BUBBLING SOLUTIONS FOR MEAN FIELD EQUATIONS WITH VARIABLE INTENSITIES ON COMPACT RIEMANN SURFACES

By

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Abstract. For an asymmetric sinh-Poisson problem arising as a mean field equation of equilibrium turbulence vortices with variable intensities of interest in hydrodynamic turbulence, we address the existence of bubbling solutions on compact Riemann surfaces. By using a Lyapunov–Schmidt reduction, we find sufficient conditions under which there exist bubbling solutions blowing up at m different points of S: positively at m_1 points and negatively at $m - m_1$ points with $m \geq 1$ and $m_1 \in \{0, 1, \dots, m\}$. Several examples in different situations illustrate our results in the sphere $\mathbb{S}^2$ and flat two-torus $\mathbb{T}$ including non-negative potentials with zero set non-empty.

1 Introduction

Let (S, g) be a compact Riemann surface and consider the problem

$$(1.1)\qquad -\Delta_g u = \lambda_1\Big(\frac{V_1(x)e^u}{\int_S V_1 e^u dv_g} - \frac{1}{|S|}\Big) - \lambda_2\tau\Big(\frac{V_2(x)e^{-\tau u}}{\int_S V_2 e^{-\tau u} dv_g} - \frac{1}{|S|}\Big),$$

where $\lambda_1, \lambda_2 \geq 0$, $\tau > 0$, V_1 and V_2 are smooth nonnegative potentials in S and $|S|$ is the area of S. Here, Δ_g is the Laplace–Beltrami operator and dv_g is the area element in (S, g). This equation has attracted a lot of attention in recent years due to its relevance in the statistical mechanics description of 2D-turbulence, as initiated by Onsager [49]. Precisely, in this context, under a deterministic assumption on the distribution of the vortex circulations, Sawada and Suzuki [56] derive the following equation:

$$(1.2)\qquad -\Delta_g u = \lambda \int_{[-1,1]} \alpha\Big(\frac{e^{\alpha u}}{\int_S e^{\alpha u} dv_g} - \frac{1}{|S|}\Big) d\mathcal{P}(\alpha) \quad \text{in } S$$

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where u is the stream function of a turbulent Euler flow, $\lambda > 0$ is a physical constant related to the inverse temperature and $\mathcal{P}$ is a Borel probability measure in $[-1, 1]$ describing the point-vortex intensities distribution.

Equation (1.2) includes several well-known problems depending on a suitable choice of $\mathcal{P}$. For instance, if $\mathcal{P} = \delta_1$ is concentrated at 1, then (1.2) is related to the classical mean field equation

$$-\Delta_g u = \lambda\Big(\frac{Ve^u}{\int_S Ve^u\,dv_g} - \frac{1}{|S|}\Big) \quad \text{in } S, \tag{1.3}$$

where V is a smooth nonnegative function on S. The latter equation has been studied in several contexts such as conformal geometry [11, 10, 40], statistical mechanics [6, 7, 12, 41] and the relativistic Chern–Simons–Higgs model when S is a flat two-torus [47, 57, 58]. Notice that solutions of (1.3) are critical points of the functional

$$J_\lambda(u) = \frac{1}{2}\int_S |\nabla u|_g^2\,dv_g - \lambda\log\Big(\int_S Ve^u\,dv_g\Big), \quad u \in \bar{H},$$

where $\bar{H} = \{u \in H^1(S) : \int_S u dv_g = 0\}$. Minimizers of J_λ for $\lambda < 8\pi$ can be found by using Moser–Trudinger's inequality. The situation in the supercritical regime $\lambda \geq 8\pi$ becomes subtler and the existence of solutions could depend on the topology and the geometry of the surface S (or the domain). A degree argument has been proved in [13, 14] by Chen and Lin, completing a program initiated by Li [43], and has received a variational counterpart in [18, 46] by means of improved forms of the Moser–Trudinger inequality.

Equation (1.1) is also related to (1.2) when $\mathcal{P} = \sigma\delta_1 + (1-\sigma)\delta_{-\tau}$ with $\tau \in [-1, 1]$ and $\sigma \in [0, 1]$. Furthermore, (1.1) is the Euler–Lagrange equation of the functional

$$\begin{aligned} J_{\lambda_1,\lambda_2}(u) = \frac{1}{2}\int_S |\nabla u|_g^2\,dv_g &- \lambda_1\log\Big(\int_S V_1e^u dv_g\Big) \\ &- \lambda_2\log\Big(\int_S V_2e^{-\tau u} dv_g\Big), \quad u \in \bar{H}. \end{aligned} \tag{1.4}$$

If $\tau = 1$ and $V_1 = V_2 \equiv 1$ problem (1.1) reduces to the mean field equation of the equilibrium turbulence, see [5, 34, 37, 48, 52], or its related sinh-Poisson version, see [3, 4, 33, 38, 39], which have received a considerable amount of interest in recent years. Precisely, in [48] a Trudinger–Moser type inequality was proved: if $\lambda_1, \lambda_2 \in [0, 8\pi)$, which can be called the subcritical case, then solutions to (1.1) are the minimizers of J_{λ_1,λ_2}, since this functional is coercive; but if $\lambda_1, \lambda_2 \in [0, 8\pi]$ and either $\lambda_1 = 8\pi$ or $\lambda_2 = 8\pi$ then the functional J_{λ_1,λ_2} still has a lower bound but it is not coercive. A minimization technique is no longer possible if $\lambda_i > 8\pi$ for some $i = 1, 2$ since J_{λ_1,λ_2} becomes unbounded from below. In general, one needs to apply variational methods to obtain the existence of critical points (generally of saddle type) for J_{λ_1,λ_2}. Several results in the supercritical case can be found in [52, 59, 60]. A quantization property was derived in [38] for a blow-up sequence $\{u_n\}_n$ to (1.1) with $\tau = 1$, one has

$$m_k(p) = \lim_{r\to 0}\lim_{n\to+\infty} \frac{\lambda_{k,n}\int_{B_r(p)} V_k e^{(-1)^{k-1}u_n}\,dv_g}{\int_S V_k e^{(-1)^{k-1}u_n}\,dv_g} \in 8\pi\mathbb{N}, \quad k = 1, 2, \tag{1.5}$$

extending the corresponding ones for (1.3) in [44] and for (1.1) with $\tau = 1$ and $V_1 = V_2 \equiv 1$ in [39].

Concerning the version of problem (1.1) on bounded domains Pistoia and Ricciardi built in [50] sequences of blowing-up solutions when $\tau > 0$ and λ_1, $\lambda_2\tau^2$ are close to 8π, while in [51] the same authors built an arbitrary large number of sign-changing blowing-up solutions when $\tau > 0$ and λ_1, $\lambda_2\tau^2$ are close to suitable (not necessarily integer) multiples of 8π. Ricciardi and Takahashi in [53] provided a complete blow-up picture for solution sequences of (1.1) and successively in [54] Ricciardi et al. constructed min-max solutions when $\lambda_1 \to 8\pi^+$ and $\lambda_2 \to 0$ on a multiply connected domain (in this case the nonlinearity $e^{-\tau u}$ may be treated as a lower-order term with respect to the main term e^u).

In a compact Riemann surface S, a blow-up analysis in subcritical case $\lambda_1 < 8\pi$ and $\lambda_2 < \frac{8\pi}{\tau^2}$, and supercritical case $\lambda_1 < 16\pi$ and $\lambda_2 < \frac{16\pi}{\tau^2}$, characterizing the blow-up masses $m_k(p)$, $k = 1, 2$, defined similarly as in (1.5), has been obtained in [36], when $0 < \tau < 1$. Furthermore, some existence results are deduced. The authors in [55] obtain the minimal blow-up masses and proved an existence result which generalizes the one obtained in [52] for $\tau = 1$.

To the extent of our knowledge, there are by now just few results concerning the existence of bubbling solutions to (1.1) and its variants in different frameworks. For instance, bubbling solutions have been constructed for a sinh-Poisson equation ($\tau = 1$) on bounded domains in [3, 4] with a Dirichlet boundary condition and recently in [32] with a Robin boundary condition. Furthermore, recently in [24] and [29], the authors have constructed blowing-up solutions on pierced domains with a Dirichlet boundary condition for any $\tau > 0$. See also [50, 51] for generalizations to $\tau > 0$ of results obtained in [3, 33] for $\tau = 1$, respectively. The construction of sign-changing bubble tower solutions for sinh-Poisson type equations on pierced domains has been addressed in [30].

By following some ideas presented in [3, 23], we are interested in constructing bubbling solutions u_{λ_1,λ_2} to (1.1) with m_1 positive bubbles and m_2 negative bubbles suitably centered at m different points of S as both $\lambda_1 \to 8\pi m_1$

and $\lambda_2\tau^2 \to 8\pi(m-m_1)$, with $m_1 \in \{0,\dots,m\}$. To this aim, introduce the Green function $G(x,p)$ with pole at $p \in S$ as the solution of

$$\begin{cases} -\Delta_g G(\cdot,p) = \delta_p - \frac{1}{|S|} & \text{in } S \\ \int_S G(x,p)dv_g = 0 \end{cases} \tag{1.6}$$

where δ_p denotes a Dirac mass in $p \in S$. Define for $\xi = (\xi_1,\dots,\xi_m) \in \tilde{S}^m \setminus \Delta$ the functional

$$\begin{aligned} \varphi_m^*(\xi) = {} & \frac{1}{4\pi}\sum_{j=1}^{m_1} \log V_1(\xi_j) + \frac{1}{4\pi\tau^2}\sum_{j=m_1+1}^{m} \log V_2(\xi_j) + \sum_{j=1}^{m_1} H(\xi_j,\xi_j) \\ & + \frac{1}{\tau^2}\sum_{j=m_1+1}^{m} H(\xi_j,\xi_j) \\ & + \sum_{j=1}^{m_1}\sum_{\substack{i=1\\ i\neq j}}^{m_1} G(\xi_i,\xi_j) - \frac{2}{\tau}\sum_{j=1}^{m_1}\sum_{i=m_1+1}^{m} G(\xi_i,\xi_j) \\ & + \frac{1}{\tau^2}\sum_{j=m_1+1}^{m}\sum_{\substack{i=m_1+1\\ i\neq j}}^{m} G(\xi_i,\xi_j), \end{aligned} \tag{1.7}$$

where $H(x,\xi)$ is the regular part of $G(x,\xi)$, $\tilde{S} = \{V_1, V_2 > 0\}$ and

$$\Delta = \{\xi \in S^m : \xi_i = \xi_j \text{ for } i \neq j\}$$

is the diagonal set in S^m with $m = m_1 + m_2$. Setting for $j \in \mathcal{J}_1 := \{1,\dots,m_1\}$

$$\rho_j(x) := V_1(x)\exp\left(8\pi H(x,\xi_j) + 8\pi\sum_{\substack{i=1\\ i\neq j}}^{m_1} G(x,\xi_i) - \frac{8\pi}{\tau}\sum_{i=m_1+1}^{m} G(x,\xi_i)\right), \tag{1.8}$$

and for $j \in \mathcal{J}_2 := \{m_1+1,\dots,m\}$

$$\rho_j(x) := V_2(x)\exp\left(8\pi H(x,\xi_j) - 8\pi\tau\sum_{i=1}^{m_1} G(x,\xi_i) + 8\pi\sum_{\substack{i=m_1+1\\ i\neq j}}^{m} G(x,\xi_i)\right), \tag{1.9}$$

both for $\xi \in S^m \setminus \Delta$ we introduce the notation

$$A_k^*(\xi) = 4\pi\sum_{j\in\mathcal{J}_k}[\Delta_g\rho_j(\xi_j) - 2K(\xi_j)\rho_j(\xi_j)], \quad k = 1,2 \tag{1.10}$$

where K is the Gaussian curvature of (S,g). The sign of A_k^*, $k = 1,2$ allows us to obtain a first existence result of bubbling solutions and several consequences; see Theorem 2.1 and Section 2. Unfortunately, there are cases where the sign of $A_k^*(\xi)$ for either $k = 1$ or $k = 2$ or both is not available, for instance, the case $S = \mathbb{T}$, $V_1 = V_2 \equiv 1$, $m_1 = m_2 = 1$ and $\tau = 1$. See also [23] for several examples in case $\lambda_2 = 0$, namely, $m_2 = 0$, that could be extended here. Following ideas presented in [23], in all these situations, a more refined analysis is necessary. To this aim, introduce the quantities for $k = 1, 2$

$$\begin{aligned} B_k^*(\xi) &\\ = &-2\pi \sum_{j\in\mathcal{J}_k} [\Delta_g \rho_j(\xi_j) - 2K(\xi_j)\rho_j(\xi_j)] \log \rho_j(\xi_j) - \frac{A_k^*(\xi)}{2} \\ &+ \lim_{r\to 0} \Bigg[8 \int_{S\setminus \bigcup_{j\in\mathcal{J}_k} B_r(\xi_j)} V_1 e^{8\pi(-\tau)^{k-1}\sum_{j=1}^{m_1} G(x,\xi_j) + 8\pi(-\tau)^{k-2}\sum_{l=m_1+1}^{m} G(x,\xi_l)} dv_g \\ &\qquad - \frac{8\pi}{r^2} \sum_{j\in\mathcal{J}_k} \rho_j(\xi_j) - A_k^*(\xi) \log\frac{1}{r} \Bigg] \end{aligned} \tag{1.11}$$

where $B_r(\xi)$ denotes the pre-image of $B_r(0)$ through the isothermal coordinate system at ξ. These types of quantities were first used and derived by Chang, Chen and Lin in [9] in the study of the mean field equation on bounded domains with a Dirichlet boundary condition; for the case of the torus see [15]. Moreover, the constant $B_k^*(\xi)$ has also been used in the construction of non-topological condensates for the relativistic abelian Chern–Simons–Higgs model as the Chern–Simons parameter tends to zero, see [19, 23, 45]. Our main result states as follows.

Theorem 1.1. *Let $\mathcal{D} \subset\subset \tilde{S}^m \setminus \Delta$ be a stable critical set of φ_m^*. Assume that*

$$\text{either } A_1^*(\xi) > 0 \,(< 0 \text{ resp.}) \quad \text{or} \quad A_1^*(\xi) = 0,\ B_1^*(\xi) > 0 \,(< 0 \text{ resp.}) \tag{1.12}$$

and

$$\text{either } A_2^*(\xi) > 0 \,(< 0 \text{ resp.}) \quad \text{or} \quad A_2^*(\xi) = 0,\ B_2^*(\xi) > 0 \,(< 0 \text{ resp.}) \tag{1.13}$$

do hold in a closed neighborhood U of $\mathcal{D}$ in $\tilde{S}^m \setminus \Delta$. Then, for all λ_1 in a small right (left resp.) neighborhood of $8\pi m_1$ and $\lambda_2\tau^2$ in a small right (left resp.) neighborhood of $8\pi m_2$ there is a solution u_{λ_1,λ_2} of (1.1) *which concentrates (along sub-sequences) at m points, positively at $q_1, \dots, q_{m_1}$ and negatively at $q_{m_1+1}, \dots, q_m$, in the sense*

$$\frac{\lambda_1 V_1 e^{u_{\lambda_1,\lambda_2}}}{\int_S V_1 e^{u_{\lambda_1,\lambda_2}} dv_g} \rightharpoonup 8\pi \sum_{j=1}^{m_1} \delta_{q_j} \quad \text{and} \quad \frac{\lambda_2 \tau^2 V_2 e^{-\tau u_{\lambda_1,\lambda_2}}}{\int_S V_2 e^{-\tau u_{\lambda_1,\lambda_2}} dv_g} \rightharpoonup 8\pi \sum_{j=m_1+1}^{m} \delta_{q_j} \tag{1.14}$$

as simultaneously $\lambda_1 \to 8\pi m_1$ and $\lambda_2\tau^2 \to 8\pi m_2$ for some $q \in \mathcal{D}$.

Notice that along with (1.14) there hold

$$(-\tau)^{k-1}u_{\lambda_1,\lambda_2} - \log \int_S V_k e^{(-\tau)^{k-1}u_{\lambda_1,\lambda_2}} \to -\infty \quad \text{in } C_{\mathrm{loc}}(S \setminus \{q_1, \dots, q_m\})$$

and

$$\sup_{\mathcal{O}_j} \left((-\tau)^{k-1}u_{\lambda_1,\lambda_2} - \log \int_S V_k e^{(-\tau)^{k-1}u_{\lambda_1,\lambda_2}} \right) \to +\infty$$

as simultaneously $\lambda_1 \to 8\pi m_1$ and $\lambda_2\tau^2 \to 8\pi m_2$, for any neighborhood $\mathcal{O}_j$ of q_j in S with $k = 1$ for $j = 1, \dots, m_1$ and $k = 2$ for $j = m_1 + 1, \dots, m$. Hence, we get that u_{λ_1,λ_2} concentrates positively at $q_1, \dots, q_{m_1}$ and negatively at $q_{m_1+1}, \dots, q_m$ as simultaneously $\lambda_1 \to 8\pi m_1$ and $\lambda_2\tau^2 \to 8\pi m_2$. As in [23], the notion of stability we are using here is the one introduced in [42]; see Definition 2.1 below. Conditions (1.12)–(1.13) on a neighborhood of $\mathcal{D}$ are required to deal with a stable critical set $\mathcal{D}$ in the sense below. Arguing as in Remark 4.5 in [23], the same conclusion of Theorem 1.1 follows under the validity of conditions (1.12)–(1.13) just on $\mathcal{D} = \{\xi_0\}$, where ξ_0 is a non-degenerate local minimum/maximum point of φ_m^*. Similarly, Theorem 1.1 is also valid in the special case $|A_k^*(\xi)| = O(|\nabla\varphi_m^*(\xi)|_g)$, $k = 1, 2$ in a neighborhood of $\mathcal{D}$ and $B_k^*(\xi) > 0$ in $\mathcal{D}$.

Now, we can address the case $S = \mathbb{T}$, $V_1 = V_2 \equiv 1$, $m_1 = m_2 = 1$ and $\tau = 1$. When $\mathbb{T}$ is a rectangle, the constants like $B_k^*(\xi)$, $k = 1, 2$, have been used by Chen, Lin and Wang [15] in the computation of the Leray–Schauder degree. Due to $H(x, x)$ being constant in $\mathbb{T}$, we deduce that $\varphi_2^*(\xi) = -2G(\xi_1, \xi_2) + \text{const.}$. Also, it is known that the Green's function satisfies $G(\xi_1, \xi_2) = G(\xi_1 - \xi_2, 0)$ and the function $G(\cdot, 0)$ has exactly three non-degenerate critical points q_1, q_2 (saddle points) and q_3 (minimum point). According to (1.11) we have that for $i, k \in \{1, 2\}$

$$B_k^*(\xi) = \lim_{r \to 0} \left[8 \int_{\mathbb{T} \setminus B_r(\xi_k)} e^{8\pi G(x,\xi_k) - 8\pi G(x,\xi_i)} - \frac{8\pi}{r^2} e^{8\pi H(\xi_k,\xi_k) - 8\pi G(\xi_i,\xi_k)} \right], \quad i \neq k.$$

Assuming that $\mathbb{T} = -\mathbb{T}$ it follows that $B_1^*(\xi) = B_2^*(\xi)$, $\xi = (\xi_1, \xi_2)$, since $G(z, 0) = G(-z, 0)$. Furthermore, it is known that $B_1^*(\xi) > 0$ when either $\xi_1 - \xi_2 = q_1$ or $\xi_1 - \xi_2 = q_2$, and $B_1^*(\xi) < 0$ when $\xi_1 - \xi_2 = q_3$. By Theorem 1.1 we deduce the existence of

- two distinct families of solutions, for λ_1, λ_2 in a small right neighborhood of 8π, concentrating positively at ξ_1 and negatively at ξ_2 with either $\xi_1 - \xi_2 = q_1$ or $\xi_1 - \xi_2 = q_2$ as $\lambda_1 \to 8\pi$ and $\lambda_2 \to 8\pi$;
- one family of solutions, for λ_1, λ_2 in a small left neighborhood of 8π, concentrating positively at ξ_1 and negatively at ξ_2 with $\xi_1 - \xi_2 = q_3$ as $\lambda_1 \to 8\pi$ and $\lambda_2 \to 8\pi$.

The case $m_2 = 0$, namely, as $\lambda_2\tau^2 \to 0^+$, can be also addressed by this approach. Thus, we have that (1.1) can be seen as a perturbation of (1.3). In this case the nonlinearity $e^{-\tau u}$ is treated as a lower-order term with respect to the main term e^u. For simplicity we denote $A(\xi)$ and $B(\xi)$ instead of $A_1^*(\xi)$ and $B_1^*(\xi)$ with $m_1 = m$ and $\mathcal{J}_2 = \varnothing$, so that we have the following result.

