi_t} f = f \circ \varphi_t.$$

Clearly, $\{C_{\varphi_t}\}_{t\geq 0}$ has the semigroup property:

(1) $C_{\varphi_0} = I$;
(2) $C_{\varphi_t} C_{\varphi_s} = C_{\varphi_{t+s}}$ for all $t, s \geq 0$.

Moreover, recall that if an operator semigroup $\{T_t\}_{t\geq 0}$ acts on a Banach space X, then it is called a **strongly continuous** or C_0**-semigroup**, if it satisfies

$$\lim_{t\to 0^+} T_t f = f$$

for any $f \in X$. Given a C_0-semigroup $\{T_t\}_{t\geq 0}$ on a Banach space X, recall that its generator is the closed and densely defined linear operator A defined by

$$Af = \lim_{t\to 0^+} \frac{T_t f - f}{t}$$

with domain $\mathcal{D}(A) = \{f \in X : \lim_{t\to 0^+} \frac{T_t f - f}{t} \text{ exists}\}$.

2 The lattice of the invariant subspaces of the Cesàro operator

The aim of this section is identifying the lattice of the invariant subspaces of the Cesàro operator $\mathcal{C}$ acting on the Hardy space $H^2(\mathbb{D})$. In particular, we will characterize the finite codimensional invariant subspaces of $\mathcal{C}$.

Our first result resembles a **Beurling–Lax Theorem** for the Cesàro operator.

Theorem 2.1. *Let $\Phi = \{\varphi_t\}_{t\geq 0}$ be the holomorphic flow given by*

$$\varphi_t(z) = e^{-t}z + 1 - e^{-t} \quad (z \in \mathbb{D}). \tag{2.1}$$

A closed subspace M in $H^2(\mathbb{D})$ is invariant under the Cesàro operator if and only if its orthogonal complement $M^\perp$ is invariant under the semigroup of composition operators induced by Φ, namely, $\{C_{\varphi_t}\}_{t\geq 0}$.

Before proceeding with the proof, note that each φ_t in (2.1) is an affine map which is a hyperbolic non-automorphism of the unit disc inducing a bounded composition operator C_{φ_t} on $H^2(\mathbb{D})$ with norm

$$\|C_{\varphi_t}\|_2 = e^{\frac{t}{2}} \tag{2.2}$$

(see, for instance, [10, Theorem 9.4]).

Likewise, the generator of the C_0-semigroup $\{C_{\varphi_t}\}_{t\geq 0}$ is given by

$$Af(z) = (1-z)f'(z) \quad (z \in \mathbb{D}),$$

(see the pioneering work by Berkson and Porta [5], for instance).

Proof. First, let us show that the cogenerator of the C_0-semigroup $\{C_{\varphi_t}\}_{t\geq 0}$ given by

$$V = (A+I)(A-I)^{-1}$$

is a well-defined bounded operator. For such a task, we will prove that $1 \in \rho(A)$, the resolvent of A, or equivalently,

$$A - I : \mathcal{D}(A) \subset H^2(\mathbb{D}) \to H^2(\mathbb{D})$$

is bijective.

Note that $A-I$ is an injective operator in $\mathcal{D}(A) \subset H^2(\mathbb{D})$. Indeed, if $(A-I)f(z)=0$ then $(1-z)f'(z) - f(z) = 0$, so $f(z) = C/(1-z)$ for some complex constant $C \in \mathbb{C}$. But for $C \neq 0$ we have that $f \notin H^2(\mathbb{D})$.

We claim that $A - I$ is also surjective. Given $g \in H^2(\mathbb{D})$, in order to find $f \in H^2(\mathbb{D})$ such that

$$(A - I)f(z) = g(z) \quad (z \in \mathbb{D})$$

or

$$(1 - z)f'(z) - f(z) = ((1 - z)f(z))' = g(z) \quad (z \in \mathbb{D}),$$

let

$$f(z) = \frac{1}{1 - z} \int_1^z g(u)\, du = \frac{1}{z - 1} \int_z^1 g(u)\, du \tag{2.3}$$

for $z \in \mathbb{D}$.

Note that the adjoint of the Cesàro operator $\mathcal{C}^*$ has the following matrix with respect to the canonical orthonormal basis of $H^2(\mathbb{D})$:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Writing

$$Tg(z) = \frac{1}{z - 1} \int_z^1 g(u)\, du,$$

for $g \in H^2(\mathbb{D})$, observe that

$$Tz^n = \frac{1}{z - 1} \left(\frac{1 - z^{n+1}}{n + 1} \right) = -\frac{1 + z + \cdots + z^n}{n + 1}.$$

Accordingly, T is a well-defined operator in $H^2(\mathbb{D})$ (as it is -$\mathcal{C}^*$). This in particular implies that the function f in (2.3) belongs to $H^2(\mathbb{D})$ and hence $A - I$ is surjective.

Accordingly, the cogenerator V of the C_0-semigroup $\{C_{\varphi_t}\}_{t \geq 0}$ is a well-defined bounded operator on $H^2(\mathbb{D})$.

Now, having in mind the norm estimate (2.2), we observe that the C_0-semigroup $\{e^{-t}C_{\varphi_{2t}}\}_{t\geq 0}$ is contractive on $H^2(\mathbb{D})$ and its generator is $2A - I$. Since

$$V - I = 2(A - I)^{-1} = 4((2A - I) - I)^{-1} = -2\mathfrak{C}^*$$

the invariant subspaces of the cogenerator are simply the common invariant subspaces of the semigroup (see [17, Chap. 10, Theorem 10.9]) and the statement of the theorem follows. □

