

# A BLOW-UP PHENOMENON FOR A NON-LOCAL LIOUVILLE-TYPE EQUATION

By

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**Abstract.** We consider the non-local Liouville equation

$$(-\Delta)^{\frac{1}{2}}u = h_\varepsilon e^u - 1 \quad \text{in } \mathbb{S}^1,$$

corresponding to the prescription of the geodesic curvature on the circle. We build a family of solutions which blow up, when  $h_\varepsilon$  approaches a function  $h$  as  $\varepsilon \rightarrow 0$ , at a critical point of the harmonic extension of  $h$  provided some generic assumptions are satisfied.

## 1 Introduction

The classical Nirenberg problem consists in finding positive functions  $h$  on the standard sphere  $(\mathbb{S}^n, g_0)$  for which there exists a metric  $g$  conformally equivalent to  $g_0$  whose scalar curvature is equal to  $h$ . In dimension  $n = 2$ , the Nirenberg problem asks what functions can be the gaussian curvature of a conformal metric on  $\mathbb{S}^2$ . It can also be rephrased in terms of solutions of a partial differential equation on the sphere. More precisely, one looks for functions  $h$  on  $\mathbb{S}^2$  for which there exists a solution  $u : \mathbb{S}^2 \rightarrow \mathbb{R}$  of the Liouville equation

$$(1) \quad -\Delta_{\mathbb{S}^2}u = h e^{2u} - 1 \quad \text{in } \mathbb{S}^2.$$

Indeed a straightforward computation shows that the gaussian curvature of the conformal metric  $g = e^{2u}g_0$  is nothing but the prescribed function  $h$ . Recently, Da Lio, Martinazzi and Rivière in [3] investigated the case  $n = 1$ . They parametrize a planar Jordan curve (i.e., a continuous closed and simple curve) through the trace

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of the Riemann mapping between the disk  $\mathbb{D}$  and the simply connected domain enclosed by the curve and find an equation similar to (1),

$$(2) \quad (-\Delta)^{\frac{1}{2}}u = he^u - 1 \quad \text{in } \mathbb{S}^1,$$

whose solutions give the curvature density  $he^u d\theta$  of the curve in this parametrization. Here  $(-\Delta)^{\frac{1}{2}}$  is the  $\frac{1}{2}$ -fractional Laplacian in  $\mathbb{S}^1$ , i.e.,

$$(-\Delta)^{\frac{1}{2}}u(z) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{S}^1} \frac{u(z) - u(w)}{|z - w|^2} dw.$$

Problem (2) is equivalent, up to a constant factor 2, to

$$(3) \quad \begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}, \\ \partial_\nu u + 2 = 2he^{\frac{u}{2}} & \text{on } \partial\mathbb{D} = \mathbb{S}^1, \end{cases}$$

where  $\nu$  is the outward-pointing normal derivative at the boundary. Problem (3) corresponds to the geometric problem of finding a flat metric  $g$  on the disk  $(\mathbb{D}, g_0)$  such that  $h$  is the geodesic curvature of  $\mathbb{S}^1$  with respect to the metric  $g = e^{2u}g_0$  which is pointwise conformal to  $g_0$ .

Necessary conditions on  $h$  to solve (2) or (3) are easily obtained. Indeed integrating (3) we get

$$(4) \quad \int_{\mathbb{S}^1} he^{\frac{u}{2}} d\sigma_{g_0} = 2\pi, \quad \text{which implies } \max_{\mathbb{S}^1} h > 0.$$

As far as we know there are few results about existence and multiplicity of solutions of (2) or (3). The first one seems due to Chang and Liu [2], who proved the existence of a solution to (3) provided  $h$  is positive, has only isolated critical points,  $\overline{h}(z) \neq 0$  whenever  $h(z) = 0$  where  $\overline{h}$  denotes the complex conjugate function of  $h$ , and a further relation between local maxima and local minima of  $h$  holds true. Later, Liu and Huang in [6] considered the case where  $h$  possesses symmetry and they found a solution to (3) if  $h$  has a minimum point  $z_0$  which satisfies  $\partial_\nu H(z_0) > 0$ , where  $H$  is the harmonic extension of  $h$  to all of  $\mathbb{D}$ , i.e.,

$$(5) \quad \begin{cases} -\Delta H = 0 & \text{in } \mathbb{D}, \\ H = h & \text{on } \mathbb{S}^1. \end{cases}$$

Finally, Zhang [8] employed a negative gradient flow method to build a solution to (3) when the necessary condition in (4) is satisfied. The existence issue is strictly related to the study of the blow-up phenomenon. Recently Jevnikar, López-Soriano, Medina and Ruiz in [5] proved that if  $u_n$  is a sequence of blow-up solutions

of (2) or (3) with  $h$  replaced by  $h_n$ ,  $h_n$  is uniformly bounded in  $C^2(\mathbb{S}^1)$  and  $e^{u_n}$  is uniformly bounded in  $L^1(\mathbb{S}^1)$ , then  $u_n$  blows-up at a unique point  $p \in \mathbb{S}^1$  such that  $h(p) > 0$  and  $\nabla H(p) = 0$  where  $H$  is the harmonic extension of  $h$  (see (5)). See also [4] for results concerning the blow-up analysis for problem (2).