Theorem 1.2. *Let $\mathcal{D} \subset\subset \tilde{S}^m \setminus \Delta$ be a stable critical set of φ_m^*. Assume that*

$$\text{(1.15)}\qquad \textit{either } A(\xi) > 0\ (< 0 \textit{ resp.}) \quad \textit{or} \quad A(\xi) = 0,\ B(\xi) > 0\ (< 0 \textit{ resp.})$$

do hold in a closed neighborhood U of $\mathcal{D}$ in $\tilde{S}^m \setminus \Delta$. Then, for all λ_1 in a small right (left resp.) neighborhood of $8\pi m_1$ and $\lambda_2\tau^2$ in a small right neighborhood of 0, there is a solution u_{λ_1,λ_2} of (1.1) which concentrates positively (along subsequences) at m points $q_1, \dots, q_m$,

$$\frac{\lambda_1 V_1 e^{u_{\lambda_1,\lambda_2}}}{\int_S V_1 e^{u_{\lambda_1,\lambda_2}} dv_g} \rightharpoonup 8\pi \sum_{j=1}^m \delta_{q_j} \quad \textit{in measure sense for some } q \in \mathcal{D}$$

$$\textit{and} \quad \frac{\lambda_2 \tau^2 V_2 e^{-\tau u_{\lambda_1,\lambda_2}}}{\int_S V_2 e^{-\tau u_{\lambda_1,\lambda_2}} dv_g} \to 0 \quad \textit{uniformly in } S.$$

Notice that a similar result can be obtained in case $m_1 = 0$ and $m_2 = m$, namely, as $\lambda_1 \to 0^+$ and $\lambda_2\tau^2 \to 8\pi m$, and u_{λ_1,λ_2} concentrates negatively at m different points of S. The same conclusion of Theorem 1.2 follows: on one hand, under the validity of condition (1.15) just on $\mathcal{D} = \{\xi_0\}$, where ξ_0 is a non-degenerate local minimum/maximum point of φ_m^*; and on the other hand, in the special case $|A(\xi)| = O(|\nabla\varphi_m^*(\xi)|_g)$ in a neighborhood of $\mathcal{D}$ and $B(\xi) > 0$ in $\mathcal{D}$. See the proof of Theorem 3.2 and Remark 4.5 in [23] for more details. Several examples for Theorem 1.2 can be derived from each example provided in [23] for the case $\lambda_2 = 0$.

The paper is organized as follows: Some consequences and examples are presented in Section 2. In Section 3, we construct a first approximation to a solution to (1.1) with the required properties and we estimate the size of the error of approximation with appropriate norms. In Section 4 we describe the scheme of our proofs, by stating the principal results we need, and we give the proof of our Theorem 1.1. Section 5 is devoted to the computation of the expansion of the energy functional on the first approximation we constructed in Section 3. The proof of Theorem 1.2 is done in Section 6. Sections 7 and 8 are devoted to proving the intermediate results we state in Section 4.

2 Consequences and examples

In this section we present several consequences of Theorem 1.1 and some examples that illustrate our results in the sphere $\mathbb{S}^2$ and flat two-torus $\mathbb{T}$. A special case of Theorem 1.1 is the following:

Theorem 2.1. *Let $\mathcal{D} \subset\subset \tilde{S}^m \setminus \Delta$ be a stable critical set of φ_m^*. Assume that $A_1^*(\xi) > 0$ (< 0 resp.) and $A_2^*(\xi) > 0$ (< 0 resp.) for all $\xi \in \mathcal{D}$. Then, for all λ_1 in a small right (left resp.) neighborhood of $8\pi m_1$ and λ_2 in a small right (left resp.) neighborhood of $\frac{8\pi m_2}{\tau^2}$ there is a solution u_{λ_1,λ_2} of (1.1) which concentrates (along sub-sequences) at m points $q_1, \dots, q_m$ in the sense of (1.14) for some $q \in \mathcal{D}$.*

The notion of stability we are using here is the following:

Definition 2.1. A critical set $\mathcal{D} \subset\subset \tilde{S}^m \setminus \Delta$ of φ_m is stable if for any closed neighborhood U of $\mathcal{D}$ in $\tilde{S}^m \setminus \Delta$ there exists $\delta > 0$ such that, if $\|G - \varphi_m\|_{C^1(U)} \le \delta$, then G has at least one critical point in U. In particular, the minimal/maximal set of φ_m is stable (if φ_m is not constant) as well as any isolated c.p. of φ_m with non-trivial local degree.

Notice that from the definition of ρ_j in (1.8)–(1.9) and $A_k^*(\xi)$ in (1.10), it is readily checked that

$$A_k^*(\xi) = 4\pi \sum_{j \in \mathfrak{I}_k} \rho_j(\xi_j) \Big[\Delta_g \log V_1(\xi_j) + (-\tau)^{k-1} \frac{8\pi}{|S|} \Big(m_1 - \frac{m_2}{\tau} \Big) - 2K(\xi_j) \Big], \quad k = 1, 2$$

for ξ a c.p. of φ_m^*, in view of $\nabla \rho_j(\xi_j) = 0$ for all $j = 1, \dots, m$. If $V_1 \ge 0$ and $V_2 \ge 0$ in S, then the function φ_2^* with $m_1 = m_2 = 1$ always attains its maximum value in $\tilde{S}^2 \setminus \Delta$ and the maximal set is clearly stable. Let us stress that V_1 and V_2 can vanish at some points of S. Thus, we have deduced the following fact.

Corollary 2.1. *Assume that $V_i \ge 0$ in S for $i = 1, 2$. If either*

$$\sup_S [2K - \Delta_g \log V_1] < \frac{8\pi}{|S|} \Big(1 - \frac{1}{\tau} \Big) \quad or \quad \inf_S [2K - \Delta_g \log V_1] > \frac{8\pi}{|S|} \Big(1 - \frac{1}{\tau} \Big)$$

and either

$$\sup_S [2K - \Delta_g \log V_2] < \frac{8\pi}{|S|} (1 - \tau) \quad or \quad \inf_S [2K - \Delta_g \log V_2] > \frac{8\pi}{|S|} (1 - \tau),$$

then there exist solutions u_{λ_1,λ_2} to (1.1) which concentrate at two points, positively at q_1 and negatively at q_2, in the sense of (1.14) as $\lambda_1 \to 8\pi$ and $\lambda_2 \tau^2 \to 8\pi$, where (q_1, q_2) is a maximum of φ_2^* in $\tilde{S}^2 \setminus \Delta$.

When $S = \mathbb{S}^2$ we have that $K = \frac{4\pi}{|\mathbb{S}^2|}$, so that, for $V_1 = V_2 \equiv 1$ and any $\tau > 0$, Corollary 2.1 then provides the existence of blow-up solutions u_{λ_1,λ_2} concentrating at two points as $\lambda_1 \to 8\pi$ and $\lambda_2\tau^2 \to 8\pi$, where λ_1 and $\lambda_2\tau^2$ belong to a small left neighborhood of 8π. In case of a flat two-torus $S = \mathbb{T}$, $K = 0$, so that for $V_1 = V_2 \equiv 1$ and any $\tau > 0$, $\tau \neq 1$, Corollary 2.1 then provides the existence of blow-up solutions u_{λ_1,λ_2} concentrating at two points as $\lambda_1 \to 8\pi$ and $\lambda_2\tau^2 \to 8\pi$, where λ_1 belongs to a small right (left resp.) neighborhood of 8π if $\tau > 1$ (< 1 resp.) and $\lambda_2\tau^2$ belongs to a small left (right resp.) neighborhood of 8π. However, the case $S = \mathbb{T}$, $V_1 = V_2 \equiv 1$, $m_1 = m_2 = 1$ and $\tau = 1$ is an example for which A_1^* and A_2^* vanish in $\mathbb{T}^2 \setminus \Delta$ and in particular at c.p.'s.

Let us mention some examples where V_1 and V_2 vanish at some points of S. Precisely, assume that

$$V_1(x) = e^{-4\pi\sum_{i=1}^{l_1} n_{1,i}G(x,p_{1,i})} \quad \text{and} \quad V_2(x) = e^{-4\pi\sum_{i=1}^{l_2} n_{2,i}G(x,p_{2,i})},$$

with $n_{1,i}, n_{2,i} > 0$ and $p_{1,i}, p_{2,j} \in S$, $i = 1, \dots, l_1$ and $j = 1, \dots, l_2$ respectively. The zero sets are $\{p_{1,1}, \dots, p_{1,l_1}\}$ for V_1 and $\{p_{2,1}, \dots, p_{2,l_2}\}$ for V_2. So, for $m_1 = m_2 = 1$, $m = 2$ we have that

$$\varphi_2^*(\xi) = -\sum_{i=1}^{l_1} n_{1,i}G(\xi_1, p_{1,i}) - \frac{1}{\tau^2}\sum_{j=1}^{l_2} n_{2,j}G(\xi_2, p_{2,j}) - \frac{2}{\tau}G(\xi_1, \xi_2),$$

and if ξ is a c.p. of φ_2^* then

$$A_k^*(\xi) = 4\pi\rho_k(\xi_k)\Big[-\frac{4\pi}{|S|}\sum_{i=1}^{l_k} n_{k,i} + \frac{8\pi}{|S|}\Big(1 - \tau^{2k-3}\Big) - 2K(\xi_k)\Big], \qquad k = 1, 2.$$

In particular, if $S = \mathbb{S}^2$ then Corollary 2.1 provides the existence of blow-up solutions u_{λ_1,λ_2} concentrating at two points as $\lambda_1 \to 8\pi$ and $\lambda_2\tau^2 \to 8\pi$ when $\sum_{i=1}^{l_1} n_{1,i} \neq 1 - \frac{2}{\tau}$ and $\sum_{j=1}^{l_2} n_{2,j} \neq 1 - 2\tau$. We deduce the same conclusion when $S = \mathbb{T}$ and $\sum_{i=1}^{l_1} n_{1,i} \neq 2 - \frac{2}{\tau}$ and $\sum_{j=1}^{l_2} n_{2,j} \neq 2 - 2\tau$. Let us stress that there is no restriction on $n_{1,i}, n_{2,j}$'s if $\tau = 1$.

Now, consider the case $m_1 = m \geq 2$ and $m_2 = 1$, namely, λ_1 close to $8\pi m$ and $\lambda_2\tau^2$ close to 8π. Roughly speaking, if u_{λ_1,λ_2} concentrates negatively at q then

$$\lambda_2\tau\Big(\frac{V_2e^{-\tau u_{\lambda_1,\lambda_2}}}{\int_S V_2e^{-\tau u_{\lambda_1,\lambda_2}}\,dv_g} - \frac{1}{|S|}\Big) \quad \text{behaves like } 4\pi\cdot\frac{2}{\tau}\Big(\delta_q - \frac{1}{|S|}\Big) \quad \text{as } \lambda_2\tau^2 \to 8\pi$$

and equation (1.1) resembles the singular mean field equation

$$-\Delta_g v = \lambda\Big(\frac{he^v}{\int_S he^v\,dv_g} - \frac{1}{|S|}\Big) - 4\pi\alpha\Big(\delta_q - \frac{1}{|S|}\Big) \quad \text{in } S,$$

with $\alpha = \frac{2}{\tau}$. According to a result of D'Aprile and Esposito [17, Theorem 1.4], it follows that the functional

$$\begin{aligned}\varphi^*_{m+1}(\xi) = \frac{1}{4\pi}\sum_{j=1}^{m}\log V_1(\xi_j) + \frac{1}{4\pi\tau^2}\log V_2(\xi_{m+1})\\ + \sum_{j=1}^{m_1} H(\xi_j,\xi_j) + \frac{1}{\tau^2}H(\xi_{m+1},\xi_{m+1})\\ + \sum_{j=1}^{m}\sum_{\substack{i=1\\ i\neq j}}^{m} G(\xi_i,\xi_j) - \frac{2}{\tau}\sum_{j=1}^{m} G(\xi_j,\xi_{m+1}),\end{aligned}$$

has a C^1-stable critical value for $\xi_{m+1} \in S$ fixed under the assumptions $S \neq \mathbb{S}^2, \mathbb{RP}^2$ and $\frac{2}{\tau} \neq 1, \dots, m-1$. Thus, we deduce the next result.

Corollary 2.2. *Assume that $V_i > 0$ in S for $i = 1, 2$, $S \neq \mathbb{S}^2, \mathbb{RP}^2$ and $\frac{2}{\tau} \neq 1, \dots, m-1$. If either*

$$\sup_S[2K - \Delta_g \log V_1] < \frac{8\pi}{|S|}\Big(m - \frac{1}{\tau}\Big) \quad \text{or} \quad \inf_S[2K - \Delta_g \log V_1] > \frac{8\pi}{|S|}\Big(m - \frac{1}{\tau}\Big)$$

and either $\sup_S[2K - \Delta_g \log V_2] < \frac{8\pi}{|S|}(1 - m\tau)$ or $\inf_S[2K - \Delta_g \log V_2] > \frac{8\pi}{|S|}(1 - m\tau)$, then there exist solutions u_{λ_1,λ_2} to (1.1) which concentrate at $m+1$ points, positively at $q_1, \dots, q_m$ and negatively at q_{m+1}, in the sense of (1.14) as $\lambda_1 \to 8\pi m$ and $\lambda_2\tau^2 \to 8\pi$, where $(q_1, \dots, q_{m+1})$ is a max-min critical point of φ^*_{m+1} in $S^{m+1} \setminus \Delta$.

When $S = \mathbb{T}$ and $V_1 = V_2 \equiv 1$, for any $\tau > 0$, $m\tau \neq 1$ and $\tau \notin \{2, 1, \frac{2}{3}, \dots, \frac{2}{m-1}\}$, Corollary 2.1 then provides the existence of blow-up solutions u_{λ_1,λ_2} concentrating at $m+1$ points as $\lambda_1 \to 8\pi m$ and $\lambda_2\tau^2 \to 8\pi$, where λ_1 belongs to a small right (left resp.) neighborhood of $8\pi m$ if $m\tau > 1$ (< 1 resp.) and $\lambda_2\tau^2$ belongs to a small left (right resp.) neighborhood of 8π. Notice that a similar result can be obtained in case $m_1 = 1$ and $m_2 = m$, namely, λ_1 close to 8π and $\lambda_2\tau^2$ close to $8\pi m$.

Observe that, on one hand, we generalize existence results of blowing-up solutions for mean field equations (1.3) in [23] to an asymmetric problem (1.1). And, on the other hand, we perform, in a compact Riemann surface S, a similar construction done for a sinh-Poisson equation in bounded domains with Dirichlet boundary conditions by [3] and extended to an asymmetric case in [50]. Both problems in [3, 50] do not contain any potential V_k and the existence of C^1-stable critical points of the corresponding φ^*_m implies the existence of blowing-up solutions. However, to prove our results is not enough to assume the existence of C^1-stable critical points of φ^*_m in (1.7). Admissibility conditions in terms of quantities either A^*_k's or B^*_k's have to be used, in the same spirit of [23]. After completion of this work, we have learned that in [1] the existence of C^1-stable critical points of vortex type Hamiltonians, including φ_m^* in (1.7), has been proved for a surface S which is not homeomorphic to the sphere nor the projective plane.

Finally, we point out that the type of arguments used to obtain our results have been also developed in several previous works by various authors. Let us quote a few papers from the vast literature concerning singular perturbation problems with nonlinearities of exponential type [8, 21, 26, 27, 31].

3 Approximation of the solution

The main idea to construct approximating solutions of (1.1), as in [23], is to use as "basic cells" the functions

$$u_{\delta,\xi}(x) = u_0\Big(\frac{|x-\xi|}{\delta}\Big) - 2\log\delta, \quad \delta > 0,\ \xi \in \mathbb{R}^2,$$

where $u_0(r) = \log \frac{8}{(1+r^2)^2}$. They are all the solutions of

$$\begin{cases} \Delta u + e^u = 0 & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^u < \infty, & \end{cases}$$

and do satisfy the following concentration property: $e^{u_{\delta,\xi}} \rightharpoonup 8\pi\delta_\xi$ in measure sense as $\delta \to 0$. We will use now isothermal coordinates to pull-back $u_{\delta,\xi}$ in S. Let us recall that every Riemann surface (S, g) is locally conformally flat, and the local coordinates in which g is conformal to the Euclidean metric are referred to as isothermal coordinates (see, for example, the simple existence proof provided by Chern [16]). For every $\xi \in S$ it amounts to finding a local chart y_ξ, with $y_\xi(\xi) = 0$, from a neighborhood of ξ onto $B_{2r_0}(0)$ (the choice of r_0 is independent of ξ) in which $g = e^{\hat\varphi_\xi(y_\xi(x))}dx$, where $\hat\varphi_\xi \in C^\infty(B_{2r_0}(0), \mathbb{R})$. In particular, $\hat\varphi_\xi$ relates with the Gaussian curvature K of (S, g) through the relation

$$\Delta\hat\varphi_\xi(y) = -2K(y_\xi^{-1}(y))e^{\hat\varphi_\xi(y)} \quad \text{for } y \in B_{2r_0}(0). \tag{3.1}$$

We can also assume that y_ξ, $\hat\varphi_\xi$ depends smoothly on ξ and that $\hat\varphi_\xi(0) = 0$, $\nabla\hat\varphi_\xi(0) = 0$. We now pull-back $u_{\delta,0}$ in $\xi \in S$, for $\delta > 0$, by simply setting

$$U_{\delta,\xi}(x) = u_{\delta,0}(y_\xi(x)) = \log\frac{8\delta^2}{(\delta^2 + |y_\xi(x)|^2)^2}$$

for $x \in y_\xi^{-1}(B_{2r_0}(0))$. Letting $\chi \in C_0^\infty(B_{2r_0}(0))$ be a radial cut-off function so that $0 \le \chi \le 1$, $\chi \equiv 1$ in $B_{r_0}(0)$, we introduce the function $PU_{\delta,\xi}$ as the unique solution of

$$\begin{cases} -\Delta_g PU_{\delta,\xi}(x) = \chi_\xi(x) e^{-\varphi_\xi(x)} e^{U_{\delta,\xi}(x)} - \frac{1}{|S|}\int_S \chi_\xi e^{-\varphi_\xi} e^{U_{\delta,\xi}} dv_g & \text{in } S, \\ \int_S PU_{\delta,\xi} dv_g = 0, \end{cases} \tag{3.2}$$

where $\chi_\xi(x) = \chi(|y_\xi(x)|)$ and $\varphi_\xi(x) = \hat{\varphi}_\xi(y_\xi(x))$. Notice that the R.H.S. in (3.2) has zero average and depends smoothly on x, and then (3.2) is uniquely solvable by a smooth solution $PU_{\delta,\xi}$.

Let us recall the transformation law for Δ_g under conformal changes: if $\tilde{g} = e^\varphi g$, then

$$\Delta_{\tilde{g}} = e^{-\varphi}\Delta_g. \tag{3.3}$$

Decompose now the Green function $G(x,\xi)$, $\xi \in S$, as

$$G(x,\xi) = -\frac{1}{2\pi}\chi_\xi(x)\log|y_\xi(x)| + H(x,\xi),$$

and by (1.6) then deduce that

$$\begin{cases} -\Delta_g H = -\frac{1}{2\pi}\Delta_g\chi_\xi \log|y_\xi(x)| - \frac{1}{\pi}\langle\nabla\chi_\xi, \nabla\log|y_\xi(x)|\rangle_g - \frac{1}{|S|} & \text{in } S, \\ \int_S H(\cdot,\xi)\, dv_g = \frac{1}{2\pi}\int_S \chi_\xi \log|y_\xi(\cdot)| dv_g. \end{cases}$$

We have used that $\Delta_g \log|y_\xi(x)| = e^{-\hat{\varphi}_\xi(y)}\Delta\log|y|\big|_{y=y_\xi(x)} = 2\pi\delta_\xi$ in view of (3.3). For $r \le 2r_0$ define $B_r(\xi) = y_\xi^{-1}(B_r(0))$, $A_r(\xi) = B_r(\xi)\setminus B_{r/2}(\xi)$, and set

$$f_\xi = \frac{\Delta_g\chi_\xi}{|y_\xi(x)|^2} + 2\langle\nabla\chi_\xi, \nabla|y_\xi(x)|^{-2}\rangle_g + \frac{2}{|S|}\int_{\mathbb{R}^2}\frac{\chi'(|y|)}{|y|^3}\, dy.$$

Setting

$$\Psi_{\delta,\xi}(x) = PU_{\delta,\xi}(x) - \chi_\xi[U_{\delta,\xi} - \log(8\delta^2)] - 8\pi H(x,\xi),$$

by the definition of f_ξ we then have that $-\Delta_g\Psi_{\delta,\xi} = -2\delta^2 f_\xi + O(\delta^4)$ in S so that

$$\int_S f_\xi dv_g = \frac{1}{2\delta^2}\int_S \Delta_g\Psi_{\delta,\xi} dv_g + O(\delta^2) = O(\delta^2)$$

for all $\delta > 0$, and hence $\int_S f_\xi dv_g = 0$. Therefore, F_ξ is well defined as the unique solution of

$$\begin{cases} -\Delta_g F_\xi = f_\xi & \text{in } S, \\ \int_S F_\xi dv_g = 0. \end{cases} \tag{3.4}$$

We have the following asymptotic expansion of $PU_{\delta,\xi}$ as $\delta \to 0$, as shown in [23]:

Lemma 3.1. *The function $PU_{\delta,\xi}$ satisfies*

$$PU_{\delta,\xi} = \chi_\xi[U_{\delta,\xi} - \log(8\delta^2)] + 8\pi H(x,\xi) + \alpha_{\delta,\xi} - 2\delta^2 F_\xi + O(\delta^4|\log\delta|)$$

uniformly in S, where F_ξ is given in (3.4) *and*

$$\alpha_{\delta,\xi} = -\frac{4\pi}{|S|}\delta^2\log\delta + 2\frac{\delta^2}{|S|}\left(\int_{\mathbb{R}^2}\chi(|y|)\frac{e^{\hat\varphi_\xi(y)}-1}{|y|^2}dy + \pi - \int_{\mathbb{R}^2}\frac{\chi'(|y|)\log|y|}{|y|}dy\right).$$

In particular, holds

$$PU_{\delta,\xi} = 8\pi G(x,\xi) - 2\frac{\delta^2\chi_\xi}{|y_\xi(x)|^2} + \alpha_{\delta,\xi} - 2\delta^2 F_\xi + O(\delta^4|\log\delta|)$$

holds locally uniformly in $S\setminus\{\xi\}$.

