

DECOMPOSING MULTITWISTS

By

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Abstract. The Decomposition Problem in the class $LIP(\mathbb{S}^2)$ is to decompose any bi-Lipschitz map $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ as a composition of finitely many maps of arbitrarily small isometric distortion. In this paper, we construct a decomposition for certain bi-Lipschitz maps which spiral around every point of a Cantor set X of Assouad dimension strictly smaller than one. These maps are constructed by considering a collection of Dehn twists on the Riemann surface $\mathbb{S}^2 \setminus X$. The decomposition is then obtained via a bi-Lipschitz path which simultaneously unwinds these Dehn twists. As part of our construction, we also show that $X \subset \mathbb{S}^2$ is uniformly disconnected if and only if the Riemann surface $\mathbb{S}^2 \setminus X$ has a pants decomposition whose cuffs have hyperbolic length uniformly bounded above, which may be of independent interest.

1 Introduction

A bi-Lipschitz homeomorphism $f : X \rightarrow Y$ between metric spaces is a homeomorphism that roughly preserves absolute distances; specifically, there exists $L \geq 1$ such that

$$L^{-1}d_X(x, y) \leq d_Y(f(x), f(y)) \leq Ld_X(x, y)$$

for all $x, y \in X$. We then say that f is an L -bi-Lipschitz map. The smallest such constant L is called the **isometric distortion** of f . Letting \mathbb{S}^n be the sphere of dimension n , we denote by $LIP(\mathbb{S}^n)$ the class of orientation preserving homeomorphisms of \mathbb{S}^n .

A central problem in bi-Lipschitz geometry is whether a bi-Lipschitz map can be decomposed into bi-Lipschitz mappings of arbitrarily small isometric distortion.

Conjecture 1.1 (Decomposition Problem). *Let $n \geq 1$ and let $f \in LIP(\mathbb{S}^n)$. Then for every $\epsilon > 0$ we can find homeomorphisms $f_k \in LIP(\mathbb{S}^n)$, for $k = 1, \dots, m$, such that f can be written as a composition $f = f_m \circ \dots \circ f_1$, where each f_k has isometric distortion at most $1 + \epsilon$.*

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The case $n = 1$ is elementary: suppose I, J are intervals in \mathbb{R} and $f : I \rightarrow J$ is an L -bi-Lipschitz map. Then f can be written as $f = f_2 \circ f_1$, where

$$f_1(x) = \int_{x_0}^x |f'(t)|^\lambda dt,$$

x_0 is fixed, $\lambda = \log_L \alpha$, f_1 is α -bi-Lipschitz and $f_2 = f \circ f_1^{-1}$ is L/α -bi-Lipschitz.

However, for $n \geq 2$, the Decomposition Problem has been so far elusive. It is clear that affine bi-Lipschitz mappings can be factored into affine mappings of small isometric distortion, but beyond this, only certain specific examples have been considered. Freedman and He [FH88] studied the logarithmic spiral map $s_k(z) = ze^{ik \log |z|}$, which is an L -bi-Lipschitz map with $|k| = L - 1/L$. Gutlyanskii and Martio [GM01] studied a related class of mappings in dimension 2, and generalized this to a class of volume preserving bi-Lipschitz automorphisms of the unit ball \mathbb{B}^3 in three dimensions.

Although in this paper we focus on $LIP(\mathbb{S}^2)$, the Decomposition Problem can also be asked for the class of quasiconformal homeomorphisms of \mathbb{S}^n . In dimension 2, the fact that every quasiconformal map arises as a solution of the Beltrami equation can be leveraged to show that the Decomposition Problem has a positive solution here; see [Leh87, Theorem 4.7]. Since every orientation preserving bi-Lipschitz map is also quasiconformal, in dimension 2 we are able to find a decomposition of bi-Lipschitz maps, but only into quasiconformal maps of small conformal distortion. Observe, however, that quasiconformal maps need not be bi-Lipschitz.

A similar problem was studied by the first named author and Markovic in [FM12]. There it was shown that C^1 diffeomorphisms of \mathbb{S}^n , for $n \geq 2$, can be decomposed into bi-Lipschitz maps of arbitrarily small isometric distortion. This solves the Decomposition Problem for C^1 bi-Lipschitz maps, but of course, bi-Lipschitz maps are only guaranteed to be differentiable almost everywhere.

In this paper, we study the Decomposition Problem for a class of maps in $LIP(\mathbb{S}^2)$ which spiral around every point of a Cantor set, with small Assouad dimension; see below for definitions. Necessarily these maps are not differentiable at any point of the Cantor set in question. This can be viewed as a generalization of the result of Freedman and He, although they were motivated to give estimates on the number of maps required in the decomposition. Our constructions will be involved enough that we will not address this question here, and be content to just find a decomposition.

Maps which spiral around every point of a Cantor set simultaneously are not new. Such mappings were constructed by Astala et al. in [AIPS15] in order to give sharp examples of the multifractal spectrum; see in particular the proof of Theorem 5.1 and Figure 7 in [AIPS15].

Here and for the rest of the paper, a Cantor set is defined to be a metric space that is homeomorphic to the standard middle-third Cantor set.

1.1 Uniformly disconnected sets and hyperbolic geometry. We identify the topological sphere \mathbb{S}^2 with the one point compactification $\mathbb{R}^2 \cup \{\infty\}$, and equip it with the chordal metric. If $X \subset \mathbb{S}^2$ is a Cantor set, then by applying a chordal isometry we may assume that $X \subset \mathbb{R}^2$. Having done this, we may then view $S := \mathbb{S}^2 \setminus X$ as a Riemann surface of infinite type.

The bi-Lipschitz maps that we will decompose arise from a collection of Dehn twists on the surface S . For the mappings we define to be bi-Lipschitz, we need some control on the ring domains on which the Dehn twists are defined. Informally, these ring domains cannot be too thin, and their boundaries cannot be too wiggly.

To address the first of these points, we recall some hyperbolic geometry. The surface S has a pants decomposition, that is, $S = \bigcup_{i=1}^{\infty} P_i$, where each P_i is a topological sphere with three disks removed. The collection of boundary curves of the pairs of pants, called the **cuffs** of the decomposition, may be enumerated by $(\alpha_j)_{j=1}^{\infty}$. Each α_j is a simple closed curve on S and generates a class $[\alpha_j]$ of simple closed curves that are freely homotopic to α_j .

We denote by $\ell_S(\alpha_j)$ the hyperbolic length of α_j and by $\ell_S[\alpha_j]$ the infimum of hyperbolic lengths of closed curves in S homotopic to α_j . We suppress the subscript S if the context is clear. It is well-known that a Cantor set $X \subset \mathbb{R}^2$ is uniformly perfect if and only if for any pants decomposition of $\mathbb{S}^2 \setminus X$, the associated cuffs $(\alpha_j)_{j=1}^{\infty}$ satisfy $\inf_j \ell_S[\alpha_j] > 0$; see [Pom79]. Recall that a non-degenerate metric space X is **uniformly perfect** if there exists a constant $C > 1$ such that for any $x \in X$ and every positive $r < \text{diam} X$, we have that $B(x, r) \setminus B(x, r/C) \neq \emptyset$. Informally, this means that any ring domain separating X cannot be too thick.

Uniform disconnectedness is, in a sense, the opposite of uniform perfectness; a metric space X is **uniformly disconnected** if there exists a constant $c > 1$ such that for any $x \in X$ and every positive $r < \text{diam} X$, there exists $X_{x,r} \subset X$ that contains x such that $\text{diam} X_{x,r} \leq r$ and

$$\text{dist}(X_{x,r}, X \setminus X_{x,r}) \geq r/c.$$

If we wish to emphasize the dependence on the constant c , we will say that X is c -uniformly disconnected. It is natural to ask whether X being uniformly disconnected implies analogous geometric properties of the surface S . Our first result gives such a characterization.

Theorem 1.2. *A Cantor set $X \subset \mathbb{R}^2$ is uniformly disconnected if and only if there exists a pants decomposition for $S = \mathbb{S}^2 \setminus X$ such that the associated cuffs $(\alpha_j)_{j=1}^{\infty}$ satisfy $\sup_j \ell[\alpha_j] < \infty$.*

By a uniformization theorem of David and Semmes [DS97], a set $X \subset \mathbb{R}^2$ is quasimetrically homeomorphic to the standard ternary Cantor set \mathcal{C} if and only if it is compact, uniformly perfect, and uniformly disconnected. Therefore, by Theorem 1.2 and [Pom79], it follows that a Cantor set $X \subset \mathbb{R}^2$ is quasimetrically homeomorphic to \mathcal{C} if and only if there exist a constant $C > 1$ and a pants decomposition for $S = \mathbb{S}^2 \setminus X$ such that the associated cuffs $(\alpha_j)_{j=1}^\infty$ satisfy

$$C^{-1} \leq \ell[\alpha_j] \leq C, \quad \text{for all } j.$$

1.2 Dehn multi-twists. Here we outline how our bi-Lipschitz mappings are constructed. Full definitions and discussion will follow in the sequel. The first step is the following proposition which is a corollary of Theorem 1.2.

Proposition 1.3. *Given $c \geq 1$, there exist $L > 1$, $k \in \mathbb{N}$, and a finite set*

$$\left\{ g_i : \overline{B}(0, 1) \setminus B\left(0, 1 - \frac{1}{L}\right) \rightarrow \mathbb{R}^2 \right\}_{i=1}^k$$

of L -bi-Lipschitz conformal maps with the following property. Let $X \subset \mathbb{R}^2$ be a c -uniformly disconnected Cantor set and let $(\alpha_j)_{j=1}^\infty$ be the cuffs from Theorem 1.2. There exist mutually disjoint closed ring domains $R_j \subset \mathbb{R}^2 \setminus X$ homotopic to α_j , and similarities $(\phi_j)_{j=1}^\infty$ of \mathbb{R}^2 such that for each $j \in \mathbb{N}$ there exists $i(j) \in \{1, \dots, k\}$ with $R_j := \phi_j \circ g_{i(j)}(\overline{B}(0, 1) \setminus B(0, 1 - \frac{1}{L}))$. Moreover, for each $j \in \mathbb{N}$, the bounded component of $\phi_j^{-1}(R_j)$ has diameter equal to 1.

This proposition says that given a pants decomposition of $\mathbb{S}^2 \setminus X$, we can find a collection of rings on which our map f will be supported with the property that, up to similarity, the rings are chosen from a finite set. This finiteness will lead to a certain uniformity in the Dehn twists that define f .

More precisely, fix a Cantor set $X \subset \mathbb{R}^2$ for which the Assouad dimension satisfies $\dim_A X < 1$. It then follows from [Luu98] that X is uniformly disconnected. Let R_j and $f_j := \phi_j \circ g_{i(j)}$ be the ring domains and conformal maps, respectively, from Proposition 1.3. Then, a Dehn twist can be defined on each $\overline{R_j}$ by

$$f|_{\overline{R_j}} := f_j \circ \mathcal{D} \circ f_j^{-1}$$

where $\mathcal{D} : \overline{B}(0, 1) \setminus B(0, 1 - \frac{1}{L}) \rightarrow \overline{B}(0, 1) \setminus B(0, 1 - \frac{1}{L})$ is the Dehn twist

$$\mathcal{D}(r, \theta) = (r, \theta + 2\pi L(1 - r)).$$

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by the Dehn twist in each $\overline{R_j}$ as above, and the identity elsewhere. The uniform bi-Lipschitz constant of the maps g_j guarantees that f is a bi-Lipschitz map; see Lemma 6.2. The main theorem of this paper reads as follows:

Theorem 1.4. *If $X \subset \mathbb{R}^2$ is a Cantor set with $\dim_A X < 1$ and if f is the bi-Lipschitz map defined above, then given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $f = f_N \circ \dots \circ f_1$, where each f_j , for $j = 1, \dots, N$, is $(1 + \epsilon)$ -bi-Lipschitz.*

It is worth pointing out that if the rings R_j can initially be chosen to be round rings, such as those constructed in [AIPS15], then the assumption $\dim_A X < 1$ can be replaced by uniform disconnectedness, and we can decompose f directly in this case. In fact, the assumption $\dim_A X < 1$ can be dropped (see Section 8 for an example), and we conjecture that it can be replaced by uniform disconnectedness.

1.3 Strategy of the proof. The crux of the proof is to construct a bi-Lipschitz path from the identity to f . Bi-Lipschitz paths were introduced in [FM12] to provide a way to deform one bi-Lipschitz mapping to another in a controlled way. Partitioning the path into small subintervals yields the required decomposition.

Consider first the special case where each of the rings R_j are round; see Figure 1. Writing V_j for the bounded component of the complement of R_j , we can unwind the Dehn twist supported in R_j in the obvious way, and extend this unwinding via the identity in the unbounded component of the complement of R_j and via a path of rotations in V_j . This unwinding can happen in each ring R_j and the corresponding domain V_j simultaneously for all j . The point is that on a given R_j , the unwinding will act via finitely many rotations (one for each ring R_k such that $R_j \subset V_k$) and then via the unwinding on R_j .

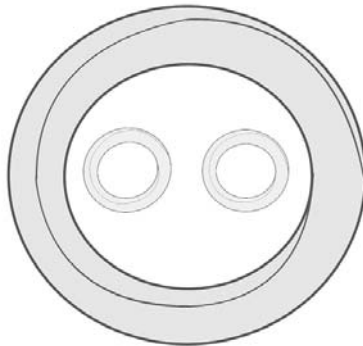


Figure 1. Round rings and Dehn twists.

This idealized case is, however, not the most general case. Complications arise once R_j are not round rings. In particular, it may certainly be the case that the

two rings R_{k_1}, R_{k_2} contained in V_j , cannot be flowed isometrically around D_j ; see Figure 2.

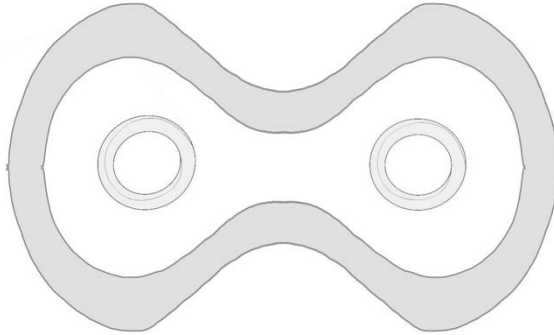


Figure 2. Rings that are not round

Our resolution to this issue is to use the hypothesis that $\dim_A X < 1$ to show that the intersection $X \cap V_j$ may be covered by small islands which can be flowed into a relatively small ball contained in V_j . The point is that while the next level of rings down from R_j may not be flowed around V_j , we can pass through finitely many levels, say N , to obtain a collection of rings which can be flowed around V_j .

Consequently, to unwind the Dehn twist in R_j , we concatenate three bi-Lipschitz paths in V_j : one to move the rings N levels down into a given disk contained in V_j , one to act as a conjugate of rotations in V_j , and then the third to undo the first path. It follows that we may apply this construction simultaneously in the collection of levels that differ by N to yield a bi-Lipschitz path. Applying this construction N times, we may concatenate the resulting bi-Lipschitz paths to obtain one path from the identity to f itself.

1.4 Outline of the paper. In Section 2, we recall the basic definitions and properties of the objects we will use. In Section 3, we prove Theorem 1.2. In Section 4, we prove some technical results on bi-Lipschitz paths. In Section 5, we study how to collapse sets of Assouad dimension less than 1 into small disks. In Section 6, we prove Proposition 1.3, and in Section 7 we prove how the map f in Theorem 1.4 can be decomposed into bi-Lipschitz mappings of small isometric distortion. Finally, in Section 8 we construct a multitwist map with a singular set of Assouad dimension close to 2 that can be decomposed using the techniques of the paper.

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2 Preliminaries

2.1 Modulus of ring domains. Given a family Γ of curves in \mathbb{R}^n , define the **conformal modulus**

$$\text{Mod}(\Gamma) = \inf_{\rho} \int_{\mathbb{R}^n} \rho(x)^n dx$$

where the infimum is taken over all Borel $\rho : \mathbb{R}^n \rightarrow [0, \infty)$ such that $\int_{\gamma} \rho ds \geq 1$ for all locally rectifiable $\gamma \in \Gamma$.

Here and for the rest, given a ring domain R in \mathbb{R}^2 with boundary components γ_1 and γ_2 , we denote by $M(R)$ the modulus of the family of curves in R that join γ_1 with γ_2 . Observe that the larger $M(R)$ is, the thinner the ring domain R is. It is well known [Loe59] that there exists a decreasing function $\psi : (0, \infty) \rightarrow (0, \infty)$ such that, if R is a ring domain with outer boundary component γ_1 and inner boundary component γ_2 , then

$$(2.1) \quad M(R) \geq \psi\left(\frac{\text{dist}(\gamma_1, \gamma_2)}{\text{diam } \gamma_2}\right).$$

2.2 Assouad dimension. A set $X \subset \mathbb{R}^N$ is **s -homogeneous** for some $s \geq 0$ if there exists $C > 0$ such that for every bounded set $A \subset X$, any $\epsilon \in (0, \text{diam } A)$, and any ϵ -separated set $V \subset A$,

$$\text{card } V \leq C(\epsilon^{-1} \text{diam } A)^s.$$

Recall that a set $V \subset A$ is ϵ -separated if for any distinct $x, y \in V$ we have $|x - y| \geq \epsilon$.

If we want to emphasize on the constant C , we say that X is **(C, s) -homogeneous**. Note that every subset of \mathbb{R}^N is N -homogeneous. Moreover, if $0 \leq s_1 \leq s_2$ and X is s_1 -homogeneous, then it is also s_2 -homogeneous. The **Assouad dimension** of a set $X \subset \mathbb{R}^N$ is defined as

$$\text{dim}_A X = \inf\{s \geq 0 : X \text{ is } s\text{-homogeneous}\}.$$

2.3 Hyperbolic geometry. Suppose $X \subset \mathbb{S}^2$ is a Cantor set and $S = \mathbb{S}^2 \setminus X$ is a hyperbolic Riemann surface with a pants decomposition. Here, we recall how the cuffs (α_j) of the decomposition can be related to the thickness of ring domains embedded in the surface; see Figure 3.