First, let us remark that a similar argument in the context of C_0-semigroups of analytic 2-isometries was also used in [19]. Likewise, recalling that the Hardy space $H^p(\mathbb{D})$, $1 \leq p < \infty$, consists of holomorphic functions f on $\mathbb{D}$ for which the norm

$$\|f\|_p = \left(\sup_{0\leq r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

is finite, we note that the previous proof also works in $H^p(\mathbb{D})$-spaces $(1 < p < \infty)$ with the natural identification of the dual space $H^p(\mathbb{D})^* \cong H^{p'}(\mathbb{D})$ where p' is the conjugate exponent: $\frac{1}{p} + \frac{1}{p'} = 1$. In this case, the bounded composition operator C_{φ_t} on $H^p(\mathbb{D})$ has norm

$$\|C_{\varphi_t}\|_p = e^{\frac{t}{p}} \tag{2.4}$$

for $1 \leq p < \infty$ (see [30, Exercise 3.12.5, pp. 56–57] and [23, Part Two, Ch. 1, Section 5 1.B pp. 165–166], for instance). Accordingly, the C_0-semigroup $\{e^{-(p'-1)t}C_{\varphi_{p't}}\}_{t\geq 0}$ is also contractive on $H^p(\mathbb{D})$ with generator $p'A - (p'-1)I$ and cogenerator

$$I + 2(p'A - (p'-1)I - I)^{-1} = I + 2(p'(A - I))^{-1}.$$

Therefore, a closed subspace M in $H^p(\mathbb{D})$ for $1 < p < \infty$ is invariant under the Cesàro operator if and only if its annihilator $M^\perp$ in $H^{p'}(\mathbb{D})$ is invariant under the C_0-semigroup $\{C_{\varphi_t}\}_{t\geq 0}$.

In this regard, it is worth noting that $(1-z)^{-1} \notin H^p(\mathbb{D})$ for any $1 \leq p \leq \infty$ since $(1 - e^{i\theta})^{-1} \notin L^p(\mathbb{T})$.

Remark 2.2. By Dunford and Schwartz [15, Thm. 11, p. 622], the resolvent can be expressed in terms of the Laplace transform of the semigroup; that is,

$$(A - I)^{-1}f(z) = \int_0^\infty e^{-t}C_{\varphi_t}f(z)\,dt \quad (z \in \mathbb{D}).$$

So

$$\mathfrak{C}^*f(z) = -\int_0^\infty e^{-t}f(e^{-t}z + 1 - e^{-t})\,dt \quad (z \in \mathbb{D}).$$

Indeed, a consequence of the previous formula is the following:

Corollary 2.3. *A closed subspace M in $H^2(\mathbb{D})$ is invariant under the Cesàro operator if and only if it is invariant under the semigroup* $\{C^*_{\varphi_t}\}_{t\geq 0}$.

Alternatively, one may use that the adjoint of the generator is the generator of the adjoint semigroup in the context of Hilbert spaces (see [17, Chap. 10], for instance).

Finally, note that the adjoint $C^*_{\varphi_t}$ in $H^2(\mathbb{D})$ in Corollary 2.3 may be explicitly computed as a weighted composition operator (see [10, Theorem 9.2]). Indeed, expressing $\varphi_t(z) = e^{-t}z + 1 - e^{-t}$ in its normal form, namely

$$\varphi_t(z) = (a_t z + b_t)/(0z + a_t^{-1})$$

where

$$a_t = e^{-t/2}, \quad b_t = \frac{1 - e^{-t}}{e^{-t/2}}$$

we deduce that $C^*_{\varphi_t} = T_{g_t} C_{\sigma_t} T^*_{h_t}$ where

$$g_t(z) = \frac{1}{-b_t z + a_t^{-1}} = \frac{e^{-t/2}}{1 - (1 - e^{-t})z} \quad (z \in \mathbb{D}),$$

$$\sigma_t(z) = \frac{a_t z}{-b_t z + a_t^{-1}} = \frac{e^{-t}z}{1 - (1 - e^{-t})z} \quad (z \in \mathbb{D}),$$

and

$$h_t(z) = a_t^{-1} = e^{t/2};$$

and T_{g_t} and T_{h_t} denote the analytic Toeplitz operators acting on $H^2(\mathbb{D})$ induced by the symbols g_t and h_t respectively.

Accordingly,

$$C^*_{\varphi_t} f(z) = \frac{1}{1 - (1 - e^{-t})z} f\left(\frac{e^{-t}z}{1 - (1 - e^{-t})z}\right) \quad (z \in \mathbb{D})$$

for $f \in H^2(\mathbb{D})$ and every $t \geq 0$.

2.1 Shift semigroups. In order to provide a characterization of the finite codimensional invariant subspaces of $\mathcal{C}$, we will make use of a semigroup of operators acting on the Hardy space of the right half-plane $\mathbb{C}_+$. Recall that the Hardy space $H^2(\mathbb{C}_+)$ consists of the functions F analytic on $\mathbb{C}_+$ with finite norm

$$\|F\|_{H^2(\mathbb{C}_+)} = \left\{ \sup_{0<x<\infty} \int_{-\infty}^{\infty} |F(x+iy)|^2 \, dy \right\}^{1/2}.$$

The classical Paley–Wiener Theorem (see [33], for instance) states that $H^2(\mathbb{C}_+)$ is isomorphic under the Laplace transform to $L^2(\mathbb{R}_+)$, the space of measurable functions square-integrable over $(0, \infty)$. More precisely, to each function $F \in H^2(\mathbb{C}_+)$ there corresponds a function $f \in L^2(\mathbb{R}_+)$ such that

$$F(s) = (\mathcal{L}f)(s) := \int_0^\infty f(x)e^{-sx}\, dx \quad (s \in \mathbb{C}_+),$$

and

$$\|F\|^2_{H^2(\mathbb{C}_+)} = 2\pi \int_0^\infty |f(x)|^2\, dx.$$

A first observation already stated in [8, Lemma 4.2] is that each

$$\varphi_t(z) = e^{-t}z + 1 - e^{-t}$$

for $z \in \mathbb{D}$ and $t > 0$ induces a composition operator in $H^2(\mathbb{D})$ which is similar under an isomorphism $\mathcal{U}$ (indeed unitarily equivalent up to a constant) to $e^t C_{\phi_t}$ in $H^2(\mathbb{C}_+)$, where

$$\phi_t(s) = e^t s + (e^t - 1) \quad (s \in \mathbb{C}_+),$$

namely,

$$\mathcal{U}C_{\varphi_t}\mathcal{U}^{-1} = e^t C_{\phi_t} \quad (t \geq 0). \tag{2.5}$$

Since we are interested in studying invariant subspaces, either for the entire semigroup or individual elements, we may disregard factors of the form $e^{\lambda t}$ for a fixed $\lambda \in \mathbb{R}$.