The ansatz will be constructed as follows. Given $m\in\mathbb{N}$, let us consider distinct points $\xi_j\in S$ and $\delta_j>0$, $j=1,\dots,m$. In order to have a good approximation, we will assume that $\exists\, C_0>1$:

$$\delta_j^2 = \begin{cases}\mu_1^2\delta^2\rho_j(\xi_j) & \text{for } j\in\{1,\dots,m_1\},\\ \mu_2^2\delta^2\rho_j(\xi_j) & \text{for } j\in\{m_1+1,\dots,m\},\end{cases}, \tag{3.5}$$

$$\text{with } 0<\mu_i\le C_0,\ i=1,2,$$

$$|\lambda_1 - 8\pi m_1| \le C_0\delta^2|\log\delta| \quad\text{and}\quad |\lambda_2\tau^2 - 8\pi m_2| \le C_0\delta^2|\log\delta|, \tag{3.6}$$

where $\delta>0$, $m_1\in\{1,\dots,m-1\}$, $m_2=m-m_1$ and ρ_j is given by (1.8)–(1.9). Up to taking r_0 smaller, we assume that the points ξ_j's are well separated and $V_1(\xi_j)$, $V_2(\xi_j)$ are uniformly away from zero, namely, we choose $\xi=(\xi_1,\dots,\xi_m)\in\Xi$, where

$$\Xi = \{(\xi_1,\dots,\xi_m)\in S^m \mid d_g(\xi_i,\xi_j)\ge 4r_0 \text{ and } V_1(\xi_j),\ V_2(\xi_j)\ge r_0\ \forall\, i,j=1,\dots,m,\ i\ne j\}.$$

Denote $U_j := U_{\delta_j,\xi_j}$ and $W_j = PU_j$, $j=1,\dots,m$, where P is the projection operator defined by (3.2). Thus, our approximating solution is

$$W(x) = \sum_{j=1}^{m_1} W_j(x) - \frac{1}{\tau}\sum_{j=m_1+1}^{m} W_j(x),$$

parametrized by $(\mu,\xi)\in\mathcal{M}\times\Xi$, with $\mu=(\mu_1,\mu_2)$ and $\mathcal{M}=(0,C_0]\times(0,C_0]$. Notice that for r_0 small enough we have that $\mathcal{D}\subset\Xi\subset\tilde S^m\setminus\Delta$. We will look for a solution u of (1.1) in the form $u = W + \phi$, for some small remainder term ϕ. In terms of ϕ, the problem (1.1) is equivalent to finding $\phi \in \overline{H}$ so that

$$L(\phi) = -[R + N(\phi)] \quad \text{in } S, \tag{3.7}$$

where the linear operator L is defined as

$$L(\phi) = \Delta_g\phi + \sum_{i=1}^{2} \lambda_i \tau^{2(i-1)} \frac{V_i(x)e^{(-\tau)^{i-1}W}}{\int_S V_i e^{(-\tau)^{i-1}W} dv_g}\left(\phi - \frac{\int_S V_i e^{(-\tau)^{i-1}W}\phi dv_g}{\int_S V_i e^{(-\tau)^{i-1}W} dv_g}\right), \tag{3.8}$$

the nonlinear part N is given by

$$N(\phi) = N_1(\phi) - N_2(\phi) \tag{3.9}$$

with

$$\begin{aligned} N_i(\phi) = \lambda_i \tau^{i-1}\Bigg(&\frac{V_i e^{(-\tau)^{i-1}(W+\phi)}}{\int_S V_i e^{(-\tau)^{i-1}(W+\phi)} dv_g} \\ &- \frac{(-\tau)^{i-1} V_i e^{(-\tau)^{i-1}W}}{\int_S V_i e^{(-\tau)^{i-1}W} dv_g}\left[\phi - \frac{\int_S V_i e^{(-\tau)^{i-1}W}\phi dv_g}{\int_S V_i e^{(-\tau)^{i-1}W} dv_g}\right] \\ &- \frac{V_i e^{(-\tau)^{i-1}W}}{\int_S V_i e^{(-\tau)^{i-1}W} dv_g}\Bigg) \end{aligned} \tag{3.10}$$

for $i = 1, 2$ and the approximation rate of W is encoded in

$$R = \Delta_g W + \lambda_1\left(\frac{V_1(x)e^W}{\int_S V_1 e^W dv_g} - \frac{1}{|S|}\right) - \lambda_2\tau\left(\frac{V_2(x)e^{-\tau W}}{\int_S V_2 e^{-\tau W} dv_g} - \frac{1}{|S|}\right). \tag{3.11}$$

Notice that for all $\phi \in \overline{H}$

$$\int_S L(\phi) dv_g = \int_S N(\phi) dv_g = \int_S R dv_g = 0.$$

In order to get the invertibility of L, let us introduce the weighted norm for any $h \in L^\infty(S)$

$$\|h\|_* = \sup_{x\in S}\left[\sum_{j=1}^{m} \frac{\delta_j^\sigma}{(\delta_j^2 + \chi_{B_{r_0}(\xi_j)}(x)|y_{\xi_j}(x)|^2 + r_0^2\chi_{S\setminus B_{r_0}(\xi_j)}(x))^{1+\sigma/2}}\right]^{-1} |h(x)|,$$

where $0 < \sigma < 1$ is a small fixed constant and χ_A denotes the characteristic function of the set A. Let us evaluate the approximation rate of W in $\|\cdot\|_*$ and recall that $m = m_1 + m_2$:

Lemma 3.2. *Assume* (3.5)–(3.6). *There exists a constant* $C > 0$*, independent of* $\delta > 0$ *small, such that*

$$\|R\|_* \le C(\delta\,|\nabla\varphi_m^*(\xi)|_g + \delta^{2-\sigma}|\log\delta|) \tag{3.12}$$

for all $\xi \in \Xi$*, where* $|\nabla\varphi_m^*(\xi)|_g^2$ *stands for* $\sum_{j=1}^m |\nabla_{\xi_j}\varphi_m^*(\xi)|_g^2$.

Proof. We shall argue in the same way as in [23, Lemma 2.1]. First, from Lemma 3.1 we note that for any $j \in \{1, \dots, m\}$,

$$W_j(x) = U_j(x) - \log(8\delta_j^2) + 8\pi H(x, \xi_j) + O(\delta^2 |\log \delta|)$$

uniformly for $x \in B_{r_0}(\xi_j)$ and

$$W_j(x) = 8\pi G(x, \xi_j) + O(\delta^2 |\log \delta|)$$

uniformly for x on compact subsets of $S \setminus \{\xi_j\}$. Since by symmetry and $\hat{\varphi}_{\xi_j}(0) = 0$ we have that

$$\int_{B_{r_0}(\xi_j)} \rho_j(x) e^{U_j} dv_g = 8\pi \rho_j(\xi_j) + O(\delta^2 |\log \delta|),$$

we then get that for $j \in \{1, \dots, m_1\}$

$$\begin{aligned}
&\int_{B_{r_0}(\xi_j)} V_1 e^W dv_g \\
(3.13) \qquad &= \frac{1}{8\delta_j^2} \int_{B_{r_0}(\xi_j)} V_1 e^{U_j + 8\pi H(x,\xi_j) + 8\pi \sum_{l=1, l\neq j}^{m_1} G(x,\xi_l) - \frac{8\pi}{\tau} \sum_{l=m_1+1}^{m} G(x,\xi_l) + O(\delta^2 |\log \delta|)} dv_g \\
&= \frac{1}{\delta_j^2} [\pi \rho_j(\xi_j) + O(\delta^2 |\log \delta|)] = \frac{\pi}{\mu_1^2 \delta^2} + O(|\log \delta|)
\end{aligned}$$

and for $j \in \{m_1 + 1, \dots, m\}$

(3.14)

$$\begin{aligned}
&\int_{B_{r_0}(\xi_j)} V_1 e^W dv_g \\
&= \int_{B_{r_0}(\xi_j)} V_1 e^{-\frac{1}{\tau}[U_j - \log(8\delta_j^2) + 8\pi H(x,\xi_j)] + 8\pi \sum_{l=1}^{m_1} G(x,\xi_l) - \frac{8\pi}{\tau} \sum_{l=m_1+1, l\neq j}^{m} G(x,\xi_l) + O(\delta^2 |\log \delta|)} dv_g \\
&= \int_{B_{r_0}(\xi_j)} V_1(x) \Big[\frac{\rho_j(x)}{V_2(x)}\Big]^{-1/\tau} (\delta_j^2 + |y_{\xi_j}(x)|^2)^{2/\tau} (1 + O(\delta^2 |\log \delta|)) dv_g \\
&= O(1).
\end{aligned}$$

So, by using (3.13)–(3.14) we have that

$$\int_S V_1 e^W dv_g = \sum_{j=1}^{m_1} \int_{B_{r_0}(\xi_j)} V_1 e^W dv_g + O(1) = \frac{\pi m_1}{\mu_1^2 \delta^2} + O(|\log \delta|). \tag{3.15}$$

Similarly, for $j \in \{1, \dots, m_1\}$ we get that

$$\int_{B_{r_0}(\xi_j)} V_2 e^{-\tau W} dv_g = O(1), \tag{3.16}$$

and for $j \in \{m_1 + 1, \dots, m\}$

$$\int_{B_{r_0}(\xi_j)} V_2 e^{-\tau W} dv_g = \frac{1}{\delta_j^2} [\pi \rho_j(\xi_j) + O(\delta^2 |\log \delta|)] = \frac{\pi}{\mu_2^2 \delta^2} + O(|\log \delta|). \tag{3.17}$$

So, by using (3.16)–(3.17) we have that

$$\int_S V_2 e^{-\tau W} dv_g = \sum_{j=m_1+1}^{m} \int_{B_{r_0}(\xi_j)} V_2 e^W dv_g + O(1) = \frac{\pi m_2}{\mu_2^2 \delta^2} + O(|\log\delta|). \tag{3.18}$$

By Lemma 3.1 and (3.5), (3.15), (3.18) we have that

- in $S \setminus \bigcup_{j=1}^m B_{r_0}(\xi_j)$, $\lambda_1 \frac{V_1 e^W}{\int_S V_1 e^W dv_g} = O(\delta^2)$ holds in view of $W(x) = O(1)$;
- in $B_{r_0}(\xi_j), j \in \{1, \dots, m_1\}$, we have

$$\begin{aligned}
&\frac{V_1 e^W}{\int_S V_1 e^W dv_g}\\
&= \frac{V_1 e^{-\log(8\delta_j^2)+8\pi H(x,\xi_j)+8\pi\sum_{l=1,l\neq j}^{m_1} G(x,\xi_l) - \frac{8\pi}{\tau}\sum_{l=m_1+1}^{m} G(x,\xi_l) + O(\delta^2|\log\delta|)}}{\pi m_1 \mu_1^{-2}\delta^{-2} + O(|\log\delta|)} e^{U_j}\\
&= \frac{1}{8\pi m_1}\Big[1 + \Big\langle \frac{\nabla(\rho_j \circ y_{\xi_j}^{-1})(0)}{\rho_j(\xi_j)}, y_{\xi_j}(x)\Big\rangle + O(|y_{\xi_j}(x)|^2 + \delta^2|\log\delta|)\Big] e^{U_j};
\end{aligned}$$

- in $B_{r_0}(\xi_j), j \in \{m_1+1, \dots, m\}$, there holds

$$\begin{aligned}
&\frac{V_1 e^W}{\int_S V_1 e^W dv_g}\\
&= \frac{V_1(x)[\rho_j(x)/V_2(x)]^{-1/\tau} + O(\delta^2|\log\delta|)}{\pi m_1 \mu_1^{-2}\delta^{-2} + O(|\log\delta|)} (\delta_j^2 + |y_{\xi_j}(x)|^2)^{2/\tau} = O(\delta^2).
\end{aligned}$$

Similarly as above, we have that

- in $S \setminus \bigcup_{j=1}^m B_{r_0}(\xi_j)$, $\lambda_2 \tau \frac{V_2 e^{-\tau W}}{\int_S V_2 e^{-\tau W} dv_g} = O(\delta^2)$ holds in view of $W(x) = O(1)$;
- in $B_{r_0}(\xi_j), j \in \{1, \dots, m_1\}$, we have

$$\begin{aligned}
&\frac{V_2 e^{-\tau W}}{\int_S V_2 e^{-\tau W} dv_g}\\
&= \frac{V_2(x)[\rho_j(x)/V_1(x)]^{-\tau} + O(\delta^2|\log\delta|)}{\pi m_2 \mu_2^{-2}\delta^{-2} + O(|\log\delta|)} (\delta_j^2 + |y_{\xi_j}(x)|^2)^{2\tau} = O(\delta^2),
\end{aligned}$$

- in $B_{r_0}(\xi_j), j \in \{m_1+1, \dots, m\}$, we have

$$\begin{aligned}
&\frac{V_2 e^{-\tau W}}{\int_S V_2 e^{-\tau W} dv_g}\\
&= \frac{1}{8\pi m_2}\Big[1 + \Big\langle \frac{\nabla(\rho_j \circ y_{\xi_j}^{-1})(0)}{\rho_j(\xi_j)}, y_{\xi_j}(x)\Big\rangle + O(|y_{\xi_j}(x)|^2 + \delta^2
\end{aligned}$$

Since as before

$$\int_S \chi_j e^{-\varphi_j} e^{U_j} dv_g = \int_{B_{r_0}(0)} \frac{8\delta_j^2}{(\delta_j^2 + |y|^2)^2} dy + O(\delta^2) = 8\pi + O(\delta^2)$$

with $\varphi_j = \varphi_{\xi_j}$, for R given by (3.11) we then have that

$$R = -\sum_{j=1}^{m_1} \chi_j e^{-\varphi_j} e^{U_j} + \frac{\lambda_1 V_1 e^W}{\int_S V_1 e^W dv_g} + \frac{8\pi m_1 - \lambda_1}{|S|} + O(\delta^2)$$
$$+ \frac{1}{\tau} \sum_{j=m_1+1}^{m} \chi_j e^{-\varphi_j} e^{U_j} - \frac{\lambda_2 \tau V_2 e^{-\tau W}}{\int_S V_2 e^{-\tau W} dv_g} + \frac{\lambda_2 \tau^2 - 8\pi m_2}{|S|\tau} + O(\delta^2),$$

where $\chi_j = \chi_{\xi_j}$. By previous computations we now deduce that $R(x) = O(\delta^2)$ in $S \setminus \bigcup_{j=1}^m B_{r_0}(\xi_j)$,

$$R = \Big[-e^{-\varphi_j} + \frac{\lambda_1}{8\pi m_1} + O(|\nabla \log(\rho_j \circ y_{\xi_j}^{-1})(0)||y_{\xi_j}(x)| + |y_{\xi_j}(x)|^2 + \delta^2 |\log \delta|)\Big] e^{U_j}$$
$$+ O(|\lambda_1 - 8\pi m_1| + |\lambda_2 \tau^2 - 8\pi m_2| + \delta^2)$$
$$= e^{U_j} O(|\nabla \log(\rho_j \circ y_{\xi_j}^{-1})(0)||y_{\xi_j}(x)| + |y_{\xi_j}(x)|^2 + |\lambda_1 - 8\pi m_1| + \delta^2 |\log \delta|)$$
$$+ O(|\lambda_1 - 8\pi m_1| + |\lambda_2 \tau^2 - 8\pi m_2| + \delta^2)$$

in $B_{r_0}(\xi_j)$, $j \in \{1, \dots, m_1\}$ and similarly,

$$R = e^{U_j} O(|\nabla \log(\rho_j \circ y_{\xi_j}^{-1})(0)||y_{\xi_j}(x)| + |y_{\xi_j}(x)|^2 + |\lambda_2 \tau^2 - 8\pi m_2| + \delta^2 |\log \delta|)$$
$$+ O(|\lambda_1 - 8\pi m_1| + |\lambda_2 \tau^2 - 8\pi m_2| + \delta^2)$$

in $B_{r_0}(\xi_j)$, $j \in \{m_1+1, \dots, m\}$, in view of $\varphi_j(\xi_j) = 0$ and $\nabla\varphi_j(\xi_j) = 0$. From the definition of $\|\cdot\|_*$ and (3.6) we deduce the validity of (3.12). This finishes the proof. □

4 Variational reduction and proof of main results

The solvability theory for the linear operator L given in (3.8), obtained as the linearization of (1.1) at the approximating solution W, is a key step in the so-called nonlinear Lyapunov–Schimdt reduction. Notice that formally the operator L approaches $\hat{L}$ defined in $\mathbb{R}^2$ as

$$\hat{L}(\phi) = \Delta\phi + \frac{8}{(1+|y|^2)^2}\left(\phi - \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\phi(z)}{(1+|z|^2)^2} dz\right),$$

by setting $y = y_{\xi_j}(x)/\delta_j$ as $\delta \to 0$. Due to the intrinsic invariances, the kernel of $\hat{L}$ in $L^\infty(\mathbb{R}^2)$ is non-empty and is spanned by 1 and Y_j, $j = 0, 1, 2$, where $Y_i(y) = \frac{4y_i}{1+|y|^2}$, $i = 1, 2$, and $Y_0(y) = 2\,\frac{1-|y|^2}{1+|y|^2}$. Since publications [20, 23, 25] it is by now rather standard to show the invertibility of L in a suitable "orthogonal" space, and a sketched proof of it will be given in Appendix A. However, as observed in [23], for Dirichlet Liouville-type equations on bounded domains as in [20, 25], the corresponding limiting operator $\tilde{L}$ takes the form $\tilde{L}(\phi) = \Delta\phi + \frac{8}{(1+|y|^2)^2}\phi$ and the function 1 does not belong to its kernel, making it possible to disregard the "dilation parameters" δ_i in the reduction. As we will see, two additional parameters μ_1 and μ_2 are needed in the reduction (one associated to all "positive bubbles" and the other one to all "negative bubbles") and in this respect our problem displays a new feature w.r.t. Dirichlet Liouville-type equations, making our situation very similar to the one arising in the study of critical problems in higher dimension. Roughly speaking, L resemble a "direct sum" of linear operators for mean field type equations.