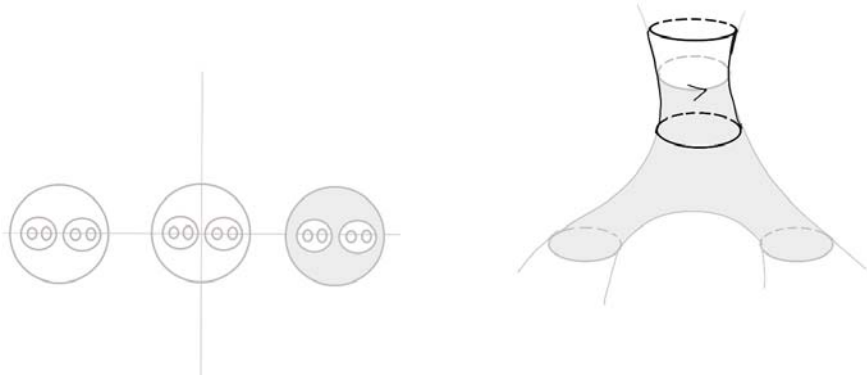


Figure 3. On the left, we have a pants decomposition for $\mathbb{S}^2 \setminus X$ with a particular pair of pants shaded. On the right, we have a topological model of a pair of pants, again shaded, with the arrowed curve a geodesic cuff α_j and the ring domain in black an example of R'_j .

Proposition 2.1. *For each j there exists a ring domain $R'_j \subset \mathbb{S}^2 \setminus X$ that contains α_j such that domains R'_j are mutually disjoint and*

$$M(R'_j) = \frac{\ell(\alpha_j)}{2 \arcsin(e^{-\ell(\alpha_j)})}.$$

This result is assuredly standard. Maskit [Mas85] proves this for finite type surfaces, but since we will be applying this to infinite type surfaces, we give a proof for the convenience of the reader. We need the following Collar Lemma.

Lemma 2.2 ([ALP⁺11, Lemma 2.2]). *There exist pairwise disjoint collars $(C_j)_j$ of cuffs $(\alpha_j)_j$ given by*

$$C_j = \{z \in S : d_S(z, \gamma) \leq B(\ell(\alpha_j))\},$$

where d_S denotes the hyperbolic metric on S and

$$B(t) = \frac{1}{2} \log \left(1 + \frac{2}{e^t - 1} \right).$$

Proof of Proposition 2.1. Let C_j be the collars from Lemma 2.2. These collars are necessarily ring domains.

Since S is a hyperbolic Riemann surface, we can consider its lift to the strip model of the hyperbolic plane. More precisely, let $\Sigma = \{z \in \mathbb{C} : |\text{Im}(z)| < \pi/2\}$.

Then the hyperbolic metric density on Σ is given by $\lambda_\Sigma(z) = \sec(\text{Im}(z))$ (see for example [BM07, Example 7.9]). Since we can identify S with Σ/G , where G is a covering group of deck transformations, we can lift α_j so that its lift is contained in the real axis in Σ . Moreover, C_j can be lifted to a rectangle in Σ whose closure is given by $R = [-r, r] \times [-s, s]$.

Here, we have $d_\Sigma(-r, r) = \ell$ and $d_\Sigma(-is, is) = 2B(\ell(\alpha_j))$. Since the hyperbolic metric and the Euclidean metric coincide on the real axis in Σ , we have $r = \ell(\alpha_j)/2$. Next,

$$2B(\ell(\alpha_j)) = d_\Sigma(-is, is) = \int_{-s}^s \sec t \, dt = 2 \ln(\sec s + \tan s).$$

Solving this for s , we see that

$$s = \arcsin(\tanh(B(\ell(\alpha_j))))$$

and hence

$$s = \arcsin(e^{-\ell(\alpha_j)}).$$

Finally, $M(C_j)$ is equal to the modulus of the path family joining the r -sides to the s -sides of the rectangle R . Thus

$$M(C_j) = \frac{r}{s} = \frac{\ell_S(\alpha_j)}{2 \arcsin(e^{-\ell_S(\alpha_j)})}. \quad \square$$

We will also need the following result of Wolpert.

Lemma 2.3 (Wolpert [Wol81]). *Let $f : S \rightarrow S'$ be a K -quasiconformal homeomorphism between hyperbolic Riemann surfaces S, S' . Let α be a closed geodesic in S , and let α' be the unique closed geodesic in S' that is homotopic to $f(\alpha)$. Then*

$$K^{-1}\ell_S[\alpha] \leq \ell_{S'}[\alpha'] \leq K\ell_S[\alpha].$$

2.4 Square thickenings We recall some terminology and notation from [Mac99]. Given $\alpha > 0$ define

$$\mathcal{G}_\alpha := \{\alpha \mathbf{n} + [0, \alpha]^2 : \mathbf{n} \in \mathbb{Z}^2\} \quad \text{and} \quad \mathcal{G}_\alpha^1 := \{e : e \text{ is an edge of some } S \in \mathcal{G}_\alpha\}.$$

Given a set $W \subset \mathbb{R}^2$ define W^α to be the collection of all squares in \mathcal{G}_α that intersect with W . For $\delta > 0$, define the δ -square thickening

$$\mathcal{T}_\delta(W) = (W^{4\delta})^\delta,$$

see Figure 4.

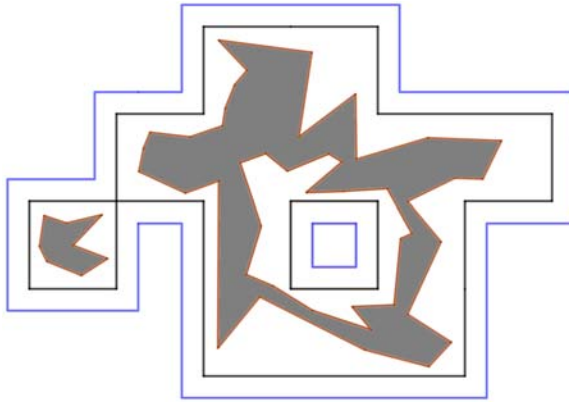


Figure 4. The shaded region is W . The black curve is the boundary of $W^{4\delta}$ and the blue curve is the boundary of $\mathcal{T}_\delta(W)$.

Lemma 2.4 ([Mac99, Lemma 2.1]). *If W is a bounded subset of the plane and $\delta > 0$, then the boundary of $\mathcal{T}_\delta(W)$ is a finite union of mutually disjoint polygonal Jordan curves made of edges in \mathcal{G}_δ^1 and*

$$(2.2) \quad \delta \leq \text{dist}(x, W) \leq 8\delta \quad \text{for all } x \in \partial\mathcal{T}_\delta.$$

2.5 Symbolic notation. At several junctures in this paper, it will be convenient to use symbolic notation to describe our constructions.

Given an integer $k \geq 0$, we denote by $\{1, 2\}^k$ the set of words formed from the alphabet $\{1, 2\}$ that have length exactly k . Conventionally, we set $\{1, 2\}^0 = \{\varepsilon\}$ where ε is the empty word. We also denote by

$$\{1, 2\}^* = \bigcup_{k \geq 0} \{1, 2\}^k$$

the set of all finite words formed from $\{1, 2\}$. Given a word $w \in \{1, 2\}^*$, we denote by $|w|$ the length of w with the convention $|\varepsilon| = 0$.

3 Uniformly disconnected Cantor sets and hyperbolic geometry

In this section, we prove Theorem 1.2. One direction of the theorem is given in Section 3.1 and the other direction is given in Section 3.2.

3.1 Assuming uniformly disconnected. Here we prove the necessary direction of Theorem 1.2.

Proposition 3.1. *Let $X \subset \mathbb{S}^2$ be a c -uniformly disconnected Cantor set. There exists $M > 0$ depending only on c , and there exists a pants decomposition for the Riemann surface $S = \mathbb{S}^2 \setminus X$ such that the associated cuffs (α_j) satisfy $\sup_j \ell[\alpha_j] < M$.*

Denote by \mathcal{C} the standard one-third Cantor set and by S_0 the Riemann surface

$$S_0 = \mathbb{S}^2 \setminus \mathcal{C}.$$

Lemma 3.2 ([Vel21, Corollary A]). *If $X \subset \mathbb{S}^2$ is a uniformly disconnected set, then there exists a quasiconformal map $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that $f(X) \subset \mathcal{C}$.*

As observed in [Shi22, p. 5], the pairs of pants in the pants decomposition of S_0 can be chosen to be conformally equivalent to one another. It follows that each such pair of pants has the same cuff lengths. To see this, suppose P and P' are two pairs of pants in this decomposition with a conformal map $h : P \rightarrow P'$. Let R and R' be the respective doubles of P and P' , that is, R and R' are genus two surfaces. Then h extends via reflection to a conformal map $\tilde{h} : R \rightarrow R'$ and hence \tilde{h} is a hyperbolic isometry. Restricting \tilde{h} to the cuffs of P , we see that P and P' have the same cuff lengths.

In particular, we conclude that there exist a constant $q > 0$ and a pair of pants decomposition of S_0 with cuffs (C_j) such that

$$(3.1) \quad \sup_j \ell_{S_0}[C_j] = q.$$

Proof of Proposition 3.1. Let $X \subset \mathbb{R}^2$ be a uniformly disconnected Cantor set, and let f be the quasiconformal map from Lemma 3.2 such that $f(X) \subset \mathcal{C}$. We will use the pants decomposition with cuffs (C_j) for S_0 . Since $f(S) \supset S_0$, we may use a subset of the (C_j) to generate a pair of pants decomposition for $f(S)$. This subset can be labelled as (C_{j_k}) and we, for brevity, will denote it by (β_k) .

Suppose (η_j) are the cuffs of a pants decomposition of S . Then each η_j is homotopic to $f^{-1}(\beta_k)$ for some k and vice versa. Hence, if we assume for a contradiction that $\ell_S[\eta_{j_m}] \rightarrow \infty$, it follows via Lemma 2.3 that $\ell_{f(S)}[\beta_{k_m}] \rightarrow \infty$.

Since $f(S) \supset S_0$, the subordination principle for the hyperbolic metric implies that if γ is any path in S_0 , then $\ell_{f(S)}(\gamma) \leq \ell_{S_0}(\gamma)$. In particular, we conclude that $\ell_{S_0}[\beta_{k_m}] \rightarrow \infty$. This contradicts (3.1). □

3.2 Towards uniformly disconnected. Here we prove the sufficient direction of Theorem 1.2.

Proposition 3.3. *Let $X \subset \mathbb{R}^2$ be a Cantor set and suppose that the Riemann surface $S = \mathbb{S}^2 \setminus X$ has a pants decomposition (P_j) where the cuffs (α_j) satisfy $\sup_j \ell[\alpha_j] < L < \infty$. Then X is c -uniformly disconnected for some c depending only on L .*

Recall the symbolic notation from Section 2.5.

Lemma 3.4. *A totally bounded metric space X is uniformly disconnected if and only if there exists a set $\mathcal{W} \subset \{1, 2\}^*$, a constant $\delta > 0$ and a collection of subsets $\{X_w : w \in \mathcal{W}\}$ with the following properties.*

- (i) *The empty word $\varepsilon \in \mathcal{W}$ and $X_\varepsilon = X$.*
- (ii) *If $wi \in \mathcal{W}$ for some $i \in \{1, 2\}$ and $w \in \{1, 2\}^*$, then $w \in \mathcal{W}$ and $X_{wi} \subset X_w$.*
- (iii) *If X_w is a point for some $w \in \mathcal{W}$, then $w1 \in \mathcal{W}$, $w2 \notin \mathcal{W}$, and $X_{w1} = X_w$.*
- (iv) *If X_w has at least two points for some $w \in \mathcal{W}$, then $w1, w2 \in \mathcal{W}$, $X_w = X_{w1} \cup X_{w2}$, and*

$$(3.2) \quad \text{dist}(X_{w1}, X_{w2}) \geq \delta \max\{\text{diam } X_{w1}, \text{diam } X_{w2}\}.$$

The constant of uniform disconnectedness and δ are quantitatively related.

Proof. Assume first that X is c -uniformly disconnected. Set $X_\varepsilon = X$. Assume now that for some $w \in \{1, 2\}^*$, we have defined a nonempty set $X_w \subset X$. If X_w is a single point, then set $X_{w1} = X_w$. Assume now that X_w contains at least two points and fix $x \in X_w$. By the uniform disconnectedness of X_w , there exists $E \subset X_w$ such that $x \in E$, $\text{diam } E \leq \frac{1}{2} \text{diam } X_w$ and $\text{dist}(E, X_w \setminus E) \geq (2c)^{-1} \text{diam } X_w$. Set $X_{w1} = E$ and $X_{w2} = X_w \setminus E$. Note that

$$\text{dist}(X_{w1}, X_{w2}) \geq (2c)^{-1} \text{diam } X_w \geq (2c)^{-1} \max\{\text{diam } X_{w1}, \text{diam } X_{w2}\}.$$

Setting \mathcal{W} to be the set of all words $w \in \{1, 2\}^*$ for which X_w has been defined, it is easy to see that $\{X_w : w \in \mathcal{W}\}$ satisfies (i)–(iv) with $\delta = (2c)^{-1}$.

Suppose now that there exists $\mathcal{W} \subset \{1, 2\}^*$, $\delta > 0$ and a collection $\{X_w : w \in \mathcal{W}\}$ satisfying (i)–(iv). We first show that if (i_n) is a sequence in $\{1, 2\}$ such that $i_1 \cdots i_n \in \mathcal{W}$ for all $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} \text{diam } X_{i_1 \dots i_n} = 0$. Assume for a contradiction that there exists $d > 0$ and a sequence (i_n) in $\{1, 2\}$ such that $i_1 \cdots i_n \in \mathcal{W}$ and $\text{diam } X_{i_1 \dots i_n} > d$ for all $n \in \mathbb{N}$. Fix $x_1 \in X_\varepsilon \setminus X_{i_1}$ and for each $n \in \mathbb{N}$ fix $x_n \in X_{i_1 \dots i_{n-1}} \setminus X_{i_1 \dots i_n}$. By (3.2), for any distinct $i, j \in \mathbb{N}$, $|x_i - x_j| \geq \delta d$. Then, the set $\{x_n : n \in \mathbb{N}\}$ is not totally bounded and we reach a contradiction.

We prove now that X is uniformly disconnected. If X contains a single point, then the claim is trivial. Assume now that $\text{diam} X > 0$ and let $x \in X$ and $r \in (0, \text{diam} X)$. Let $w \in \mathcal{W}$ be the maximal word (in word-length) such that $x \in X_w$ and $\text{diam} X_w \geq r$. Write $w = i_1 \cdots i_n$ and assume that $x \in X_{w_{i_{n+1}}}$ where $i_{n+1} \in \{1, 2\}$. Setting $E = X_{w_{i_{n+1}}}$, we have that $\text{diam} E < r$ while

$$\begin{aligned} \text{dist}(E, X \setminus E) &= \min_{j=1, \dots, n+1} \text{dist}(X_{i_1 \dots i_j}, X_{i_1 \dots i_{j-1}} \setminus X_{i_1 \dots i_j}) \\ &\geq \min \left\{ \text{dist}(X_{w_1}, X_{w_2}), \delta \min_{j=1, \dots, n} \text{diam} X_{i_1 \dots i_j} \right\} \\ &= \min \{ \text{dist}(X_{w_1}, X_{w_2}), \delta \text{diam} X_w \}. \end{aligned}$$

By the triangle inequality and (3.2),

$$\text{diam} X_w \leq \text{diam} X_{w_1} + \text{dist}(X_{w_1}, X_{w_2}) + \text{diam} X_{w_2} \leq (1 + 2\delta^{-1}) \text{dist}(X_{w_1}, X_{w_2}).$$

Therefore,

$$\text{dist}(E, X \setminus E) \geq \min\{\delta, (1 + 2\delta^{-1})^{-1}\} \text{diam} X_w \geq (1 + 2\delta^{-1})^{-1} r$$

and X is c -uniformly disconnected with $c = (1 + 2\delta^{-1})^{-1}$. □

We now show Proposition 3.3 and thus complete the proof of Theorem 1.2.

Proof of Proposition 3.3. We assume that each α_j has been chosen to minimize $\ell_S(\alpha_j)$ in its homotopy class. By Proposition 2.1 there exist disjoint ring domains R'_j in $\mathbb{R}^2 \setminus X$ containing the cuffs α_j such that $\sup_j M(R'_j) \leq m$ for some m depending only on L . For each j let V'_j and U'_j be the bounded and unbounded, respectively, components of $\mathbb{R}^2 \setminus R'_j$. We relabel the ring domains R'_j in the following way.

Firstly, we remark that there exist three indices j_1, j_2, j_3 such that $V'_{j_1}, V'_{j_2}, V'_{j_3}$ are mutually disjoint and X is contained in $V'_{j_1} \cup V'_{j_2} \cup V'_{j_3}$. For $l = 1, 2, 3$, we denote $R'_{l,e} := R'_{j_l}$. Inductively, assume that for some $l \in \{1, 2, 3\}$ and some $w \in \{1, 2\}^*$ we have defined $R'_{l,w} = R'_j$ for an index j . There exist two indices i_1, i_2 such that

- (i) $V'_{i_1}, V'_{i_2} \subset V'_j$ and $V'_{i_1} \cap V'_{i_2} = \emptyset$;
- (ii) if $V'_i \subset V'_j$ for some i , then $V'_i \subset V'_{i_1}$ or $V'_i \subset V'_{i_2}$.

Set now $R'_{l,w1} := R'_{i_1}$ and $R'_{l,w2} := R'_{i_2}$.

If for some $j, l \in \{1, 2, 3\}$ and $w \in \{1, 2\}^*$ we have defined $R'_{l,w} = R'_j$, then define $V'_{l,w} := V'_j$ and $U'_{l,w} := U'_j$. Set also $X_{l,w} = X \cap V'_{l,w}$. It is easy to see that for each $l = 1, 2, 3$, the collection $\{X_{l,w} : w \in \{1, 2\}^*\}$ satisfies (i)–(iii) of Lemma 3.4 for $X_{l,e}$.

Fix now $l \in \{1, 2, 3\}$. By (2.1) we have that there exists $d > 0$ depending only on m such that for all $w \in \{1, 2\}^*$

$$\text{dist}(U'_{l,w}, V'_{l,w}) \geq d \text{diam} V'_{l,w}.$$

Therefore, for each $w \in \{1, 2\}^*$

$$\begin{aligned} \text{dist}(X_{l,w1}, X_{l,w2}) &\geq \max_{i=1,2} \text{dist}(X_{l,wi}, \partial V'_{l,wi}) \geq \max_{i=1,2} \text{dist}(V'_{l,wi}, U'_{l,wi}) \\ &\geq d \max_{i=1,2} \text{diam } V'_{l,wi} \\ &\geq d \max_{i=1,2} \text{diam } X_{l,wi}. \end{aligned}$$

Working as above, we can deduce that for all distinct $l, l' \in \{1, 2, 3\}$, we have

$$\text{dist}(X_{l,\epsilon}, X_{l',\epsilon}) \geq d \max\{\text{diam } X_{l,\epsilon}, \text{diam } X_{l',\epsilon}\}.$$

Since X is compact, by Lemma 3.4, X is C -uniformly disconnected with C depending only on d , hence only on m , hence only on L . □

4 Bi-Lipschitz paths

Our strategy to proving Theorem 1.4 is to use bi-Lipschitz paths to yield the required decomposition. We recall the following definition from [FM12].