The goal of this article is to provide the first example of the blow-up phenomenon for the problems (2) and (3). To do so we will construct a family of solutions to some approximated problems, with  $h$  perturbed as a function  $h_\varepsilon \rightarrow h$  in  $C^1(\mathbb{S}^1)$  as  $\varepsilon \rightarrow 0$ , which concentrate at one point as  $\varepsilon \rightarrow 0$ . In particular, our result also gives the first multiplicity result to (2)/(3). More precisely, consider

$$(6) \quad h_\varepsilon(z) := h(z) + \varepsilon k(z),$$

with  $\varepsilon > 0$  a small parameter,  $h \in C^{2,\alpha}(\mathbb{S}^1)$  for some  $\alpha > 0$  and  $k \in C^1(\mathbb{S}^1)$ , and the problem

$$(7) \quad (-\Delta)^{\frac{1}{2}}u = h_\varepsilon(z)e^u - 1 \quad \text{in } \mathbb{S}^1,$$

or equivalently

$$(8) \quad \begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}, \\ \partial_\nu u + 2 = 2h_\varepsilon(z)e^{\frac{u}{2}} & \text{on } \mathbb{S}^1. \end{cases}$$

We will build a family of solutions to (7) or (8) which blow-up at any point  $\zeta_0 \in \mathbb{S}^1$  around which  $h$  satisfies suitable conditions. For the sake of simplicity, we will assume that  $\zeta_0 = 1$ . According to the blow-up analysis performed in [5] we need to assume

$$(9) \quad h(1) > 0 \quad \text{and} \quad h'(1) = (-\Delta)^{\frac{1}{2}}h(1) = 0,$$

where  $h'$  stands for the tangential derivative of  $h$ .

In fact, Theorem 1.1 from [5] proves that any blowing-up sequence of

$$\begin{cases} -\Delta u = K_\varepsilon(z)e^u & \text{in } \mathbb{D}, \\ \partial_\nu u + 2 = 2h_\varepsilon(z)e^{\frac{u}{2}} & \text{on } \mathbb{S}^1, \end{cases}$$

with  $K_0 = K$ , must blow at a boundary point  $\zeta_0$  satisfying:

$$\begin{aligned} h^+(\zeta_0)^2 + K(\zeta_0) &> 0, \\ h'(\zeta_0) &= -\frac{\partial_{\xi_2} K(\zeta_0)}{h(\zeta_0) + \sqrt{h(\zeta_0)^2 + K(\zeta_0)}}, \\ (-\Delta)^{\frac{1}{2}}h(\zeta_0) &= -\frac{\partial_{\xi_1} K(\zeta_0)}{h(\zeta_0) + \sqrt{h(\zeta_0)^2 + K(\zeta_0)}}; \end{aligned}$$

that is, since  $\zeta_0 = 1$  and  $K \equiv 0$ , equation (9).

We also require the following non-degeneracy condition at the point 1,

$$(10) \quad \left( h''(1) - \frac{Q(h)}{\pi^2 h(1)} \right) h''(1) + ((-\Delta)^{\frac{1}{2}} h'(1))^2 \neq 0,$$

where

$$(11) \quad Q(h) := \int_{\mathbb{S}^1 \times \mathbb{S}^1} \log \frac{1}{|z-w|} \frac{h(z) - h(1)}{|z-1|^2} \frac{h(w) - h(1)}{|w-1|^2} dwdz.$$

This condition is not restrictive since, if the equality holds, we can replace  $h(z)$  with  $h(z) + c$ ,  $c$  being a small constant, and (10) holds provided  $c$  is small enough. We need  $h''$  to be Hölder continuous in order for  $Q(h)$  to appear in the main term in the energy expansion (see Propositions 4.1, 4.2).

Equation (10) involves not only the second derivatives of  $h$ , but also the quadratic form  $Q(h)$  defined in (11), which has an interesting interpretation. First, we notice that the function

$$\widehat{h}(z) := \frac{h(z) - h(1)}{|z-1|^2}$$

is bounded and has zero average, due to (9); therefore, we may consider the (zero-average) solution  $\widetilde{h}$  to

$$(-\Delta)^{\frac{1}{2}} \widetilde{h} = \widehat{h} \quad \text{in } \mathbb{S}^1.$$

Using Green’s representation, we can write

$$\begin{aligned} Q(h) &= \int_{\mathbb{S}^1} \widehat{h}(z) \left( \int_{\mathbb{S}^1} \log \frac{1}{|z-w|} \widehat{h}(w) dw \right) dz \\ &= \int_{\mathbb{S}^1} ((-\Delta)^{\frac{1}{2}} \widetilde{h}(z)) (2\pi \widetilde{h}(z)) dz \\ &= 2\pi \int_{\mathbb{S}^1} |(-\Delta)^{\frac{1}{4}} \widetilde{h}(z)|^2 dz \\ &> 0. \end{aligned}$$

For the sake of simplicity, we can assume the perturbative term  $k$  to vanish at the point 1, i.e.,  $k(1) = 0$ . Finally, we assume the transversality condition

$$(12) \quad h''(1)(-\Delta)^{\frac{1}{2}} k(1) - k'(1)(-\Delta)^{\frac{1}{2}} h'(1) \neq 0.$$