To be more precise, for $i = 0, 1, 2$ and $j = 1, \ldots, m$ introduce the functions

$$Z_{ij}(x) = Y_i\Big(\frac{y_{\xi_j}(x)}{\delta_j}\Big) = \begin{cases} 2\frac{\delta_j^2 - |y_{\xi_j}(x)|^2}{\delta_j^2 + |y_{\xi_j}(x)|^2} & \text{for } i = 0, \\ \frac{4\delta_j (y_{\xi_j}(x))_i}{\delta_j^2 + |y_{\xi_j}(x)|^2} & \text{for } i = 1, 2, \end{cases}$$

and set $Z_1 = \sum_{l=1}^{m_1} Z_{0l}$ and $Z_2 = \sum_{l=m_1+1}^{m} Z_{0l}$. For $i = 1, 2$ and $j = 1, \ldots, m$, let PZ_i and PZ_{ij} be the projections of Z_i, Z_{ij} as the solutions in $\overline{H}$ of

$$\begin{aligned} \Delta_g PZ_i &= \chi_j \Delta_g Z_i - \frac{1}{|S|}\int_S \chi_j \Delta_g Z_i dv_g, \\ \Delta_g PZ_{ij} &= \chi_j \Delta_g Z_{ij} - \frac{1}{|S|}\int_S \chi_j \Delta_g Z_{ij} dv_g. \end{aligned} \tag{4.1}$$

In Appendix A we prove the following result:

Proposition 4.1. *There exists $\delta_0 > 0$ so that for all $0 < \delta \le \delta_0$, $h \in C(S)$ with $\int_S h\, dv_g = 0$, $\mu \in \mathcal{M}$, $\xi \in \Xi$ there is a unique solution $\phi \in \overline{H} \cap W^{2,2}(S)$ and $c_{0i}, c_{ij} \in \mathbb{R}$ of*

$$\begin{cases} L(\phi) = h + \sum_{i=1}^{2}[c_{0i}\Delta_g PZ_i + \sum_{j=1}^{m} c_{ij}\Delta_g PZ_{ij}] & \text{in } S, \\ \int_S \phi \Delta_g PZ_i dv_g = \int_S \phi\Delta_g PZ_{ij} dv_g = 0 & \\ & \forall\, i = 1, 2,\ j = 1, \ldots, m. \end{cases} \tag{4.2}$$

Moreover, the map $(\mu, \xi) \mapsto (\phi, c_{0i}, c_{ij})$ is twice-differentiable in μ and once-differentiable in ξ with

$$\|\phi\|_\infty \le C|\log\delta|\|h\|_*, \quad \sum_{i=1}^{2}\Big[|c_{0i}| + \sum_{j=1}^{m}|c_{ij}|\Big] \le C\|h\|_*, \tag{4.3}$$

$$\sum_{i=1}^{2}\Big[\|\partial_{\mu_i}\phi\|_\infty + \sum_{k=1}^{2}\frac{1}{|\log\delta|}\|\partial_{\mu_i\mu_k}\phi\|_\infty + \sum_{j=1}^{m}\delta\|\partial_{(\xi_j)_i}\phi\|_\infty\Big] \le C|\log\delta|^2\|h\|_* \tag{4.4}$$

for some $C > 0$.

Let us recall that $u = W + \phi$ solves (1.1) if $\phi \in \bar{H}$ does satisfy (3.7). Since the operator L is not fully invertible, in view of Proposition 4.1 one can solve the nonlinear problem (3.7) just up to a linear combination of $\Delta_g PZ_1$, $\Delta_g PZ_2$ and $\Delta_g PZ_{ij}$, as explained in the following (see Appendix B for the proof):

Proposition 4.2. *There exists* $\delta_0 > 0$ *so that for all* $0 < \delta \le \delta_0$, $\mu \in \mathcal{M}$, $\xi \in \Xi$ *the problem*

$$
\begin{cases} L(\phi) = -[R + N(\phi)] + \sum_{i=1}^{2} [c_{0i}\Delta_g PZ_i + \sum_{j=1}^{m} c_{ij}\Delta_g PZ_{ij}] \quad \text{in } S, \\ \int_S \phi \Delta_g PZ_i dv_g = \int_S \phi \Delta_g PZ_{ij} dv_g = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, i = 1, 2,\ j = 1, \dots, m \end{cases} \tag{4.5}
$$

admits a unique solution $\phi(\mu, \xi) \in \bar{H} \cap W^{2,2}(S)$ *and* $c_{0i}(\mu, \xi)$, $c_{ij}(\mu, \xi) \in I\!R$, $i = 1, 2$ *and* $j = 1, \dots, m$, *where* $\delta_j > 0$ *are as in* (3.5) *and* N, R *are given by* (3.9), (3.11)*, respectively. Moreover, the map* $(\mu, \xi) \mapsto (\phi(\mu, \xi), c_{0i}(\mu, \xi), c_{ij}(\mu, \xi))$ *is twice-differentiable in* μ *and once-differentiable in* ξ *with*

$$
\|\phi\|_\infty \le C(\delta|\log\delta|\,|\nabla\varphi_m(\xi)|_g + \delta^{2-\sigma}|\log\delta|^2), \tag{4.6}
$$

$$
\sum_{i=1}^{2}\left[\|\partial_{\mu_i}\phi\|_\infty + \sum_{j=1}^{m}\delta\|\partial_{(\xi_j)_i}\phi\|_\infty + \sum_{k=1}^{2}\frac{\|\partial_{\mu_i\mu_k}\phi\|_\infty}{|\log\delta|}\right] \le C(\delta|\log\delta|^2|\nabla\varphi_m(\xi)|_g + \delta^{2-\sigma}|\log\delta|^3). \tag{4.7}
$$

The function $[W + \phi](\mu, \xi)$ will be a true solution of (3.7) if $\mu \in \mathcal{M}$ and $\xi \in \Xi$ are such that $c_{0i}(\mu, \xi) = c_{ij}(\mu, \xi) = 0$ for all $i = 1, 2$, and $j = 1, \dots, m$. This problem is equivalent to finding critical points of the reduced energy

$$
E_{\lambda_1,\lambda_2}(\mu, \xi) = J_{\lambda_1,\lambda_2}([W + \phi](\mu, \xi)),
$$

where J_{λ_1,λ_2} is given by (1.4), as stated in (we omit its proof):

Lemma 4.1. *There exists* δ_0 *such that, if* $(\mu, \xi) \in \mathcal{M} \times \Xi$ *is a critical point of* E_{λ_1,λ_2} *for* $0 < \delta \le \delta_0$, *then* $u = W(\mu, \xi) + \phi(\mu, \xi)$ *is a solution to* (1.1)*, where* δ_i *are given by* (3.5).

Once equation (1.1) has been reduced to the search of c.p.'s for E_{λ_1,λ_2}, it becomes crucial to show that the main asymptotic term of E_{λ_1,λ_2} is given by $J_{\lambda_1,\lambda_2}(W)$, for which an expansion has been given in Theorem 5.1. More precisely, by estimates in Appendix B we have

Theorem 4.1. *Assume* (3.5)–(3.6). *The following expansion does hold:*

$$
\begin{aligned}
&E_{\lambda_1,\lambda_2}(\mu,\xi)\\
&= -8\pi\Big(m_1+\frac{m_2}{\tau^2}\Big) - \lambda_1\log(\pi m_1) - \lambda_2\log(\pi m_2) + 2(\lambda_1-8\pi m_1)\log\delta\\
&\quad+\frac{2}{\tau^2}(\lambda_2\tau^2-8\pi m_2)\log\delta - 32\pi^2\varphi_m^*(\xi) + 2(\lambda_1-8\pi m_1)\log\mu_1\\
&\quad+A_1^*(\xi)\mu_1^2\delta^2\log\delta + [A_1^*(\xi)\mu_1^2\log\mu_1 - B_1^*(\xi)\mu_1^2]\delta^2\\
&\quad+\frac{1}{\tau^2}\{2(\lambda_2\tau^2-8\pi m_2)\log\mu_2\\
&\qquad+A_2^*(\xi)\mu_2^2\delta^2\log\delta + [A_2^*(\xi)\mu_2^2\log\mu_2 - B_2^*(\xi)\mu_2^2]\delta^2\}\\
&\quad+o(\delta^2) + r_{\lambda_1,\lambda_2}(\mu,\xi)
\end{aligned}
\tag{4.8}
$$

in $C^2(\mathbb{R}^2)$ *and* $C^1(\Xi)$ *as* $\delta\to 0^+$, *where* $\varphi_m^*(\xi)$, $A_k^*(\xi)$ *and* $B_k^*(\xi)$, $k=1,2$ *are given by* (1.7), (1.10) *and* (1.11), $k=1,2$, *respectively. The term* $r_{\lambda_1,\lambda_2}(\mu,\xi)$ *satisfies*

$$
\begin{aligned}
&|r_{\lambda_1,\lambda_2}(\mu,\xi)| + \frac{\delta|\nabla_\xi r_{\lambda_1,\lambda_2}(\mu,\xi)|}{|\log\delta|} + \frac{|\nabla_\mu r_{\lambda_1,\lambda_2}(\mu,\xi)|}{|\log\delta|}\\
&\qquad+\frac{|D_\mu^2 r_{\lambda_1,\lambda_2}(\mu,\xi)|}{|\log\delta|^2} \le C(\delta^2|\log\delta|\,|\nabla\varphi_m^*(\xi)|_g^2 + \delta^{3-\sigma}|\log\delta|^2)
\end{aligned}
\tag{4.9}
$$

for some $C>0$ *independent of* $(\mu,\xi)\in\mathcal{M}\times\Xi$.

We are now in position to establish the main result stated in the Introduction. We shall argue similarly to [23, Theorem 1.5].

Proof of Theorem 1.1. According to Lemma 4.1, we just need to find a critical point of $E=E_{\lambda_1,\lambda_2}(\mu,\xi)$ with $\mu=(\mu_1,\mu_2)$. Recall that $\tau>0$ is fixed. Assumptions (1.12) and (1.13) allow us to choose $\mu_k=\mu_k(\lambda_k,\xi)$ for $\lambda_k\tau^{2(k-1)}$ close to $8\pi m_k$, $k=1,2$, respectively. Precisely, fixing $k\in\{1,2\}$ we choose $\lambda_k\tau^{2(k-1)}-8\pi m_k=\delta^2$ ($-\delta^2$ resp.) if either $A_k^*(\xi)>0$ (<0 resp.) or $A^*(\xi)=0$, $B_k^*(\xi)>0$ (<0 resp.) in U. Thus, we deduce the expansions

$$
\begin{aligned}
\frac{\tau^{2(k-1)}\partial_{\mu_k}E(\mu,\xi)}{\lambda_k\tau^{2(k-1)}-8\pi m_k} &= \frac{2}{\mu_k} + 2A_k^*(\xi)\mu_k\log\delta + A_k^*(\xi)(2\mu_k\log\mu_k+\mu_k) - 2B_k^*(\xi)\mu_k\\
&\quad+o(1)+O(|\log\delta|^2\,|\nabla\varphi_m^*(\xi)|_g^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\tau^{2(k-1)}\partial_{\mu_k\mu_k}E(\mu,\xi)}{\lambda_k\tau^{2(k-1)}-8\pi m_k} &= -\frac{2}{\mu_k^2} + 2A_k^*(\xi)\log\delta + A_k^*(\xi)(2\log\mu_k+3) - 2B_k^*(\xi)\\
&\quad+o(1)+O(|\log\delta|^3\,|\nabla\varphi_m^*(\xi)|_g^2),
\end{aligned}
$$

as $\delta \to 0^+$. Arguing in the same way as in the proof of Theorem 3.2 in [23], we conclude the existence of a C^1 map $\mu_k = \mu_k(\lambda_k, \xi)$ satisfying

$$\partial_{\mu_k} E(\mu(\lambda, \xi), \xi) = 0,$$

with $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ for all $\xi \in U$. Now, considering

$$\tilde{E}(\xi) = E_\lambda(\mu_1(\lambda_1, \xi), \mu_2(\lambda_2, \xi), \xi)$$

and again arguing in the same way as in the proof of Theorem 3.2 in [23] it follows that $\tilde{E}(\xi) = -32\pi^2\varphi_m^*(\xi) + O(\delta^2|\log\delta|)$,

$$\begin{aligned}\nabla_\xi \tilde{E}(\xi) &= \nabla_\xi E(\mu_1(\lambda_1, \xi), \mu_2(\lambda_2, \xi), \xi)\\ &\quad + \nabla_\mu E(\mu_1(\lambda_1, \xi), \mu_2(\lambda_2, \xi), \xi)\nabla_\xi \mu(\lambda, \xi)\\ &= -32\pi^2\nabla\varphi_m^*(\xi) + O(\delta|\log\delta|^2)\end{aligned}$$

uniformly in $\xi \in U$ and there exists a critical point $\xi_{\lambda_1,\lambda_2} = \xi_\delta \in U$ of $\tilde{E}(\xi)$, since $\mathcal{D}$ is a stable critical set of φ_m^* (see Definition 2.1). Up to taking U smaller so that $\nabla\varphi_m^*(\xi) \neq 0$ for all $\xi \in U \setminus \mathcal{D}$, it can be deduced that the pair $(\mu(\lambda_1, \lambda_2, \xi_\delta), \xi_\delta)$ is a c.p. of $E(\mu, \xi)$ and, along a sub-sequence, $\xi_\delta \to q \in \mathcal{D}$ as $\delta \to 0$, namely, as $\lambda_1 \to 8\pi m_1$ and $\lambda_2\tau^2 \to 8\pi m_2$. By construction, the corresponding solution has the required asymptotic properties (1.14). See proof of Theorem 1.5 in [23] for more details. This completes the proof. □

5 The reduced energy

The purpose of this section is to give an asymptotic expansion of the "reduced energy" $J_{\lambda_1,\lambda_2}(W)$, where J_{λ_1,λ_2} is the energy functional given by (1.4). For technical reasons, we will be concerned with establishing it in a C^2-sense in μ and just in a C^1-sense in ξ. To this aim, the following result will be very useful; see [23, Lemma 3.1] for a proof.

Lemma 5.1. *Letting $f \in C^{2,\gamma}(S)$ (possibly depending in ξ), $0 < \gamma < 1$, denote by $P_2(f)$ the second-order Taylor expansion of $f(x)$ at ξ:*

$$P_2 f(x) = f(\xi) + \langle \nabla(f \circ y_\xi^{-1})(0), y_\xi(x)\rangle + \frac{1}{2}\langle D^2(f \circ y_\xi^{-1})(0) y_\xi(x), y_\xi(x)\rangle.$$

The following expansions do hold as $\delta \to 0$:

$$\begin{aligned}\int_S \chi_\xi e^{-\varphi_\xi} f(x) e^{U_{\delta,\xi}} dv_g &= 8\pi f(\xi) - 2\delta^2 \Delta_g f(\xi)\left[2\pi \log\delta + \int_{\mathbb{R}^2} \frac{\chi'(|y|)\log|y|}{|y|} dy + \pi\right] \\ &\quad + 8\delta^2 \int_S \chi_\xi e^{-\varphi_\xi} \frac{f(x) - P_2(f)(x)}{|y_\xi(x)|^4} dv_g + 4\delta^2 f(\xi)\int_{\mathbb{R}^2} \frac{\chi'(|y|)}{|y|^3} dy + o(\delta^2),\end{aligned}$$

$$\int_S \chi_\xi e^{-\varphi_\xi} f(x) e^{U_{\delta,\xi}} \frac{dv_g}{\delta^2 + |y_\xi(x)|^2} = \frac{4\pi}{\delta^2} f(\xi) + \pi \Delta_g f(\xi) + O(\delta^\gamma)$$

and

$$\int_S \chi_\xi e^{-\varphi_\xi} f(x) e^{U_{\delta,\xi}} \frac{a\delta^2 - |y_\xi(x)|^2}{(\delta^2 + |y_\xi(x)|^2)^2} dv_g = \frac{4\pi}{3\delta^2}(2a-1) f(\xi) + (a-2)\frac{\pi}{3}\Delta_g f(\xi) + O(\delta^\gamma)$$

for $a \in \mathbb{R}$.

We are now ready to establish the expansion of $J_{\lambda_1,\lambda_2}(W)$:

Theorem 5.1. *Assume* (3.5)–(3.6). *The following expansion does hold*

$$\begin{aligned} J_{\lambda_1,\lambda_2}(W) &= -8\pi\left(m_1 + \frac{m_2}{\tau^2}\right) - \lambda_1 \log(\pi m_1) - \lambda_2 \log(\pi m_2) + 2(\lambda_1 - 8\pi m_1)\log\delta \\ &\quad + \frac{2}{\tau^2}(\lambda_2 \tau^2 - 8\pi m_2)\log\delta - 32\pi^2 \varphi_m^*(\xi) + 2(\lambda_1 - 8\pi m_1)\log\mu_1 \\ &\quad + A_1^*(\xi)\mu_1^2\delta^2\log\delta + [A_1^*(\xi)\mu_1^2 \log\mu_1 - B_1^*(\xi)\mu_1^2]\delta^2 \\ &\quad + \frac{1}{\tau^2}\{2(\lambda_2\tau^2 - 8\pi m_2)\log\mu_2 \\ &\qquad + A_2^*(\xi)\mu_2^2\delta^2\log\delta + [A_2^*(\xi)\mu_2^2\log\mu_2 - B_2^*(\xi)\mu_2^2]\delta^2\} + o(\delta^2) \end{aligned} \tag{5.1}$$

in $C^2(\mathbb{R}^2)$ and $C^1(\Xi)$ as $\delta \to 0^+$, where $\varphi_m^*(\xi)$, $A_1^*(\xi)$, $A_2^*(\xi)$, $B_1^*(\xi)$ and $B_2^*(\xi)$ are given by (1.7), (1.10) *and* (1.11), *$k = 1, 2$, respectively.*

As in [23, Theorem 3.2], the proof will be divided into several steps.

Proof of 5.1 in $C(\mathbb{R}^2 \times \Xi)$. First, let us consider the term. Integrating by parts we have that

$$\int_S |\nabla W|_g^2 dv_g = \sum_{j,l=1}^{m_1} \int_S \chi_j e^{-\varphi_j} e^{U_j} W_l dv_g - \frac{1}{\tau} \sum_{j=1}^{m_1} \sum_{l=m_1+1}^{m} \int_S \chi_j e^{-\varphi_j} e^{U_j} W_l dv_g$$
$$- \frac{1}{\tau} \sum_{j=m_1+1}^{m} \sum_{l=1}^{m_1} \int_S \chi_j e^{-\varphi_j} e^{U_j} W_l dv_g$$
$$+ \frac{1}{\tau^2} \sum_{j,l=m_1+1}^{m} \int_S \chi_j e^{-\varphi_j} e^{U_j} W_l dv_g$$

in view of $\int_S W dv_g = 0$. Since by (1.6) and (3.2)

$$\int_S \chi_j e^{-\varphi_j} e^{U_j} G(x, \xi_l) dv_g = \int_S (-\Delta_g PU_j) G(x, \xi_l) dv_g = PU_j(\xi_l) \tag{5.2}$$

for all $j, l = 1, \ldots, m$, by Lemmata 3.1, 5.1, (5.2) and computations done in the proof of [23, Theorem 3.2], we have that for $l = j$

$$\int_S \chi_j e^{-\varphi_j} e^{U_j} W_j dv_g$$
$$= -16\pi - 32\pi \log \delta_j + 64\pi^2 H(\xi_j, \xi_j) + 16\pi \alpha_{\delta_j,\xi_j}$$
$$- 32\pi \delta_j^2 F_{\xi_j}(\xi_j) + O(\delta^4 |\log \delta|^2).$$

Similarly, by Lemmata 3.1, 5.1 and (5.2) we have that for $l \neq j$

$$\int_S \chi_j e^{-\varphi_j} e^{U_j} W_l dv_g$$
$$= 64\pi^2 G(\xi_l, \xi_j) + 8\pi(\alpha_{\delta_j,\xi_j} + \alpha_{\delta_l,\xi_l})$$
$$- 16\pi(\delta_j^2 F_{\xi_j}(\xi_l) + \delta_l^2 F_{\xi_l}(\xi_j) + O(\delta^4 |\log \delta|^2).$$

Setting

$$\alpha_{1,\delta,\xi} = \sum_{j=1}^{m_1} \alpha_{\delta_j,\xi_j},$$
$$\alpha_{2,\delta,\xi} = \sum_{j=m_1+1}^{m} \alpha_{\delta_j,\xi_j},$$
$$F_{1,\delta,\xi}(x) = \sum_{j=1}^{m_1} \delta_j^2 F_{\xi_j}(x)$$

and

$$F_{2,\delta,\xi}(x) = \sum_{j=m_1+1}^{m} \delta_j^2 F_{\xi_j}(x),$$

we find that

$$\begin{aligned}
&\sum_{j,l=1}^{m_1}\int_S \chi_j e^{-\varphi_j} e^{U_j} W_l dv_g\\
&\quad = -16\pi m_1\\
&\qquad + \sum_{j=1}^{m_1}\Big[-32\pi\log(\mu_1\delta) - 16\pi\log V_1(\xi_j) - 64\pi^2 H(\xi_j,\xi_j)\\
&\qquad\qquad - 64\pi^2\sum_{\substack{i=1\\ i\neq j}}^{m_1} G(\xi_j,\xi_i) + \frac{128\pi^2}{\tau}\sum_{i=m_1+1}^{m} G(\xi_j,\xi_i)\Big]\\
&\qquad + 16\pi m_1\alpha_{1,\delta,\xi} - 32\pi\sum_{j=1}^{m_1} F_{1,\delta,\xi}(\xi_j) + O(\delta^4|\log\delta|^2),
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{m_1}\sum_{l=m_1+1}^{m}\int_S \chi_j e^{-\varphi_j} e^{U_j} W_l dv_g\\
&\quad = 64\pi^2\sum_{j=1}^{m_1}\sum_{l=m_1+1}^{m} G(\xi_j,\xi_l) + 8\pi m_2\alpha_{1,\delta,\xi} + 8\pi m_1\alpha_{2,\delta,\xi}\\
&\qquad - 16\pi\sum_{j=m_1+1}^{m} F_{1,\delta,\xi}(\xi_j) - 16\pi\sum_{j=1}^{m_1} F_{2,\delta,\xi}(\xi_j) + O(\delta^4|\log\delta|^2),
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=m_1+1}^{m}\sum_{l=1}^{m_1}\int_S \chi_j e^{-\varphi_j} e^{U_j} W_l dv_g\\
&\quad = 64\pi^2\sum_{j=m_1+1}^{m}\sum_{l=1}^{m_1} G(\xi_j,\xi_l) + 8\pi m_2\alpha_{1,\delta,\xi} + 8\pi m_1\alpha_{2,\delta,\xi}\\
&\qquad - 16\pi\sum_{j=m_1+1}^{m} F_{1,\delta,\xi}(\xi_j) - 16\pi\sum_{j=1}^{m_1} F_{2,\delta,\xi}(\xi_j) + O(\delta^4|\log\delta|^2),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j,l=m_1+1}^{m}\int_S \chi_j e^{-\varphi_j} e^{U_j} W_l dv_g\\
&\quad = -16\pi m_2 + \sum_{j=m_1+1}^{m}\Big[-32\pi\log(\mu_2\delta) - 16\pi\log V_2(\xi_j) - 64\pi^2 H(\xi_j,\xi_j)\\
&\qquad\qquad - 128\pi^2\tau\sum_{i=1}^{m_1} G(\xi_j,\xi_i) - 64\pi^2\sum_{\substack{i=m_1+1\\ i\neq j}}^{m} G(\xi_j,\xi_i)\Big]\\
&\qquad + 16\pi m_2\alpha_{2,\delta,\xi} - 32\pi\sum_{j=m_1+1}^{m} F_{2,\delta,\xi}(\xi_j) + O(\delta^4|\log\delta|^2)
\end{aligned}$$

in view of (3.5). Now, setting

$$\alpha_{\delta,\xi} = \alpha_{1,\delta_j,\xi_j} - \frac{1}{\tau}\alpha_{2,\delta,\xi} \quad \text{and} \quad F_{\delta,\xi}(x) = F_{1,\delta,\xi}(x) - \frac{1}{\tau}F_{2,\delta,\xi}(x),$$

summing up the four previous expansions, for the gradient term we get that

$$\begin{aligned}(5.3)\qquad &\frac{1}{2}\int_S |\nabla W|_g^2 dv_g \\ &= -8\pi\Big(m_1 + \frac{m_2}{\tau^2}\Big) - 16\pi\Big(m_1 \log(\mu_1\delta) + \frac{m_2}{\tau^2}\log(\mu_2\delta)\Big) - 32\pi^2\varphi_m^*(\xi) \\ &\quad + 8\pi\Big(m_1 - \frac{m_2}{\tau}\Big)\alpha_{\delta,\xi} - 16\pi\sum_{j=1}^{m_1} F_{\delta,\xi}(\xi_j) + \frac{16\pi}{\tau}\sum_{j=m_1+1}^{m} F_{\delta,\xi}(\xi_j) + o(\delta^2)\end{aligned}$$

in view of (1.7).