Definition 4.1. Let (X, d_X) be a metric space. A path $H : [0, 1] \rightarrow LIP(X)$ is called a **bi-Lipschitz path** if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $s, t \in [0, 1]$ with $|s - t| < \delta$, the following two conditions hold:

- (i) for all $x \in X$, $d_X(H_s \circ H_t^{-1}(x), x) < \epsilon$;
- (ii) we have that $H_s \circ H_t^{-1}$ is $(1 + \epsilon)$ -bi-Lipschitz with respect to d_X .

In this paper bi-Lipschitz paths are denoted by capital letters F, G, H, \dots . Given two bi-Lipschitz maps $f, g : X \rightarrow X$, two bi-Lipschitz paths $F, G : [0, 1] \rightarrow LIP(X)$, and a subset $E \subset X$, we define

- (i) the concatenation of F with G to be the bi-Lipschitz path $H : [0, 1] \rightarrow LIP(X)$ with $H_t = F_{2t}$ for $t \in [0, 1/2]$ and $H_t = G_{2t-1}$ for $t \in [1/2, 1]$, and we may then concatenate finitely many bi-Lipschitz paths in the obvious way;
- (ii) the restriction $F|E : [0, 1] \rightarrow LIP(E)$ by $(F|E)_t = F_t|E$;
- (iii) the composition $F \circ G$ by $(F \circ G)_t = F_t \circ G_t$ for all $t \in [0, 1]$;
- (iv) the composition $f \circ F \circ g$ by $(f \circ F \circ g)_t = f \circ F_t \circ g$ for all $t \in [0, 1]$.

We emphasize that in (iii) and (iv) here, the compositions need not be bi-Lipschitz paths. Much of our work will involve showing that our constructions are made carefully enough that when we do need to compose or conjugate, we do still have a bi-Lipschitz path. For illustrative purposes, we include examples where (iii) and (iv) fail to give a bi-Lipschitz path.

Example 4.2. Let $L > 1$, $B = B(0, 1/3) \subset \mathbb{R}^2$ and let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an L -bi-Lipschitz map which is the identity on $\mathbb{R}^2 \setminus B$. Then define

$$\tilde{f}(z) = \begin{cases} f(z - n) + n, & z \in B(n, 1/3), n \in \{0, 1, 2, \dots\}, \\ z, & \text{otherwise.} \end{cases}$$

Clearly \tilde{f} is also an L -bi-Lipschitz map.

Set $F_t = e^{int}z$ for $t \in [0, 1]$ and $H = \text{Id} \circ F \circ \tilde{f}$ where $\text{Id} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the identity map. Then,

$$H_t^{-1} \circ H_s(z) = \tilde{f}^{-1}(e^{i(s-t)\pi}\tilde{f}(z)).$$

Suppose $\epsilon < L - 1$, $\delta > 0$ and $|s - t| = \delta$. Then there exists $N \in \mathbb{N}$ large enough that $B(Ne^{i(s-t)\pi}, 1/3) \cap B(N, 1/3) = \emptyset$. Hence on $B(N, 1/3)$ we have that $H_t^{-1} \circ H_s$ agrees with a composition of a rotation and \tilde{f} . This means that $H_t^{-1} \circ H_s$ is not $(1+\epsilon)$ -bi-Lipschitz and hence H is not a bi-Lipschitz path. We conclude that $\text{Id} \circ F \circ \tilde{f}$ is not a bi-Lipschitz path.

Using the same example and setting $G : [0, 1] \rightarrow \text{LIP}(\mathbb{R}^2)$ to be the constant path \tilde{f} we see that $F \circ G$ is not a bi-Lipschitz path. Hence, compositions of bi-Lipschitz paths are not always bi-Lipschitz paths.

It is worth pointing out that in a bi-Lipschitz path, the elements are bi-Lipschitz with uniform constant.

Lemma 4.3. *Suppose $H : [0, 1] \rightarrow \text{LIP}(X)$ is a bi-Lipschitz path. Then there exists $L > 1$ such that H_t is an L -bi-Lipschitz map for each $t \in [0, 1]$.*

Proof. Clearly H_0 is an L_0 -bi-Lipschitz map for some $L_0 > 1$. Set $\epsilon = 1$ and the corresponding $\delta > 0$ so that condition (ii) holds. In particular, for every $t_1 \in (0, \delta)$, by condition (ii) applied to $(H_{t_1} \circ H_0^{-1}) \circ H_0$, the map H_{t_1} is $2L_0$ -bi-Lipschitz. Next, for every $t_2 \in [\delta, 2\delta)$, there exists $t_1 \in (0, \delta)$ with $|t_2 - t_1| < \delta$. Applying condition (ii) to $(H_{t_2} \circ H_{t_1}^{-1}) \circ (H_{t_1} \circ H_0^{-1}) \circ H_0$, we see that H_{t_2} is 2^2L_0 -bi-Lipschitz.

Continuing inductively, we see that for any $t \in [0, 1]$, H_t is $2^{\lceil 1/\delta \rceil + 1}L_0$ -bi-Lipschitz. □

Next we show that if the restrictions of H on three sets whose union is \mathbb{R}^2 are bi-Lipschitz paths, then H is a bi-Lipschitz path quantitatively.

Lemma 4.4. *Let $A, B, C \subset \mathbb{R}^2$ be closed sets such that $\mathbb{R}^2 = A \cup B \cup C$. Suppose that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $s, t \in [0, 1]$ with $|s - t| < \delta$, then the two conditions in Definition 4.1 hold simultaneously for $H|_A, H|_B, H|_C$. Then the two conditions in Definition 4.1 also hold for H with the same δ .*

Proof. Fix $\epsilon > 0$ and let $\delta > 0$ such that the two conditions in Definition 4.1 hold simultaneously for $H|A, H|B, H|C$. Let $s, t \in [0, 1]$ such that $|s - t| < \delta$. If $x \in \mathbb{R}^2$, then without loss of generality we may assume that $x \in A$ and we have

$$|H_s \circ H_t^{-1}(x) - x| = |(H|A)_s \circ (H|A)_t^{-1}(x) - x| < \epsilon$$

and H satisfies (i).

For (ii), it suffices to show that $H_s \circ H_t^{-1}$ is $(1 + \epsilon)$ -Lipschitz. Let $x, y \in \mathbb{R}^2$. If both x and y belong to the same set from A, B, C , then (ii) follows immediately. Assume without loss of generality that $x \in A$ and $y \in B \setminus A$. Let $z \in [x, y] \cap A$ such that $A \cap ([y, z] \setminus \{z\}) = \emptyset$. Since $y \in B \setminus A$, we have that $z \neq y$.

There are now two cases. First, if $z \in B$ then

$$\begin{aligned} |H_s \circ H_t^{-1}(x) - H_s \circ H_t^{-1}(y)| &\leq |H_s \circ H_t^{-1}(x) - H_s \circ H_t^{-1}(z)| \\ &\quad + |H_s \circ H_t^{-1}(z) - H_s \circ H_t^{-1}(y)| \\ &\leq |(H|A)_s \circ (H|A)_t^{-1}(x) - (H|A)_s \circ (H|A)_t^{-1}(z)| \\ &\quad + |(H|B)_s \circ (H|B)_t^{-1}(z) - (H|B)_s \circ (H|B)_t^{-1}(y)| \\ &\leq (1 + \epsilon)(|x - z| + |z - y|) \\ &= (1 + \epsilon)|x - y|. \end{aligned}$$

Second, if $z \in C$ then we have two sub-cases. If also $y \in C$, we have the same argument as above, with the role of B played by C . If $y \notin C$, then let $w \in C \cap [z, y]$ be such that $C \cap ([w, y] \setminus \{w\}) = \emptyset$. We have $w \neq y$ by construction. Since $z \in A \cap C$ and $w \in B \cap C$, we have

$$\begin{aligned} |H_s \circ H_t^{-1}(x) - H_s \circ H_t^{-1}(y)| &\leq |H_s \circ H_t^{-1}(x) - H_s \circ H_t^{-1}(z)| \\ &\quad + |H_s \circ H_t^{-1}(z) - H_s \circ H_t^{-1}(w)| \\ &\quad + |H_s \circ H_t^{-1}(w) - H_s \circ H_t^{-1}(y)| \\ &\leq |(H|A)_s \circ (H|A)_t^{-1}(x) - (H|A)_s \circ (H|A)_t^{-1}(z)| \\ &\quad + |(H|C)_s \circ (H|C)_t^{-1}(z) - (H|C)_s \circ (H|C)_t^{-1}(w)| \\ &\quad + |(H|B)_s \circ (H|B)_t^{-1}(w) - (H|B)_s \circ (H|B)_t^{-1}(y)| \\ &\leq (1 + \epsilon)(|x - z| + |z - w| + |w - y|) \\ &= (1 + \epsilon)|x - y|. \end{aligned} \quad \square$$

Our next result involves the removability of a Cantor set for a bi-Lipschitz path.

Proposition 4.5. *Let $X \subset \mathbb{R}^2$ be a Cantor set. For each $0 \leq t \leq 1$, suppose $F_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a continuous mapping such that $F_t|_{\mathbb{R}^2 \setminus X}$ is a bi-Lipschitz path. Then F extends to a bi-Lipschitz path on \mathbb{R}^2 .*

Proof. First, it is well-known that a bi-Lipschitz map on U can be extended to a bi-Lipschitz map on the metric closure of U . Hence the hypothesis that F is a bi-Lipschitz path on $\mathbb{R}^2 \setminus X$, and Lemma 4.3 imply that there exists $L \geq 1$ so that each F_t is L -bi-Lipschitz on \mathbb{R}^2 .

Next, we show that property (i) holds in the definition of a bi-Lipschitz path. Suppose $\epsilon > 0$ is given and find $\delta > 0$ so that if $|s - t| < \delta$ then for any $z \in \mathbb{R}^2 \setminus X$,

$$|F_t(z) - F_s(z)| < \epsilon/2.$$

If $x \in X$, find a sequence (x_n) in $\mathbb{R}^2 \setminus X$ with $x_n \rightarrow x$. Then if we choose $N \in \mathbb{N}$ so that $|x - x_n| < \epsilon/4L$ for $n \geq N$, we have

$$\begin{aligned} |F_t(x) - F_s(x)| &\leq |F_t(x) - F_t(x_n)| + |F_t(x_n) - F_s(x_n)| + |F_s(x_n) - F_s(x)| \\ &\leq 2L|x - x_n| + \epsilon/2 \\ &< \epsilon. \end{aligned}$$

Hence condition (i) is satisfied.

Turning now to property (ii), let $x \in X$ and $y \in \mathbb{R}^2$ and find sequences $(x_n), (y_n)$ in $\mathbb{R}^2 \setminus X$ with $x_n \rightarrow x$ and $y_n \rightarrow y$. Given $\epsilon > 0$, find $\delta > 0$ so that if $|s - t| < \delta$ and $z, w \in \mathbb{R}^2 \setminus X$ then

$$|F_t(z) - F_t(w)| < (1 + \epsilon/3)|F_s(z) - F_s(w)|.$$

Next, choose $N \in \mathbb{N}$ large enough that if $n \geq N$ then $|x - x_n| < L^{-2}\epsilon|x - y|/3$ and $|y - y_n| < \epsilon L^{-2}|x - y|/3$. Hence

$$|F_t(x) - F_t(x_n)| \leq L|x - x_n| < \epsilon(3L)^{-1}|x - y| \leq \epsilon|F_s(x) - F_s(y)|/3$$

and

$$|F_t(y_n) - F_t(y)| \leq L|y_n - y| < \epsilon(3L)^{-1}|x - y| \leq \epsilon|F_s(x) - F_s(y)|/3.$$

It follows that

$$\begin{aligned} |F_t(x) - F_t(y)| &\leq |F_t(x) - F_t(x_n)| + |F_t(x_n) - F_t(y_n)| + |F_t(y_n) - F_t(y)| \\ &< (1 + \epsilon)|F_s(x) - F_s(y)|. \end{aligned}$$

We conclude that property (ii) is satisfied and hence F is a bi-Lipschitz path on \mathbb{R}^2 . □

4.1 Uniform families of bi-Lipschitz paths. For the construction in the proof of Theorem 1.4, it will be useful to consider collections of bi-Lipschitz paths with uniform control. To that end we make the following definition.

Definition 4.6. A collection \mathcal{H} of bi-Lipschitz paths $H : [0, 1] \rightarrow LIP(X)$ on a common metric space X is a **uniform family of bi-Lipschitz paths** if

- (i) there exists $L \geq 1$ so that H_0 has isometric distortion bounded above by L for all $H \in \mathcal{H}$,
- (ii) given $\epsilon > 0$, there exists $\delta > 0$ so that if $s, t \in [0, 1]$ with $|s - t| < \delta$ then the two conditions in Definition 4.1 hold simultaneously for all $H \in \mathcal{H}$.

It is clear from Definition 4.6 and Lemma 4.3 that there is a uniform bound on the isometric distortion of any map from any path in a uniform family of bi-Lipschitz paths. We have the following composition result.

Lemma 4.7. *Let \mathcal{H} and \mathcal{G} be two uniform families of bi-Lipschitz paths so that for each $G \in \mathcal{G}$ and each $t \in [0, 1]$, G_t is an isometry. Then the family $\mathcal{F} = \{G \circ H : G \in \mathcal{G}, H \in \mathcal{H}\}$ is a uniform family of bi-Lipschitz paths.*

Proof. Given $\epsilon > 0$, find $\delta > 0$ so that both conditions in Definition 4.1 and Definition 4.6 hold for $|s - t| < \delta$, all $H \in \mathcal{H}$ and all $G \in \mathcal{G}$. Fixing $G \in \mathcal{G}$ and $H \in \mathcal{H}$, let $F = G \circ H$. Then using the fact that G_t is an isometry,

$$\begin{aligned} |F_t(z) - F_s(z)| &= |G_t(H_t(z)) - G_s(H_s(z))| \\ &= |G_t(H_t(z)) - G_t(H_s(z)) + G_t(H_s(z)) - G_s(H_s(z))| \\ &\leq |H_t(z) - H_s(z)| + |G_t(H_s(z)) - G_s(H_s(z))| \\ &\leq 2\epsilon, \end{aligned}$$

which verifies that condition (i) of Definition 4.1 holds uniformly for all paths in \mathcal{F} . For condition (ii), we have

$$\begin{aligned} |F_t(z) - F_t(w)| &= |G_t(H_t(z)) - G_t(H_t(w))| \\ &= |H_t(z) - H_t(w)| \\ &\leq (1 + \epsilon)|H_s(z) - H_s(w)| \\ &= (1 + \epsilon)|G_s(H_s(z)) - G_s(H_s(w))| \\ &= (1 + \epsilon)|F_s(z) - F_s(w)|, \end{aligned}$$

which verifies that condition (ii) of Definition 4.1 holds uniformly for all paths in \mathcal{F} . Finally, since G_0 is an isometry, and the isometric distortion of H_0 is uniformly bounded above, it follows that the same is true for any $F \in \mathcal{F}$. □

For our next result, we see that a family of conjugates of a bi-Lipschitz path by controlled dilations is uniform.

Lemma 4.8. *Let $F : [0, 1] \rightarrow LIP(\mathbb{R}^n)$ be a bi-Lipschitz path and let $c \geq 1$. The family*

$$\mathcal{F} = \{ \phi \circ F \circ \phi^{-1} : \phi \text{ is a similarity of } \mathbb{R}^n \text{ with scaling factor at most } c \}$$

is a uniform family of bi-Lipschitz paths.

Proof. Fix $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to be a similarity of scaling factor $\lambda \leq c$. First, suppose F_0 has isometric distortion L_0 . Then we have

$$| \phi \circ F_0 \circ \phi^{-1}(z) - \phi \circ F_0 \circ \phi^{-1}(w) | = \lambda | F_0 \circ \phi^{-1}(z) - F_0 \circ \phi^{-1}(w) |$$

from which it easily follows that $\phi \circ F_0 \circ \phi^{-1}$ is L_0 -bi-Lipschitz.

Next, given $\epsilon > 0$, find $\delta > 0$ so that the two conditions in Definition 4.1 hold for F_t . If $s, t \in [0, 1]$ with $|s - t| < \delta$, then

$$| \phi \circ F_t \circ \phi^{-1}(z) - \phi \circ F_s \circ \phi^{-1}(z) | = \lambda | F_t \circ \phi^{-1}(z) - F_s \circ \phi^{-1}(z) | < \lambda \epsilon \leq c \epsilon$$

and we conclude that property (i) of Definition 4.1 holds uniformly in \mathcal{F} . Finally,

$$\begin{aligned} | \phi \circ F_t \circ \phi^{-1}(z) - \phi \circ F_t \circ \phi^{-1}(w) | &= \lambda | F_t \circ \phi^{-1}(z) - F_t \circ \phi^{-1}(w) | \\ &\leq \lambda(1 + \epsilon) | F_s \circ \phi^{-1}(z) - F_s \circ \phi^{-1}(w) | \\ &= (1 + \epsilon) | \phi \circ F_s \circ \phi^{-1}(z) - \phi \circ F_s \circ \phi^{-1}(w) |, \end{aligned}$$

from which we conclude condition (ii) in Definition 4.1 holds uniformly in \mathcal{F} . \square

4.2 Bi-Lipschitz paths on triangles. As part of our construction, we will be using specific bi-Lipschitz paths which deform triangles in \mathbb{R}^2 . Let T be a triangle in \mathbb{C} . If the vertices are w_1, w_2, w_3 , taken in counterclockwise order, then we may also denote this triangle by $T(w_1, w_2, w_3)$. In our construction, there will be two triangles T_1 and T_2 which share a vertex, and a bi-Lipschitz path $G : [0, 1] \rightarrow LIP(\mathbb{R}^2)$ such that G_0 is the identity in T_1 and G_1 is an affine map from T_1 onto T_2 . We will focus on constructing G_t inside T_1 .