This is a rather natural requirement for the perturbation. In fact, after a simple computation, one can see that (12) is equivalent to requiring that the matrix

$$\begin{pmatrix} \partial_{\xi_1 \varepsilon}^2 (H + \varepsilon K)(\zeta) & \partial_{\xi_2 \varepsilon}^2 (H + \varepsilon K)(\zeta) \\ \partial_{\xi_1 \xi_2}^2 (H + \varepsilon K)(\zeta) & \partial_{\xi_2 \xi_2}^2 (H + \varepsilon K)(\zeta) \end{pmatrix}_{\varepsilon=0, \zeta=1}$$

is nonsingular, with  $H$  and  $K$  being the harmonic extensions of  $h$  and  $k$ , respectively.

In other words, the curve  $\gamma(\varepsilon)$  of critical points to  $H + \varepsilon K$  crosses the boundary in  $\zeta = 1$  at  $\varepsilon = 0$  not vertically but rather crossing the boundary.

Our main result reads as

**Theorem 1.1.** *Assume  $h \in C^{2,\alpha}(\mathbb{S}^1)$  and  $k \in C^1(\mathbb{S}^1)$ . Suppose that  $\zeta_0 = 1 \in \mathbb{S}^1$  satisfies (9) and (10). If (12) holds true, then there exists  $\varepsilon_0 > 0$  such that, for every  $\varepsilon \in (0, \varepsilon_0)$  or for every  $\varepsilon \in (-\varepsilon_0, 0)$ , there exists a solution  $u_\varepsilon$  of*

$$(-\Delta)^{\frac{1}{2}}u = h_\varepsilon(z)e^u - 1 \quad \text{in } \mathbb{S}^1,$$

blowing-up at  $\zeta_0 = 1$  as  $\varepsilon \rightarrow 0$ , with  $h_\varepsilon$  defined at (6).

Furthermore, there exist  $\delta_\varepsilon > 0$  and  $\zeta_\varepsilon \in \mathbb{S}^1$  with  $\delta_\varepsilon = O(\varepsilon)$  and  $\zeta_\varepsilon = 1 + O(\varepsilon)$  such that

$$u_\varepsilon(f_{\delta_\varepsilon, \zeta_\varepsilon}(z)) + \log |f'_{\delta_\varepsilon, \zeta_\varepsilon}(z)| + \log h(1) = O(\varepsilon) \text{ in } L^p(\mathbb{S}^1) \quad \forall p \in [1, +\infty),$$

where  $f_{\delta, \zeta}$  is the conformal map

$$(13) \quad f = f_{\delta, \zeta}(z) := \frac{z + (1 - \delta)\zeta}{1 + (1 - \delta)\overline{\zeta}z}.$$

This theorem completes our previous work [1], where we studied the problem of prescribing the gaussian and geodesic curvatures for a conformal metric on the unit disk, which turns out to be equivalent to solving the problem

$$(14) \quad \begin{cases} -\Delta u = 2K(z)e^u & \text{in } \mathbb{D}, \\ \partial_\nu u + 2 = 2h(z)e^{\frac{u}{2}} & \text{on } \mathbb{S}^1, \end{cases}$$

where  $K, h$  are the prescribed curvatures. The reader can find an exhaustive list of references concerning this problem in [1]. In particular, there we built a family of conformal metrics with curvatures  $K_\varepsilon, h_\varepsilon$  converging to  $K, h$  respectively as  $\varepsilon$  goes to 0, which blows up at one boundary point under some generic assumptions. The strategy we follow in the present paper is similar, but it requires some careful estimates of the error term.

Briefly, we consider the conformal map given in (13) and we take

$$(15) \quad \delta = \delta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \eta = \eta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \zeta = \zeta_\varepsilon = e^{i\eta_\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 1 \in \mathbb{S}^1$$

so that  $(1 - \delta)\zeta \in \mathbb{D}$ . Thus, letting  $v(z) := u(f(z)) + 2 \log |f'(z)|$ , we rewrite problem (7) as

$$(16) \quad (-\Delta)^{\frac{1}{2}}v = h_\varepsilon(f(z))e^v - 1 \quad \text{in } \mathbb{S}^1,$$

and using a Ljapunov–Schmidt procedure we find a solution of (16) as

$$(17) \quad v(z) \sim V(z) + W(z) + \tau,$$

where the first order term is just a constant

$$(18) \quad V(z) := V_\xi(z) \equiv -\log h(\xi),$$

solving

$$(19) \quad (-\Delta)^{\frac{1}{2}} V = h(\xi)e^V - 1 \quad \text{on } \mathbb{S}^1.$$

Non-trivial solutions to (19) have been classified by Ou [7] and Zhang [9]. The second-order term, which is the key of the ansatz, is defined as

$$(20) \quad W(z) := W_\xi(z) = \frac{1}{\pi} \int_{\mathbb{S}^1} \log \frac{1}{|z-w|} (h(f(w)) - h(\xi))e^V dw,$$

solving

$$(-\Delta)^{\frac{1}{2}} W = (h(f(z)) - h(\xi))e^V - \frac{1}{2\pi} \int_{\mathbb{S}^1} (h(f(w)) - h(\xi))e^V dw \quad \text{in } \mathbb{S}^1;$$

finally,  $\tau$  is a small constant.