By means of the inverse Laplace transform we are led to consider the semigroup on $L^2(\mathbb{R}_+)$

$$V_t g(x) = e^{-t}e^{-(1-e^{-t})x}g(e^{-t}x) \quad (x > 0,\ t \geq 0). \tag{2.6}$$

Now, proceeding as in [18], we may find a further equivalence with an operator on $L^2(\mathbb{R})$ using the unitary mapping $T : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+)$ defined by

$$Th(x) = x^{-1/2}h(\log x) \quad (x > 0)$$

and

$$T^{-1}g(y) = e^{y/2}g(e^y) \quad (y \in \mathbb{R}).$$

Accordingly,

$$T^{-1}V_t Th(y) = e^{-t/2}e^{-(1-e^{-t})e^y}h(y - t) \quad (y \in \mathbb{R}) \tag{2.7}$$

for $t \geq 0$.

Denoting by $\{S_t : t \geq 0\}$ the right-shift semigroup on $L^2(\mathbb{R})$:

$$S_t f(y) = f(y-t) \quad (y \in \mathbb{R})$$

and recalling that if w denotes a positive measurable function in $\mathbb{R}$ the space $L^2(\mathbb{R}, w(y)dy)$ consists of measurable functions in $\mathbb{R}$ square-integrable respect to the measure $w(y)dy$, a key observation is the following

Proposition 2.4. *The semigroup $\{\sigma_t : t \geq 0\}$ in $L^2(\mathbb{R})$ given by*

$$\sigma_t h(y) = e^{-(1-e^{-t})e^y} h(y-t) \quad (y \in \mathbb{R}) \tag{2.8}$$

for $h \in L^2(\mathbb{R})$ is unitarily equivalent to the right-shift semigroup $\{S_t : t \geq 0\}$ acting on the weighted Lebesgue space $L^2(\mathbb{R}, e^{-2(e^y-1)}\, dy)$.

Proof. Let us denote the weight $w(y) = e^{-2(e^y-1)}$ for $y \in \mathbb{R}$ and consider the unitary mapping $W : L^2(\mathbb{R}) \to L^2(\mathbb{R}, w(y)\, dy)$ given by

$$Wh(y) = h(y)/\sqrt{w(y)} \quad (y \in \mathbb{R}),$$

for $h \in L^2(\mathbb{R})$. A computation shows that for any function $f \in L^2(\mathbb{R}, w(y)\, dy)$ and $t > 0$

$$\begin{aligned} W\sigma_t W^{-1} f(y) &= W\sigma_t f(y) e^{-(e^y-1)} \\ &= We^{-(1-e^{-t})e^y} f(y-t) e^{-(e^{y-t}-1)} \\ &= e^{e^y-1} e^{-(1-e^{-t})e^y} f(y-t) e^{-(e^{y-t}-1)} \\ &= f(y-t), \end{aligned}$$

for $y \in \mathbb{R}$. This yields the statement of the proposition. □

Figure 1 shows a plot of the weight function described in Proposition 2.4.

As a by-product of equations (2.5), (2.6), (2.7), Proposition 2.4 and Theorem 2.1, if we denote by $\mathfrak{F}$ the unitary isomorphism $\mathfrak{F} = WT^{-1}\mathcal{L}^{-1}\mathfrak{U}$ from $H^2(\mathbb{D})$ onto $L^2(\mathbb{R}, e^{-2(e^y-1)}\, dy)$, the following result regarding the lattice of invariant subspaces of the Cesàro operator holds:

Theorem 2.5. *A closed subspace M in $H^2(\mathbb{D})$ is invariant under the Cesàro operator if and only if $\mathfrak{F}M^\perp$ in $L^2(\mathbb{R}, e^{-2(e^y-1)}\, dy)$ is invariant under the right-shift semigroup $\{S_t : t \geq 0\}$.*

Accordingly, characterizing the lattice of invariant subspaces of the Cesàro operator in the Hardy space reduces to characterizing the lattice of the right-shift semigroup in $L^2(\mathbb{R}, e^{-2(e^y-1)}\, dy)$.

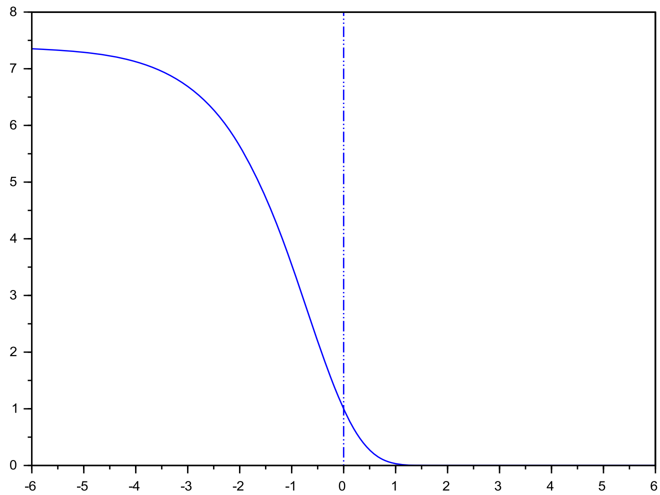

Figure 1. The weight $e^{-2(e^y-1)}$ as a function of y.

Though the lattice of the invariant subspaces of the right-shift semigroup acting on weighted Lebesgue spaces is only characterized for a very restricted subclass of weights (for instance, the Beurling–Lax Theorem provides a characterization in $L^2(\mathbb{R}_+)$, where the weight is the characteristic function $\chi_{(0,+\infty)}$ of $(0,+\infty)$), the question if such a lattice contains non-standard invariant subspaces has been extensively studied (see [11, 12, 13, 14], [20, 21], [27] or [31], for instance).