After conjugating by an affine map, we may assume that T_1 and T_2 share 0 as a vertex, that $T_1 = T(0, 1, a)$ and $T_2 = T(0, c, b)$ for $0 < \arg a < \pi$, $0 \leq \arg c < \arg b \leq \pi$ and $\arg b - \arg c < \pi$. These restrictions ensure that neither T_1 nor T_2 degenerate to line segments.

Proposition 4.9. *There exists a bi-Lipschitz path $G : [0, 1] \rightarrow LIP(T_1)$ such that G_0 is the identity and G_1 is the map given by*

$$G_1(z) = \left(\frac{b - \bar{a}c}{a - \bar{a}}\right)z + \left(\frac{ac - b}{a - \bar{a}}\right)\bar{z}.$$

Proof. First, every real-linear map in \mathbb{C} is of the form $Az + B\bar{z}$ for $A, B \in \mathbb{C}$, and since we require our maps to be orientation-preserving, we have $|A| > |B|$. Given $T_1 = T(0, 1, a)$ and $T_2 = (0, c, b)$, it is elementary to check that

$$g_1(z) = \left(\frac{b - \bar{a}c}{a - \bar{a}}\right)z + \left(\frac{ac - b}{a - \bar{a}}\right)\bar{z}$$

fixes 0, maps 1 to c and maps a to b .

Set $\gamma_1(t) = ct + (1 - t)$ for $0 \leq t \leq 1$ and $\gamma_2(t) = bt + (1 - t)a$ for $0 \leq t \leq 1$. Then define

$$G_t(z) = \left(\frac{\gamma_2(t) - \bar{a}\gamma_1(t)}{a - \bar{a}}\right)z + \left(\frac{a\gamma_1(t) - \gamma_2(t)}{a - \bar{a}}\right)\bar{z}.$$

For any $t \in [0, 1]$, G_t maps T_1 onto the triangle with vertices 0, $\gamma_1(t)$ and $\gamma_2(t)$.

For ease of notation, we define the map

$$h := G_t \circ G_s^{-1},$$

which maps $T(0, \gamma_1(s), \gamma_2(s))$ onto $T(0, \gamma_1(t), \gamma_2(t))$. Since $h(z) = \gamma_1(t)\alpha(z/\gamma_1(s))$, where α maps $T(0, 1, \gamma_2(s)/\gamma_1(s))$ onto $T(0, 1, \gamma_2(t)/\gamma_1(t))$, we can compute that

$$\begin{aligned} (4.1) \quad h(z) &= \left(\frac{\gamma_2(t)/\gamma_1(t) - \overline{\gamma_2(s)/\gamma_1(s)}}{\gamma_2(s)/\gamma_1(s) - \overline{\gamma_2(s)/\gamma_1(s)}}\right) \left(\frac{\gamma_1(t)z}{\gamma_1(s)}\right) \\ &+ \left(\frac{\gamma_2(s)/\gamma_1(s) - \overline{\gamma_2(t)/\gamma_1(t)}}{\gamma_2(s)/\gamma_1(s) - \overline{\gamma_2(s)/\gamma_1(s)}}\right) \left(\frac{\gamma_1(t)\bar{z}}{\gamma_1(s)}\right). \end{aligned}$$

We collect some estimates that we will need. First we may suppose there exists $R > 0$ so that

$$(4.2) \quad \frac{1}{R} \leq |\gamma_i(t)| \leq R,$$

for all $t \in [0, 1]$ and $i \in \{1, 2\}$. Note also that $|z| \leq R$ for $z \in T(0, \gamma_1(s), \gamma_2(s))$. Next, since the interior of every open triangle $G_t(T(0, 1, a))$ is contained in the upper half-plane and the triangles do not degenerate, there exists $r > 0$ so that

$$(4.3) \quad \sup_{t \in [0, 1]} \left| \frac{\gamma_2(t)}{\gamma_1(t)} - \overline{\left(\frac{\gamma_2(t)}{\gamma_1(t)}\right)} \right| > r.$$

We also observe that

$$(4.4) \quad |\gamma_1(t) - \gamma_1(s)| = |c - 1| \cdot |t - s|, \quad |\gamma_2(t) - \gamma_2(s)| = |b - a| \cdot |t - s|,$$

and hence that

$$(4.5) \quad \left| \frac{\gamma_1(t)}{\gamma_1(s)} \right| \leq 1 + \frac{|\gamma_1(t) - \gamma_1(s)|}{|\gamma_1(s)|} \leq 1 + R|c - 1||t - s|.$$

Finally, via an elementary calculation we have

$$\frac{\gamma_2(s)}{\gamma_1(s)} - \frac{\gamma_2(t)}{\gamma_1(t)} = \frac{(b - ac)(s - t)}{(1 + t(c - 1))(1 + s(c - 1))}$$

and hence by (4.2) we have

$$(4.6) \quad \left| \frac{\gamma_2(s)}{\gamma_1(s)} - \frac{\gamma_2(t)}{\gamma_1(t)} \right| \leq R^2|b - ac||s - t|.$$

We can now prove property (i) for showing G is a bi-Lipschitz path. Given $\epsilon > 0$, we choose $\delta < \epsilon/\xi$, with ξ chosen below, so that if $|t - s| < \delta$ with $s, t \in [0, 1]$, then by (4.1),

$$\begin{aligned} |h(z) - z| &= \left| z \left(\frac{\gamma_2(t) - \gamma_2(s)}{\gamma_1(s)} + \left(\frac{\overline{\gamma_2(s)}}{|\gamma_1(s)|^2} \right) \cdot (\gamma_1(s) - \gamma_1(t)) \right) \left(\frac{\gamma_2(s)}{\gamma_1(s)} - \frac{\overline{\gamma_2(s)}}{\gamma_1(s)} \right)^{-1} \right. \\ &\quad \left. + \overline{z} \left(\frac{\gamma_2(s)/\gamma_1(s) - \gamma_2(t)/\gamma_1(t)}{\gamma_2(s)/\gamma_1(s) - \overline{\gamma_2(s)}/\gamma_1(s)} \right) \left(\frac{\gamma_1(t)}{\gamma_1(s)} \right) \right|. \end{aligned}$$

Using (4.2), (4.3), (4.4) and (4.6), we obtain

$$|h(z) - z| \leq \frac{R}{r}(R|b - a||t - s| + R^3|c - 1||t - s|) + \frac{R^5|b - ac|}{r}|t - s|.$$

By choosing

$$\xi = \frac{R^2|b - a|}{r} + \frac{R^4|c - 1|}{r} + \frac{R^5|b - ac|}{r},$$

we obtain $|h(z) - z| < \epsilon$ for $z \in T(0, \gamma_1(s), \gamma_2(s))$ as required.

Next, we prove property (ii). If $h(z) = Az + B\overline{z}$ is orientation preserving, then $|A| > |B|$ and h is bi-Lipschitz with isometric distortion given by

$$(4.7) \quad \max \left\{ |A| + |B|, \frac{1}{|A| - |B|} \right\}.$$

In our setting,

$$|A| = \left| \frac{\gamma_1(t)}{\gamma_1(s)} \right| \cdot \left| \frac{\gamma_2(t)}{\gamma_1(t)} - \frac{\overline{\gamma_2(s)}}{\gamma_1(s)} \right| \cdot \left| \frac{\gamma_2(s)}{\gamma_1(s)} - \frac{\overline{\gamma_2(s)}}{\gamma_1(s)} \right|^{-1}.$$

We can compute that

$$\frac{\gamma_2(t)/\gamma_1(t) - \overline{\gamma_2(s)}/\gamma_1(s)}{\gamma_2(s)/\gamma_1(s) - \overline{\gamma_2(s)}/\gamma_1(s)} = 1 + \frac{\gamma_2(t)/\gamma_1(t) - \gamma_2(s)/\gamma_1(s)}{\gamma_2(s)/\gamma_1(s) - \overline{\gamma_2(s)}/\gamma_1(s)}$$

and hence by (4.5) and (4.6) we obtain

$$(4.8) \quad |A| \leq (1 + R|c - 1||t - s|) \left(1 + \frac{R^2|b - ac|}{r}|t - s| \right).$$

We also have that

$$|B| = \left| \frac{\gamma_1(t)}{\gamma_1(s)} \right| \cdot \left| \frac{\gamma_2(t)}{\gamma_1(t)} - \frac{\gamma_2(s)}{\gamma_1(s)} \right| \cdot \left| \frac{\gamma_2(s)}{\gamma_1(s)} - \overline{\frac{\gamma_2(s)}{\gamma_1(s)}} \right|^{-1}$$

and it follows from the comment after (4.2), (4.3) and (4.6) that

$$(4.9) \quad |B| \leq \frac{R^4|b - ac|}{r}|s - t|.$$

It then follows easily from (4.7), (4.8) and (4.9) that given $\epsilon > 0$ we can choose $\delta > 0$ so that if $s, t \in [0, 1]$ with $|s - t| < \delta$, then $G_t \circ G_s^{-1}$ is $(1 + \epsilon)$ -bi-Lipschitz, completing the proof that G_t is a bi-Lipschitz path. \square

4.3 Dehn twists and conjugates. Suppose $R \subset \mathbb{R}^2$ is a ring domain. Then we can define a Dehn twist \mathfrak{D} in R as follows. There exists $\eta \in (0, 1)$ and a conformal map bijection $g : S \rightarrow R$, where $S = B(0, 1) \setminus \overline{B}(0, 1 - \eta)$ is a round ring. By the conformal invariance of the modulus of ring domains, η is uniquely defined. The Dehn twist in S is given in polar coordinates by

$$\mathfrak{D}(r, \theta) = \left(r, \theta + 2\pi \frac{1}{\eta}(1 - r) \right),$$

and then the Dehn twist in R is given by $g \circ \mathfrak{D} \circ g^{-1}$.

Lemma 4.10. *Let $\eta > 0$, and consider S and \mathfrak{D} as above. For $0 \leq t \leq 1$ and $re^{i\theta} \in S$, set*

$$D_t(r, \theta) = \left(r, \theta + 2\pi t \frac{1}{\eta}(1 - r) \right).$$

Then D is a bi-Lipschitz path in S connecting the identity to \mathfrak{D} .

Proof. For convenience, for $1 - \eta \leq r \leq 1$, set $h(r) = \frac{1}{\eta}(1 - r)$. Let $z = re^{i\theta} \in S$. If $s, t \in [0, 1]$, we have

$$|D_t(z) - D_s(z)| = |ze^{2\pi i t h(|z|)} - ze^{2\pi i s h(|z|)}| = |z| |1 - e^{2\pi i (s-t) h(|z|)}|.$$

Since $|z| \leq 1$ and $h(|z|) \in [0, 1]$, it is clear that the first condition in the definition of a bi-Lipschitz path is satisfied.

Next, it is clear that

$$D_t^{-1}(re^{i\theta}) = re^{i(\theta - 2\pi t h(r))},$$

and hence for $z, w \in S$ and $s, t \in [0, 1]$ we have

$$\begin{aligned} |D_s \circ D_t^{-1}(z) - D_s \circ D_t^{-1}(w)| &= |ze^{2\pi i h(|z|)(s-t)} - we^{2\pi i h(|w|)(s-t)}| \\ &= |z - we^{2\pi i (s-t)(h(|w|) - h(|z|))}|. \end{aligned}$$

Since h is linear, it follows that

$$|h(|w|) - h(|z|)| = \frac{1}{\eta} ||w| - |z|| \leq \frac{1}{\eta} |w - z|.$$

We conclude that there exists $C > 0$ independent of z, w such that

$$\begin{aligned} |D_s \circ D_t^{-1}(z) - D_s \circ D_t^{-1}(w)| &= |z - we^{2\pi i (s-t)(h(|w|) - h(|z|))}| \\ &\leq |z - w| + |w| |1 - e^{2\pi i (s-t)(h(|w|) - h(|z|))}| \\ &\leq |z - w|(1 + C|s - t|), \end{aligned}$$

from which the second condition in the definition of a bi-Lipschitz path is satisfied. \square

We will need to know that conformal conjugates of D_t are also bi-Lipschitz paths. As was observed in [FM12, Remark 2.6], the conjugate of a bi-Lipschitz path on a closed manifold by a conformal map is a bi-Lipschitz path and this cannot be weakened to conjugation by a diffeomorphism. However, here we have closed ring domains, and so this remark does not immediately apply.

Proposition 4.11. *Suppose S is a round annulus, R is a ring domain with smooth boundary components, $g: S \rightarrow R$ is a conformal map and $F: [0, 1] \rightarrow LIP(S)$ is a bi-Lipschitz path such that $F_t(S) = S$ for all $t \in [0, 1]$. If $H = g \circ F \circ g^{-1}$, then H is a bi-Lipschitz path with $H_t(R) = R$ for each $t \in [0, 1]$.*

Proof. We start with condition (i) from Definition 4.1. Since the boundary components of R are assumed smooth, the Riemann map g and all its derivatives extend continuously to ∂S , see [BK87, p. 24]. In particular, there exists an upper bound M for both $|g'|$ and $|(g^{-1})'|$ on \bar{S} and \bar{R} , respectively. Hence g and g^{-1} are M -bi-Lipschitz maps.

Given $\epsilon > 0$, find $\delta > 0$ so that if $s, t \in [0, 1]$ satisfy $|s - t| < \delta$ then

$$|F_s \circ F_t^{-1}(z) - z| < \epsilon/M$$

for all $z \in R$. Then

$$\begin{aligned} |H_s \circ H_t^{-1}(z) - z| &= |g \circ F_s \circ F_t^{-1} \circ g^{-1}(z) - g(g^{-1}(z))| \\ &\leq M |F_s \circ F_t^{-1} \circ g^{-1}(z) - g^{-1}(z)| \\ &\leq \epsilon. \end{aligned}$$

Hence condition (i) holds.

Next, for condition (ii), given $\epsilon > 0$, find $\delta > 0$ so that if $s, t \in [0, 1]$ with

$$(4.10) \quad |s - t| < \delta, \quad \text{then } F_s \circ F_t^{-1} \text{ is } (1 + \epsilon)\text{-bi-Lipschitz.}$$

Consider the functions $p : \bar{S} \times \bar{S} \rightarrow \mathbb{C}$ and $q : \bar{R} \times \bar{R} \rightarrow \mathbb{C}$ defined by

$$(4.11) \quad p(z, w) = \begin{cases} \frac{g(z) - g(w)}{z - w} - g'(w) & \text{if } z \neq w \\ 0 & \text{if } z = w \end{cases}$$

and

$$(4.12) \quad q(z, w) = \begin{cases} \frac{g^{-1}(z) - g^{-1}(w)}{z - w} - (g^{-1})'(w) & \text{if } z \neq w, \\ 0 & \text{if } z = w. \end{cases}$$

By differentiability of g and g^{-1} , both p and q are continuous functions on compact sets in \mathbb{C}^2 and hence bounded. That is, there exists $C > 0$ so that $|p(z, w)| \leq C$ for all $(z, w) \in \bar{S} \times \bar{S}$, and $|q(z, w)| \leq C$, for all $(z, w) \in \bar{R} \times \bar{R}$. Hence given $\epsilon' > 0$, there exists $r > 0$ so that if $z, w \in \bar{S}$ with $|z - w| < r$, then

$$(4.13) \quad |p(z, w)| < \epsilon'.$$

By reducing r if necessary, by the same reasoning we can also assume that if $z, w \in \bar{R}$ with $|z - w| < r$ then

$$(4.14) \quad |q(z, w)| < \epsilon'.$$

Now, let $z, w \in R$ with $|z - w| < r/[M(1 + \epsilon)]$. Set $u = F_s \circ F_t^{-1} \circ g^{-1}(z)$ and $v = F_s \circ F_t^{-1} \circ g^{-1}(w)$. Then since $F_s \circ F_t^{-1}$ is $(1 + \epsilon)$ -bi-Lipschitz and g^{-1} is M -bi-Lipschitz, we have $|u - v| < r$. Hence by (4.11) we have

$$(4.15) \quad |H_s \circ H_t^{-1}(z) - H_s \circ H_t^{-1}(w)| = |g(u) - g(v)| = |g'(v) + p(u, v)| \cdot |u - v|.$$

Next, again using the fact that $F_s \circ F_t^{-1}$ is $(1 + \epsilon)$ -bi-Lipschitz, we obtain

$$(4.16) \quad |u - v| \leq (1 + \epsilon)|g^{-1}(z) - g^{-1}(w)|.$$

Using (4.12), we have

$$(4.17) \quad |g^{-1}(z) - g^{-1}(w)| = |(g^{-1})'(w) + q(z, w)| \cdot |z - w|.$$

Combining (4.15), (4.16) and (4.17), we obtain

$$(4.18) \quad \begin{aligned} &|H_s \circ H_t^{-1}(z) - H_s \circ H_t^{-1}(w)| \\ &\leq |g'(v) + p(u, v)| \cdot (1 + \epsilon) \cdot |(g^{-1})'(w) + q(z, w)| \cdot |z - w|. \end{aligned}$$

Next, we have

$$\begin{aligned}
 & |g'(v)(g^{-1})'(w)| \\
 &= |g'(g^{-1}(w))(g^{-1})'(w) + [g'(F_s(F_t^{-1}(g^{-1}(w)))) - g'(g^{-1}(w))](g^{-1})'(w)|.
 \end{aligned}$$

Using condition (i) of F being a bi-Lipschitz path, the fact that $|(g^{-1})'|$ is bounded and the fact that g' is uniformly continuous on \bar{S} , by shrinking δ if necessary, we may conclude that

$$(4.19) \quad |g'(v)(g^{-1})'(w)| \leq 1 + \epsilon.$$

Combining (4.10), (4.13), (4.14), (4.18), (4.19), and the bounds for the derivatives of g, g^{-1} we obtain

$$\begin{aligned}
 & |H_s \circ H_t^{-1}(z) - H_s \circ H_t^{-1}(w)| \\
 & \leq (1 + \epsilon)(|g'(v)(g^{-1})'(w)| + |p(u, v)|((g^{-1})'(w)| \\
 & \quad + |q(z, w)g'(v)| + |p(u, v)q(z, w)|)|z - w| \\
 & \leq (1 + \epsilon)((1 + \epsilon) + 2M\epsilon' + (\epsilon')^2)|z - w|.
 \end{aligned}$$