Regarding the case considered in [1], the construction of the solutions needs to be more careful. Actually, the non-degeneracy condition corresponding to (10) for problem (14) involves the second-order derivatives of the interior curvature  $K$  (see (7) and (8) in [1]), excluding the case  $K \equiv 0$  considered here.

In fact, in order to find concentrating solutions to (14) with  $K_\varepsilon = K + \varepsilon G$ ,  $h_\varepsilon = h + \varepsilon I$ , we need to find solutions  $(s, d, t)$  to a system of the form: (see [1], Section 6)

$$\begin{cases} a_{11}s + a_{12}d + b_1 + o(1) = 0, \\ a_{21}s + b_2 + o(1) = 0 \\ a_{32}d + a_{33}t + o(1) = 0, \end{cases}$$

with  $a_{12} = \Delta K(1) + 4|\nabla H(1)|^2$ ; since now  $K \equiv 0$  and we need to assume  $\nabla H(1) = 0$ , then  $a_{22} = 0$ , hence the first two equations of the system may be incompatible and more careful estimates are needed.

Quite surprisingly, assumptions in Theorem 1.1 look simpler and more natural compared to [1, Theorem 1.1].

**Example 1.2.** Let us consider

$$h(e^{i\theta}) = (1 - \cos \theta) \cos(N\theta) + h(1), \quad \text{with } N \in \mathbb{N}, h(1) > 0.$$

One easily sees that  $h'(1) = (-\Delta)^{\frac{1}{2}}h(1) = 0$ , hence (9) is satisfied. Moreover, by explicit computations one gets  $h''(1) = 1$ ,  $(-\Delta)^{\frac{1}{2}}h'(1) = 0$  and, since

$$\widehat{h}(e^{i\theta}) = \frac{h(e^{i\theta}) - h(1)}{|e^{i\theta} - 1|^2} = \frac{\cos(N\theta)}{2},$$

then

$$Q(h) = 2\pi \int_{\mathbb{S}^1} \frac{\cos(N\theta)}{2} \frac{\cos(N\theta)}{2N} d\theta = \frac{\pi^2}{2N},$$

therefore (10) reads  $1 - \frac{1}{2Nh(1)} \neq 0$  and is satisfied if  $h(1) \neq \frac{1}{2N}$ . Finally, transversality condition (12) is satisfied as long as  $(-\Delta)^{\frac{1}{2}}k(1) \neq 0$ , which means we can choose a generic perturbation vanishing at 1, namely

$$k(e^{i\theta}) = \sum_{n=0}^{+\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)), \quad \text{with } \sum_{n=0}^{+\infty} a_n = 0 \neq \sum_{n=0}^{+\infty} na_n$$

and  $a_n, b_n$  decaying fast enough in order for  $k$  to be in  $C^{2,\alpha}$ .

Any of these choices of  $h, k$  satisfies the assumptions of Theorem 1.1 and therefore gives rise to a blowing-up family of solutions to the problem (7).

The plan of the paper is as follows: in Section 2 we provide crucial estimates for the main term  $W$  in the ansatz; in Section 3 we develop the linear theory and solve the auxiliary fixed-point problem for  $\phi$ ; in Section 4 we evaluate the projections on the kernels of the linearized operator and we conclude the proof of Theorem 1.1.

## 2 Ansatz and error estimates

We look for a solution of (16) as  $v(z) = V(z) + W(z) + \tau + \phi(z)$ , where  $V$  and  $W$  are defined in (18) and (20),  $\tau = \tau_\varepsilon \rightarrow 0$  ( $\varepsilon \rightarrow 0$ ) is a constant and  $\phi(z) = \phi_{\xi, \delta, \tau}(z)$  is a small function to be found.

In fact, we want to find  $\xi, \delta, \tau$  such that  $\phi$  solves

$$(-\Delta)^{\frac{1}{2}}(V + W + \tau + \phi) = h_\varepsilon(f(z))e^{V+W+\tau+\phi} - 1 \quad \text{in } \mathbb{S}^1,$$

that is

$$\begin{aligned} (-\Delta)^{\frac{1}{2}}\phi - h(\xi)e^V\phi &= (h_\varepsilon(f(z))e^{W+\tau} - h(f(z)))e^V + \frac{1}{2\pi} \int_{\mathbb{S}^1} (h(f(w)) - h(\xi))e^V dw \\ &\quad + (h_\varepsilon(f(z))e^{W+\tau} - h(\xi))e^V\phi \\ &\quad + h_\varepsilon(f(z))e^{V+W+\tau}(e^\phi - 1 - \phi) \quad \text{in } \mathbb{S}^1. \end{aligned}$$