Recall that given $a \in \mathbb{R}\cup\{-\infty\}\cup\{\infty\}$, the "**standard invariant subspaces**" of $\{S_t : t \geq 0\}$ are given by

$$L^2((a,\infty), w(y)\,dy) = \{f \in L^2(\mathbb{R}, w(y)\,dy) : f(y) = 0 \text{ for a.e. } y \leq a\}.$$

In [11, Equation (8)], Domar proved that if the weight satisfies

$$\underline{\lim}_{y\to-\infty} \frac{\log w(y)}{y} > -\infty$$

then the lattice of invariant subspaces of $\{S_t : t \geq 0\}$ in $L^2(\mathbb{R}, w(y)\,dy)$ contains non-standard invariant subspaces.

A word about notation: Domar denotes by $L^2(\mathbb{R}, w(y)\,dy)$ the space of measurable functions f in $\mathbb{R}$ such that $f\,w \in L^2(\mathbb{R})$. Note that this does not affect the previous equation since it is enough to consider the positive function $w^{1/2}$.

In our case, $w(y) = e^{-2(e^y-1)}$ for $y \in \mathbb{R}$ and consequently, $\{S_t : t \geq 0\}$ has non-standard invariant subspaces in $L^2(\mathbb{R}, e^{-2(e^y-1)}\,dy)$. Indeed, it is possible to exhibit many non-standard invariant subspaces in this case. In order to show them recall that, by means of the unitary equivalence $\mathcal{L} : L^2(\mathbb{R}_+, \sqrt{2\pi}\,dt) \to H^2(\mathbb{C}_+)$, the Beurling–Lax Theorem asserts that a closed subspace $\mathcal{M}$ of $L^2(\mathbb{R}_+)$ is invariant under every truncated right-shift to $L^2(\mathbb{R}_+)$

$$S_{\mathbb{R}_+,\,\tau} f(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \tau, \\ f(t-\tau) & \text{if } t > \tau, \end{cases} \qquad \tau \geq 0,$$

if and only if there exists an inner function $\Theta \in H^\infty(\mathbb{C}_+)$ such that $\mathcal{L}\mathcal{M} = \Theta H^2(\mathbb{C}_+)$ (see [30, Cor. 6.5.5(2), p. 149], for instance). Here, recall that an inner function Θ is an analytic function in $\mathbb{C}_+$ with $|\Theta(z)| \leq 1$ for $z \in \mathbb{C}_+$, such that the non-tangential limits exist and are of modulus 1 almost everywhere on the imaginary axis.

Example 2.6. Let $T \in \mathbb{R}$ be fixed and write

$$L^2(\mathbb{R}, w(y)\,dy) = L^2((-\infty, T), w(y)\,dy) \oplus L^2((T, \infty), w(y)\,dy),$$

the orthogonal direct sum of closed subspaces. Note that

(1) $\exp(-2(e^T - 1)) \leq w(y) \leq e^2$ on $(-\infty, T)$, so $L^2((-\infty, T), w(y)\,dy)$ is naturally isomorphic to $L^2(-\infty, T)$.

(2) If M is a closed subspace of $L^2((-\infty, T), w(y)\,dy)$ invariant under all truncated right-shifts on $L^2((-\infty, T), w(y)\,dy)$, i.e.,

$$S_{(-\infty,T),\,\tau} f(y) = \begin{cases} 0 & \text{if } y-\tau > T, \\ f(y-\tau) & \text{if } y-\tau \leq T, \end{cases}$$

for $\tau \geq 0$, then $M \oplus L^2((T, \infty), w(y)\,dy)$ is a closed subspace of $L^2(\mathbb{R}, w(y)\,dy)$ invariant under all right shifts.

Now, the Beurling–Lax Theorem provides a large class of non-standard invariant subspaces M: take the "twisted" Laplace transform

$$\tilde{\mathcal{L}} f(s) = \int_{-T}^{\infty} e^{-su} f(-u)\,du \tag{2.9}$$

which gives an isomorphism from $L^2((-\infty, T), w(y)\,dy)$ onto $e^{sT} H^2(\mathbb{C}_+)$. Then any subspace of the form $\tilde{\mathcal{L}}^{-1} e^{sT} K_\Theta$ is invariant under all truncated right shifts $S_{(-\infty,T),\,\tau}$ where $\Theta \in H^\infty(\mathbb{C}_+)$ is inner and $K_\Theta = H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+)$ is the associated model space (these calculations are easiest to follow when $T = 0$, and the general case is a shifted version).

As an explicit example, let K_Θ be spanned by the reproducing kernel

$$s \mapsto 1/(s+\overline{\lambda})$$

for $\lambda \in \mathbb{C}_+$, so that M is the one-dimensional space of $L^2((-\infty, T), w(y)\,dy)$ spanned by $e^{\overline{\lambda}t}$; then $M \oplus L^2((T, \infty), w(y)\,dy)$ is a non-standard invariant subspace for all right shifts.

While a theorem of Aleman and Koreblum [2] asserts that the analytic Volterra operator is unicellular in H^p-spaces, as a consequence of the previous considerations we have a new deduction of the following known result:

Corollary 2.7. *The Cesàro operator $\mathcal{C}$ is not a unicellular operator in $H^2(\mathbb{D})$.*

In this regard, the feature that the Cesàro operator $\mathcal{C}$ is not a unicellular operator on $H^2(\mathbb{D})$ can be also deduced from [7], as the referee kindly pointed out to us. Indeed, it follows from the result that the point spectrum of $I - \mathcal{C}^*$ in $H^2(\mathbb{D})$ is $\mathbb{D}$ along with the fact that any operator on a Hilbert space which has at least two eigenvalues cannot be unicellular. Likewise, in [25, Corollary 6], the authors constructed two non-zero invariant subspaces of $\mathcal{C}$ whose intersection is zero space.

On the other hand, it is worth pointing out that classifying the invariant subspaces turns out to be completely different if one considers other semigroups studied in the context of Cesàro-like operators, as in the following remark:

Remark 2.8. In [4], the authors considered the composition operator group on $H^2(\mathbb{C}_+)$ corresponding to the flow on $\mathbb{C}_+$ given by

$$\phi_t(s) = e^{-t}s, \quad s \in \mathbb{C}_+,\ t \in \mathbb{R},$$

in a broader context of studying Cesàro-like operators.