In particular, given $\eta > 0$ we can find $\delta > 0$ and $r > 0$ so that if $s, t \in [0, 1]$ with $|s - t| < \delta$, then for any $z, w \in \bar{R}$ with $|z - w| < r$ we have

$$(4.20) \quad |H_s \circ H_t^{-1}(z) - H_s \circ H_t^{-1}(w)| \leq (1 + \eta)|z - w|.$$

To show that condition (ii) holds, suppose for a contradiction that it does not. Then we can find $\eta > 0$ and sequences s_n, t_n in $[0, 1]$ with $|s_n - t_n| \rightarrow 0$ and sequences z_n, w_n in \bar{R} for which

$$(4.21) \quad \left| \frac{H_{s_n} \circ H_{t_n}^{-1}(z_n) - H_{s_n} \circ H_{t_n}^{-1}(w_n)}{z_n - w_n} \right| > 1 + \eta$$

for all n . By passing to subsequences, we may assume that $z_n \rightarrow z_0$ and $w_n \rightarrow w_0$. If $z_0 = w_0$ then we obtain a contradiction to (4.20). Otherwise, suppose $|z_0 - w_0| = \zeta$ and find $N \in \mathbb{N}$ so that if $n \geq N$ then $|z_n - w_n| > \zeta/2$. By condition (i), we have

$$\begin{aligned}
 & |H_{s_n} \circ H_{t_n}^{-1}(z_n) - H_{s_n} \circ H_{t_n}^{-1}(w_n)| \\
 & \leq |H_{s_n} \circ H_{t_n}^{-1}(z_n) - z_n| + |z_n - w_n| + |H_{s_n} \circ H_{t_n}^{-1}(w_n) - w_n| \\
 & \leq |z_n - w_n| + 2\epsilon \\
 & \leq \left(1 + \frac{4\epsilon}{\zeta}\right)|z_n - w_n|.
 \end{aligned}$$

Since $|s_n - t_n| \rightarrow 0$, we can choose n large enough so that $4\epsilon/\zeta < \eta$ and hence contradict (4.21). We conclude that condition (ii) holds and hence H_t is a bi-Lipschitz path. □

4.4 Interpolation in an annulus. In this subsection, we will prove the following interpolation result.

Proposition 4.12. *Suppose $T > 1$, and let $R = \{z \in \mathbb{C} : 1 \leq |z| \leq T\}$ with boundary components $S_1 = \{z : |z| = 1\}$ and $S_T = \{z : |z| = T\}$. Let $P : [0, 1] \rightarrow LIP(S_1)$ and $Q : [0, 1] \rightarrow LIP(S_T)$ be bi-Lipschitz paths such that $P_0 = P_1$ is the identity on S_1 , $Q_0 = Q_1$ is the identity on S_T , and $\arg P_t, \arg Q_t$ are strictly increasing in t . Then there exists a bi-Lipschitz path $F : [0, 1] \rightarrow LIP(R)$, with $F_0 = F_1$ the identity on R and F_t extends P_t and Q_t for each t .*

We start with the following fairly elementary estimate.

Lemma 4.13. *Suppose that $c_1, c_2 > 0$, $c_3 \in [-3, 3]$ and $a \in \mathbb{R}$. For any $\epsilon > 0$ and any $\delta_1, \delta_2 \in (-\epsilon, \epsilon)$, $\delta_3 \in (-2\epsilon, 2\epsilon)$, we have*

$$|a + i(c_1 + c_2 + c_3x) + i(\delta_1c_1 + \delta_2c_2 + \delta_3a)| \leq (1 + 8\epsilon)|a + i(c_1 + c_2 + c_3a)|.$$

Proof. We consider three cases.

Case 1: $ac_3 \geq 0$. Then,

$$\begin{aligned} |a + i(c_1 + c_2 + c_3a) + i(\delta_1c_1 + \delta_2c_2 + \delta_3a)| &\leq |a + i(c_1 + c_2 + c_3a)| + |\delta_1|c_1 + |\delta_2|c_2 + |\delta_3||a| \\ &\leq |a + i(c_1 + c_2 + c_3a)| + (2\epsilon)|i(c_1 + c_2 + c_3a)| + 2\epsilon|a| \\ &\leq (1 + 4\epsilon)|a + i(c_1 + c_2 + c_3a)|. \end{aligned}$$

Case 2: $ac_3 < 0$ and $c_1 + c_2 + c_3a \leq \frac{1}{2}(c_1 + c_2)$. We have that

$$|a| > \frac{1}{2|c_3|}(c_1 + c_2) \geq \frac{1}{6}(c_1 + c_2).$$

Therefore,

$$\begin{aligned} |a + i(c_1 + c_2 + c_3a) + i(\delta_1c_1 + \delta_2c_2 + \delta_3a)| &\leq |a + i(c_1 + c_2 + c_3a)| + |\delta_1|c_1 + |\delta_2|c_2 + |\delta_3||a| \\ &\leq |a + i(c_1 + c_2 + c_3a)| + \epsilon(c_1 + c_2) + 2\epsilon|a| \\ &\leq |a + i(c_1 + c_2 + c_3a)| + 8\epsilon|a| \\ &\leq (1 + 8\epsilon)|a + i(c_1 + c_2 + c_3a)|. \end{aligned}$$

Case 3: $ac_3 < 0$ and $c_1 + c_2 + c_3a > \frac{1}{2}(c_1 + c_2)$. We have that

$$\begin{aligned}
 &|a + i(c_1 + c_2 + c_3a) + i(\delta_1c_1 + \delta_2c_2 + \delta_3a)| \\
 &\leq |a + i(c_1 + c_2 + c_3a)| + |\delta_1|c_1 + |\delta_2|c_2 + |\delta_3||a| \\
 &\leq |a + i(c_1 + c_2 + c_3a)| + \epsilon(c_1 + c_2) + 2\epsilon|a| \\
 &\leq |a + i(c_1 + c_2 + c_3a)| + 2\epsilon|i(c_1 + c_2 + c_3a)| + 2\epsilon|a| \\
 &\leq (1 + 4\epsilon)|a + i(c_1 + c_2 + c_3a)|.
 \end{aligned}$$

□

Next, we prove an interpolation result on strips.

Lemma 4.14. *Suppose that $F, G : [0, 1] \rightarrow LIP(\mathbb{R})$ be bi-Lipschitz paths with $F_0(x) = G_0(x) = x$ for all $x \in \mathbb{R}$, $F_1(x) = G_1(x) = x + 2\pi$ for all $x \in \mathbb{R}$, F_t, G_t are 2π -periodic for all $t \in [0, 1]$ and $F_t(x), G_t(x)$ are both strictly increasing in t for a fixed x . Let $M > 0$ and let S be the strip $S = \{z \in \mathbb{C} : 0 < \text{Re}(z) < M\}$. Then there exists a bi-Lipschitz path $H : [0, 1] \rightarrow LIP(\overline{S})$ which extends to ∂S with $H_t(iy) = iF_t(y)$ and $H_t(M + iy) = M + iG_t(y)$ for $0 \leq t \leq 1$ and $y \in \mathbb{R}$. Moreover, $H_0(z) = z$ and $H_1(z) = z + 2\pi$ for all $z \in S$.*

Proof. We define H_t via the obvious convex interpolation in S . That is, we set

$$H_t(x + iy) = x + i(G_t(y) + (1 - x/M)(F_t(y) - G_t(y)))$$

for $0 \leq t \leq 1$, $0 \leq x \leq M$ and $y \in \mathbb{R}$. Clearly H_0 is the identity and H_1 is a translation by $2\pi i$. We need to show that H_t is a bi-Lipschitz path.

We start by showing that each H_t is a bi-Lipschitz map. Using Lemma 4.3, suppose that F_t is L -bi-Lipschitz and G_t is λ -bi-Lipschitz for all $t \in [0, 1]$. Setting $z = x + iy$ and $w = x' + iy'$, we have

$$\begin{aligned}
 &|H_t(z) - H_t(w)| \\
 &\leq |x - x'| + \left| \frac{x}{M}G_t(y) - \frac{x'}{M}G_t(y') + \left(1 - \frac{x}{M}\right)F_t(y) - \left(1 - \frac{x'}{M}\right)F_t(y') \right| \\
 &\leq |x - x'| + \left(1 - \frac{x}{M}\right)|F_t(y) - F_t(y')| + \frac{x}{M}|G_t(y) - G_t(y')| \\
 &\quad + \frac{|x - x'|}{M}|G_t(y') - F_t(y')| \\
 &\leq (1 + 2\pi)|x - x'| + \max\{L, \lambda\}|y - y'| \\
 &\leq \max\{L, \lambda, 1 + 2\pi\}(|x - x'| + |y - y'|) \\
 &\leq 2 \max\{L, \lambda, 1 + 2\pi\}|z - w|.
 \end{aligned}$$

For the lower bound, we consider two cases. First, set $C = \min\{\frac{M}{8\pi L}, \frac{M}{8\pi\lambda}, \frac{1}{2}\}$.

Case 1. Suppose that $|x - x'| \geq C|y - y'|$. It follows that

$$|H_t(z) - H_t(w)| \geq |x - x'| \geq \frac{C}{2}(|x - x'| + |y - y'|) \geq \frac{C}{2}|z - w|.$$

Case 2. Suppose that $|x - x'| < C|y - y'|$. Without loss of generality, assume that $y' \leq y$. Then,

$$\begin{aligned} & |H_t(z) - H_t(w)| \\ & \geq \left| \left(1 - \frac{x}{M}\right)(F_t(y) - F_t(y')) + \frac{x}{M}(G_t(y) - G_t(y')) + (G_t(y') - F_t(y'))(x - x')/M \right| \\ & \geq \left(1 - \frac{x}{M}\right)(F_t(y) - F_t(y')) + \frac{x}{M}(G_t(y) - G_t(y')) - |G_t(y') - F_t(y')||x - x'|/M \\ & \geq \min\{L^{-1}, \lambda^{-1}\}|y - y'| - 2\pi|x - x'|/M \\ & \geq \left(\min\{L^{-1}, \lambda^{-1}\} - \frac{2\pi C}{M}\right)|y - y'| \\ & \geq \frac{2\pi C}{M}|y - y'| \\ & \geq \frac{\pi C}{M}(|x - x'| + |y - y'|) \\ & \geq \frac{\pi C}{M}|z - w|. \end{aligned}$$

Next, we show that h_t satisfies condition (i) of Definition 4.1. From Definition 4.1 (i), by setting $u = F_t^{-1}(x)$, it follows that given $\epsilon > 0$, we may find $\delta > 0$ so that if $|s - t| < \delta$, then $|F_s(u) - F_t(u)| < \epsilon$ for all $u \in \mathbb{R}$. The same holds true for G_t . Now suppose $z \in S$ and $h_t^{-1}(z) = x + iy$. Then we have

$$z = x + i\left(G_t(y) + \left(1 - \frac{x}{M}\right)(F_t(y) - G_t(y))\right)$$

and

$$H_s \circ H_t^{-1}(z) = x + i\left(G_s(y) + \left(1 - \frac{x}{M}\right)(F_s(y) - G_s(y))\right).$$

Therefore, we obtain

$$|H_s \circ H_t^{-1}(z) - z| = \left|x(G_t(y) - G_s(y)) + \left(1 - \frac{x}{M}\right)(F_s(y) - F_t(y))\right| < \epsilon.$$

Hence condition (i) of Definition 4.1 is satisfied.

Finally, we show that h_t satisfies condition (ii) of Definition 4.1. Note that

$$\begin{aligned} & H_s(z) - H_s(w) \\ & = (x - x') + i\left[\left(1 - \frac{x}{M}\right)F_s(y) - \left(1 - \frac{x'}{M}\right)F_s(y')\right] + i[xG_s(y) - x'G_s(y')]/M \\ & = (x - x') \\ & \quad + i\left[\left(1 - \frac{x}{M}\right)(F_s(y) - F_s(y')) + \frac{x}{M}(G_s(y) - G_s(y')) + (G_s(y') - F_s(y'))(x - x')/M\right]. \end{aligned}$$

Fix $\epsilon > 0$. We know that there exists $\delta > 0$ such that if $|t - s| < \delta$, then

$$\begin{aligned} |F_s(y) - F_s(y')| &\leq (1 + \epsilon)|F_t(y) - F_t(y')|, \\ |G_s(y) - G_s(y')| &\leq (1 + \epsilon)|G_t(y) - G_t(y')| \end{aligned}$$

and

$$|F_s(y) - F_t(y)| \leq \epsilon, \quad |G_s(y) - G_t(y)| \leq \epsilon.$$

Therefore,

$$\begin{aligned} H_s(z) - H_s(w) &= (x - x') + i \left[\left(1 - \frac{x}{M} \right) (F_t(y) - F_t(y')) (1 + \delta_1) + \frac{x}{M} (G_t(y) - G_t(y')) (1 + \delta_2) \right. \\ &\quad \left. + (G_t(y') - F_t(y') + \delta_3)(x - x')/M \right] \\ &= a + i(c_1 + c_2 + c_3 a + \delta_1 c_1 + \delta_2 c_2 + \delta_3 a) \\ &= a + i(c_1 + c_2 + c_3 a) + (\delta_1 c_1 + \delta_2 c_2 + \delta_3 a)i \\ &= H_t(z) - H_t(w) + (\delta_1 c_1 + \delta_2 c_2 + \delta_3 a)i \end{aligned}$$

where

$$\begin{aligned} a &= x - x', \\ c_1 &= \left(1 - \frac{x}{M} \right) (F_t(y) - F_t(y')), \\ c_2 &= x(G_t(y) - G_t(y'))/M, \\ c_3 &= (G_t(y') - F_t(y'))/M, \end{aligned}$$

and $\delta_1, \delta_2, \delta_3$ are functions of x, y, x', y', s, t satisfying

$$|\delta_1| < \epsilon, \quad |\delta_2| < \epsilon, \quad |\delta_3| < 2\epsilon.$$

Now, it follows from Lemma 4.13 that

$$|H_s(z) - H_s(w)| \leq (1 + 8\epsilon)|H_t(z) - H_t(w)|$$

and condition (ii) of Definition 4.1 is satisfied. □

We are now in a position to prove Proposition 4.12.

Proof of Proposition 4.12. The idea is to lift via the exponential function and then use Lemma 4.14. To that end, define \tilde{P} and \tilde{Q} via the functional equations

$$P \circ \exp = \exp \circ \tilde{P} \quad \text{and} \quad Q \circ \exp = \exp \circ \tilde{Q}.$$

Since the exponential function is conformal and has uniformly bounded derivative on the strip $S = \{z : 0 \leq \operatorname{Re}(z) \leq \ln T\}$, we conclude via the same argument as in Proposition 4.11 that \tilde{P} and \tilde{Q} are bi-Lipschitz paths in the lines $\{z : \operatorname{Re}(z) = 0\}$ and $\{z : \operatorname{Re}(z) = \ln T\}$ respectively.

Applying Lemma 4.14 to the strip $S = \{z : 0 \leq \operatorname{Re}(z) \leq \ln T\}$ with boundary bi-Lipschitz path \tilde{P} and \tilde{Q} , we obtain a bi-Lipschitz path \tilde{F} which extends the boundary bi-Lipschitz paths.

Since \tilde{F}_t is $2\pi i$ -periodic by construction, we obtain the required bi-Lipschitz path F via $F \circ \exp = \exp \circ \tilde{F}$, again using the fact that the exponential function has uniformly bounded derivative in S . □

5 Bi-Lipschitz collapsing for sets of small Assouad dimension

The goal in this section is to show that for a Cantor set $X \subset \mathbb{R}^2$ with $\dim_A X < 1$, we can cover it by small topological disks that can then be collapsed via a bi-Lipschitz path into a small disk. This is the content of Proposition 5.1 below; see Figure 5 for a schematic.

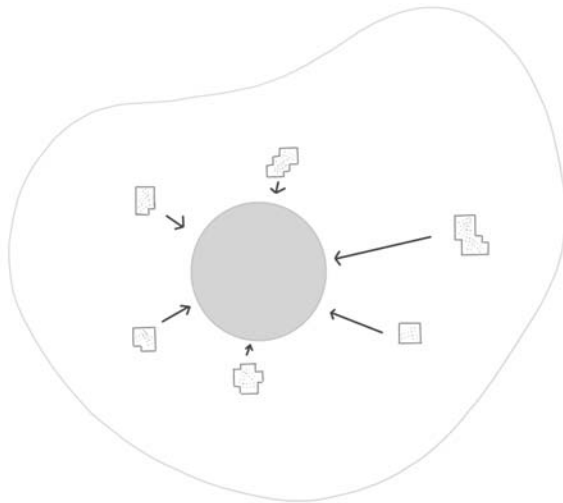


Figure 5. The larger domain is Ω , the shaded ball is B , the PL curves give the boundaries of the components of $\mathcal{T}_\delta(X)$ and the arrows indicate that the bi-Lipschitz path H_t constructed in Proposition 5.1 moves these components into B in an isometric way.

Proposition 5.1. *Let $C > 0$, $c \geq 1$, $s \in [0, 1)$, $\eta \in (0, 1)$ and let $\Omega \subset \mathbb{R}^2$ be a domain with $\text{diam } \Omega = 1$ such that for any $x, y \in \Omega$ there exists a path $\gamma_{x,y} : [0, 1] \rightarrow \Omega$ such that $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$ and*

$$(5.1) \quad \text{dist}(\gamma_{x,y}, \partial\Omega) \geq (2c)^{-1} \min\{\text{dist}(x, \partial\Omega), \text{dist}(y, \partial\Omega)\}.$$

Let $X \subset \Omega$ be (C, s) -homogeneous with $\text{dist}(X, \partial\Omega) > \eta$. There exists $\epsilon > 0$ so that if $z, w \in \Omega$ have distance at least 2ϵ from $\partial\Omega$, then the disk $B(z, \epsilon)$ can be deformed continuously and isometrically to $B(w, \epsilon)$ in Ω . There exists $\delta > 0$ so that if $B = B(z, \epsilon) \subset \Omega$ is a disk of radius ϵ with center z satisfying $\text{dist}(z, \partial\Omega) \geq 2\epsilon$, there exists a bi-Lipschitz path $H : [0, 1] \rightarrow \text{LIP}(\Omega)$ such that

- (i) H_1 maps the closed neighborhood $\mathcal{T}_\delta(X)$ of X into B ;
- (ii) for each $t \in [0, 1]$ and each component D of $\mathcal{T}_\delta(X)$, the map $H_t|_D$ is an isometry.

A couple of remarks are in order.