This can be rewritten as

$$(21) \quad \mathcal{L}_0\phi = \mathcal{E} + \mathcal{L}\phi + \mathcal{N}(\phi) \quad \text{in } \mathbb{S}^1,$$

with

$$\begin{aligned}
 \mathcal{L}_0\phi &:= (-\Delta)^{\frac{1}{2}}\phi - h(\zeta)e^V\phi, \\
 \mathcal{E} &:= (h_\varepsilon(f(z))e^{W+\tau} - h(f(z)))e^V + \frac{1}{2\pi} \int_{\mathbb{S}^1} (h(f(w)) - h(\zeta))e^V dw, \\
 \mathcal{L}\phi &:= (h_\varepsilon(f(z))e^{W+\tau} - h(\zeta))e^V\phi, \\
 \mathcal{N}(\phi) &:= h_\varepsilon(f(z))e^{V+W+\tau}(e^\phi - 1 - \phi).
 \end{aligned}
 \tag{22}$$

The following two auxiliary results will be useful along the paper, and can be found at [1, Proposition 7.1] and [1, Proposition 7.2] respectively.

**Proposition 2.1.** *Let  $\zeta \in \mathbb{S}^1$ . For any  $z \in \mathbb{S}^1$  one has*

$$|f(z) - \zeta| = O\left(\frac{\delta}{\delta + |z + \zeta|}\right),$$

and in particular

$$\|f(z) - \zeta\|_{L^p} = \begin{cases} O(\delta \log \frac{1}{\delta}) & p = 1, \\ O(\delta^{\frac{1}{p}}) & p > 1. \end{cases}$$

Moreover, if  $z \in \mathbb{S}^1$  and  $h \in C^2(\mathbb{S}^1)$ , then

$$h(f(z)) - h(\zeta) = \delta h'(\zeta)\Theta(z) + O\left(\frac{\delta^2}{(\delta + |z + \zeta|)^2}\right),$$

with

$$\Theta(z) = \Theta_{\delta,\zeta}(z) := \frac{2\langle z, \zeta^\perp \rangle}{1 + (1 - \delta)^2 + 2(1 - \delta)\langle z, \zeta \rangle}.$$

**Sketch of the proof.** From the definition of  $f(z)$ , we have

$$|f(z) - \zeta| = \frac{\delta|z - \zeta|}{|(1 - \delta)z + \zeta|} = O\left(\frac{\delta}{|(1 - \delta)z + \zeta|}\right).$$

Moreover, since  $z, \zeta \in \partial\mathbb{D}^2$ , then

$$|(1 - \delta)z + \zeta|^2 = \delta^2 + (1 - \delta)|z + \zeta|^2 \geq \frac{(\delta + |z + \zeta|)^2}{4},$$

hence  $\delta + |z + \zeta| = O(|(1 - \delta)z + \zeta|)$ , which proves the first statement.

Now, by the chain rule, one has

$$h(f(z)) - h(\zeta) = h'(\zeta)\langle f(z) - \zeta, \zeta^\perp \rangle + O(|f(z) - \zeta|^2),$$

and simple computations show that

$$\langle f(z) - \zeta, \zeta^\perp \rangle = \delta\left(1 - \frac{\delta}{2}\right)\Theta(z),$$

which proves the Lemma. □



**Proposition 2.2.** *Given  $\zeta \in \mathbb{S}^1$ ,*

$$\int_{\mathbb{S}^1} (h(f(z)) - h(\zeta)) dz = -2\pi\delta(-\Delta)^{\frac{1}{2}}h(\zeta) + O(\delta^2).$$

We can now give estimates on the correction term  $W$  given in (20).

**Lemma 2.3.** *The function  $W$  satisfies*

$$(23) \quad W(z) = O\left(\delta|\eta|(1 + \log|z + \zeta|) + \frac{\delta^2}{\delta + |z + \zeta|} \left(1 + \left|\log \frac{|z + \zeta|}{\delta}\right|\right)\right),$$

with  $\delta, \eta, \zeta$  as in (15), and

$$\|W\|_{L^p} = O(\delta^{1+\frac{1}{p}} + \delta|\eta|), \quad \|e^W\|_{L^p} = O(1),$$

for every  $p \in [1, +\infty)$ .

**Proof.** Estimate (23) follows by [1, Lemma 2.1] noticing that, as a consequence of (9),

$$|h'(\zeta)| + |(-\Delta)^{\frac{1}{2}}h(\zeta)| = O(|\eta|).$$

The  $L^p$  estimates are straightforward. □

**Proposition 2.4.** *The correction term  $W$  satisfies*

$$(24) \quad W(z) = \frac{2}{\pi h(\zeta)} \delta \int_{\mathbb{S}^1} \log \frac{|w - \zeta|}{|f(z) - w|} \frac{h(w) - h(\zeta) - h'(\zeta)\langle w, \zeta^\perp \rangle}{|w - \zeta|^2} dw + O\left(\delta \log \frac{1}{\delta} (\delta + |\eta|) \left(1 + \log^2 \frac{1}{\delta + |z + \zeta|}\right)\right).$$