Proceeding similarly as before, the transformed semigroup on $L^2(0, \infty)$ is given by

$$\tilde{V}_t g(x) = e^t g(e^t x) \quad (x > 0,\ t \in \mathbb{R} \text{ and } g \in L^2(0, \infty)),$$

which transferred to $L^2(\mathbb{R})$ is

$$T^{-1}\tilde{V}_t T h(y) = e^{t/2} h(y+t) \quad (y \in \mathbb{R},\ t \in \mathbb{R}),$$

for $h \in L^2(\mathbb{R})$.

The subspaces M invariant under the group $(\tau_t)_{t\in\mathbb{R}} = (T^{-1}\tilde{V}_t T)_{t\in\mathbb{R}}$ were essentially classified by Lax—the factors $e^{t/2}$ are irrelevant—and can be found, with a slightly different notation, in [30, Cor. 6.5.4, p. 149]. There are two types:

(1) 1-invariant subspaces, i.e., $\tau_t M \subset M$ for all $t < 0$ but not for all $t \in \mathbb{R}$. These have the form $M = \mathcal{F} q H^2(\Pi^+)$, where q is measurable with $|q| = 1$ almost everywhere and here Π^+ denotes the upper half-plane;

(2) 2-invariant subspaces, i.e., $\tau_t M \subset M$ for all $t \in \mathbb{R}$. These have the form $M = \mathcal{F} \chi_E L^2(\mathbb{R})$ for some measurable subset $E \subset \mathbb{R}$.

Here $\mathcal{F}$ denotes the Fourier transform but, alternatively, one can use the bilateral Laplace transform and express the subspaces in terms of $L^2(i\mathbb{R})$ and the space $H^2(\mathbb{C}_+)$ of the right half-plane.

Likewise, in this case the invariant subspaces of the form $\tilde{\mathcal{L}}^{-1} K_\Theta \oplus L^2(\mathbb{R}_+)$ are 1-invariant subspaces as described above, corresponding in $L^2(i\mathbb{R})$ to

$$\overline{\Theta} K_\Theta \oplus H^2(\mathbb{C}_+) = \overline{\Theta} H^2(\mathbb{C}_+).$$

Finally, as an application of Theorem 2.5, we present a characterization of the finite codimensional invariant subspaces of the Cesàro operator $\mathcal{C}$ in $H^2(\mathbb{D})$. Of particular relevance will be a theorem of Domar [12] which states that the lattice of the invariant subspaces of $\{S_\tau : \tau \geq 0\}$ consists of just the standard invariant subspaces in $L^2(\mathbb{R}_+, w(x)\,dx)$ whenever:

(1) w is a positive continuous function in $\mathbb{R}_+$ such that $\log w$ is concave in $[c, \infty)$ for some $c > 0$.

(2) $\lim_{x\to\infty} \frac{-\log w(x)}{x} = \infty$.

(3) $\lim_{x\to\infty} \frac{\log|\log w(x)| - \log x}{\sqrt{\log x}} = \infty$.

Theorem 2.9. *A finite codimensional closed subspace M in $H^2(\mathbb{D})$ is invariant under the Cesàro operator if and only if $\mathfrak{F} M^\perp$ in $L^2(\mathbb{R}, e^{-2(e^y-1)}\,dy)$ is spanned by a finite subset of functions given by*

$$\bigcup_{\lambda\in\Lambda} \{y^k e^{\lambda y} : \ k = 0, 1, 2, \ldots, n_\lambda\}$$

where $\Lambda \subset \mathbb{C}_+$ is a finite set and $n_\lambda \geq 0$ for each $\lambda \in \Lambda$.

Before proceeding with the proof, let us introduce the notation

$$f_{\lambda,k}(y) = y^k e^{\lambda y}$$

for $y \in \mathbb{R}$, $\lambda \in \mathbb{C}_+$ and $k = 0, 1, 2, \ldots$. Note that

$$(S_t f_{\lambda,k})(y) = f_{\lambda,k}(y - t) = \sum_{j=0}^{k} \binom{k}{j} (-t)^{k-j} e^{-\lambda t} y^j e^{\lambda y},$$

so

$$S_t f_{\lambda,k} = \sum_{j=0}^{k} \binom{k}{j} (-t)^{k-j} e^{-\lambda t} f_{\lambda,j} \tag{2.10}$$

for any $\lambda \in \mathbb{C}_+$ and $k = 0, 1, 2, \ldots$.

Proof. Suppose first that M is a finite codimensional closed subspace in $H^2(\mathbb{D})$ invariant under $\mathcal{C}$. Theorem 2.5 yields that $N = \mathfrak{F}M^{\perp}$ is a finite dimensional subspace of $L^2(\mathbb{R}, e^{-2(e^y-1)}\,dy)$ invariant under all right shifts. Thus, $P_{-}N$, the projection onto $L^2((-\infty, 0), e^{-2(e^y-1)}\,dy) \cong L^2(-\infty, 0)$, is a finite dimensional subspace invariant under all the truncated right shifts.

Thus, by the Beurling–Lax Theorem, $P_{-}N$ corresponds to a model space and in particular is spanned by a finite set of functions of the form

$$y^k e^{\lambda y}, \text{ for } y \in (-\infty, 0)$$

where $k = 0, 1, 2, \ldots, n_\lambda$ for $\lambda \in \Lambda \subset \mathbb{C}_+$. We now show that N is spanned by what we shall call the "natural extension" to $\mathbb{R}$ of such functions as elements of $L^2(\mathbb{R}, e^{-2(e^y-1)}\,dy)$, namely, $y^k e^{\lambda y}$ for $y \in \mathbb{R}$.