Remark 5.2. First, condition (5.1) on Ω is inspired by, but slightly weaker than, the well-known c -John property. Second, for the rest of this section, we call curves $\gamma_{x,y}$ c -cigar curves. Finally, if $c' > c$, then there exists a piecewise linear (abbrv. PL) c' -cigar curve σ joining x with y in Ω . In light of this observation, we will assume from now on that all cigar curves are PL.

5.1 Convex sets. Given a set $E \subset \mathbb{R}^N$, we denote by $\text{Hull}(E)$ the *closed convex hull* of E , that is, the intersection of all closed convex sets that contain E . Such a set is itself convex and $\text{diam}(\text{Hull}(E)) = \text{diam}(E)$.

Lemma 5.3. *Let $E \subset \mathbb{R}^N$ be a bounded set. If $x, y \in \text{Hull}(E)$ and*

$$|x - y| = \text{diam}(\text{Hull}(E)),$$

then $x, y \in \overline{E}$.

Proof. For a contradiction, assume that x is not in \overline{E} . That is, $r := \text{dist}(x, \overline{E}) > 0$. Let $P \subset \mathbb{R}^N$ be the $(N - 1)$ -plane that contains x and is orthogonal to the line segment $[x, y]$. Then, since $|x - y| = \text{diam}(\text{Hull}(E))$, it follows that $\text{Hull}(E)$ lies on \overline{H} where H is one of the two components of $\mathbb{R}^N \setminus P$. Therefore,

$$E \subset (H \cap B(y, |x - y|)) \setminus B(x, r).$$

Then, setting $\delta = \text{dist}(\partial B(x, r) \cap \partial B(y, |x - y|), P)$ we have that the set

$$\{z \in H : \text{dist}(z, P) \geq \delta\} \cap \text{Hull}(E)$$

is a convex set which contains E and is a proper subset of $\text{Hull}(E)$, which is a contradiction. □

Lemma 5.4. *Let E_1, \dots, E_n be sets in \mathbb{R}^N . There exists $l \in \{1, \dots, n\}$ and there exist mutually disjoint convex closed sets $\Delta_1, \dots, \Delta_l$ in \mathbb{R}^N such that each E_i is contained in some Δ_j and*

$$\sum_{j=1}^l \text{diam } \Delta_j \leq \sum_{i=1}^n \text{diam } E_i.$$

Proof. If one of the sets E_i is unbounded, then set $l = 1$, $\Delta_1 = \mathbb{R}^N$ and the claim is trivial.

Assume now that all sets E_i are bounded. In this case, the construction of the convex sets Δ_j is in an inductive fashion.

Step 1. For each $i \in \{1, \dots, n\}$, let $\Delta_i^{(1)} = \text{Hull}(E_i)$. If the sets $\Delta_i^{(1)}$ are mutually disjoint, then set $\Delta_i = \Delta_i^{(1)}$ and the procedure terminates; if some intersect, proceed to the next step.

Inductive Step. Suppose that for some $k \in \{1, \dots, n - 1\}$ we have defined closed convex sets $\Delta_1^{(k)}, \dots, \Delta_{n-k+1}^{(k)}$ such that at least two of them intersect. In particular, let $1 \leq i_0 < j_0 \leq n - k + 1$ be such that

$$\Delta_{i_0}^{(k)} \cap \Delta_{j_0}^{(k)} \neq \emptyset.$$

We now define $\Delta_i^{(k+1)}$ for $i \in \{1, \dots, n - k\}$ as follows:

- if $i < i_0$ or if $i_0 < i < j_0$, then set $\Delta_i^{(k+1)} = \Delta_i^{(k)}$;
- if $i = i_0$, then set $\Delta_{i_0}^{(k+1)} = \text{Hull}(\Delta_{i_0}^{(k)} \cup \Delta_{j_0}^{(k)})$;
- if $j_0 \leq i \leq n - k$, then set $\Delta_i^{(k+1)} = \Delta_{i+1}^{(k)}$.

Note that

$$\text{diam } \Delta_{i_0}^{(k+1)} = \text{diam}(\Delta_{i_0}^{(k)} \cup \Delta_{j_0}^{(k)}) \leq \text{diam } \Delta_{i_0}^{(k)} + \text{diam } \Delta_{j_0}^{(k)}.$$

If the sets $\Delta_i^{(k+1)}$ are mutually disjoint, then set $\Delta_i = \Delta_i^{(k+1)}$ and the procedure terminates; if some intersect, proceed to the next step.

It is clear that the procedure above will terminate in m steps for some $m \in \{1, \dots, n\}$. The sets $\Delta_1, \dots, \Delta_{n-m+1}$ produced are convex, mutually disjoint, and each E_i is contained in some Δ_j . It remains to show that

$$(5.2) \quad \sum_{i=1}^{n-m+1} \text{diam } \Delta_i \leq \sum_{i=1}^n \text{diam } E_i.$$

To prove (5.2), first note that for all $i \in \{1, \dots, n\}$, $\text{diam } E_i = \text{diam } \Delta_i^{(1)}$. Therefore, if $m = 1$, then (5.2) follows.

Suppose now that $m \geq 2$. Fix $k \in \{1, \dots, m-1\}$ and let $i_0, j_0 \in \{1, \dots, n-k+1\}$ be as in the construction of domains $\Delta_i^{(k+1)}$. Then,

$$\begin{aligned} \sum_{i=1}^{n-k+1} \text{diam } \Delta_i^{(k)} &= \sum_{i \in \{1, \dots, n-k+1\} \setminus \{i_0, j_0\}} \text{diam } \Delta_i^{(k)} + \text{diam } \Delta_{i_0}^{(k)} + \text{diam } \Delta_{j_0}^{(k)} \\ &\geq \sum_{i \in \{1, \dots, n-k\} \setminus \{i_0\}} \text{diam } \Delta_i^{(k+1)} + \text{diam } \Delta_{i_0}^{(k+1)} \\ &= \sum_{i=1}^{n-k} \text{diam } \Delta_i^{(k+1)}. \end{aligned}$$

Now by induction, (5.2) follows. □

Lemma 5.5. *Let $\Omega \subset \mathbb{R}^2$ be a domain with nonempty boundary, and let $\Delta \subset \Omega$ be a compact convex set with PL boundary. Let $\delta \in (0, 1)$, let*

$$0 < r < (1 - \delta) \text{dist}(\Delta, \partial\Omega),$$

and let $\gamma : [0, 1] \rightarrow \Omega$ be a PL curve in Ω with $\gamma(0) \in \Delta$ and $|\gamma(t) - \gamma(0)| < r$ for all $t \in [0, 1]$. Then there exists a bi-Lipschitz path $H : [0, 1] \rightarrow \text{LIP}(\Omega)$ such that

- (i) for each $t \in [0, 1]$, $H_t|_{\partial\Omega}$ is the identity;
- (ii) for each $t \in [0, 1]$, $H_t|\Delta$ is a translation mapping with $H_t(\gamma(0)) = \gamma(t)$.

Proof. Without loss of generality, we may assume that γ is a straight line segment; in the general case of PL curves γ , concatenate the bi-Lipschitz paths from the various segments of γ and re-parameterize if necessary. Assume then, that $\gamma : [0, 1] \rightarrow \Omega$ with $\gamma(t) = \gamma(0) + tv$ for some $v \in \mathbb{C}$ with $|v| < r$.

By the hypotheses, Δ is a convex polygon with vertices v_1, \dots, v_n . Fix $z_0 \in \Delta$ and for $i \in \{1, \dots, n\}$ let w_i be the point on the ray from z_0 through v_i that is distance $(1 - \delta/2) \text{dist}(\Delta, \partial\Omega)$ away from v_i (and outside Δ). Let Y be the convex hull of w_1, \dots, w_n and set $d = \text{dist}(\Delta, \partial Y) > 0$.

Triangulate the PL ring domain $\overline{Y} \setminus \overline{\Delta}$ via triangles T_1, \dots, T_m which have, alternately, one or two vertices contained in ∂Y .

Given a direction $e^{i\theta}$, we will construct a bi-Lipschitz path which moves Δ onto $\Delta_1 = \{z : z = z' + de^{i\theta}/2, z' \in \Delta\}$. For $z \in \Delta$ we just define

$$H_t(z) = (z + de^{i\theta}/2)t + (1 - t)z.$$

If T_i has two vertices on ∂Y and third vertex $\zeta_1 \in \partial\Delta$, then we apply the bi-Lipschitz path from Proposition 4.9 (conjugated by a suitable similarity) which fixes the two vertices in ∂Y and moves ζ_1 to $\zeta_1 + de^{i\theta}/2 \in Y$.

If T_i has one vertex on ∂Y and two vertices ξ_1, ξ_2 in ∂X , then we apply the bi-Lipschitz path from Proposition 4.9 (again conjugated by a suitable similarity) which fixes the vertex in ∂Y and moves ξ_j to $\xi_j + de^{i\theta}/2$ for $j = 1, 2$.

This piecewise construction yields a bi-Lipschitz path which moves Δ to Δ_1 and fixes every point of ∂Y and hence can be extended to fix every point of $\Omega \setminus Y$. By concatenating a finite number of bi-Lipschitz paths, we may move X along any PL path in Ω , as long as we avoid $\partial\Omega$, such that the path acts as a translation on X . □

For the rest of the paper, given a bounded set $X \subset \mathbb{R}^2$, a number $r > 0$ and a curve $\gamma : [0, a] \rightarrow \mathbb{R}^2$ with $\gamma(0) \in X$, we denote

$$\mathcal{N}(X, \gamma, r) := \bigcup_{t \in [0, a]} (\gamma(t) - \gamma(0) + N(X, r)).$$

5.2 Proof of Proposition 5.1. The first claim about the existence of such an ϵ follows by following a c -cigar curve from z to w . Henceforth, fix $B = B(z_0, \epsilon)$.

Suppose first that $\text{diam} X = 0$, that is $X = \{x_0\}$ for some $x_0 \in \Omega$. Let γ be a PL c -cigar path that joins x_0 with x_0 in Ω . Let Δ be a compact convex set with PL boundary contained in $B(x_0, r)$ with $r < \min\{\epsilon, \frac{1}{3} \text{dist}(x_0, \partial\Omega)\}$. We then apply Lemma 5.5 to find the required bi-Lipschitz path $H : [0, 1] \rightarrow LIP(\Omega)$ such that for any $t \in [0, 1]$, $H_t(x_0) = \gamma(t)$.

Suppose now and for the rest of the proof of Proposition 5.1 that $\text{diam} X > 0$. Set

$$(5.3) \quad \delta = \left(\frac{\min\{\eta, \epsilon\}}{216cC} \right)^{\frac{1}{1-s}}.$$

We may assume that $C > 1$, hence δ is less than 1. Then let V be a δ -net of X and let D_1, \dots, D_n be the components of $\mathcal{T}_\delta(X)$.

Since $\delta < \eta/20$, we have that

$$\text{dist}(\mathcal{T}_\delta(X), \partial\Omega) \geq \text{dist}(X, \partial\Omega) - \text{dist}_H(\mathcal{T}_\delta(X), X) \geq \eta - 8\delta \geq \eta/2$$

where dist_H denotes the Hausdorff distance.

Let $i \in \{1, \dots, n\}$. For each $x \in \partial D_i$ there exists $z \in X$ such that $|x - z| \leq 8\delta$ and there exists $v \in V$ such that $|z - v| \leq \delta$. Therefore, for every $x \in \partial D_i$, $\text{dist}(x, V) \leq 9\delta$ and it follows that

$$(5.4) \quad \text{diam} D_i \leq 18\delta \text{card}(V \cap D_i).$$

Therefore,

$$(5.5) \quad \sum_{i=1}^n \text{diam} D_i \leq 18\delta \text{card}(V) \leq 18C\delta^{1-s} = (12c)^{-1} \min\{\eta, \epsilon\}.$$

The construction of the bi-Lipschitz path H consists of two parts. In the first part we construct at most $n - 1$ many bi-Lipschitz paths that “gather the sets D_i together” and in the second part we construct a bi-Lipschitz path that leads the cluster of gathered sets D_i into the disk B .

5.2.1 Part 1. The construction in this part is in an inductive manner.

Step 0. Apply Lemma 5.4 for the sets D_1, \dots, D_n and obtain closed mutually disjoint convex sets $\Delta_1^{(1)}, \dots, \Delta_{k_0}^{(0)}$ for some positive integer $k_0 \in \{1, \dots, n\}$. Note that

$$\sum_{i=1}^{k_0} \text{diam } \Delta_i^{(0)} \leq \sum_{i=1}^n \text{diam } D_i \leq (12c)^{-1} \min\{\eta, \epsilon\}.$$

Moreover, the sets $\Delta_1^{(0)}, \dots, \Delta_{k_0}^{(0)}$ are contained in Ω and for each $i \in \{1, \dots, k_0\}$

$$\text{dist}(\Delta_i^{(0)}, \partial\Omega) \geq \text{dist}(\mathcal{J}_\delta(X), \partial\Omega) - \text{diam } \Delta_i^{(0)} \geq \eta/2 - (12c)^{-1}\eta > \frac{1}{3}\eta.$$

If $k_0 = 1$, then the procedure terminates and we proceed to Part 2; otherwise proceed to the next step.

Inductive step. Suppose that for some positive integer $m \in \{0, \dots, n - 2\}$ we have defined disjoint closed convex sets $\Delta_1^{(m-1)}, \dots, \Delta_{k_{m-1}}^{(m-1)} \subset \Omega$ such that $2 \leq k_{m-1} \leq n - m + 1$ and the following three properties hold.

(P1) For each $i \in \{1, \dots, k_{m-1}\}$ there exists $j \in \{1, \dots, n\}$ with $D_j \subset \Delta_i^{(m-1)}$.

(P2) We have

$$\sum_{i=1}^{k_{m-1}} \text{diam } \Delta_i^{(m-1)} < (6c)^{-1} \min\{\eta, \epsilon\}.$$

(P3) For each $i \in \{1, \dots, k_{m-1}\}$, $\text{dist}(\Delta_i^{(m-1)}, \partial\Omega) > \eta/3$.

Let $\gamma_m : [0, 1] \rightarrow \Omega$ be a PL c -cigar curve with $\gamma_m(0) \in X \cap \Delta_1^{(m-1)}$ and $\gamma_m(1) \in X \cap \Delta_2^{(m-1)}$. By (5.1) and inductive assumption (P3), we have that for all $t \in [0, 1]$,

$$(5.6) \quad \text{dist}(\gamma_m(t), \partial\Omega) \geq (2c)^{-1} \min\{\text{dist}(\gamma_m(0), \partial\Omega), \text{dist}(\gamma_m(1), \partial\Omega)\} \geq (2c)^{-1}\eta.$$

Using inductive assumption (P2), we can find a number $r_m > 0$ such that

- (i) $r_m < \frac{1}{3} \text{dist}(\Delta_1^{(m-1)}, \Delta_i^{(m-1)})$ for all $i \in \{2, \dots, k_{m-1}\}$,
- (ii) $r_m < (6c)^{-1} \min\{\eta, \epsilon\} - \sum_{i=1}^{k_{m-1}} \text{diam } \Delta_i^{(m-1)}$.

The second property of r_m implies that

$$r_m < (2c)^{-1} \text{dist}(\Delta_1^{(m-1)}, \partial\Omega) - \text{diam } \Delta_1^{(m-1)}$$

which, along with (5.6), implies that $\mathcal{N}(\Delta_1^{(m-1)}, \gamma_m, 2r_m) \subset \Omega$. Let

$$T_m = \sup \left\{ t \in [0, 1] : \mathcal{N}(\Delta_1^{(m-1)}, \gamma_m|_{[0,t]}, r_m) \cap \bigcup_{j=2}^{k_{m-1}} \Delta_j^{(m-1)} = \emptyset \right\}.$$

Since $\text{dist}(\Delta_1^{(m-1)}, \Delta_i^{(m-1)}) \geq 3r_m$ for all $i \neq 1$, we have that $T_m > 0$. Let $i_0 \in \{2, \dots, k_{m-1}\}$ be such that

$$\mathcal{N}(\Delta_1^{(m-1)}, \gamma_m|_{[0,T_m]}, r_m) \cap \Delta_{i_0}^{(m-1)} \neq \emptyset.$$

For simplicity, we may assume that $i_0 = 2$. Denote by $H^{(m)}$ the bi-Lipschitz path given from Lemma 5.5 for the curve $\gamma = \gamma_m|_{[0,T_m]}$. Consider now the disjoint closed sets

$$E_1 = H_1^{(m)}(\Delta_1^{(m-1)}) \cup \Delta_2^{(m-1)}, E_2 = \Delta_3^{(m-1)}, \dots, E_{k_m-1} = \Delta_{k_{m-1}}^{(m-1)}$$

and apply Lemma 5.4 to the sets E_j to obtain mutually disjoint closed convex sets $\Delta_1^{(m)}, \dots, \Delta_{k_m}^{(m)}$ with $k_m \leq k_{m-1} - 1$. We note that

- (i) for each $i \in \{1, \dots, k_m\}$ there exists $j \in \{1, \dots, n\}$ with $D_j \subset \Delta_i^{(m)}$;
- (ii)

$$\sum_{i=1}^{k_m} \text{diam } \Delta_i^{(m)} \leq \sum_{i=1}^{k_{m-1}} \text{diam } \Delta_i^{(m-1)} + r_m < (6c)^{-1} \min\{\eta, \epsilon\}.$$

It follows that $\Delta_1^{(m)}, \dots, \Delta_{k_m}^{(m)}$ are contained in Ω and, in fact, for each $i \in \{1, \dots, k_m\}$

$$\text{dist}(\Delta_i^{(2)}, \partial\Omega) \geq \text{dist}(\mathcal{T}_\delta(X), \partial\Omega) - \text{diam } \Delta_i^{(m)} \geq \eta/2 - (6c)^{-1} > \eta/3.$$

Therefore, we have verified that inductive assumptions (P1)–(P3) hold for m . If $k_m = 1$ the procedure terminates and we proceed to Part 2; otherwise proceed to the next step.

After p steps, for some $p \in \{0, \dots, n - 1\}$, we have $k_p = 1$. By the choice of δ and numbers r_1, \dots, r_p , the final convex set $\Delta_1^{(p)}$ satisfies properties (P1)–(P3); precisely, we have

- (i) $\text{diam } \Delta_1^{(p)} < (6c)^{-1} \min\{\eta, \epsilon\}$,
- (ii) there exists $i \in \{1, \dots, n\}$ such that $D_i \subset \Delta_1^{(p)}$,
- (iii) $\Delta_1^{(p)} \subset \Omega$ and $\text{dist}(\Delta_1^{(p)}, \partial\Omega) > \eta/3$.