**Proof.** We split the function into three parts,

$$(25) \quad W(z) = \underbrace{-\frac{1}{\pi h(\zeta)} \log |(1 - \delta)z + \zeta| \int_{\mathbb{S}^1} (h(f(w)) - h(\zeta)) dw}_{=: W_1(z)} + \underbrace{\frac{1}{\pi h(\zeta)} \int_{\mathbb{S}^1} \log \frac{|(1 - \delta)z + \zeta|}{|z - w|} \left( h'(\zeta) \frac{(2 - \delta)\delta \langle w, \zeta^\perp \rangle}{\delta^2 + (1 - \delta)|w + \zeta|^2} \right) dw}_{=: W_2(z)}$$

$$(26) \quad + \frac{1}{\pi h(\zeta)} \int_{\mathbb{S}^1} \log \frac{|(1 - \delta)z + \zeta|}{|z - w|} \times \left( h(f(w)) - h(\zeta) - h'(\zeta) \frac{(2 - \delta)\delta \langle w, \zeta^\perp \rangle}{\delta^2 + (1 - \delta)|w + \zeta|^2} \right) dw,$$

and we denote (25) and (26) by  $W_3(z)$ . By Proposition 2.2 the first one can be estimated as

$$\begin{aligned} W_1(z) &= -\frac{1}{\pi h(\xi)} \log \sqrt{\delta^2 + (1 - \delta)|z + \xi|^2} (\delta(-\Delta)^{\frac{1}{2}} h(\xi) + O(\delta^2)) \\ &= O\left(\left(1 + \log \frac{1}{\delta + |z + \xi|}\right) \delta(\delta + |\eta|)\right). \end{aligned}$$

Furthermore, defining

$$k_2(z) := -\frac{2\pi}{1 - \delta} \arctan \frac{(1 - \delta)\langle z, \xi^\perp \rangle}{1 + (1 - \delta)\langle z, \xi \rangle}$$

we get that  $k_2$  is harmonic, hence by an explicit computation we get

$$(-\Delta)^{\frac{1}{2}} k_2 = -\frac{2\pi \langle w, \xi^\perp \rangle}{\delta^2 + (1 - \delta)|w + \xi|^2} \quad \text{in } \mathbb{S}^1;$$

therefore, by Green’s representation formula,

$$\begin{aligned} W_2(z) &= \frac{\delta(2 - \delta)h'(\xi)}{\pi h(\xi)} \int_{\mathbb{S}^1} \log |z - w| \frac{\langle w, \xi^\perp \rangle}{\delta^2 + (1 - \delta)|w + \xi|^2} dw \\ &= O(\delta|\eta|) \left( -\frac{2\pi}{1 - \delta} \arctan \frac{(1 - \delta)\langle w, \xi^\perp \rangle}{1 + (1 - \delta)\langle z, \xi \rangle} \right) \\ &= O(\delta|\eta|). \end{aligned}$$

To estimate  $W_3$  we write, thanks to (13),

$$\frac{(1 - \delta)z + \xi}{z - w} = \frac{(1 - \delta)f(w) - \xi}{f(w) - f(z)},$$

and thus

$$\begin{aligned} W_3(z) &= \frac{1}{\pi h(\xi)} \int_{\mathbb{S}^1} \log \frac{|(1 - \delta)v - \xi|}{|v - f(z)|} (h(v) - h(\xi) - h'(\xi)\langle v, \xi^\perp \rangle) \\ &\quad \times \frac{\delta(2 - \delta)}{\delta^2 + (1 - \delta)|v - \xi|^2} dv \\ &= \frac{2}{\pi h(\xi)} \delta \int_{\mathbb{S}^1} \log \frac{|v - \xi|}{|v - f(z)|} \frac{h(v) - h(\xi) - h'(\xi)\langle v, \xi^\perp \rangle}{|v - \xi|^2} dv \\ (27) \quad &+ \frac{1}{\pi h(\xi)} \delta \int_{\mathbb{S}^1} \log \frac{|v - f(z)|}{|v - \xi|} (h(v) - h(\xi) - h'(\xi)\langle v, \xi^\perp \rangle) \\ (28) \quad &\times \left( \frac{2}{|v - \xi|^2} - \frac{2 - \delta}{\delta^2 + (1 - \delta)|v - \xi|^2} \right) dv \\ (29) \quad &+ \frac{1}{\pi h(\xi)} \int_{\mathbb{S}^1} \log \frac{|(1 - \delta)v - \xi|}{|v - \xi|} (h(v) - h(\xi) - h'(\xi)\langle v, \xi^\perp \rangle) \\ (30) \quad &\times \frac{\delta(2 - \delta)}{\delta^2 + (1 - \delta)|v - \xi|^2} dv, \end{aligned}$$

and we denote (27) and (28) by  $W_{3,1}(z)$  and (29) and (30) by  $W_{3,2}(z)$ . To analyze  $W_{3,2}$  we use that, since  $|v| = |\xi| = 1$ , then

$$|(1 - \delta)v - \xi|^2 = (1 - \delta)|v - \xi|^2 + \delta^2,$$

and moreover

$$\log(1 + t^2) = \begin{cases} O(\log t) & t > 1 \\ O(t^2) & t \leq 1 \end{cases},$$