Let us now expand the potential terms in $J_{\lambda_1,\lambda_2}(W)$, similarly to the proof of [23, Theorem 3.2]. By Lemma 3.1 for any $j = 1, \dots, m_1$ we find that

$$\begin{aligned}&\int_{B_{r_0}(\xi_j)} V_1 e^W dv_g \\ &= \frac{e^{\alpha_{\delta,\xi}}}{8\delta_j^2}\bigg[\int_S \chi_j e^{U_j}\rho_j e^{-2F_{\delta,\xi}} dv_g - 8\delta_j^2 \int_{A_{2r_0}(\xi_j)} \frac{\chi_j\rho_j}{|y_{\xi_j}(x)|^4} dv_g + O(\delta^4|\log\delta|)\bigg].\end{aligned}$$

By Lemma 5.1 (with $f(x) = e^{\varphi_j}\rho_j e^{\alpha_{\delta,\xi}-2F_{\delta,\xi}}$) we can now deduce that

$$\begin{aligned}&8\delta_j^2 e^{-\alpha_{\delta,\xi}}\int_{B_{r_0}(\xi_j)} V_1 e^W dv_g \\ &= 8\pi\rho_j(\xi_j)e^{-2F_{\delta,\xi}(\xi_j)} - 4\pi(\Delta_g\rho_j(\xi_j) - 2K(\xi_j)\rho_j(\xi_j))\delta_j^2\log\delta_j \\ &\quad - 2(\Delta_g\rho_j(\xi_j) - 2K(\xi_j)\rho_j(\xi_j))\Big(\int_{\mathbb{R}^2}\frac{\chi'(|y|)\log|y|}{|y|}dy + \pi\Big)\delta_j^2 \\ &\quad + 4\delta_j^2\rho_j(\xi_j)\int_{\mathbb{R}^2}\frac{\chi'(|y|)}{|y|^3}dy \\ &\quad + 8\delta_j^2\int_{B_{r_0}(\xi_j)}\bigg[V_1 e^{8\pi\sum_{j=1}^m G(x,\xi_j) - \frac{8\pi}{\tau}\sum_{i=m_1+1}^m G(x,\xi_i)} - e^{-\varphi_j}\frac{P_2(e^{\varphi_j}\rho_j)}{|y_{\xi_j}(x)|^4}\bigg]dv_g \\ &\quad - 8\delta_j^2\int_{A_{2r_0}(\xi_j)}\chi_j e^{-\varphi_j}\frac{P_2(e^{\varphi_j}\rho_j)}{|y_{\xi_j}(x)|^4}dv_g + o(\delta^2)\end{aligned}$$

in view of $\frac{\rho_j(x)}{|y_{\xi_j}(x)|^4} = V_1 e^{8\pi\sum_{j=1}^{m_1} G(x,\xi_j) - \frac{8\pi}{\tau}\sum_{i=m_1+1}^m G(x,\xi_i)}$ in $B_{r_0}(\xi_j)$ and by (3.1)

$$(5.4)\qquad \Delta_g[e^{\varphi_j}\rho_j](\xi_j) = \Delta_g\rho_j(\xi_j) - 2K(\xi_j)\rho_j(\xi_j).$$

Now, by Lemma 3.1 for any $j = m_1 + 1, \dots, m$ we find that

$$\int_{B_{r_0}(\xi_j)} V_1 e^W dv_g = \int_{B_{r_0}(\xi_j)} V_1 \left[\frac{\rho_j}{V_2}\right]^{-1/\tau} e^{-\frac{1}{\tau}[U_j - \log(8\delta_j^2)] + \alpha_{\delta,\xi} + O(\delta^2)} dv_g$$
$$= e^{\alpha_{\delta,\xi}} \left[\int_{B_{r_0}(\xi_j)} V_1 e^{8\pi \sum_{j=1}^{m_1} G(x,\xi_j) - \frac{8\pi}{\tau} \sum_{i=m_1+1}^{m} G(x,\xi_i)} dv_g + O(\delta^2) \right].$$

On the other hand, we have that

$$\int_{S \setminus \bigcup_{j=1}^m B_{r_0}(\xi_j)} V_1 e^W dv_g$$
$$= e^{\alpha_{\delta,\xi}} \left[\int_{S \setminus \bigcup_{j=1}^m B_{r_0}(\xi_j)} V_1 e^{8\pi \sum_{j=1}^{m_1} G(x,\xi_j) - \frac{8\pi}{\tau} \sum_{i=m_1+1}^{m} G(x,\xi_i)} dv_g + O(\delta^2) \right].$$

Since

$$\sum_{j=1}^{m_1} e^{-2F_{\delta,\xi}(\xi_j)} = m_1 - 2 \sum_{j=1}^{m_1} F_{\delta,\xi}(\xi_j) + O(\delta^4) \tag{5.5}$$

and by (3.5)

$$\delta_j^2 \log \delta_j = \rho_j(\xi_j)\mu_i^2 \delta^2 \log \delta + \rho_j(\xi_j)\mu_i^2\delta^2 \log \mu_i + \frac{1}{2}\rho_j(\xi_j) \log \rho_j(\xi_j) \mu_i^2 \delta^2$$

holds, we then we then obtain that

$$\begin{aligned} &\frac{1}{\pi} e^{-\alpha_{\delta,\xi}} \mu_1^2 \delta^2 \int_S V_1 e^W dv_g \\ &\quad = m_1 - \frac{A_1^*(\xi)}{8\pi} \mu_1^2 \delta^2 \log(\mu_1 \delta) + \frac{B_{1,\chi}(\xi)}{8\pi} \mu_1^2 \delta^2 - 2 \sum_{j=1}^{m_1} F_{\delta,\xi}(\xi_j) + o(\delta^2), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} B_{1,\chi}(\xi) = &-2\pi \sum_{j=1}^{m_1} [\Delta_g \rho_j(\xi_j) - 2K(\xi_j)\rho_j(\xi_j)] \log \rho_j(\xi_j) \\ &- \frac{A_1^*(\xi)}{2\pi} \left(\int_{\mathbb{R}^2} \frac{\chi'(|y|) \log |y|}{|y|} dy + \pi \right) + 4 \int_{\mathbb{R}^2} \frac{\chi'(|y|)}{|y|^3} dy \sum_{j=1}^{m_1} \rho_j(\xi_j) \\ &+ 8 \int_S \left[V_1 e^{8\pi \sum_{j=1}^{m_1} G(x,\xi_j) - \frac{8\pi}{\tau} \sum_{l=m_1+1}^{m} G(x,\xi_l)} - \sum_{j=1}^{m_1} \chi_j e^{-\varphi_j} \frac{P_2(e^{\varphi_j}\rho_j)}{|y_{\xi_j}(x)|^4} \right] dv_g. \end{aligned}$$

By integration by parts on integrals involving χ and the splitting of S as the union of $\bigcup_{j=1}^{m_1} B_r(\xi_j)$ and $S \setminus \bigcup_{j=1}^{m_1} B_r(\xi_j)$, $r \le r_0$, we easily deduce that

$$\begin{aligned} B_{1,\chi}(\xi) = & -2\pi \sum_{j=1}^{m_1} [\Delta_g \rho_j(\xi_j) - 2K(\xi_j)\rho_j(\xi_j)] \log \rho_j(\xi_j) - \frac{A_1^*(\xi)}{2} \\ & + 8 \int_{S \setminus \bigcup_{j=1}^{m_1} B_r(\xi_j)} V_1 e^{8\pi \sum_{j=1}^m G(x,\xi_j) - \frac{8\pi}{\tau} \sum_{l=m_1+1}^m G(x,\xi_l)} dv_g - \frac{8\pi}{r^2} \sum_{j=1}^{m_1} \rho_j(\xi_j) \\ & - A_1^*(\xi) \log \frac{1}{r} \\ & + 8 \sum_{j=1}^{m_1} \int_{B_r(\xi_j)} \frac{e^{\varphi_j(x)}\rho_j(x) - P_2(e^{\varphi_j}\rho_j)(x)}{|y_{\xi_j}(x)|^4} e^{-\varphi_j(x)} dv_g \end{aligned}$$

in view of (5.4) and the definitions of $A_1^*(\xi)$, $P_2(e^{\varphi_j}\rho_j)$. As a by-product we have that $B_{1,\chi}(\xi)$ does not depend on χ and $r \le r_0$. Since

$$\lim_{r\to 0} \int_{B_r(\xi_j)} \frac{e^{\varphi_j(x)}\rho_j(x) - P_2(e^{\varphi_j}\rho_j)(x)}{|y_{\xi_j}(x)|^4} e^{-\varphi_j(x)} dv_g = 0$$

in view of $e^{\varphi_j(x)}\rho_j(x) - P_2(e^{\varphi_j}\rho_j)(x) = o(|y_{\xi_j}(x)|^2)$ as $x \to \xi_j$, we have that $B_{1,\chi}(\xi)$ coincides with $B_1^*(\xi)$ as defined in (1.11) with $k = 1$.

Similar to the above, by Lemmata 3.1, 5.1 (with $f(x) = e^{\varphi_j}\rho_j e^{-\tau\alpha_{\delta,\xi}+2\tau F_{\delta,\xi}}$), (5.4),

$$\sum_{j=m_1+1}^m e^{2\tau F_{\delta,\xi}(\xi_j)} = m_2 + 2\tau \sum_{j=m_1+1}^m F_{\delta,\xi}(\xi_j) + O(\delta^4) \tag{5.7}$$

and by (3.5), we then obtain that

$$\begin{aligned} \frac{1}{\pi} e^{\tau\alpha_{\delta,\xi}} \mu_2^2\delta^2 \int_S V_2 e^{-\tau W} dv_g = m_2 - \frac{A_2^*(\xi)}{8\pi}\mu_2^2\delta^2 \log(\mu_2\delta) + \frac{B_{2,\chi}(\xi)}{8\pi}\mu_2^2\delta^2 \\ + 2\tau \sum_{j=m_1+1}^m F_{\delta,\xi}(\xi_j) + o(\delta^2), \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} B_{2,\chi}(\xi) = & -2\pi \sum_{j=m_1+1}^m [\Delta_g \rho_j(\xi_j) - 2K(\xi_j)\rho_j(\xi_j)] \log \rho_j(\xi_j) - \frac{A_2^*(\xi)}{2} \\ & + 8 \int_{S \setminus \bigcup_{j=m_1+1}^m B_r(\xi_j)} V_2 e^{-8\pi\tau \sum_{j=1}^m G(x,\xi_j) + 8\pi \sum_{l=m_1+1}^m G(x,\xi_l)} dv_g \\ & - \frac{8\pi}{r^2} \sum_{j=m_1+1}^m \rho_j(\xi_j) \\ & - A_2^*(\xi) \log \frac{1}{r} + 8 \sum_{j=m_1+1}^m \int_{B_r(\xi_j)} \frac{e^{\varphi_j(x)}\rho_j(x) - P_2(e^{\varphi_j}\rho_j)(x)}{|y_{\xi_j}(x)|^4} e^{-\varphi_j(x)} dv_g, \end{aligned}$$

Now $B_{2,\chi}(\xi)$ does not depend on χ and $r \le r_0$, and coincides with $B_2^*(\xi)$ as defined in (1.11) with $k = 2$.

Finally, from (3.6), expansions (5.3), (5.6) and (5.8) and Taylor's expansion for $a \ge 1$, $\log(a+t) = \log a + \frac{t}{a} + O(t^2)$ as $t \to 0$, we get the expansion (5.1) as $\delta \to 0$ and the proof is complete. □

We establish now expansion (5.1) in a C^1-sense in ξ, where the derivatives in ξ are with respect to a given coordinate system. Recall we use ideas in [23, Theorem 3.2].

Proof of (5.1) in $C^1(\Xi)$. We just need to expand the derivatives of $J_{\lambda_1,\lambda_2}(W)$ in ξ. Let us fix $i \in \{1, 2\}$ and $j \in \{1, \dots, m\}$. We have that

$$\partial_{(\xi_j)_i}[J_{\lambda_1,\lambda_2}(W)] = -\int_S \Big[\Delta_g W + \frac{\lambda_1 V_1 e^W}{\int_S V_1 e^W dv_g} - \frac{\lambda_1 \tau V_2 e^{-\tau W}}{\int_S V_2 e^{-\tau W} dv_g}\Big] \partial_{(\xi_j)_i} W dv_g.$$

Notice that as in Lemma 3.1, it follows that

$$\begin{aligned}\partial_{(\xi_j)_i} W_q = &-2\frac{\chi_q}{\delta_q^2 + |y_{\xi_q}(x)|^2}\Big[\partial_{(\xi_j)_i}|y_{\xi_q}(x)|^2 + \delta_q^2 \partial_{(\xi_j)_i}(\log \rho_q(\xi_q))\Big] \\ &- 4\log|y_{\xi_q}(x)|\partial_{(\xi_j)_i}\chi_q + 8\pi\partial_{(\xi_j)_i}H(x,\xi_q) + O(\delta^2|\log\delta|)\end{aligned} \tag{5.9}$$

does hold uniformly in S. Hence, by using (5.9) and expansions in the proof of (35) in $C^1(\Xi)$ in [23, Theorem 3.2], we deduce that

$$\begin{aligned}&-\int_S \Delta_g W \partial_{(\xi_j)_i} W dv_g \\ &\qquad = \sum_{l=1}^{m_1} \int_S \chi_l e^{-\varphi_l} e^{U_l} \partial_{(\xi_j)_i} W dv_g - \frac{1}{\tau} \sum_{l=m_1+1}^{m} \int_S \chi_l e^{-\varphi_l} e^{U_l} \partial_{(\xi_j)_i} W dv_g \\ &\qquad = -32\pi^2 \partial_{(\xi_j)_i} \varphi_m^*(\xi) + O(\delta^2|\log\delta|)\end{aligned} \tag{5.10}$$

for $j \in \{1, \dots, m_1\}$. Similarly, for $j \in \{m_1 + 1, \dots, m\}$ we compute

$$-\int_S \Delta_g W \partial_{(\xi_j)_i} W dv_g = -32\pi^2 \partial_{(\xi_j)_i} \varphi_m(\xi) + O(\delta^2|\log\delta|).$$

In order to give an expansion of the second term in $\partial_{(\xi_j)_i}[J_\lambda(W)]$, first observe that by Lemma 3.1 we have

$$V_1 e^W = \frac{e^{\alpha_{\delta,\xi} - 2F_{\delta,\xi}(x)}}{8\delta_j^2} \rho_j e^{U_j}[1 + O(\delta^4|\log\delta|)] \quad \text{uniformly in } B_{r_0}(\xi_j), j = 1 \dots, m_1 \tag{5.11}$$

$$V_1 e^W = O(1) \text{ uniformly in } S \setminus \bigcup_{j=1}^{m_1} B_{r_0}(\xi_j), \tag{5.12}$$

$$V_2 e^{-\tau W} = \frac{e^{-\tau\alpha_{\delta,\xi}+2\tau F_{\delta,\xi}(x)}}{8\delta_j^2}\rho_j e^{U_j}[1+O(\delta^4|\log\delta|)] \tag{5.13}$$
$$\text{uniformly in } B_{r_0}(\xi_j),\, j=m_1+1\dots,m$$

$$\text{and } V_2 e^{-\tau W} = O(1) \text{ uniformly in } S\setminus\bigcup_{j=m_1+1}^{m} B_{r_0}(\xi_j). \tag{5.14}$$

So, arguing in the same way as in the proof of (35) in $C^1(\Xi)$ in [23, Theorem 3.2] and taking into account that for $k = 1, 2$

$$\int_S V_k e^{(-\tau)^{k-1}W}\partial_{(\xi_j)_i} W dv_g = \sum_{l=1}^{m_1}\int_S V_k e^{(-\tau)^{k-1}W}\partial_{(\xi_j)_i} W_l - \frac{1}{\tau}\sum_{l=m_1+1}^{m}\int_S V_k e^{(-\tau)^{k-1}W}\partial_{(\xi_j)_i} W_l,$$

we have that

$$\int_S \frac{V_k e^{(-\tau)^{k-1}W}}{\int_S V_k e^{(-\tau)^{k-1}W}dv_g}\partial_{(\xi_j)_i} W dv_g = O(\delta^2|\log\delta|), \quad k=1,2. \tag{5.15}$$

In conclusion, by (5.10)–(5.15) we can write

$$\partial_{(\xi_j)_i}[J_{\lambda_1,\lambda_2}(W)] = -32\pi^2\partial_{(\xi_j)_i}\varphi_m^*(\xi) + O(\delta^2|\log\delta|)$$

and the proof is complete. □

Finally, we address the expansions for the derivatives of $J_{\lambda_1,\lambda_2}(W)$ in μ. Recall that we argue similarly to the proof of (35) in $C^2(\mathbb{R})$ in [23, Theorem 3.2].