5.2.2 Part 2. Let $z_0 \in \Omega$ be the center of B and let $\gamma_{p+1} : [0, 1] \rightarrow \Omega$ be a PL c -cigar curve in Ω with $\gamma_{p+1}(0) \in X \cap \Delta_1^{(p)}$ and $\gamma_{p+1}(1) = z_0$. If $z_0 \in X \cap \Delta_1^{(p)}$, then we can choose γ_{p+1} to be constant. By (5.1), we have that for all $t \in [0, 1]$,

$$(5.7) \quad \begin{aligned} \text{dist}(\gamma_{p+1}(t), \partial\Omega) &\geq (2c)^{-1} \min\{\text{dist}(\gamma_{p+1}(0), \partial\Omega), \text{dist}(\gamma_{p+1}(1), \partial\Omega)\} \\ &\geq (2c)^{-1} \min\{\epsilon, \eta\}. \end{aligned}$$

Let r_{m+1} be a positive number with $r_{m+1} < (6c)^{-1} \min\{\eta, \epsilon\}$. Then (5.7) implies that

$$\mathcal{N}(\Delta_1^{(p)}, \gamma_{p+1}, r_{p+1}) \subset \Omega.$$

Let now $H^{(p+1)}$ be the bi-Lipschitz path given from Lemma 5.5 for $\gamma = \gamma_{p+1}$. If $p = 0$, then we define $H : [0, 1] \rightarrow LIP(\mathbb{R}^N)$ with $H = H^{(p+1)}$. If $p \geq 1$, we concatenate the bi-Lipschitz paths $H^{(1)}, \dots, H^{(p+1)}$ and we obtain the desired bi-Lipschitz path H .

6 A multitwist bi-Lipschitz map

In §6.1 we prove Proposition 1.3 while in §6.2 we show that the multitwist map in Theorem 7.1 is bi-Lipschitz.

6.1 Proof of Proposition 1.3. In this subsection we prove Proposition 1.3. To that end, we require the following “egg-yolk principle” lemma which is a simple application of Koebe’s Distortion Theorem.

Lemma 6.1. *Given $\delta > 0$, there exists $L_0 > 1$ with the following property. If U is a domain in \mathbb{R}^2 , $K \subset U$ is a compact connected set with $\text{dist}(K, \partial U) \geq \delta \text{diam } K$, $x_0 \in K$ is a point, and $f : U \rightarrow \mathbb{R}^2$ is a conformal map, then for all $x, y \in K$,*

$$L_0^{-1}|f'(x_0)||x - y| \leq |f(x) - f(y)| \leq L_0|f'(x_0)||x - y|.$$

Proof. If K is a single point, the claim is trivial. Assume for the rest that $\text{diam } K = d > 0$. Let V be a maximal $(\delta d/4)$ -separated subset of K containing x_0 . By the doubling property of \mathbb{R}^2 , there exists $N \in \mathbb{N}$ depending only on δ such that $\text{card } V \leq N$.

By the Koebe Distortion Theorem (see for example [GM05, Theorem I.4.5] and [Pom92, Theorem 1.3]), there exists a universal $A > 1$ such that for any $z \in K$ and for any $w, w_1, w_2 \in B(z, \frac{1}{2}\delta d)$ we have

$$(6.1) \quad A^{-1}|f'(w)||w_1 - w_2| \leq |f(w_1) - f(w_2)| \leq A|f'(w)||w_1 - w_2|,$$

$$(6.2) \quad A^{-1}|f'(z)| \leq |f'(w)| \leq A|f'(z)|,$$

$$(6.3) \quad \text{dist}(f(z), \partial f(U)) \geq A^{-1}\delta d|f'(z)|.$$

By (6.2), we have that for all $x \in K$,

$$(6.4) \quad A^{-N}|f'(x_0)| \leq |f'(x)| \leq A^N|f'(x_0)|.$$

We show that $f|K$ is $(L_1|f'(x_0)|)$ -Lipschitz for some $L_1 > 0$ depending only on δ . Fix $x, y \in K$ and consider two cases. If $|x - y| < \delta d/2$, then by (6.1) and (6.4)

$$A^{-N}|f'(x_0)||x - y| \leq |f(x) - f(y)| \leq A^N|f'(x_0)||x - y|.$$

Suppose now that $|x - y| \geq \delta d/2$. Then, there exist $z, z' \in V$ such that $x \in B(z, \delta d/4)$ and $y \in B(z', \delta d/4)$, and by connectedness of K , there exist distinct $z_1, \dots, z_l \in V$ such that $z_1 = z, z_l = z'$, and for all $j \in \{1, \dots, l - 1\}$, $|z_j - z_{j+1}| < \delta d/2$. Therefore,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(z)| + \sum_{i=1}^{l-1} |f(z_{i+1}) - f(z_i)| + |f(z') - f(z)| \\ &\leq |f'(x_0)|(N + 1)A^N(\delta d/2) \\ &\leq |f'(x_0)|(N + 1)A^N|x - y|. \end{aligned}$$

By (6.3) we have that $\text{dist}(w, \partial f(U)) \geq A^{-N}\delta d|f'(x_0)|$ for all $w \in f(K)$. On the other hand, since $f|K$ is L_1 -Lipschitz, we have that $\text{diam} f(K) \leq L_1|f'(x_0)|d$. Therefore,

$$\text{dist}(f(K), \partial f(U)) \geq (L_1A^N)^{-1}\delta \text{diam} f(K).$$

Then, working as above, we can find $L_2 > 0$ depending only on L_1 and N (hence only on δ) such that $f^{-1}|f(K)$ is $(L_2|(f^{-1})'(f(x_0))|)$ -Lipschitz. Therefore, for all $x, y \in K$

$$|x - y| \leq \frac{L_2}{|f'(x_0)|}|f(x) - f(y)|$$

and the proof is complete. □

We can now prove Proposition 1.3.

Proof of Proposition 1.3. Let $X \subset \mathbb{R}^2$ be a c -uniformly disconnected set. By Theorem 1.2 we know that there is a geodesic pants decomposition of the hyperbolic Riemann surface $S := \mathbb{S}^2 \setminus X$ so that the cuffs (α_j) have uniformly bounded hyperbolic length. By Proposition 2.1, there exist mutually disjoint ring domains (R'_j) which are thickenings of (α_j) with a uniform upper bound M_0 on their moduli.

For each j , denote by V'_j and U'_j the bounded and unbounded, respectively, components of $\mathbb{R}^2 \setminus R'_j$. Let ζ_j be a similarity of \mathbb{R}^2 such that $\text{diam} \zeta_j^{-1}(V'_j) = 1$ and $0 \in \zeta_j^{-1}(V'_j)$. By (2.1), there exists ϵ_0 depending only on M_0 (hence only on c) such that $\text{dist}(\partial \zeta_j^{-1}(U'_j), \partial \zeta_j^{-1}(V'_j)) \geq \epsilon_0$. By Lemma 2.4, there exists a polygonal Jordan curve γ_j with edges in $\mathcal{G}_{\epsilon_0/16}^1$ which encloses $\zeta_j^{-1}(V'_j)$ and satisfies

$$\epsilon_0/16 \leq \text{dist}(x, \zeta_j^{-1}(V'_j)) \leq \epsilon_0/2, \quad \text{for all } x \in \gamma_j.$$

Applying Lemma 2.4, there exists a polygonal Jordan curve Γ_j with edges in $\mathcal{G}_{\epsilon_0/32}^1$ which encloses γ_j and satisfies

$$\epsilon_0/32 \leq \text{dist}(x, \gamma_j) \leq \epsilon_0/4, \quad \text{for all } x \in \Gamma_j.$$

The ring domain R_j'' bounded by γ_j and Γ_j satisfies

- (i) $\text{dist}(\gamma_j, \Gamma_j) \geq \epsilon_0/32$,
- (ii) $1 \leq \text{diam } R_j'' \leq 1 + \frac{3}{2}\epsilon_0$ and
- (iii) $\text{dist}(x, \partial\zeta_j(R_j'')) \geq \epsilon_0/16$, for all $x \in R_j''$.

It follows that $R_j'' \subset [-1 - \frac{3}{2}\epsilon_0, 1 + \frac{3}{2}\epsilon_0]^2$ and since the boundary curves of R_j'' are made of edges in $\mathcal{G}_{\epsilon_0/32}^1$, there are at most k many different domains R_j'' , with k depending only on ϵ_0 , hence only on c .

There exists $\delta_0 \in (0, 1)$ depending only on ϵ_0 (hence only on c) and for each j there exists $\delta_j \in (0, 1 - \delta_0)$, and there exists a conformal map

$$\psi_j : B(0, 1) \setminus \overline{B}(0, \delta_j) \rightarrow R_j''.$$

Setting

$$K := \left\{ 1 - \frac{3}{4}\delta_0 \leq |x| \leq 1 - \frac{1}{4}\delta_0 \right\} \subset U := B(0, 1) \setminus \overline{B}(0, \delta_j),$$

we have $\text{dist}(K, \partial U) \geq \delta_0/4$ and $\text{diam } K = 2 - \delta_0/2$. Hence by Lemma 6.1, we have that $|\psi_j'(1 - \frac{3}{4}\delta_0)|^{-1}\psi_j$ restricted on K is a L_0 -bi-Lipschitz, where L_0 depends only on δ_0 (hence only on c). Moreover,

$$\begin{aligned} \frac{1}{L_0(2 - \delta_0/2)} &\leq \frac{\text{diam } \psi_j(K)}{L_0 \text{diam } K} \leq \left| \psi_j' \left(1 - \frac{3}{4}\delta_0 \right) \right| \leq L_0 \frac{\text{diam } \psi_j(K)}{\text{diam } K} \\ &\leq L_0 \frac{\sqrt{2}(2 + 3\epsilon_0)}{2 - \delta_0/2}. \end{aligned}$$

For each $j \in \mathbb{N}$, let $\lambda_j = \text{diam } \psi_j(\partial B(0, 1 - \frac{3}{4}\delta_0)) \in [1, 1 + 3\epsilon_0/2]$. It follows that the map

$$(\lambda_j)^{-1}\psi_j|_K$$

is L_1 -bi-Lipschitz for some L_1 depending on $L_0, \epsilon_0, \delta_0$, hence only on c .

To complete the proof set

$$L = \max \left\{ \frac{4 - \delta_0}{2\delta_0}, \frac{4L_1}{4 - \delta_0} \right\},$$

define conformal maps

$$g_j : \overline{B}(0, 1) \setminus B(0, 1 - 1/L) \rightarrow \mathbb{R}^2 \quad \text{with} \quad g_j(x) = (\lambda_j)^{-1}\psi_j|_K((1 - \delta_0/4)x),$$

and define similarities

$$\phi_j : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{with } \phi_j(x) = (\lambda_j)^{-1} \zeta_j^{-1}(x).$$

Since $L \geq \frac{4-\delta_0}{2\delta_0}$, we have that $(1 - \delta_0/4)x \in K$ for all $x \in \overline{B(0, 1)} \setminus B(0, 1 - 1/L)$. Moreover, since $L \geq \frac{4L_1}{4-\delta_0}$, we have that g_j is L -bi-Lipschitz. Since there are at most k many domains R'_j , there are at most k many conformal maps g_j . \square

Setting

$$f_j = \phi_j \circ g_{i(j)} \quad \text{and} \quad R_j = f_j(\overline{B(0, 1)} \setminus B(0, 1 - 1/L))$$

where ϕ_j and $g_{i(j)}$ are as in the statement of Proposition 1.3, and applying Lemma 6.1 to the ring

$$K' = \overline{B(0, 1 - \epsilon_0/8)} \setminus B(0, 1 - 7\epsilon_0/8),$$

we see that there exists $\zeta > 0$ so that

$$(6.5) \quad \text{dist}(\partial R'_j, R_j) \geq \text{dist}(\partial K_j, R_j) \geq \zeta \text{diam } R_j$$

for all j .

6.2 A multitwist bi-Lipschitz map. For the rest of this section we fix a c -uniformly disconnected Cantor set $X \subset \mathbb{R}^2$. By Proposition 1.3, we obtain $k \in \mathbb{N}$, $L > 1$, a finite set $\{g_1, \dots, g_k\}$ of L -bi-Lipschitz conformal maps defined on $\overline{B(0, 1)} \setminus B(0, 1 - \frac{1}{L})$, similarities $(\phi_j)_{j \in \mathbb{N}}$ and ring domains R_j such that for each $j \in \mathbb{N}$ there exists $i(j) \in \{1, \dots, k\}$

$$(6.6) \quad R_j = f_j\left(\overline{B(0, 1)} \setminus B\left(0, 1 - \frac{1}{L}\right)\right) \quad \text{with } f_j = \phi_j \circ g_{i(j)}.$$

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a map such that f is the identity outside of the union of $\overline{R_j}$, while for each $j \in \mathbb{N}$, $f|_{R_j} = f_j \circ \mathfrak{D} \circ f_j^{-1}$ with

$$\mathfrak{D}(r, \theta) = (r, \theta + 2\pi L(1 - r)).$$

Lemma 6.2. *The map f is L_0 -bi-Lipschitz with L_0 depending only on c .*

Proof. It is fairly elementary to see that \mathfrak{D} is L_1 -bi-Lipschitz for some $L_1 > 1$ depending only on L (hence only on c). It follows that for each $j \in \mathbb{N}$, $f|_{R_j}$ is $L^2 L_1$ -bi-Lipschitz. Since f is the identity outside of the union of $\overline{R_j}$ (and hence bi-Lipschitz), we get that f is an L_2 -bounded length distortion map for some $L_2 > 1$ depending only on L . That is,

$$L_2^{-1} \ell(\gamma) \leq \ell(f(\gamma)) \leq L_2 \ell(\gamma)$$

for any rectifiable curve γ , with ℓ denoting length. The proof is completed by recalling that every bounded length distortion homeomorphism of \mathbb{R}^2 (or any quasiconvex space) is bi-Lipschitz quantitatively. \square

7 Decomposition and proof of Theorem 1.4

In this section we will prove the following result, which immediately implies Theorem 1.4.

Theorem 7.1. *Suppose the Assouad dimension of X is less than 1 and f is the bi-Lipschitz map from §6.2. Then there exists a bi-Lipschitz path $H : [0, 1] \rightarrow LIP(\mathbb{R}^2)$ such that $H_0 = f$ and H_1 is the identity.*

The proof comprises of 4 steps. In the first step we relabel the ring domains R_j obtained from Proposition 1.3. In the second step we use Proposition 5.1 to unwind the Dehn twists in each R_j without changing small neighborhoods of X . In the third step we compose the bi-Lipschitz paths from the second step to perform unwindings arbitrarily close to X . Finally, in the fourth step, we use the uniformity of our maps to take a limit in the sequence of bi-Lipschitz paths obtained from the third step and recover the desired bi-Lipschitz path.

For the rest, we denote by $(R_j)_{j \in \mathbb{N}}$, $(\phi_j)_{j \in \mathbb{N}}$, $\{g_1, \dots, g_k\}$, and

$$(f_j)_{j \in \mathbb{N}} = (\phi_j \circ g_{i(j)})_{j \in \mathbb{N}}$$

the ring domains, similarities, and conformal maps, respectively, from Proposition 1.3.

7.1 Step 1: Relabelling the ring domains R_j . This step is similar to the proof of Proposition 3.3.

For each $j \in \mathbb{N}$ let V_j and U_j be the bounded and unbounded, respectively, components of $\mathbb{R}^2 \setminus \overline{R_j}$.

Let ε be the empty word. There exist three distinct $l_1, l_2, l_3 \in \mathbb{N}$ such that

- (i) for all $j \in \mathbb{N}$, there exists $i \in \{1, 2, 3\}$ such that $R_j \subset V_{l_i}$ and
- (ii) for all $j \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$, $R_{l_i} \cap V_j = \emptyset$.

For each $l \in \{1, 2, 3\}$, we denote $R_{l,\varepsilon} = R_{l_i}$ where ε denotes the empty word.

Inductively, suppose that for some $l \in \{1, 2, 3\}$ and for some finite word $w \in \{1, 2\}^*$ we have labelled $R_{l,w} = R_{j_0}$ where $j_0 \in \mathbb{N}$. Then there exist exactly two distinct $j_1, j_2 \in \mathbb{N}$ such that

- (i) $R_{j_1}, R_{j_2} \subset V_{j_0}$ and
- (ii) for all $j \in \mathbb{N} \setminus \{j_1, j_2\}$ with $R_j \subset V_{j_0}$, either $R_j \subset V_{j_1}$, or $R_j \subset V_{j_2}$.

We denote $R_{l,w1} = R_{j_1}$ and $R_{l,w2} = R_{j_2}$.

Thus, we have that $\{R_j : j \in \mathbb{N}\} = \{R_{l,w} : l \in \{1, 2, 3\}, w \in \{1, 2\}^*\}$. Given $l \in \{1, 2, 3\}$ and $w \in \{1, 2\}^*$ we denote by $V_{l,w}$ and $U_{l,w}$ the bounded and unbounded, respectively, components of $\mathbb{R}^2 \setminus \overline{R_{l,w}}$. Further, denote by $X_{l,w}$ the intersection $X_{l,w} = X \cap V_{l,w}$.

Moreover, if $R_j = R_{l,w}$ we set $\phi_{l,w} = \phi_j$ and $f_{l,w} = f_j$. In particular,

$$f_{l,w} = \phi_{l,w} \circ g_{i(l,w)}.$$

By Proposition 1.3 we have that for all $l \in \{1, 2, 3\}$, $w \in \{1, 2\}^*$ and $i \in \{1, 2\}$

$$(7.1) \quad \frac{\text{diam } R_{l,wi}}{\text{diam } R_{l,w}} \leq \frac{\text{diam } V_{l,w}}{\text{diam } R_{l,w}} \leq \frac{\text{diam } R_{l,w} - 2 \text{dist}(V_{l,w}, U_{l,w})}{\text{diam } R_{l,w}}.$$

Suppose $\text{dist}(V_{l,w}, U_{l,w})$ is realized by $|x - y|$. Then since $x, y \in \partial R_{l,w}$ and f is the identity there, we have by (6.6) that for some $j \in \mathbb{N}$,

$$(7.2) \quad \begin{aligned} \text{dist}(V_{l,w}, U_{l,w}) &= |f(x) - f(y)| \geq \frac{\text{diam } R_{l,w}}{L} |(f_{l,w}^{-1})'(x) - (f_{l,w}^{-1})'(y)| \\ &\geq \frac{\text{diam } R_{l,w}}{L^2}. \end{aligned}$$

We conclude via (7.1) that

$$(7.3) \quad \frac{\text{diam } R_{l,wi}}{\text{diam } R_{l,w}} \leq 1 - \frac{1}{L^2}.$$

7.2 Step 2: Unwinding the Dehn twist in $R_{l,w}$ while acting as isometries on neighborhoods of $X_{l,w}$. For each $l \in \{1, 2, 3\}$ and $w \in \{1, 2\}^*$ we define a bi-Lipschitz path $H_{l,w} : [0, 1] \rightarrow LIP(\mathbb{R}^2)$ as follows.