therefore

$$\begin{aligned} \log \frac{|(1 - \delta)v - \xi|}{|v - \xi|} &= \frac{1}{2} \left( \log(1 - \delta) + \log \left( 1 + \frac{\delta^2}{(1 - \delta)|v - \xi|^2} \right) \right) \\ &= \begin{cases} O(1 + \log \frac{\delta}{|v - \xi|}) & |v - \xi| \leq \delta, \\ O(\delta + \frac{\delta^2}{|v - \xi|^2}) & |v - \xi| > \delta; \end{cases} \end{aligned}$$

in addition,

$$\begin{aligned} (h(v) - h(\xi) - h'(\xi)\langle v, \xi^\perp \rangle) \frac{\delta(2 - \delta)}{\delta^2 + (1 - \delta)|v - \xi|^2} &= O\left( \delta \frac{|v - \xi|^2}{\delta^2 + (1 - \delta)|v - \xi|^2} \right) \\ &= \begin{cases} O(\frac{|v - \xi|^2}{\delta}) & |v - \xi| \leq \delta, \\ O(\delta) & |v - \xi| > \delta. \end{cases} \end{aligned}$$

From this we obtain

$$\begin{aligned} W_{3,2} &= O\left( \int_{|v - \xi| \leq \delta} \left( 1 + \log \frac{\delta}{|v - \xi|} \right) \frac{|v - \xi|^2}{\delta} dv + \int_{|v - \xi| > \delta} \left( \delta + \frac{\delta^2}{|v - \xi|^2} \right) \delta dv \right) \\ &= O\left( \delta^2 \int_0^1 \left( 1 + \log \frac{1}{t} \right) t^2 dt + \delta^2 \int_1^{O(\frac{1}{\delta})} \left( \delta + \frac{1}{t^2} \right) dt \right) \\ &= O(\delta^2). \end{aligned}$$

To estimate  $W_{3,1}$  we observe first that

$$\begin{aligned} (h(v) - h(\xi) - h'(\xi)\langle v, \xi^\perp \rangle) &\left( \frac{2}{|v - \xi|^2} - \frac{2 - \delta}{\delta^2 + (1 - \delta)|v - \xi|^2} \right) \\ (31) \quad &= \frac{(h(v) - h(\xi) - h'(\xi)\langle v, \xi^\perp \rangle)}{|v - \xi|^2} \frac{2\delta^2 - \delta|v - \xi|^2}{\delta^2 + (1 - \delta)|v - \xi|^2} \\ &= O\left( \frac{\delta}{\delta + |v - \xi|} \right), \end{aligned}$$

and we will divide the integral in three regions, depending on whether  $v$  is closer to  $\zeta$  than to  $f(z)$ , further, or at a comparable distance. Notice that if

$$\frac{|v - \zeta|}{|v - f(z)|} \leq \frac{1}{2},$$

then

$$|v - \zeta| \leq 2|f(z) - \zeta|$$

and

$$\left| \log \frac{|v - \zeta|}{|v - f(z)|} \right| = \log \frac{|v - f(z)|}{|v - \zeta|} = O\left(1 + \log \frac{|f(z) - \zeta|}{|v - \zeta|}\right).$$

On the contrary, if

$$\frac{1}{2} < \frac{|v - \zeta|}{|v - f(z)|} \leq 2,$$

then

$$|f(z) - \zeta| \leq 3|v - f(z)|$$

and

$$\left| \log \frac{|v - \zeta|}{|v - f(z)|} \right| = O\left(\left| \frac{|v - \zeta|}{|v - f(z)|} - 1 \right|\right) = O\left(\frac{|f(z) - \zeta|}{|v - f(z)|}\right).$$

Finally, if

$$\frac{|v - \zeta|}{|v - f(z)|} > 2,$$

then

$$|v - f(z)| \leq |f(z) - \zeta|$$

and

$$\left| \log \frac{|v - \zeta|}{|v - f(z)|} \right| = \log \frac{|v - \zeta|}{|v - f(z)|} = O\left(1 + \log \frac{|f(z) - \zeta|}{|v - f(z)|}\right),$$