Proof (of (5.1) in $C^2(\mathbb{R}^2)$). We just focus on the first and second derivative of $J_{\lambda_1,\lambda_2}(W)$ in μ_i, $i = 1, 2$. Since $\partial_{\mu_i} = \delta\rho_l^{\frac{1}{2}}(\xi_l)\partial_{\delta_l}$, $i = 1$ for $l \in \{1, \dots, m_1\}$ and $i = 2$ for $l \in \{m_1+1, \dots, m\}$, in view of (3.5), arguing as in Lemma 3.1, it is easy to show that

$$\delta^{-1}\rho_l^{-\frac{1}{2}}(\xi_l)\partial_{\mu_i} W_l = -\chi_l\frac{4\delta_l}{\delta_l^2+|y_{\xi_l}(x)|^2} + \beta_{\delta_l,\xi_l} - 4\delta_l F_{\xi_l} + O(\delta^3|\log\delta|), \tag{5.16}$$

$$\delta^{-2}\rho_l^{-1}(\xi_l)\partial_{\mu_i\mu_i} W_l = 4\chi_l\frac{\delta_l^2-|y_{\xi_l}(x)|^2}{(\delta_l^2+|y_{\xi_l}(x)|^2)^2} + \gamma_{\delta_l,\xi_l} - 4F_{\xi_l} + O(\delta^2|\log\delta|) \tag{5.17}$$

do hold uniformly in S, where

$$\beta_{\delta_l,\xi_l} = -\frac{8\pi}{|S|}\delta_l\log\delta_l + \frac{4\delta_l}{|S|}\left(\int_{\mathbb{R}^2}\chi(|y|)\frac{e^{\hat\varphi_\xi(y)}-1}{|y|^2}dy - \int_{\mathbb{R}^2}\frac{\chi'(|y|)\log|y|}{|y|}dy\right)$$

and

$$\gamma_{\delta_l,\xi_l} = -\frac{8\pi}{|S|}\log\delta_l + \frac{4}{|S|}\left(\int_{\mathbb{R}^2}\chi(|y|)\frac{e^{\hat\varphi_\xi(y)}-1}{|y|^2}dy - 2\pi - \int_{\mathbb{R}^2}\frac{\chi'(|y|)\log|y|}{|y|}dy\right).$$

Note that $\partial_{\mu_i}W_l = 0$ either if $i = 1$ and $l \in \{m_1+1,\dots,m\}$ or $i = 2$ and $l \in \{1,\dots,m\}$. Let us stress that $\partial_{\mu_i\mu_k}W_l = 0$ for all $l = 1,\dots,m$ and $i \neq k$, so that $\partial_{\mu_i\mu_k}W = 0$ for $i \neq k$. By Lemma 5.1 we then have that either for $i = 1$, $l \in \{1,\dots,m_1\}$ or $i = 2$, $l \in \{m_1+1,\dots,m\}$

$$\delta^{-1}\rho_l^{-\frac{1}{2}}(\xi_l)\int_S \chi_j e^{-\varphi_j}e^{U_j}\partial_{\mu_i}W_l dv_g = -\frac{16\pi}{\delta_j}\delta_{jl} + 8\pi\beta_{\delta_l,\xi_l} - 32\pi\delta_l F_{\xi_l}(\xi_j) + O(\delta^3|\log\delta|^2),$$

$$\delta^{-2}\rho_l^{-1}(\xi_l)\int_S \chi_j e^{-\varphi_j}e^{U_j}\partial_{\mu_i\mu_i}W_l dv_g = \frac{16\pi}{3\delta_j^2}\delta_{jl} + 8\pi\gamma_{\delta_l,\xi_l} - 32\pi F_{\xi_l}(\xi_j) + O(\delta^2|\log\delta|^2) \tag{5.18}$$

and for either $k = 1, j \in \{1,\dots,m_1\}$ or $k = 2$, $l \in \{m_1+1,\dots,m\}$

$$\begin{aligned}\delta^{-1}\rho_l^{-\frac{1}{2}}(\xi_l)&\int_S \chi_j e^{-\varphi_j}e^{U_j}\partial_{\mu_k}U_j\partial_{\mu_i}W_l dv_g\\ &= \frac{2}{\mu_k}\delta^{-1}\rho_l^{-\frac{1}{2}}(\xi_l)\int_S \chi_j e^{-\varphi_j}e^{U_j}\frac{|y_{\xi_j}(x)|^2-\delta_j^2}{\delta_j^2+|y_{\xi_j}(x)|^2}\partial_{\mu_i}W_l dv_g\\ &= \frac{32\pi}{3\delta_j^2}\delta\rho_j(\xi_j)^{\frac{1}{2}}\delta_{jl} + O(\delta^\gamma)\end{aligned} \tag{5.19}$$

in view of $\int_{\mathbb{R}^2}\frac{|y|^2-1}{(1+|y|^2)^3}dy = 0$, where δ_{jl} denotes the Kronecker's symbol. Note that $\partial_{\mu_k}U_j = 0$ for either $k = 1$ and $j \in \{m_1+1,\dots,m\}$ or $k = 2$ and $j \in \{1,\dots,m_1\}$. Since $\int_S \partial_{\mu_i}W dv_g = \int_S \partial_{\mu_i\mu_k}W dv_g = 0$, we then deduce the following expansions:

$$\begin{aligned}\int_S(-\Delta_g W)&\partial_{\mu_1}W dv_g\\ &= \sum_{j,l=1}^{m_1}\int_S \chi_j e^{-\varphi_j}e^{U_j}\partial_{\mu_1}W_l dv_g - \frac{1}{\tau}\sum_{j=m_1+1}^{m}\sum_{l=1}^{m_1}\int_S \chi_j e^{-\varphi_j}e^{U_j}\partial_{\mu_1}W_l dv_g\\ &= -\frac{16\pi m_1}{\mu_1} + 8\pi m_1\delta\sum_{l=1}^{m_1}\rho_l^{\frac{1}{2}}(\xi_l)\beta_{\delta_l,\xi_l} - 32\pi\mu_1\delta^2\sum_{j,l=1}^{m_1}\rho_l(\xi_l)F_{\xi_l}(\xi_j)\\ &\quad - \frac{8\pi m_2}{\tau}\delta\sum_{l=1}^{m_1}\rho_l^{\frac{1}{2}}(\xi_l)\beta_{\delta_l,\xi_l}\\ &\quad + \frac{32\pi}{\tau}\mu_1\delta^2\sum_{j=m_1+1}^{m}\sum_{l=1}^{m_1}\rho_l(\xi_l)F_{\xi_l}(\xi_j) + O(\delta^4|\log\delta|^2),\end{aligned} \tag{5.20}$$

and

$$(5.21)\quad \begin{aligned}&\int_S(-\Delta_g W)\partial_{\mu_2}W dv_g\\&=-\frac{1}{\tau}\sum_{j=1}^{m_1}\sum_{l=m_1+1}^{m}\int_S \chi_j e^{-\varphi_j}e^{U_j}\partial_{\mu_2}W_l dv_g\\&\quad+\frac{1}{\tau^2}\sum_{j,l=m_1+1}^{m}\int_S \chi_j e^{-\varphi_j}e^{U_j}\partial_{\mu_2}W_l dv_g\\&=-\frac{16\pi m_2}{\mu_2\tau^2}-\frac{8\pi m_1}{\tau}\delta\sum_{l=1}^{m_1}\rho_l^{\frac12}(\xi_l)\beta_{\delta_l,\xi_l}+\frac{32\pi}{\tau}\mu_2\delta^2\sum_{j=1}^{m_1}\sum_{l=m_1+1}^{m}\rho_l(\xi_l)F_{\xi_l}(\xi_j)\\&\quad+\frac{8\pi m_2}{\tau^2}\delta\sum_{l=m_1+1}^{m}\rho_l^{\frac12}(\xi_l)\beta_{\delta_l,\xi_l}\\&\quad-\frac{32\pi}{\tau^2}\mu_2\delta^2\sum_{j,l=m_1+1}^{m}\rho_l(\xi_l)F_{\xi_l}(\xi_j)+O(\delta^4|\log\delta|^2),\end{aligned}$$

as $\delta\to 0$. Since by Lemma 3.1 (5.11) and (5.12) hold and $\partial_{\mu_1}W=O(\delta^2|\log\delta|)$ uniformly in $S\setminus\bigcup_{j=1}^{m_1}B_{r_0}(\xi_j)$, by Lemma 5.1 we can write that

$$\begin{aligned}&\int_S V_1e^W\partial_{\mu_1}W dv_g\\&=\sum_{j,l=1}^{m_1}\int_{B_{r_0}(\xi_j)}V_1e^W\partial_{\mu_1}W_l dv_g+O(\delta^2|\log\delta|)\\&=-\sum_{j=1}^{m_1}\frac{e^{\alpha_{\delta,\xi}}}{2\mu_1}\int_{B_{r_0}(\xi_j)}e^{-2F_{\delta,\xi}(x)}\frac{\rho_je^{U_j}}{\delta_j^2+|y_{\xi_j}(x)|^2}dv_g\\&\quad+\pi\frac{e^{\alpha_{\delta,\xi}}}{\mu_1^2\delta}\left(m_1\sum_{l=1}^{m}\rho_l^{\frac12}(\xi_l)\beta_{\delta_l,\xi_l}-4\sum_{j,l=1}^{m_1}\rho_l^{\frac12}(\xi_l)\delta_lF_{\xi_l}(\xi_j)\right)+O(\delta|\log\delta|)\\&=\pi\frac{e^{\alpha_{\delta,\xi}}}{\mu_1^2\delta^2}\bigg(-\frac{2m_1}{\mu_1}+m_1\delta\sum_{l=1}^{m_1}\rho_l^{\frac12}(\xi_l)\beta_{\delta_l,\xi_l}-\frac{\mu_1\delta^2}{8\pi}A_1^*(\xi)\\&\qquad\qquad-\frac{4}{\mu_1\tau}\sum_{j=1}^{m_1}F_{2,\delta,\xi}(\xi_j)+O(\delta^{2+\gamma})\bigg)\end{aligned}$$

in view of (5.4) and from (5.5)

$$\sum_{j=1}^{m_1}e^{-2F_{\delta,\xi}(\xi_j)}=m_1-2\sum_{j,l=1}^{m_1}\delta_l^2F_{\xi_l}(\xi_j)+\frac{2}{\tau}\sum_{j=1}^{m_1}F_{2,\delta,\xi}(\xi_j)+O(\delta^4).$$

Combining with (5.6) we then get that

$$\frac{\int_S V_1 e^W \partial_{\mu_1} W dv_g}{\int_S V_1 e^W dv_g} \tag{5.22}$$
$$= -\frac{2}{\mu_1} + \delta \sum_{l=1}^{m_1} \rho_l^{\frac{1}{2}}(\xi_l)\beta_{\delta_l,\xi_l} - \frac{\delta^2 A_1^*(\xi)}{8\pi m_1}[\mu_1 + 2\mu_1 \log \mu_1]$$
$$- \frac{A_1^*(\xi)}{4\pi m_1}\mu_1 \delta^2 \log \delta + \frac{B_1^*(\xi)}{4\pi m_1}\mu_1 \delta^2 - \frac{4}{m_1 \mu_1}\sum_{j=1}^{m_1} F_{1,\delta,\xi}(\xi_j) + o(\delta^2).$$

Similarly as above, (5.13) and (5.14) hold and $\partial_{\mu_1} W = O(\delta^2 |\log \delta|)$ uniformly in $S \setminus \bigcup_{j=1}^{m_1} B_{r_0}(\xi_j)$, so that

$$\int_S V_2 e^{-\tau W} \partial_{\mu_1} W dv_g$$
$$= \sum_{j,l=1}^{m_1} \int_{B_{r_0}(\xi_j)} V_2 e^{-\tau W} \partial_{\mu_1} W_l dv_g + \sum_{j=m_1+1}^{m} \sum_{l=1}^{m_1} \int_{B_{r_0}(\xi_j)} V_2 e^{-\tau W} \partial_{\mu_1} W_l dv_g$$
$$+ O(\delta^2 |\log \delta|)$$
$$= \pi \frac{e^{-\tau \alpha_{\delta,\xi}}}{\mu_2^2 \delta^2}\left(m_2 \delta \sum_{l=1}^{m_1} \rho_l^{\frac{1}{2}}(\xi_l)\beta_{\delta_l,\xi_l} - \frac{4}{\mu_1}\sum_{j=m_1+1}^{m} F_{1,\delta,\xi}(\xi_j) + O(\delta^4 |\log \delta|)\right)$$

in view of $\tau > 0$, (5.7) and

$$\int_{B_{r_0}(\xi_j)} \frac{V_2 e^{-\tau W}}{\delta_j^2 + |y_{\xi_j}(x)|^2} dv_g = O\left(\int_{B_{r_0}(\xi_j)} (\delta_j^2 + |y_{\xi_j}(x)|^2)^{\tau - 1} dv_g \right) = O(1).$$

Combining with (5.8) we then get that

$$\frac{\int_S V_2 e^{-\tau W} \partial_{\mu_1} W dv_g}{\int_S V_2 e^{-\tau W} dv_g} = \delta \sum_{l=1}^{m_1} \rho_l^{\frac{1}{2}}(\xi_l)\beta_{\delta_l,\xi_l} - \frac{4}{m_2 \mu_1}\sum_{j=1}^{m_1} F_{1,\delta,\xi}(\xi_j) + o(\delta^2), \tag{5.23}$$

which yields

$$\partial_{\mu_1}[J_{\lambda_1,\lambda_2}(W)]$$
$$= \int_S (-\Delta_g W)\partial_{\mu_1} W dv_g - \lambda_1 \frac{\int_S V_1 e^W \partial_{\mu_1} W dv_g}{\int_S V_1 e^W dv_g} + \lambda_2 \tau \frac{\int_S V_2 e^{-\tau W} \partial_{\mu_1} W dv_g}{\int_S V_2 e^{-\tau W} dv_g} \tag{5.24}$$
$$= \frac{2(\lambda_1 - 8\pi m_1)}{\mu_1} + 2A_1^*(\xi)\mu_1 \delta^2 \log \delta + [A_1^*(\xi)\{\mu_1 + 2\mu_1 \log \mu_1\}$$
$$- 2B_1^*(\xi)\mu_1]\delta^2 + o(\delta^2)$$

in view of (5.20), so that we deduce the validity of (5.1) for the first derivative in μ_1. Now, for the first derivative in μ_2, similarly as above we have that

$$\frac{\int_S V_1 e^W \partial_{\mu_2} W dv_g}{\int_S V_1 e^W dv_g} = -\frac{\delta}{\tau}\sum_{l=m_1+1}^{m} \rho_l^{\frac{1}{2}}(\xi_l)\beta_{\delta_l,\xi_l} + \frac{4}{m_1 \mu_2 \tau}\sum_{j=1}^{m_1} F_{2,\delta,\xi}(\xi_j) + o(\delta^2). \tag{5.25}$$

in view of (5.6), and

$$\frac{\int_S V_2 e^{-\tau W}\partial_{\mu_2} W dv_g}{\int_S V_2 e^{-\tau W} dv_g} \tag{5.26}$$
$$= \frac{2}{\mu_2\tau} - \frac{\delta}{\tau}\sum_{l=m_1+1}^{m} \rho_l^{\frac{1}{2}}(\xi_l)\beta_{\delta_l,\xi_l} + \frac{\delta^2 A_2^*(\xi)}{8\pi m_2\tau}[\mu_2 + 2\mu_2\log\mu_2]$$
$$+ \frac{A_2^*(\xi)}{4\pi m_2\tau}\mu_2\delta^2\log\delta - \frac{B_2^*(\xi)}{4\pi m_2\tau}\mu_2\delta^2 + \frac{4}{m_2\mu_2\tau}\sum_{j=m_1+1}^{m} F_{2,\delta,\xi}(\xi_j) + o(\delta^2).$$

by using (5.7) and combining with (5.8). Thus, by using (5.21) we conclude the validity of (5.1) for the first derivative in μ_2:

$$\begin{aligned}\partial_{\mu_2}[J_{\lambda_1,\lambda_2}(W)] &= \frac{2(\lambda_2\tau^2 - 8\pi m_2)}{\mu_2\tau^2} + \frac{2A_2^*(\xi)}{\tau^2}\mu_2\delta^2\log\delta \\ &\quad + [A_2^*(\xi)\{\mu_2 + 2\mu_2\log\mu_2\} - 2B_2^*(\xi)\mu_2]\frac{\delta^2}{\tau^2} \\ &\quad + o(\delta^2)\end{aligned} \tag{5.27}$$

Towards the expansion of the second derivatives in μ, we proceed in a similar way to obtain (5.24) and (5.27) with the aid of the expansions (5.16) for $\partial_{\mu_i} W$ and (5.17) for $\partial_{\mu_i\mu_i} W_l$, (5.18) and (5.19) (see also the validity of expansion (35) in $C^2(\mathbb{R})$ in [23, Theorem 3.2]). We omit the details, so we conclude the validity of (5.1) also for the second derivatives in μ and the proof is complete. □

6 Proof of Theorem 1.2

In this section, we shall study the existence of blowing-up solutions as $\lambda_1 \to 8\pi m_1$ and $\lambda_2\tau^2 \to 0$, which resembles the equation (1.3). For simplicity, we shall denote $m_1 = m$ so that our approximating solution is $W(x) = \sum_{j=1}^m W_j(x)$, and we look for solutions to (1.1) in the form $u = W + \phi$. Assumptions (3.5)–(3.6) are replaced by

$$\begin{aligned}&\delta_j^2 = \mu^2\delta^2\rho_j(\xi_j), \quad j = 1, \dots, m \quad \text{with } 0 < \mu \le C_0, \\ &|\lambda_1 - 8\pi m| \le C\delta^2|\log\delta| \quad \text{and} \quad 0 < \lambda_2\tau^2 \le C\delta^2|\log\delta|.\end{aligned} \tag{6.1}$$

Notice that from similar computations above to obtain (5.8), we have that

$$\int_S V_2 e^{-\tau W} dv_g = e^{-\tau\alpha_{\delta,\xi}}\left[\int_S V_2 e^{-8\pi\tau\sum_{j=1}^m G(x,\xi_j)} dv_g + O(\delta^2)\right] \ge \eta_0 > 0$$

for some $\eta_0 > 0$. By conditions (6.1) we get that

$$\frac{\lambda_2\tau V_2 e^{-\tau W}}{\int_S V_2 e^{-\tau W}} = O(\delta^2|\log\delta|) \quad \text{uniformly in } S. \tag{6.2}$$

Hence, estimate (3.12) follows. Now, denote $Z = \sum_{l=1}^m Z_{0l}$ and PZ its projection according to (4.1). By using (6.2) and similar arguments used in the proofs of [23, Proposition 4.1] and Proposition 4.1, the invertibility of L in (3.8) follows in this case (as $\lambda_1 \to 8\pi m$ and $\lambda_2\tau^2 \to 0$), and we have deduced the following fact.

Proposition 6.1. *There exists $\delta_0 > 0$ so that for all $0 < \delta \le \delta_0$, $\mu \in (0, C_0]$, $\xi \in \Xi$, the problem*

$$\begin{cases} L(\phi) = -[R + N(\phi)] + c_0 \Delta_g PZ + \sum_{i=1}^2 \sum_{j=1}^m c_{ij} \Delta_g PZ_{ij} \quad \text{in } S, \\ \int_S \phi \Delta_g PZ dv_g = \int_S \phi \Delta_g PZ_{ij} dv_g = 0 \\ \qquad\qquad\qquad\qquad \forall\, i = 1, 2,\ j = 1, \dots, m \end{cases}$$

admits a unique solution $\phi(\mu, \xi) \in \bar{H} \cap W^{2,2}(S)$ and $c_0(\mu, \xi)$, $c_{ij}(\mu, \xi) \in I\!R$, $i = 1, 2$ and $j = 1, \dots, m$, where $\delta_j > 0$ are as in (6.1) *and N, R are given by* (3.9), (3.11)*, respectively. Moreover, the map $(\mu, \xi) \mapsto (\phi(\mu, \xi), c_0(\mu, \xi), c_{ij}(\mu, \xi))$ is twice-differentiable in μ and once-differentiable in ξ with*

$$\|\phi\|_\infty + \frac{\|\partial_\mu \phi\|_\infty}{|\log \delta|} + \sum_{i=1}^2 \sum_{j=1}^m \frac{\delta \|\partial_{(\xi_j)_i} \phi\|_\infty}{|\log \delta|} + \frac{\|\partial_{\mu\mu}\phi\|_\infty}{|\log \delta|^2} \le C(\delta |\log \delta| |\nabla \varphi_m^*(\xi)|_g + \delta^{2-\sigma} |\log \delta|^2).$$

As in the case $m_2 \ge 1$, the function $[W + \phi](\mu, \xi)$ will be a true solution of (3.7) if $\mu \in [C_0^{-1}, C_0]$ and $\xi \in \Xi$ are such that $c_0(\mu, \xi) = c_{ij}(\mu, \xi) = 0$ for all $i = 1, 2$, and $j = 1, \dots, m$. Similarly to Lemma 4.1, this problem is equivalent to finding critical points of the reduced energy $E_{\lambda_1,\lambda_2}(\mu, \xi) = J_{\lambda_1,\lambda_2}([W + \phi](\mu, \xi))$, where J_{λ_1,λ_2} is given by (1.4). Notice that

$$\lambda_2 \log\left(\int_S V_2 e^{-\tau W} dv_g\right) = -\lambda_2 \tau \alpha_{\delta,\xi} + \lambda_2 \log\left(\int_S V_2 e^{-8\pi\tau \sum_{j=1}^m G(x,\xi_j)} dv_g\right) + O(\delta^4 |\log \delta|).$$

Let us stress that $\lambda_2 \log(\int_S V_2 e^{-8\pi\tau \sum_{j=1}^m G(x,\xi_j)} dv_g)$ is independent of μ. Taking into account computations in the proof of [23, Theorem 3.2] and similar ones in the proof of Theorem 5.1, we have that

$$\begin{aligned} &J_{\lambda_1,\lambda_2}(W) \\ &= -8\pi m - \lambda_1 \log(\pi m) + 2(\lambda_1 - 8\pi m) \log(\mu\delta) - 32\pi^2 \varphi_m^*(\xi) + A(\xi)\mu^2\delta^2 \log \delta \\ &\quad + [A(\xi)\mu^2 \log \mu - B(\xi)\mu^2]\delta^2 - \lambda_2 \log\left(\int_S V_2 e^{-8\pi\tau \sum_{j=1}^m G(x,\xi_j)} dv_g\right) + o(\delta^2). \end{aligned}$$

Consequently, from estimates in Appendix B we obtain

Theorem 6.1. *Assume* (6.1). *The following expansion holds:*

$$
\begin{aligned}
&E_{\lambda_1,\lambda_2}(\mu,\xi)\\
&\quad= -8\pi m - \lambda_1 \log(\pi m) - 2(\lambda_1 - 8\pi m)\log\delta - 32\pi^2\varphi_m^*(\xi)\\
&\qquad + 2(\lambda_1 - 8\pi m)\log\mu + A(\xi)\mu^2\delta^2\log\delta + [A(\xi)\mu^2\log\mu - B(\xi)\mu^2]\delta^2\\
&\qquad - \lambda_2\log\left(\int_S V_2 e^{-8\pi\tau\sum_{j=1}^m G(x,\xi_j)}dv_g\right) + o(\delta^2) + r_{\lambda_1,\lambda_2}(\mu,\xi)
\end{aligned}
$$

in $C^2(\mathbb{R})$ *and* $C^1(\Xi)$ *as* $\delta \to 0^+$, *where* $\varphi_m^*(\xi)$, $A(\xi)$ *and* $B(\xi)$ *are given by* (1.7), (1.10) *and* (1.11) *with* $k = 1$, *respectively. The term* $r_{\lambda_1,\lambda_2}(\mu,\xi)$ *satisfies* (4.9) *for some* $C > 0$ *independent of* $(\mu,\xi) \in (0, C_0] \times \Xi$.