Observe that since N is finite dimensional, by the aforementioned Domar theorem,

$$N \cap L^2((0, \infty), e^{-2(e^y-1)}\,dy) = \{0\}.$$

Therefore, there exist $h_{\lambda,k} \in N$ such that $P_{-}f_{\lambda,k} = h_{\lambda,k}$ and N is spanned by $h_{\lambda,k}$. Let N_1 be spanned by $f_{\lambda,k}$ with the same k and λ. By (2.10), N_1 is invariant under all right shifts S_t. Then span $\{N, N_1\}$ is a finite dimensional invariant subspace of all right shifts S_t. Upon applying Domar's theorem again,

$$\text{span}\,\{N, N_1\} \cap L^2((0, \infty), e^{-2(e^y-1)}\,dy) = \{0\}.$$

Since

$$h_{\lambda,k} - f_{\lambda,k} \in \text{span}\,\{N, N_1\} \cap L^2((0, \infty), e^{-2(e^y-1)}\,dy),$$

we conclude that $h_{\lambda,k} = f_{\lambda,k}$. Thus, N is spanned by $f_{\lambda,k}$.

For the converse, assume $N = \mathfrak{F}M^{\perp}$ in $L^2(\mathbb{R}, e^{-2(e^y-1)}\,dy)$ is a finite dimensional subspace spanned by a finite subset of the form $\bigcup_{\lambda\in\Lambda}\{y^k e^{\lambda y} : k = 0, 1, 2, \ldots, n_\lambda\}$ where $\lambda \in \Lambda \subset \mathbb{C}_+$ is finite. Equation (2.10) yields that N is invariant under all right shifts S_t. Now, Theorem 2.5 and the fact that $\mathfrak{F}$ is an isomorphism yield that M is a finite codimensional closed subspace invariant under $\mathcal{C}$, which completes the proof. $\square$

Remark 2.10. Note that in the case that $N = \mathfrak{F}M^{\perp}$ is infinite-dimensional, the arguments above giving the structure of N in terms of the structure of $P_{-}N$ fail because we can no longer assume that $P_{-}N$ is closed. However, it is of interest to note that the closure $\overline{P_{-}N}$ in $L^2(-\infty, 0)$ has the same property of invariance under all truncated right shifts, and corresponds to a model space.

For instance, if B is a Blaschke product in $\mathbb{C}_{+}$ with the set of zeros Λ and multiplicities $n_\lambda + 1$ for $\lambda \in \Lambda$ with $n_\lambda \in \{0, 1, 2, \ldots\}$ and we set

$$N_B = \overline{\operatorname{span}\,\{f_{\lambda,k} : \lambda \in \Lambda, k = 0, 1, \ldots, n_\lambda\}}^{L^2(\mathbb{R}, e^{-2(e^y-1)}\,dy)},$$

clearly N_B is invariant under all right shifts. Nevertheless,

$$\overline{\tilde{\mathcal{L}}P_{-}N_B} = \overline{\operatorname{span}\,\{\tilde{\mathcal{L}}P_{-}f_{\lambda,k} : \lambda \in \Lambda, k = 0, 1, \ldots, n_\lambda\}}^{H^2(\mathbb{C}_{+})} = K_B,$$

where $\tilde{\mathcal{L}}$ is defined in (2.9) with $T = 0$. Consequently, $N_B \neq L^2(\mathbb{R}, e^{-2(e^y-1)}\,dy)$, and if $B_1 \neq B_2$ are two Blaschke products, then $N_{B_1} \neq N_{B_2}$.

2.2 A final remark regarding the lattice of the invariant subspaces of $\mathcal{C}$. As we have just noted, the approach addressed in the previous theorem fails if $P_{-}N$ is not closed. Indeed, the following example shows that $P_{-}N$ need not be closed even if N is a closed shift-invariant subspace of $L^2(\mathbb{R}, e^{-2(e^y-1)}\,dy)$ showing, somehow, the limits of such an approach.

Let $\lambda > 0$ and denote by e_λ the function

$$e_\lambda : y \in \mathbb{R} \mapsto e^{\lambda y}.$$

Since $1 < e^{-2(e^y-1)} < e^2$ for $y < 0$, we have

$$\|e_\lambda\|^2_{L^2((-\infty,0), e^{-2(e^y-1)}\,dy)} \approx \|e_\lambda\|^2_{L^2(-\infty,0)} \approx 1/\lambda. \tag{2.11}$$

On the other hand,

$$\|e_\lambda\|^2_{L^2(\mathbb{R}, e^{-2(e^y-1)}\,dy)} \geq \int_1^2 e^{2\lambda y} e^{-2(e^y-1)}\,dy \geq e^{2\lambda} e^{-2(e^2-1)}. \tag{2.12}$$

Now take N to be the closed linear span in $L^2(\mathbb{R}, e^{-2(e^y-1)}\,dy)$ of $\{e_{n^2} : n \in \mathbb{N}\}$, that is,

$$N = \overline{\operatorname{span}\,\{e_{n^2} : n \in \mathbb{N}\}}^{L^2(\mathbb{R}, e^{-2(e^y-1)}\,dy)}.$$

Now the twisted Laplace transform given in (2.9), with $T = 0$, provides an isomorphism from $L^2((-\infty, 0), e^{-2(e^y-1)}\,dy)$ onto $H^2(\mathbb{C}_{+})$ transforming e_λ to $1/(s+\lambda)$. By an argument similar to that used in proving the classical Müntz–Szász theorem it follows that $\overline{P_-N} \subsetneq L^2((-\infty,0), e^{-2(e^y-1)}\,dy)$ since there are functions orthogonal to each $1/(s+n^2)$, for example, $1/(s+2)$ times a Blaschke product with zeros at $\{n^2 : n \in \mathbb{N}\}$.

Now $(\ker P_-) \cap N$ is a closed shift-invariant subspace of $L^2((0,\infty), e^{-2(e^y-1)}\,dy)$ and hence, by Domar's theorem, a standard subspace. It must be $\{0\}$ (again this follows from a Müntz–Szász argument) so the restriction

$$P_- : N \to L^2((-\infty,0), e^{-2(e^y-1)}\,dy)$$

is injective. Finally, by Banach's open mapping theorem the norm estimates in (2.11) and (2.12) show that it cannot have a closed range.

The following proposition characterizes when P_-N is a closed subspace of $L^2((-\infty,0), e^{-2(e^y-1)}\,dy)$.