First, set $H_{l,w}|_{U_{l,w}}$ to be the identity. Second, define $H_{l,w}|_{R_{l,w}}$ so that for each $t \in [0, 1]$

$$(H_{l,w}|_{R_{l,w}})_t = f_{l,w} \circ D_{1-t} \circ (f_{l,w})^{-1}$$

recalling D_t from Lemma 4.10.

Lemma 7.2. *The family of bi-Lipschitz paths*

$$\mathcal{F} := \{H_{l,w}|_{R_{l,w}} : l \in \{1, 2, 3\}, w \in \{1, 2\}^*\},$$

which unwinds the Dehn twist in each $R_{l,w}$, is a uniform family of bi-Lipschitz paths.

Proof. For each $i \in \{1, \dots, k\}$ and each $t \in [0, 1]$ set $H_t^i = g_i \circ D_{1-t} \circ g_i^{-1}$. By Proposition 4.11, each H^i is a bi-Lipschitz path. Now for each $i \in \{1, \dots, k\}$ let

$$\mathcal{G}^i = \{\phi_{l,w} \circ H^i \circ (\phi_{l,w})^{-1} : l \in \{1, 2, 3\}, w \in \{1, 2\}^*\}.$$

Since X is bounded, there exists c depending on the diameter of X such that each $\phi_{l,w}$ has a scaling factor at most c . Therefore, by Lemma 4.8, \mathcal{G}^i is a uniform family of bi-Lipschitz paths. Note that $\mathcal{F} \subset \bigcup_{i=1}^k \mathcal{G}^i$ so \mathcal{F} is a uniform family of bi-Lipschitz paths as a finite union of uniform families of bi-Lipschitz paths. \square

Before defining $H_{l,w}|_{V_{l,w}}$ we make some remarks.

First, there exist $C > 0$ and $s \in (0, 1)$ such that for any $l \in \{1, 2, 3\}$ and $w \in \{1, 2\}^*$ the set $\phi_{l,w}^{-1}(X_{l,w})$ is (C, s) -homogeneous.

Second, since $\{\phi_{l,w}^{-1}(V_{l,w})\}_{l,w}$ is a finite collection of Jordan domains with smooth boundary, there exists $c > 1$ such that for all $l \in \{1, 2, 3\}$ and $w \in \{1, 2\}^*$, the domain $\phi_{l,w}^{-1}(V_{l,w})$ satisfies (5.1) with constant c .

Third, by the bi-Lipschitz Schoenflies Theorem [Tuk80, Theorem A], there exists $L' > 1$ depending only on L such that every g_i extends to be an L' -bi-Lipschitz map on $\overline{B}(0, 1)$. Therefore, for each $l \in \{1, 2, 3\}$ and $w \in \{1, 2\}^*$ there exists a disk $B_{l,w} \subset \phi_{l,w}^{-1}(V_{l,w})$ such that

$$\text{radius}(B_{l,w}) \geq \epsilon \quad \text{and} \quad \text{dist}(B_{l,w}, \partial\phi_{l,w}^{-1}(V_{l,w})) \geq \epsilon$$

with $\epsilon := (2L')^{-1}(1 - L^{-1})$.

Fourth, by (6.5), there exists $\eta > 0$ such that for all $l \in \{1, 2, 3\}$ and $w \in \{1, 2\}^*$

$$\text{dist}(\phi_{l,w}^{-1}(X_{l,w}), \partial\phi_{l,w}^{-1}(V_{l,w})) \geq \eta \text{diam} \phi_{l,w}^{-1}(V_{l,w}) \geq \eta.$$

Let δ be the constant given in (5.3) depending only on C, s, c, η, ϵ above. Recall from the proof of Proposition 1.3 that for all $l \in \{1, 2, 3\}$ and $w \in \{1, 2\}^*$,

$$\phi_{l,w}^{-1}(V_{l,w}) \subset \left[-1 - \frac{3}{2}\epsilon_0, 1 + \frac{3}{2}\epsilon_0 \right]^2.$$

Therefore, there exist at most k_1 different configurations for $\mathcal{T}_\delta(\phi_{l,w}^{-1}(X_{l,w}))$ inside $\phi_{l,w}^{-1}(V_{l,w})$. Applying Proposition 5.1 for each of these finitely many cases we obtain bi-Lipschitz paths $\{H_1, \dots, H_{k_1}\}$ such that for each $l \in \{1, 2, 3\}$ and $w \in \{1, 2\}^*$, there exists $j(l, w) \in \{1, \dots, k_1\}$ for which

- (i) $H_{j(l,w)} : [0, 1] \rightarrow LIP(\phi_{l,w}^{-1}(V_{l,w}))$,
- (ii) $H_{j(l,w)}$ is an isometry on each component of $\mathcal{T}_\delta(\phi_{l,w}^{-1}(X_{l,w}))$,
- (iii) $(H_{j(l,w)})_1$ maps $\mathcal{T}_\delta(\phi_{l,w}^{-1}(X_{l,w}))$ onto $B_{l,w}$.

By (7.3) there exists $p \in \mathbb{N}$, so that if $u \in \{1, 2\}^p$ then

$$R_{l,wu} \subset \phi_{l,w}(\mathcal{T}_\delta(\phi_{l,w}^{-1}(X_{l,w}))).$$

We define $H_{l,w}|_{V_{l,w}}$ as follows.

(a) For $0 \leq t \leq 1/3$, we set

$$(H_{l,w}|V_{l,w})_t = \phi_{l,w} \circ (H_{i(l,w)})_{3t} \circ \phi_{l,w}^{-1}$$

to be the path which moves $\phi_{l,w}(\mathcal{J}_\delta(\phi_{l,w}^{-1}(X_{l,w})))$ into the disk $\phi_{l,w}(B_{l,w})$.

(b) For $2/3 \leq t \leq 1$, we set

$$(H_{l,w}|V_{l,w})_t = (H_w^l|V_{l,w})_{1-t}.$$

(c) For $1/3 \leq t \leq 2/3$, we define $H_{l,w}|V_{l,w}$ as a path of rotations. Fix l, w and suppose that $B_{l,w} = B(z_0, r)$. Find a conformal map

$$\psi_{l,w} : \phi_{l,w}^{-1}(V_{l,w}) \setminus \overline{B_{l,w}} \rightarrow \{z : 1 < |z| < \rho_{l,w}\}$$

for some $\rho_{l,w} > 1$. Since the boundary $\phi_{l,w}^{-1}(V_{l,w})$ is smooth, $\psi_{l,w}$ extends smoothly on $\partial\phi_{l,w}^{-1}(V_{l,w})$. We apply Proposition 4.12 with P, Q given by

$$\begin{aligned} Q_t(z) &= \psi_{l,w} \circ \phi_{l,w}^{-1} \circ (H_{l,w}|\partial V_{l,w})(z), \\ P_t(z) &= \psi_{l,w}(z_0 + (z - z_0)e^{2\pi(1-3t)i}). \end{aligned}$$

Here $H_{l,w}|\partial V_{l,w}$ agrees with $H_{l,w}$ on the inner boundary component of $R_{l,w}$, recalling the construction in Lemma 7.2. This yields a bi-Lipschitz path

$$P_{l,w} : [0, 1] \rightarrow LIP(\{z : 1 \leq |z| \leq \rho_{l,w}\}).$$

By Proposition 4.11, $G_{l,w} := (\psi_w^l)^{-1} \circ P_w^l \circ \psi_w^l$ is a bi-Lipschitz path. Since there are finitely many different pairs $(\phi_{l,w}^{-1}(V_{l,w}), B_{l,w})$, the set

$$\{G_{l,w} : l \in \{1, 2, 3\}, w \in \{1, 2\}^*\}$$

is finite. Set now for $1/3 \leq t \leq 2/3$,

$$(H_{l,w}|V_{l,w})_t = \phi_{l,w} \circ (G_{l,w})_{3t-1} \circ \phi_{l,w}^{-1}.$$

By the finiteness of the family $\{G_{l,w}\}_{l,w}$, and working as in Lemma 7.2, we see that $\{H_{l,w}|V_{l,w} : l \in \{1, 2, 3\}, w \in \{1, 2\}^*\}$ is a uniform family of bi-Lipschitz paths.

By Lemma 4.4, $\{H_{l,w} : l \in \{1, 2, 3\}, w \in \{1, 2\}^*\}$ is a uniform family of bi-Lipschitz paths. The key point in the construction of $H_{l,w}$ is that it unwinds the Dehn twist in $R_{l,w}$ and acts as an isometry on $R_{l,wu}$ for any $u \in \{1, 2\}^p$.

7.3 Step 3: Composing unwindings in a controlled way. The next step is to combine the paths $H_{l,w}$ defined above. Let $k \in \{0, 1, \dots, p - 1\}$. Define

$$(F_0^k)_t(z) = \begin{cases} (H_{l,w})_t(z), & z \in R_{l,w} \cup V_{l,w} \text{ for } |w| = k, l \in \{1, 2, 3\}, \\ z, & \text{otherwise.} \end{cases}$$

This is a bi-Lipschitz path. For example, for $k = 0$, this path unwinds the Dehn twists in the three outermost rings $R_{1,\epsilon}, R_{2,\epsilon}, R_{3,\epsilon}$. Then for $j \in \mathbb{N}$, suppose that F_{j-1}^k has been defined. We then define

$$(7.4) \quad (F_j^k)_t(z) = \begin{cases} (F_{j-1}^k)_t \circ (H_{l,w})_t(z), & z \in R_{l,w} \cup V_{l,w}, |w| = k + jp, \\ & l \in \{1, 2, 3\}, \\ (F_{j-1}^k)_t(z), & \text{otherwise.} \end{cases}$$

If $|w| = k + jp$ and $l \in \{1, 2, 3\}$, then $(F_{j-1}^k)_t$ acts as an isometry on $R_{l,w} \cup V_{l,w}$. Hence Lemma 4.7 implies that the composition in (7.4) gives a bi-Lipschitz path, and we conclude that F_j^k is a bi-Lipschitz path which unwinds the Dehn twists in $R_{l,w}$ for $|w| = k, k + p, k + 2p, \dots, k + jp$ and $l = 1, 2, 3$.

7.4 Step 4: Taking a limit. Set F^k by $(F^k)_t = \lim_{j \rightarrow \infty} (F_j^k)_t$ for all $t \in [0, 1]$. We claim that F^k is a bi-Lipschitz path. To that end, first consider, for $n \in \mathbb{N}$, the domain

$$\mathcal{U}_n := \bigcup_{\substack{|w|=k+np \\ l \in \{1,2,3\}}} U_{l,w}.$$

By construction, on this set we have $F^k|_{\mathcal{U}_n} = F_n^k|_{\mathcal{U}_n}$, and hence $F^k|_{\mathcal{U}_n}$ is a bi-Lipschitz path.

Next, note from (7.4) that F_j^k is obtained from F_{j-1}^k by modifications from a uniform family of bi-Lipschitz paths (namely, the family $\{H_{l,w}\}_{l,w}$) on a region where F_{j-1}^k acts as a family of isometries in a uniform way. By Lemma 4.7, it follows that the family

$$\left\{ F_j^k \Big| \bigcup_{n \in \mathbb{N}} \mathcal{U}_n : j \in \mathbb{N} \right\}$$

is a uniform family of bi-Lipschitz paths. Hence $F^k|_{\bigcup_{n \in \mathbb{N}} \mathcal{U}_n}$ is a bi-Lipschitz path.

Since $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n = \mathbb{R}^2 \setminus X$, an application of Proposition 4.5 shows that F^k is in fact a bi-Lipschitz path on all of \mathbb{R}^2 which unwinds the Dehn twists in $R_{l,w}$ for $|w| \in k + p\mathbb{N}, l = 1, 2, 3$. Hence the concatenation of the finitely many paths F^0, F^1, \dots, F^{p-1} yields a bi-Lipschitz path which connects f to the identity.

8 A decomposable multitwist with singular set of large Assouad dimension

Let D_ε be the rectangle $[-\sqrt{2}, \sqrt{2}] \times [-1, 1]$, let $\alpha \in (0, 1)$ and let

$$D_1 = \left[-\sqrt{2}\left(1 - \frac{1}{2}\alpha\right), -\sqrt{2}\frac{1}{2}\alpha \right] \times [\alpha - 1, 1 - \alpha],$$

$$D_2 = \left[\sqrt{2}\frac{1}{2}\alpha, \sqrt{2}\left(1 - \frac{1}{2}\alpha\right) \right] \times [\alpha - 1, 1 - \alpha],$$

as in Figure 6. Here ε denotes the empty word. For each $i \in \{1, 2\}$ let ϕ_i be the similarity of \mathbb{R}^2 mapping D_ε onto D_i with scaling factor $\frac{1}{\sqrt{2}}(1 - \alpha)$. Let X be the Cantor set attractor of the iterated function system $\{\phi_1, \phi_2\}$.

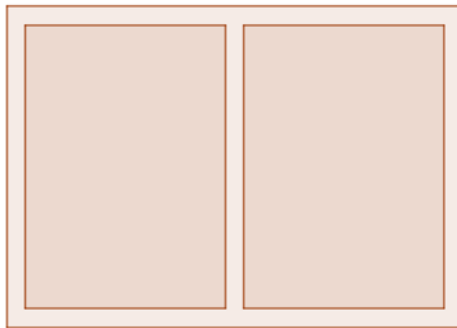


Figure 6. The first two steps in the construction of X .

By self-similarity, X is uniformly disconnected and its Assouad dimension is

$$\dim_A X = \frac{\log 2}{\log \sqrt{2} - \log(1 - \alpha)}$$

which is greater than 1 when α is sufficiently small. Moreover, there exists a multitwist bi-Lipschitz map f as in Section 6.2, and by self-similarity, the set of maps $\{g_j\}$ in Proposition 1.3 contains one single element.

We claim that the map f is decomposable. To prove the claim, we follow the arguments in Section 7. We may assume that the domains $\{V_{l,w}\}$ are exactly the interiors of the rectangles $\{D_w\}$. For simplicity, we drop the index l . The only step in the proof that we need to check (and the only one that requires the assumption on the Assouad dimension) is the existence of bi-Lipschitz paths $H_{j(w)}$. Since the collection $\{g_j\}$ contains only one element, we only need to construct for each $\epsilon > 0$ a ‘‘collapsing’’ bi-Lipschitz path $H : [0, 1] \rightarrow D_\varepsilon$ which, for some small $\delta > 0$, is

an isometry on each component of $\mathcal{T}_\delta(X)$ and maps $\mathcal{T}_\delta(X)$ into a ball B in D_ϵ of radius ϵ .

We give a rough sketch of the construction of H and leave the details to the reader. Fix $\epsilon > 0$. Choose $\beta \in (0, 1)$ such that $(1 - \alpha)(1 + \beta) < 1$ and choose $n \in \mathbb{N}$ such that

$$(1 + \beta)^n(1 - \alpha)^n < \frac{1}{4}\epsilon.$$

The bi-Lipschitz path H is a concatenation of n bi-Lipschitz paths H_1, \dots, H_n . Let H_1 be the bi-Lipschitz path that is identity outside of $\bigcup_{w \in \{1, 2\}^{n-1}} D_w$ and for each $w \in \{1, 2\}^{n-1}$, it moves D_{w1} towards D_{w2} so that they both end up in a rectangle D'_w with sides parallel to the axes and side-lengths

$$4(1 + \beta)\left(\frac{1}{\sqrt{2}}(1 - \alpha)\right)^n, \quad 2\sqrt{2}(1 + \beta)\left(\frac{1}{\sqrt{2}}(1 - \alpha)\right)^n.$$

The choice of β ensures that D'_w is contained in D_w . Moreover, H_1 acts as an isometry on D_{wi} for all $wi \in \{1, 2\}^n$.

Assume now that for some $m \in \{1, \dots, n - 1\}$ we have defined the paths H_1, \dots, H_m and assume that

- (i) the concatenation of these paths is the identity outside of $\bigcup_{w \in \{1, 2\}^{n-m}} D_w$,
- (ii) for each $w \in \{1, 2\}^{n-m}$, the concatenation has moved $X \cap D_w$ inside a rectangle $D'_w \subset D_w$ with sides parallel to the axes and side-lengths

$$2\sqrt{2}(\sqrt{2})^m(1 + \beta)^m\left(\frac{1}{\sqrt{2}}(1 - \alpha)\right)^n, \quad 2(\sqrt{2})^m(1 + \beta)^m\left(\frac{1}{\sqrt{2}}(1 - \alpha)\right)^n,$$

- (iii) for each $u \in \{1, 2\}^n$ the concatenation of these paths acts as an isometry on D_u .

Let H_{m+1} be the bi-Lipschitz path that is identity outside of $\bigcup_{w \in \{1, 2\}^{n-m-1}} D_w$ and for each $w \in \{1, 2\}^{n-m-1}$, it moves D'_{w1} towards D'_{w2} so that they both end up in a rectangle D'_w with sides parallel to the axes and side-lengths

$$2\sqrt{2}(\sqrt{2})^{m+1}(1 + \beta)^{m+1}\left(\frac{1}{\sqrt{2}}(1 - \alpha)\right)^n, \quad 2(\sqrt{2})^{m+1}(1 + \beta)^{m+1}\left(\frac{1}{\sqrt{2}}(1 - \alpha)\right)^n.$$

Note that H_{m+1} acts as an isometry on D_u for all $u \in \{1, 2\}^n$.

Finally, the concatenation H of paths H_1, \dots, H_n is the identity outside of D , acts as an isometry on D_u for all $u \in \{1, 2\}^n$, and $H(X)$ is contained in a rectangle $D' \subset D$ with side-lengths

$$2\sqrt{2}(1 + \beta)^n(1 - \alpha)^n, \quad 2(1 + \beta)^n(1 - \alpha)^n.$$

By the choice of n , the rectangle D' has diameter less than ϵ and the proof is complete.

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