Proof (of Theorem 1.2). We argue in the same way as in the proof of Theorem 1.1 with $k = 1$. □

7 Appendix A

We shall argue in the same way as in Appendix A in [23]. We first address a-priori estimates for the operator L when all the c_{ij}'s vanish:

Proposition 7.1. *There exist* $\delta_0 > 0$ *and* $C > 0$ *so that, for all* $0 < \delta \le \delta_0$, $h \in C(S)$ *with* $\int_S h dv_g = 0$, $\xi \in \Xi$ *and* $\phi \in H_0^1(S) \cap W^{2,2}(S)$ *a solution of* (4.2) *with* $c_{0i} = c_{ij} = 0$, $i = 1, 2$ *and* $j = 1, \dots, m$, *one has*

$$\|\phi\|_\infty \le C|\log\delta|\|h\|_*.$$

Proof. By contradiction, assume the existence of sequences $\delta \to 0$, $\mu = (\mu_1, \mu_2)$ with $\mu \to \mu^*$, points $\xi \in \Xi$ with $\xi \to \xi^*$, functions h with $|\log\delta|\|h\|_* = o(1)$ and solutions ϕ with $\|\phi\|_\infty = 1$. Recall that $\delta_j^2 = \mu_i\delta^2\rho_j(\xi_j)$. Setting

$$\mathcal{K}_i = \frac{\lambda_i\tau^{2(i-1)}V_i e^{(-\tau)^{i-1}W}}{\int_S V_i e^{(-\tau)^{i-1}W}dv_g}, \quad \psi_i = \phi + \tilde{c}_i(\phi), \quad \tilde{c}_i(\phi) = -\frac{\int_S V_i e^{(-\tau)^{i-1}W}\phi dv_g}{\int_S V_i e^{(-\tau)^{i-1}W}dv_g} \quad \text{for } i = 1, 2,$$

we have that

$$\psi_1 - \tilde{c}_1(\phi) = \psi_2 - \tilde{c}_2(\phi),$$

$$\Delta_g\psi_1 + \mathcal{K}_1\psi_1 + \mathcal{K}_2[\psi_1 - \tilde{c}_1(\phi) + \tilde{c}_2(\phi)] = h$$

and

$$\Delta_g\psi_2 + \mathcal{K}_1[\psi_2 - \tilde{c}_2(\phi) + \tilde{c}_1(\phi)] + \mathcal{K}_2\psi_2 = h$$

in S and ψ_i, $i = 1, 2$ does satisfy the same orthogonality conditions as ϕ.

Since $\|\psi_{i,n}\|_\infty \le 2\|\phi_n\|_\infty \le 2$ and $\Delta_g \psi_i = o(1)$ in $C_{\rm loc}(S \setminus \{\xi_1^*, \dots, \xi_m^*\})$, we can assume that $\psi_{i,n} \to \psi_{i,\infty}$ in $C^1_{\rm loc}(S \setminus \{\xi_1^*, \dots, \xi_m^*\})$. Since $\psi_{i,\infty}$ is bounded, it extends to an harmonic function in S, and then

$$\psi_{i,\infty} = \tilde c_{i,0} := -\lim \frac{\int_S V_i e^{(-\tau)^{i-1}W} \phi dv_g}{\int_S V_i e^{(-\tau)^{i-1}W} dv_g}$$

in view of $\frac{1}{|S|}\int_S \psi_{i,n} dv_g = \tilde c_{i,n}(\phi)$.

The function $\Psi_{i,j} = \psi_i(y_{\xi_j}^{-1}(\delta_j y))$ $i = 1$, for $j = 1, \dots, m_1$ and $i = 2$ for $j = m_1 + 1, \dots, m$ satisfy $\Delta \Psi_{1,j} + \tilde{\mathcal{K}}_{1,j}\Psi_{1,j} + \tilde{\mathcal{K}}_{2,j}[\Psi_{1,j} - \tilde c_1 + \tilde c_2] = \tilde h_j$ and

$$\Delta \Psi_{2,j} + \tilde{\mathcal{K}}_{1,j}[\Psi_{2,j} - \tilde c_2 + \tilde c_1] + \tilde{\mathcal{K}}_{2,j}\Psi_{2,j} = \tilde h_j \quad \text{in } B_{\frac{2r_0}{\delta_j}}(0),$$

where $\tilde{\mathcal{K}}_{i,j} = \delta_j^2 \mathcal{K}_i(y_{\xi_j}^{-1}(\delta_j y))$ and $\tilde h_j(y) = \delta_j^2 h(y_{\xi_j}^{-1}(\delta_j y))$. Since $|\tilde h_j| \le C\|h\|_*$,

$$\tilde{\mathcal{K}}_{1,j} = \begin{cases} \frac{8}{(1+|y|^2)^2}(1 + O(\delta^2 |\log \delta|)) & \text{for } j = 1, \dots, m_1, \\ O(\delta_j^2) & \text{for } j = m_1 + 1, \dots, m, \end{cases}$$

and

$$\tilde{\mathcal{K}}_{2,j} = \begin{cases} O(\delta_j^2) & \text{for } j = 1, \dots, m_1, \\ \frac{8}{(1+|y|^2)^2}(1 + O(\delta^2 |\log \delta|)) & \text{for } j = m_1 + 1, \dots, m, \end{cases}$$

uniformly in $B_{\frac{2r_0}{\delta}}(0)$, in view of Lemma 3.1, (5.6) and (5.8), up to a sub-sequence, by elliptic estimates $\Psi_{i,j} \to \Psi_{j,\infty}$ with $i = 1$ if $j = 1, \dots, m_1$ and $i = 2$ if $j = m_1 + 1, \dots, m$ in $C^1_{\rm loc}(\mathbb{R}^2)$, where $\Psi_{j,\infty}$ is a bounded solution of

$$\Delta \Psi_{j,\infty} + \frac{8}{(1 + |y|^2)^2}\Psi_{j,\infty} = 0$$

of the form $\Psi_{j,\infty} = \sum_{i=0}^2 a_{ij} Y_i$ (see for example [2]). Since

$$-\Delta_g PZ_{ij} = \chi_j e^{-\varphi_j} e^{U_j} Z_{ij} - \frac{1}{|S|}\int_S \chi_j e^{-\varphi_j} e^{U_j} Z_{ij} dv_g$$

in view of (4.1) and $\Delta_g = e^{-\varphi_j}\Delta$ in $B_{2r_0}(\xi_j)$ through y_{ξ_j}, we have that

$$\begin{aligned} 0 &= -\int_S \psi_l \Delta_g PZ_{ij} \\ &= 32\int_{\mathbb{R}^2} \Psi_{l,j}\frac{y_i}{(1 + |y|^2)^3}dy - \frac{32}{|S|}\int_{\mathbb{R}^2}\frac{y_i}{(1 + |y|^2)^3}dy \int_S \psi_{l,n} + O(\delta^3), \end{aligned}$$

with $l = 1$ if $j = 1, \dots, m_1$ and $l = 2$ if $j = m_1 + 1, \dots, m$. Since then

$$\int_{\mathbb{R}^2} \Psi_{j,\infty}\frac{y_i}{(1 + |y|^2)^3}dy = 0,$$

we deduce that $a_{1j} = a_{2j} = 0$. By the orthogonality condition $\int_S \phi \Delta_g PZ_1 = 0$, similarly we deduce that

$$\begin{aligned} 0 &= -\sum_{j=1}^{m_1} \int_S \psi_1 \Delta_g PZ_{0j} dv_g \\ &= 16 \sum_{j=1}^{m_1} \int_{\mathbb{R}^2} \Psi_j \frac{1-|y|^2}{(1+|y|^2)^3} dy - \frac{16}{|S|} m_1 \int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^3} dy \int_S \psi_{1,n} + O(\delta^2), \end{aligned}$$

which implies $\sum_{j=1}^{m_1} a_{0j} = 0$ in view of

$$\int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^3} dy = 0.$$

By using the same argument, the orthogonality condition $\int_S \phi \Delta_g PZ_2 = 0$ implies that $\sum_{j=m_1+1}^{m} a_{0j} = 0$. By dominated convergence we have that

$$\begin{aligned} &\int_S G(y,\xi_j) \mathcal{K}_1 \psi_1 dv_g \\ &= -\frac{1}{2\pi} \log \delta \int_{B_{r_0}(\xi_j)} \mathcal{K}_1 \psi_1 dv_g + \int_{\mathbb{R}^2} \Big[-\frac{1}{2\pi} \log |y| + H(\xi_j,\xi_j) \Big] \frac{8}{(1+|y|^2)^2} \Psi_{j,\infty} dy \\ &\quad + \sum_{\substack{i=1 \\ i \neq j}}^{m_1} G(\xi_i,\xi_j) \int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} \Psi_{i,\infty} dy + o(1) \\ &= -\frac{1}{2\pi} \log \delta \int_{B_{r_0}(\xi_j)} \mathcal{K}_1 \psi_1 dv_g + 4a_{0j} + o(1) \end{aligned}$$

in view of $\int_{\mathbb{R}^2} \log |y| \frac{1-|y|^2}{(1+|y|^2)^3} dy = -\frac{\pi}{2}$ and

$$\begin{aligned} \int_S G(y,\xi_j) \mathcal{K}_2 \psi_2 dv_g = &\sum_{i=m_1+1}^{m} G(\xi_i,\xi_j) \int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} \Psi_{i,\infty}(y)\, dy \\ &+ O\Big(\delta^2 \int_{B_{r_0}(\xi_j)} |G(y,\xi_j)| dv_g \Big) + o(1) = o(1) \end{aligned}$$

for $j = 1, \dots, m_1$. In view of $\int_S \mathcal{K}_l \psi_l = 0$, $l = 1, 2$ and

$$\begin{aligned} \Big| \int_S G(y,\xi_j) h dv_g \Big| &\le C |\log \delta| \int_S |h| dv_g + \frac{\|h\|_*}{\delta^2} \Big| \int_{B_\delta(\xi_j)} G(y,\xi_j) dv_g \Big| \\ &\le C' |\log \delta| \|h\|_* = o(1), \end{aligned}$$

by the Green's representation formula

$$\begin{aligned}\sum_{j=1}^{m_1}\Psi_j(0)=\sum_{j=1}^{m_1}\psi_1(\xi_j)&=\frac{m_1}{|S|}\int_S\psi_1 dv_g+\sum_{j=1}^{m_1}\int_S G(y,\xi_j)[\mathcal{K}_1\psi_1+\mathcal{K}_2\psi_2-h]dv_g\\&=m_1\tilde c_{1,0}+4\sum_{j=1}^{m_1}a_{0j}+o(1)\end{aligned}$$

which gives

$$2\sum_{j=1}^{m_1}a_{0j}=m_1\tilde c_{1,0}+4\sum_{j=1}^{m_1}a_{0j}$$

as $n\to+\infty$. Since $\sum_{j=1}^{m_1}a_{0j}=0$, we get that $\tilde c_{1,0}=0$. By using a similar argument, we obtain that

$$\int_S G(y,\xi_j)\mathcal{K}_1\psi_1 dv_g=o(1)\quad\text{for } j=1,\dots,m_1$$

and

$$\int_S G(y,\xi_j)\mathcal{K}_2\psi_2 dv_g=-\frac{1}{2\pi}\log\delta\int_{B_{r_0}(\xi_j)}\mathcal{K}_2\psi_2 dv_g+4a_{0j}+o(1)$$

for $j=m_1+1,\dots,m$, so that, from the Green's representation formula for $\Psi_j(0)$ for $j=m_1+1,\dots,m$ we get that $\tilde c_{2,0}=0$.

Following [25], let $P\hat Z_j\in H_0^1(S)$ be s.t. $\Delta_g P\hat Z_j=\chi_j\Delta_g\hat Z_j-\frac{1}{|S|}\int_S\chi_j\Delta_g\hat Z_j dv_g$ in S, where

$$\hat Z_j(x)=\beta_j\Big(\frac{y_{\xi_j}(x)}{\delta_j}\Big),\quad \beta_j(y)=\frac{4}{3}[2\log\delta_j+\log(1+|y|^2)]\frac{1-|y|^2}{1+|y|^2}+\frac{8}{3}\frac{1}{1+|y|^2},$$

satisfies $e^{\varphi_j}\Delta_g\hat Z_j+e^{U_j}\hat Z_j=e^{U_j}Z_{0j}$ in $B_{2r_0}(\xi_j)$. Since it is easily seen that

$$P\hat Z_j=\chi_j\hat Z_j+\frac{16\pi}{3}H(\cdot,\xi_j)+O(\delta^2|\log\delta|^2)$$

uniformly in S, we test the equation of ψ_1 against $P\hat Z_j$, $j=1,\dots,m_1$ to get:

$$\begin{aligned}o(1)=\int_S hP\hat Z_j&=\int_S\psi_1\left[\chi_j\Delta_g\hat Z_j-\frac{1}{|S|}\int_S\chi_j\Delta_g\hat Z_j dv_g\right]dv_g\\&\quad+\int_S[\mathcal{K}_1\psi_1+\mathcal{K}_2(\psi_1-\tilde c_1+\tilde c_2)]P\hat Z_j dv_g\\&=\int_S\chi_j\psi_1[\Delta_g\hat Z_j+\mathcal{K}_1\hat Z_j]dv_g+o(1)=\int_S\chi_j\psi e^{U_j}Z_{0j}dv_g+o(1)\\&=16\int_{\mathbb{R}^2}\Psi_j\frac{1-|y|^2}{(1+|y|^2)^3}dy+o(1)\end{aligned}$$

in view of

$$\int_S \mathcal{K}_1 \psi_1 dv_g = 0,$$
$$\int_S \mathcal{K}_2[\psi_1 - \tilde{c}_1 + \tilde{c}_2] P\hat{Z}_j dv_g = o(1),$$
$$\int_S \psi_1 dv_g = o(1),$$
$$\int_S \chi_j \Delta_g \hat{Z}_j dv_g = O(1),$$
$$\int_S \chi_j \psi_1 [\mathcal{K}_1 - e^{U_j}] \hat{Z}_j dv_g = O(\delta^2 |\log \delta|^2)$$

and

$$\int_S hP\hat{Z}_j = O(|\log \delta| \|h\|_*) = o(1), \quad j = 1, \dots, m_1.$$

Since $\int_{\mathbb{R}^2} \Psi_j \frac{1-|y|^2}{(1+|y|^2)^3} dy = 0$ we have that $a_{0j} = 0$, $j = 1, \dots, m_1$. Now, testing the equation of ψ_2 against $P\hat{Z}_j$, $j = m_1 + 1, \dots, m$, leads us to deduce that $a_{0j} = 0$, $j = m_1 + 1, \dots, m$. So far, we have shown that $\psi_i \to 0$ in $C_{\text{loc}}(S \setminus \{\xi_1^*, \dots, \xi_m^*\})$ and uniformly in $\bigcup_{j=1}^m B_{R\delta_j}(\xi_j)$, for all $R > 0$ for both $i = 1, 2$, in view of $\psi_1 - \tilde{c}_1 = \psi_2 - \tilde{c}_2$.

Setting $\hat{\psi}_{i,j}(y) = \psi_i(y_{\xi_j}^{-1}(y))$, $\hat{\mathcal{K}}_j(y) = [\mathcal{K}_1 + \mathcal{K}_2](y_{\xi_j}^{-1}(y))$ and $\hat{h}_j(y) = h(y_{\xi_j}^{-1}(y))$ for $y \in B_{2r_0}(0)$, we have that $e^{\hat{\varphi}_j} \Delta \hat{\psi}_{1,j} + \hat{\mathcal{K}}_j \hat{\psi}_{1,j} = \hat{h}_j + \mathcal{K}_2(y_{\xi_j}^{-1}(y))[\tilde{c}_1 - \tilde{c}_2]$. By now it is rather standard to show that the operator $\hat{L}_j = e^{\varphi_j} \Delta + \hat{\mathcal{K}}_j$ satisfies the maximum principle in $B_r(0) \setminus B_{R\delta_j}(0)$ for R large and $r > 0$ small enough, see for example [20]. As a consequence, we get that $\psi_1 \to 0$ in $L^\infty(S)$. Similarly, we also get that $\psi_2 \to 0$ in $L^\infty(S)$. Since $\psi_i = \phi + \tilde{c}_i(\phi)$ and $\tilde{c}_i(\phi) \to \tilde{c}_{i,0} = 0$ along a sub-sequence, $\|\psi_i\|_\infty \to 0$ implies $\phi \to 0$ in $L^\infty(S)$, in contradiction to $\|\phi\|_\infty = 1$. This completes the proof. □

We are now ready for

Proof of Proposition 4.1. Since $\|\Delta_g PZ_{ij}\|_* \leq C$ for all $i = 0, 1, 2$, $j = 1, \dots, m$, by Proposition 7.1 any solution of (4.2) satisfies

$$\|\phi\|_\infty \leq C|\log \delta| \left[\|h\|_* + \sum_{i=1}^2 \left(|c_{0i}| + \sum_{j=1}^m |c_{ij}| \right) \right].$$

To estimate the values of the c_{ij}'s, test equation (4.2) against PZ_{ij}, $i = 1, 2$ and $j = 1, \dots, m$:

$$\int_S \phi L(PZ_{ij}) dv_g$$
$$= \int_S hPZ_{ij} dv_g + \sum_{k=1}^2 \left[c_{0k} \sum_{l=0}^m \int_S \Delta_g PZ_{0l} PZ_{ij} dv_g + \sum_{l=1}^m c_{kl} \int_S \Delta_g PZ_{kl} PZ_{ij} dv_g \right].$$