Proposition 2.11. *With the above notation, P_-N is closed if and only if the (not necessarily direct) sum of the two closed subspaces N and $L^2((0,\infty), e^{-2(e^y-1)}\,dy)$ is closed.*

Proof. It is well known that if $P : H \to K$ is a projection onto a subspace K of a Hilbert space H then K is automatically closed: since if (x_n) is a sequence in K tending to $x \in H$, we have $x_n = Px_n \to Px$, so $x = Px \in K$.

Now if $H := N + L^2((0,\infty), e^{-2(e^y-1)}\,dy)$ is closed, the projection P_- maps H to itself and its image is P_-N. Conversely, if P_-N is closed then

$$N + L^2((0,\infty), e^{-2(e^y-1)}\,dy) = P_-N \oplus L^2((0,\infty), e^{-2(e^y-1)}\,dy),$$

the orthogonal direct sum of two closed subspaces, and is therefore closed. □

3 Functions of the Cesàro operator

Suppose that $(T(t))_{t\geq 0}$ is a C_0-semigroup with $\|T(t)\| \leq e^{mt}$ for some $m < 1$ and infinitesimal generator A.

It is a standard fact that

$$\int_0^\infty e^{-\lambda t} T(t)\,dt = (\lambda - A)^{-1}$$

provided that $\operatorname{Re}\lambda > m$.

Now consider the operator

$$B = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\lambda t}}{\sqrt{t}} T(t)\,dt.$$

Then

$$B^2 = \frac{1}{\pi}\int_0^\infty \frac{e^{-\lambda t}}{\sqrt{t}} T(t)\,dt \int_0^\infty \frac{e^{-\lambda u}}{\sqrt{u}} T(u)\,du$$
$$= \frac{1}{\pi}\int_{w=0}^\infty e^{-\lambda w} T(w)\,dw \int_{t=0}^w \frac{1}{\sqrt{t}\sqrt{w-t}}\,dt$$

with $w = t + u$. The second integral is π (use the substitution $t = w\sin^2\theta$) and so $B^2 = (\lambda - A)^{-1}$. Similarly for $(-B)^2$, of course.

This gives an alternative way of looking at a result in [28] on the square roots of the Cesàro operator, by using the composition semigroup and the observations in Section 2. Not surprisingly, it is linked with the fact that the Laplace transform of e^{at} is $1/(s-a)$ and the Laplace transform of $e^{at}/\sqrt{t}$ is $\sqrt{\pi}/\sqrt{s-a}$. There is a more general functional calculus available here, but this calculation at least can be done directly.

Indeed the Phillips functional calculus [22, Rem. 3.3.3] allows us, given a bounded semigroup $(T(t))_{t\geq 0}$ of operators on a Banach space X, to associate an operator $f(A)$ to a function f that is the Laplace transform of a Borel measure μ on $[0,\infty)$ of bounded variation, by the formula

$$f(A)x = \int_{[0,\infty)} T(t)x\,d\mu(t) \quad (x \in X).$$

Note that the convention in [22] is that the generator is $-A$, rather than A, and we have allowed for that in the discussion below. In particular, we have

$$\int_0^\infty \frac{e^{-\lambda t}}{t^{1-\beta}}\,dt = \lambda^{-\beta}\Gamma(\beta) \tag{3.1}$$

for $\operatorname{Re}\lambda, \operatorname{Re}\beta > 0$, so that

$$\int_0^\infty \frac{e^{-t}e^{at}}{t^{1-\beta}}\,dt = (1-a)^{-\beta}\Gamma(\beta)$$

for $\operatorname{Re} a < 1$ and $\operatorname{Re}\beta > 0$, from which we obtain

$$\int_0^\infty \frac{e^{-t}T(t)}{t^{1-\beta}}\,dt = (I-A)^{-\beta}\Gamma(\beta) \tag{3.2}$$

for $\operatorname{Re}\beta > 0$. A similar formula holds on replacing $T(t)$ by $T(t)^*$ and A by A^*.

Recalling that $\mathcal{C}^* = (I-A)^{-1}$, we have the following:

Theorem 3.1. *For* $\operatorname{Re}\beta > 0$ *let* M_β *denote the matrix of* $\mathcal{C}^\beta$ *(as defined using* (3.2)*) with respect to the standard orthonormal basis* $(z^j)_{j=0}^\infty$. *Then*

$$(M_\beta)_{i,j} = \begin{cases} 0 & \text{if } i < j, \\ \binom{i}{j}\sum_{k=0}^{i-j}(-1)^k\binom{i-j}{k}(j+k+1)^{-\beta} & \text{if } i \geq j. \end{cases} \tag{3.3}$$

Proof. The (j, i) entry of the matrix for $(I - A)^{-\beta}$ is the coefficient of z^j in

$$\frac{1}{\Gamma(\beta)} \int_0^\infty \frac{e^{-t}}{t^{1-\beta}} (e^{-t} z + (1 - e^{-t}))^i \, dt,$$

namely, 0 for $j > i$ and otherwise

$$\frac{1}{\Gamma(\beta)} \int_0^\infty \frac{e^{-t}}{t^{1-\beta}} \binom{i}{j} e^{-jt} (1 - e^{-t})^{i-j} \, dt$$
$$= \frac{1}{\Gamma(\beta)} \binom{i}{j} \int_0^\infty \frac{e^{(-1-j)t}}{t^{1-\beta}} \sum_{k=0}^{i-j} \binom{i-j}{k} (-1)^k e^{-kt} \, dt.$$

Now, using (3.1) we obtain (3.3). □

In the case $\beta = 1/2$ this agrees with the formula in [28].

Remark 3.2. It is clear from the functional calculus that every subspace for $\mathcal{C}$ is also an invariant subspace for $\mathcal{C}^\beta$ for $\operatorname{Re} \beta > 0$. Since invariant subspaces for $\mathcal{C}^{1/n}$ are clearly invariant subspaces for $\mathcal{C}$ for $n = 1, 2, \ldots$, we may conclude that $\mathcal{C}$ and $\mathcal{C}^{1/n}$ have the same lattice of invariant subspaces.

Acknowledgements. The authors are grateful to a referee for carefully reading the manuscript and providing some extremely helpful comments which improved its readability.

Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

References

[1] A. Aleman, *A class of integral operators on spaces of analytic functions*, in *Topics in Complex Analysis and Operator Theory*, Universidad de Málaga, Málaga, 2007, pp. 3–30.

[2] A. Aleman and B. Korenblum, *Volterra invariant subspaces of H^p*, Bull. Sci. Math. **132** (2008), 510–528.

[3] A. Aleman and A. G. Siskakis, *Integration operators on Bergman spaces*, Indiana Univ. Math. J. **46** (1997), 337–356.

[4] A. G. Arvanitidis and A. G. Siskakis, *Cesàro operators on the Hardy spaces of the half-plane*, Canad. Math. Bull. **56** (2013), 229–240.

[5] E. Berkson and H. Porta, *Semigroups of analytic functions and composition operators*, Michigan Math. J. **25** (1978), 101–115.

[6] F. Bracci, M. D. Contreras and S. Díaz-Madrigal, Continuous Semigroups of Holomorphic Self-Maps of the Unit Disc, Springer, Cham, 2020.

[7] A. Brown, P. R. Halmos and A. L. Shields, *Cesàro operators*, Acta Sci. Math. (Szeged) **26** (1965), 125–137.

[8] J.R. Carmo and S. W. Noor, *Universal composition operators*, J. Operator Theory **87** (2022), 137–156.

[9] C. C. Cowen, *Subnormality of the Cesàro operator and a semigroup of composition operators*, Indiana Univ. Math. J. **33** (1984), 305–318.

[10] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[11] Y. Domar, *Translation invariant subspaces of weighted ℓ^p and L^p spaces*, Math. Scand. **49** (1981), 133–144.

[12] Y. Domar, *Extensions of the Titchmarsh convolution theorem with application in the theory of invariant subspaces*, Proc. London Math. Soc. **46** (1983), 288–300.

[13] Y. Domar, *A solution of the translation-invariant subspace problem for weighted L^p on $\mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{Z}$*, in *Radical Banach Algebras and Automatic Continuity*, Springer, Berlin–Heidelberg, 1983, pp. 214–226.

[14] Y. Domar, *Translation-Invariant Subspaces of Weighted L^p*, American Mathematical Society, Providence, RI, 1989.

[15] N. Dunford and J.T. Schwartz, *Linear Operators. Part I*, John Wiley & Sons, New York, 1988.

[16] C. Foiaş, *Spectral maximal spaces and decomposable operators in Banach space*, Arch. Math. **14** (1963), 341–349.

[17] P.A. Fuhrmann, *Linear Systems and Operators in Hilbert Space*, McGraw-Hill International, New York, 1981.

[18] E. A. Gallardo-Gutiérrez and J. R. Partington, *Invariant subspaces for translation, dilation and multiplication semigroups*, J. Anal. Math. **107** (2009), 65–78.

[19] E. A. Gallardo-Gutiérrez and J. R. Partington, *C_0-semigroups of 2-isometries and Dirichlet spaces*, Rev. Mat. Iberoamericana **34** (2018), 1415–1425.

[20] E. A. Gallardo-Gutiérrez, J. R. Partington and D. J. Rodríguez, *An extension of a theorem of Domar on invariant subspaces*, Acta Sci. Math. (Szeged) **83** (2017), 271–290.

[21] E. A. Gallardo-Gutiérrez and D. J. Rodríguez, *Non-standard translation-invariant subspaces for weighted L^2 on $\mathbb{R}_+$*, in *Harmonic Analysis, Function Theory, Operator Theory, and their Applications*, Theta, Bucharest, 2017, pp. 125–132.

[22] M. Haase, *The Functional Calculus for Sectorial Operators*, Birkhäuser, Basel, 2006.

[23] V. Havin and B. Jöricke, *The Uncertainty Principle in Harmonic Analysis*, Springer, Berlin, 1994.

[24] T. L. Kriete and D. Trutt, *The Cesàro operator in ℓ^2 is subnormal*, Amer. J. Math. **93** (1971), 215–225.

[25] T. L. Kriete and D. Trutt, *On the Cesàro operator*, Indiana Univ. Math. J. **24** (1974/75), 197–214.

[26] K. Laursen and M. Neumann, *An Introduction to Local Spectral Theory*, The Clarendon Press, Oxford University Press, New York, 2000.

[27] P. D. Lax, *Translation invariant subspaces*, Acta Math. **101** (1959), 163–178.

[28] J. Mashreghi, M. Ptak and W. T. Ross, *Square roots of some classical operators*, Studia Math. **269** (2023), 83–106.

[29] V. G. Miller, T. L. Miller and R. C. Smith, *Bishop's property* (β) *and the Cesàro operator*, J. London Math. Soc. (2) **58** (1998), 197–207.

[30] N. K. Nikolski, *Operators, Functions, and Systems: an Easy Reading. Vol. 1. Hardy, Hankel, and Toeplitz*, American Mathematical Society, Providence, RI, 2002.

[31] N. K. Nikolskii, *Unicellularity and non-unicellularity of weighted shift operators*, Dokl. Ak. Nauk SSR **172** (1967), 287–290.

[32] A.-M. Persson, *On the spectrum of the Cesàro operator on spaces of analytic functions*, J. Math. Anal. Appl. **340** (2008), 1180–1203.

[33] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1987.

[34] A. G. Siskakis, *Composition semigroups and the Cesàro operator on* H^p, J. London Math. Soc. (2) **36** (1987), 153–164.

[35] A. G. Siskakis, *The Cesàro operator is bounded on* H^1, Proc. Amer. Math. Soc. **110** (1990),461–462.

Eva A. Gallardo-Gutiérrez
Departamento de Análisis Matemático y Matemática Aplicada
Facultad de Matemáticas
Universidad Complutense de Madrid
Plaza de Ciencias 3, 28040 Madrid, Spain
and
Instituto de Ciencias Matemáticas ICMAT
28049 Madrid, Spain
email: eva.gallardo@mat.ucm.es

Jonathan R. Partington
School of Mathematics
University of Leeds
Leeds LS2 9JT, United Kingdom
email: J.R.Partington@leeds.ac.uk

(Received June 22, 2022 and in revised form September 25, 2022)