Since for $j = 1, \dots, m$ we have the following estimates in $C(S)$:

$$PZ_{ij} = \chi_j Z_{ij} + O(\delta),\ i = 1,2 \quad PZ_{0j} = \chi_j(Z_{0j} + 2) + O(\delta^2|\log\delta|), \tag{7.1}$$

it readily follows that $\int_S \Delta_g PZ_{kl} PZ_{ij} dv_g = -\frac{32\pi}{3}\delta_{ki}\delta_{lj} + O(\delta)$, where the δ_{ij}'s are the Kronecker's symbols. By Lemma 3.1, (3.5), (5.6), (5.8) and (7.1) we have that for $i = 1, 2$

$$\begin{aligned} L(PZ_{ij}) &= \chi_j \Delta_g Z_{ij} + e^{U_j} PZ_{ij} + O\Big(\delta^2 + \delta \sum_{k=1}^m e^{U_k}\Big) \\ &= e^{U_j}[PZ_{ij} - e^{-\varphi_j}\chi_j Z_{ij}] + O\Big(\delta^2 + \delta\sum_{k=1}^m e^{U_k}\Big) \end{aligned}$$

in view of

$$\frac{\int_S V_1 e^W PZ_{ij} dv_g}{\int_S V_1 e^W dv_g} = O(\delta) \quad \text{and} \quad \frac{\int_S V_2 e^{-\tau W} PZ_{ij} dv_g}{\int_S V_2 e^{-\tau W} dv_g} = O(\delta) \quad \text{for all } j = 1, \dots, m,$$

leading to $\|L(PZ_{ij})\|_* = O(\delta)$. Similarly, we have that

$$L(PZ_1) = \sum_{j=1}^{m_1} e^{U_j}[PZ_{0j} - \chi_j e^{-\varphi_j} Z_{0j} - 2\chi_j] + O(\delta^2) + O\Big(\delta\sum_{k=1}^m e^{U_k}\Big)$$

in view of $\frac{\int_S V_1 e^W PZ_{0j} dv_g}{\int_S V_1 e^W dv_g} = \frac{2}{m_1} + O(\delta^2|\log\delta|)$ and $\frac{\int_S V_2 e^{-\tau W} PZ_{0j} dv_g}{\int_S V_2 e^{-\tau W} dv_g} = O(\delta^2|\log\delta|)$ for $j = 1, \dots, m_1$, leading to $\|L(PZ_1)\|_* = O(\delta)$. Also, by using a similar argument for $j = m_1 + 1, \dots, m$, we find that $\|L(PZ_2)\|_* = O(\delta)$. Hence, we get that

$$\sum_{i=1}^2 \Big[|c_{0i}| + \sum_{j=1}^m |c_{ij}|\Big] \le C'\|h\|_* + \delta|\log\delta| O\Big(\sum_{i=1}^2 \Big[|c_{0i}| + \sum_{j=1}^m |c_{ij}|\Big]\Big),$$

yielding the desired estimates $\|\phi\|_\infty = O(|\log\delta|\|h\|_*)$ and

$$\sum_{i=1}^2 [|c_{0i}| + \sum_{j=1}^m |c_{ij}|] = O(\|h\|_*).$$

To prove the solvability assertion, problem (4.2) is equivalent to finding $\phi \in H$ such that

$$\begin{aligned} &\int_S \langle\nabla\phi, \nabla\psi\rangle_g dv_g \\ &= \int_S \Big[\frac{\lambda_1 V_1 e^W}{\int_S V_1 e^W dv_g}\Big(\phi - \frac{\int_S V_1 e^W \phi dv_g}{\int_S V_1 e^W dv_g}\Big) \\ &\qquad + \frac{\lambda_2 \tau^2 V_2 e^{-\tau W}}{\int_S V_2 e^{-\tau W} dv_g}\Big(\phi - \frac{\int_S V_2 e^{-\tau W}\phi dv_g}{\int_S V_2 e^{-\tau W} dv_g}\Big) - h\Big]\psi dv_g \quad \forall\, \psi \in \mathcal{H}, \end{aligned}$$

where

$$\mathcal{H} = \left\{ \phi \in H_0^1(S) : \int_S \phi \Delta_g PZ_{ij} dv_g = \int_S \phi \Delta_g PZ_i dv_g = 0,\ i = 1, 2,\ j = 1, \dots, m \right\}.$$

With the aid of Riesz representation theorem, the Fredholm's alternative guarantees unique solvability for any h provided that the homogeneous equation has only the trivial solution: for (4.2) with $h = 0$, the a-priori estimate (4.3) gives that $\phi = 0$.

So far, we have seen that, if $T(h)$ denotes the unique solution ϕ of (4.2), the operator T is a continuous linear map from $\{h \in L^\infty(S) : \int_S h dv_g = 0\}$, endowed with the $\|\cdot\|_*$-norm, into $\{\phi \in L^\infty(S) : \int_S \phi dv_g = 0\}$, endowed with $\|\cdot\|_\infty$-norm. The argument below is heuristic but can be made completely rigorous. The operator T and the coefficients c_{0i}, c_{ij} are differentiable w.r.t. ξ_l, $l = 1, \dots, m$, or μ_k, $k = 1, 2$. We shall argue in the same way to obtain (57) in [23, Appendix A]; differentiating equation (4.2), we formally get that $X = \partial_\beta \phi$, where $\beta = \xi_l$ with $l = 1, \dots, m$ or $\beta = \mu_k$, $k = 1, 2$, satisfies

$$L(X) = \tilde{h}(\phi) + \sum_i d_{0i} \Delta_g PZ + \sum_{i,j} d_{ij} \Delta_g PZ_{ij},$$

for a suitable choice of $\tilde{h}(\phi)$, $d_{0i} = \partial_\beta c_{0i}$, $d_{ij} = \partial_\beta c_{ij}$, and the orthogonality conditions become

$$\int_S X \Delta_g PZ_{ij} dv_g = - \int_S \phi \partial_\beta (\Delta_g PZ_{ij}) dv_g, \qquad \int_S X \Delta_g PZ_i dv_g = - \int_S \phi \partial_\beta (\Delta_g PZ_i) dv_g.$$

Now, finding and estimating suitable coefficients b_{0i}, b_{ij} so that

$$Y = X + \sum_k b_{0k} PZ_k + \sum_{k,l} b_{kl} PZ_{kl}$$

satisfies the orthogonality conditions

$$\int_S Y \Delta_g PZ_i dv_g = \int_S Y \Delta_g PZ_{ij} dv_g = 0,$$

the function X can be uniquely expressed as

$$X = T(f) - \sum_i b_0 PZ_i - \sum_{i,j} b_{ij} PZ_{ij},$$

where

$$f = \tilde{h}(\phi) + \sum_i b_{0i} L(PZ_i) + \sum_{i,j} b_{ij} L(PZ_{ij}).$$

Moreover, we find that $\|f\|_* \le C \frac{|\log \delta|}{\delta} \|h\|_*$ for $\beta = \xi_l$ and $\|f\|_* \le C |\log \delta| \|h\|_*$ for $\beta = \mu_k$. By (4.3) we deduce that for any first derivative

$$\|\partial_{\xi_l} \phi\|_\infty \le C \Big[|\log \delta| \|f\|_* + \frac{\|\phi\|_\infty}{\delta} \Big] \le C' \frac{|\log \delta|^2}{\delta} \|h\|_*.$$

and $\|\partial_{\mu_k}\phi\|_\infty \le C|\log\delta|^2\|h\|_*$. Differentiating once more in μ_j the equation satisfied by $\partial_{\mu_i}\phi$ and arguing as above, we finally obtain that

$$\|\partial_{\mu_i\mu_j}\phi\|_\infty \le C|\log\delta|^3\|h\|_*,$$

and the proof is complete. □

8 Appendix B

We shall argue in the same way as [23, Proposition 4.2], so that by Proposition 4.1 we now deduce the following.

Proof of Proposition 6.1. In terms of the operator T, problem (4.5) takes the form $\mathcal{A}(\phi) = \phi$, where $\mathcal{A}(\phi) := -T(R + N(\phi))$. After [20, 23, 24, 25, 28], a standard fixed point argument can be used to obtain that $\mathcal{A}$ is a contraction mapping of $\mathcal{F}_\nu$ into itself, where

$$\mathcal{F}_\nu = \left\{\phi \in C(S) : \|\phi\|_\infty \le \nu\left[\delta|\log\delta|\sum_{j=1}^m |\nabla\log(\rho_j\circ y_{\xi_j}^{-1})(0)| + \delta^{2-\sigma}|\log\delta|^2\right]\right\}.$$

Therefore it has a unique fixed point $\phi \in \mathcal{F}_\nu$.

By the Implicit Function Theorem it follows that the map

$$(\mu,\xi) \to (\phi(\mu,\xi), c_{0i}(\mu,\xi), c_{ij}(\mu,\xi))$$

is (at least) twice-differentiable in μ and once-differentiable in ξ. Differentiating $\phi = -T(R + N(\phi))$ w.r.t. $\beta = \xi_l$, $l = 1,\dots,m$, or $\beta = \mu$, we get that

$$\partial_\beta\phi = -\partial_\beta T(R + N(\phi)) - T(\partial_\beta R + \partial_\beta N(\phi)).$$

By Lemma 3.2 and (4.4) we have that

$$\begin{aligned}\|\partial_{\xi_l}T(R + N(\phi))\|_\infty &\le C\frac{|\log\delta|^2}{\delta}(\|R\|_* + \|N(\phi)\|_*)\\ &= O\left(|\log\delta|^2\sum_{j=1}^m |\nabla\log(\rho_j\circ y_{\xi_j}^{-1})(0)| + \delta^{1-\sigma}|\log\delta|^3\right),\end{aligned}$$

for $l = 1,\dots,m$, in view of $\|\partial_{\xi_l}W\|_\infty \le \frac{C}{\delta}$ and

$$\begin{aligned}\|\partial_\mu T(R + N(\phi))\|_\infty &\le C|\log\delta|^2(\|R\|_* + \|N(\phi)\|_*)\\ &= O\left(\delta|\log\delta|^2\sum_{j=1}^m |\nabla\log(\rho_j\circ y_{\xi_j}^{-1})(0)| + \delta^{2-\sigma}|\log\delta|^3\right),\end{aligned}$$

in view of $\|\partial_\mu W\|_\infty \le C$. So, differentiating $\partial_\beta N_i(\phi)$ as in [23, Appendix A] with $N_i(\phi)$ in (3.10), we find that

$$\|\partial_\beta N(\phi)\|_* \le C[\|\partial_\beta W\|_\infty \|\phi\|_\infty^2 + \|\phi\|_\infty \|\partial_\beta \phi\|_\infty] \tag{8.1}$$

and

$$\|\partial_{\xi_l} N(\phi)\|_* = O\Big(\delta|\log\delta|^2 \sum_{j=1}^m |\nabla \log(\rho_j \circ y_{\xi_j}^{-1})(0)|^2 + \delta^{3-2\sigma}|\log\delta|^4\Big) + o\Big(\frac{\|\partial_{\xi_l}\phi\|_\infty}{|\log\delta|}\Big),$$
$$\|\partial_\mu N(\phi)\|_* = O\Big(\delta^2|\log\delta|^2 \sum_{j=1}^m |\nabla \log(\rho_j \circ y_{\xi_j}^{-1})(0)|^2 + \delta^{4-2\sigma}|\log\delta|^4\Big) + o\Big(\frac{\|\partial_\mu\phi\|_\infty}{|\log\delta|}\Big).$$

Since

$$\int_S \chi_j e^{-\varphi_j} e^{U_j} dv_g = \int_{\mathbb{R}^2} \chi(|y|) \frac{8\mu_k^2\delta^2\rho_j(\xi_j)}{(\mu_k^2\delta^2\rho_j(\xi_j) + |y|^2)^2} dy,$$

if either $k = 1$ for $j = 1, \dots, m_1$ or $k = 2$ for $j = m_1 + 1, \dots, m$, we have that

$$\partial_{\xi_l}\Big(\int_S \chi_j e^{-\varphi_j} e^{U_j} dv_g\Big) = 8\partial_{\xi_l} \log\rho_j(\xi_j) \int_{\mathbb{R}^2} \frac{1 - |y|^2}{(1 + |y|^2)^3} + O(\delta^2) = O(\delta^2)$$

and similarly,

$$\partial_{\mu_k}\Big(\int_S \chi_j e^{-\varphi_j} e^{U_j} dv_g\Big) = \int_{\mathbb{R}^2} \chi(|y|) \frac{16\mu_k\delta^2\rho_j(\xi_j)(|y|^2 - \mu_k^2\delta^2\rho_j(\xi_j))}{(\mu_k^2\delta^2\rho_j(\xi_j) + |y|^2)^3} dy = O(\delta^2).$$

Since $\varphi_j(\xi_j) = 0$ and $\nabla\varphi_j(\xi_j) = 0$, we have that $e^{-\varphi_j} = 1 + O(|y_{\xi_j}(x)|^2)$ and $\partial_\beta(\chi_j e^{-\varphi_j}(x)) = O(|y_{\xi_j}(x)|)$, and then

$$\Delta_g \partial_{\xi_l} W = -\sum_{j=1}^{m_1} \chi_j e^{U_j} \partial_{\xi_l} U_j + \frac{1}{\tau} \sum_{j=m_1+1}^m \chi_j e^{U_j} \partial_{\xi_l} U_j + O(\delta^{1-\sigma}),$$

in view of $|\partial_{\xi_l} U_j| = O(\frac{1}{\delta})$, $l = 1, \dots, m$ and

$$\Delta_g \partial_\mu W = -\sum_{j=1}^{m} \chi_j e^{U_j} \partial_\mu U_j + \frac{1}{\tau} \sum_{j=m_1+1}^m \chi_j e^{U_j} \partial_\mu U_j + O(\delta^{2-\sigma}),$$

in view of $|\partial_\mu U_j| = O(1)$, where the big O is estimated in $\|\cdot\|_*$-norm. Note that in $B_{r_0}(\xi_j)$

$$\partial_{\xi_l} W = \begin{cases} \partial_{\xi_l} U_j + O(\delta^2|\log\delta| + |y_{\xi_j}(x)| + |\nabla \log(\rho_j \circ y_{\xi_j}^{-1})(0)|), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } j \in \{1, \dots, m_1\}, \\ -\frac{1}{\tau}[\partial_{\xi_l} U_j + O(\delta^2|\log\delta| + |y_{\xi_j}(x)| + |\nabla \log(\rho_j \circ y_{\xi_j}^{-1})(0)|)], \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } j \in \{m_1+1, \dots, m\}, \end{cases}$$

and

$$\partial_{\mu_k} W = \begin{cases} \partial_{\mu_k} U_j - \frac{2}{\mu_k} + O(\delta^2|\log\delta|), & \text{for } j \in \{1, \dots, m_1\}, \\ -\frac{1}{\tau}[\partial_{\mu_k} U_j - \frac{2}{\mu_k} + O(\delta^2|\log\delta|)], & \text{for } j \in \{m_1+1, \dots, m\}. \end{cases}$$

Furthermore, $\partial_{\xi_l} W = O(1)$ and $\partial_{\mu_k} W = O(\delta^2|\log\delta|)$ in $S \setminus \bigcup_{j=1}^m B_{r_0}(\xi_j)$. From computations in the proof of Lemma 3.1 we find that

$$\begin{aligned} &\frac{\lambda_1 V_1 e^W}{\int_S V_1 e^W dv_g} \\ &= \frac{\lambda_1}{8\pi m_1} \sum_{j=1}^{m_1} \chi_j \\ &\qquad \times \Big[1 + \Big\langle \frac{\nabla(\rho_j \circ y_{\xi_j}^{-1})(0)}{\rho_j(\xi_j)}, y_{\xi_j}(x)\Big\rangle + O(|y_{\xi_j}(x)|^2 + \delta^2|\log\delta|)\Big] e^{U_j} \\ &\quad + O(\delta^2)\chi_{S\setminus\bigcup_{j=1}^{m_1} B_{r_0}(\xi_j)}, \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} &\frac{\lambda_2 \tau V_2 e^{-\tau W}}{\int_S V_2 e^{-\tau W} dv_g} \\ &= \frac{\lambda_2 \tau}{8\pi m_2} \sum_{j=m_1+1}^{m} \chi_j \\ &\qquad \times \Big[1 + \Big\langle \frac{\nabla(\rho_j \circ y_{\xi_j}^{-1})(0)}{\rho_j(\xi_j)}, y_{\xi_j}(x)\Big\rangle + O(|y_{\xi_j}(x)|^2 + \delta^2|\log\delta|)\Big] e^{U_j} \\ &\quad + O(\delta^2)\chi_{S\setminus\bigcup_{j=m_1+1}^{m} B_{r_0}(\xi_j)}. \end{aligned} \tag{8.3}$$

By (5.15), (5.22), (5.23), (5.25), (5.26), (8.2) and (8.3) we deduce for $\partial_\beta R$ the estimate

$$|\partial_{\xi_l} R\|_* + \frac{1}{\delta}\|\partial_{\mu_k} R\|_* = O\bigg(\sum_{j=1}^m |\nabla \log(\rho_j \circ y_{\xi_j}^{-1})(0)| + \delta^{1-\sigma}|\log\delta|\bigg),$$
$$l = 1, \dots, m, \; k = 1, 2.$$

Combining all the estimates, we then get that

$$\|\partial_{\xi_l}\phi\|_\infty = O\left(|\log\delta|^2\sum_{j=1}^m |\nabla\log(\rho_j\circ y_{\xi_j}^{-1})(0)| + \delta^{1-\sigma}|\log\delta|^3\right) + o(\|\partial_{\xi_l}\phi\|_\infty)$$

and

$$\|\partial_{\mu_k}\phi\|_\infty = O\left(\delta|\log\delta|^2\sum_{j=1}^m |\nabla\log(\rho_j\circ y_{\xi_j}^{-1})(0)| + \delta^{2-\sigma}|\log\delta|^3\right) + o(\|\partial_{\mu_k}\phi\|_\infty),$$

which in turn provides the validity of (4.7). We proceed in the same way to obtain the estimate (4.7) on $\partial_{\mu_i\mu_j}\phi$, and the proof is complete. □

Lemma 4.1 is rather standard and we will omit its proof. Since the problem has been reduced to finding c.p.'s of the reduced energy

$$E_{\lambda_1,\lambda_2}(\mu,\xi) = J_{\lambda_1,\lambda_2}(W+\phi(\mu,\xi)),$$

where J_{λ_1,λ_2} is given by (1.4), the last key step is to show that the main asymptotic term of E_{λ_1,λ_2} is given by $J_{\lambda_1,\lambda_2}(W)$.

Proof of Theorem 4.1. We argue in the same way as in the proof of [23, Theorem 4.4]. For simplicity we write J instead of J_{λ_1,λ_2}. Thus, we get that

$$\begin{aligned}
&J(W+\phi) - J(W)\\
&\quad = -\frac{1}{2}\int_S [R\phi - N(\phi)\phi]dv_g + \int_0^1\int_0^1 [D^2J(W+ts\phi) - D^2J(W)][\phi,\phi]\, t\,dsdt,
\end{aligned}$$

so that it is straighforward to deduce that

$$\begin{aligned}
|J(W+\phi) - J(W)| &= O(\|R\|_*\|\phi\|_\infty + \|\phi\|_\infty^3)\\
&= O(\delta^2|\log\delta|\,|\nabla\varphi_m^*(\xi)|^2 + \delta^{3-\sigma}|\log\delta|^2)
\end{aligned}$$

in view of (4.6),

$$4\pi\nabla_{\xi_j}\varphi_m^*(\xi) = \nabla\log(\rho_j\circ y_{\xi_j}^{-1})(0) \quad \text{for } j = 1,\dots,m_1$$

and

$$4\pi\tau^2\nabla_{\xi_j}\varphi_m^*(\xi) = \nabla\log(\rho_j\circ y_{\xi_j}^{-1})(0) \quad \text{for } j = m_1+1,\dots,m.$$

Now, differentiating w.r.t. $\beta = \xi_l$, $l = 1,\dots,m$, or $\beta = \mu_k$, $k = 1, 2$ we get that

$$\begin{aligned}
&|\partial_\beta[J(W+\phi) - J(W)]|\\
&\quad = O(\|\partial_\beta R\|_*\|\phi\|_\infty + \|R\|_*\|\partial_\beta\phi\|_\infty + \|\phi\|_\infty^2\|\partial_\beta\phi\|_\infty + \|\phi\|_\infty^3\|\partial_\beta W\|_\infty)
\end{aligned}$$

by using (8.1), so that,

$$|\partial_{\xi_l}[J(W+\phi)-J(W)]| = O\Big([\delta^2|\log\delta|\,|\nabla\varphi_m^*(\xi)|^2+\delta^{3-\sigma}|\log\delta|^2]\frac{|\log\delta|}{\delta}\Big)$$

and $|\partial_{\mu_k}[J(W+\phi)-J(W)]| = O([\delta^2|\log\delta|\,|\nabla\varphi_m^*(\xi)|^2+\delta^{3-\sigma}|\log\delta|^2]|\log\delta|)$ in view of (4.6)–(4.7), $\|\partial_{\xi_l}W\|_\infty = O(\frac{1}{\delta})$ and $\|\partial_{\mu_k}W\|_\infty = O(1)$. Arguing similarly for the second derivative in μ, we get that

$$|\partial_{\mu_i\mu_k}[J(W+\phi)-J(W)]| = O([\delta^2|\log\delta|\,|\nabla\varphi_m^*(\xi)|^2+\delta^{3-\sigma}|\log\delta|^2]|\log\delta|^2).$$

Combining the previous estimates on the difference $J(W+\phi)-J(W)$ with the expansion of $J(W)=J_{\lambda_1,\lambda_2}(W)$ contained in Theorem 5.1, we deduce the validity of the expansion (4.8) with an error term which can be estimated (in $C^2(\mathbb{R}^2)$ and $C^1(\Xi)$) like $o(\delta^2)+r_{\lambda_1,\lambda_2}(\mu,\xi)$ as $\delta\to 0$, where $r_{\lambda_1,\lambda_2}(\mu,\xi)$ does satisfy (4.9). □

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