and, putting all this information together, we conclude that

$$\begin{aligned}
 W_{3,1}(z) &= O\left(\delta \int_{\frac{|v-\zeta|}{|v-f(z)|} \leq \frac{1}{2}} \left| \log \frac{|v-\zeta|}{|v-f(z)|} \right| \frac{\delta}{\delta+|v-\zeta|} dv \right. \\
 &\quad + \delta \int_{\frac{1}{2} < \frac{|v-\zeta|}{|v-f(z)|} \leq 2} \left| \log \frac{|v-\zeta|}{|v-f(z)|} \right| \frac{\delta}{\delta+|v-\zeta|} dv \\
 &\quad \left. + \delta \int_{\frac{|v-\zeta|}{|v-f(z)|} > 2} \left| \log \frac{|v-\zeta|}{|v-f(z)|} \right| \frac{\delta}{\delta+|v-\zeta|} dv \right) \\
 &= O\left(\delta^2 \int_{|v-\zeta| \leq |f(z)-\zeta|} \left(1 + \log \frac{|f(z)-\zeta|}{|v-\zeta|}\right) \frac{dv}{\delta+|v-\zeta|} \right. \\
 &\quad + \delta^2 \int_{|v-f(z)| \geq \frac{|f(z)-\zeta|}{3}} \frac{|f(z)-\zeta|}{|v-f(z)|} \frac{2dv}{|v-f(z)|} \\
 &\quad \left. + \delta^2 \int_{|v-f(z)| \leq |f(z)-\zeta|} \left(1 + \log \frac{|f(z)-\zeta|}{|v-f(z)|}\right) \frac{dv}{\delta+|v-f(z)|} \right) \\
 &= O\left(\delta^2 \int_{|u| \leq |f(z)-\zeta|} \left(1 + \log \frac{|f(z)-\zeta|}{|u|}\right) \frac{du}{\delta+u} + \delta^2 |f(z)-\zeta| \int_{\frac{|f(z)-\zeta|}{3}}^2 \frac{dt}{t^2}\right) \\
 &= O\left(\delta^2 |f(z)-\zeta| \int_0^1 \left(1 + \log \frac{1}{t}\right) \frac{dt}{\delta+|f(z)-\zeta|t} + \delta^2\right) \\
 &= O\left(\delta |f(z)-\zeta| \int_0^{\frac{\delta}{|f(z)-\zeta|}} \left(1 + \log \frac{1}{t}\right) dt + \delta^2 \int_{\frac{\delta}{|f(z)-\zeta|}}^1 \left(1 + \log \frac{1}{t}\right) \frac{dt}{t} + \delta^2\right) \\
 &= O\left(\delta^2 \left(1 + \log^2 \frac{|f(z)-\zeta|}{\delta}\right)\right). \quad \square
 \end{aligned}$$

**Proposition 2.5.** For any  $1 < p < 2$ ,

$$\|\mathcal{E}\|_{L^p} = O(\delta^{\frac{5}{4}} + \delta|\eta| + \delta^{\frac{1}{4}}\varepsilon + |\eta|\varepsilon + |\tau|).$$

**Proof.** Recalling the definition of  $h_\varepsilon$ , we split the error term in three parts,

$$\begin{aligned}
 \mathcal{E} &= \underbrace{\frac{1}{h(\zeta)} h(f(z))(e^{W+\tau} - 1)}_{=: \mathcal{E}_1} + \underbrace{\frac{1}{h(\zeta)} \varepsilon k(f(z))e^{W+\tau}}_{=: \mathcal{E}_2} \\
 &\quad + \underbrace{\frac{1}{2\pi h(\zeta)} \int_{\mathbb{S}^1} (h(f(w)) - h(\zeta)) dw}_{=: \mathcal{E}_3}.
 \end{aligned}
 \tag{32}$$

For the first one, by the inequality  $|e^t - 1| \leq (1 + e^t)|t|$  we obtain

$$\begin{aligned}
 \|\mathcal{E}_1\|_{L^p} &= O(\|e^{W+\tau} - 1\|_{L^p}) = O(\|1 + e^{W+\tau}\|_{L^{2p}} \|W + \tau\|_{L^{2p}}) \\
 &= O(\delta^{1+\frac{1}{2p}} + \delta|\eta| + |\tau|).
 \end{aligned}
 \tag{33}$$

Since  $k(1) = 0$ , one has  $k(f(z)) = O(|f(z) - \zeta| + |\eta|)$ , and hence, using Proposition 2.1 and Lemma 2.3,

$$(34) \quad \|\mathcal{E}_2\|_{L^p} = O(\varepsilon \| |f(z) - \zeta| + |\eta| \|_{L^{2p}} \|e^{W+\tau}\|_{L^{2p}}) = O(\varepsilon(\delta^{\frac{1}{2p}} + |\eta|)).$$

Finally, applying Proposition 2.2 we conclude that

$$\mathcal{E}_3 = -\frac{1}{h(\zeta)}(\delta(-\Delta)^{\frac{1}{2}}h(\zeta) + O(\delta^2)) = O(\delta^2 + \delta|\eta|),$$

and the result follows. □

### 3 The projected problem

The results contained in this section follow from [1, Section 3 and Section 4] by simplifying to the case  $K \equiv 0$ . We enunciate here the statements traslated to this context.

Define

$$\mathfrak{C} := \left\{ \zeta \in \partial\mathbb{D} : h(\zeta) \neq 0 \right\}.$$

Notice that, since we are assuming  $h(1) > 0$ , we will have  $\zeta \in \mathfrak{C}$  for any  $\zeta$  close enough to 1 (that is, for  $\delta$  small enough). We consider the Hilbert space

$$\mathbf{H} := \left\{ \phi \in H^1(\mathbb{D}) : \int_{\mathbb{S}^1} \phi = 0 \right\},$$

equipped with the scalar product and the corresponding norm

$$\langle u, v \rangle := \int_{\mathbb{D}} \nabla u \nabla v \quad \text{and} \quad \|u\| := \|\nabla u\|_{L^2(\mathbb{D})} = \left( \int_{\mathbb{D}} |\nabla u|^2 \right)^{\frac{1}{2}}.$$

We start by stating a linear invertibility result. Consider the functions

$$(35) \quad \mathcal{Z}_1(z) := \frac{\langle z, \zeta \rangle}{4h(\zeta)^2}, \quad \mathcal{Z}_2(z) := \frac{\langle z, \zeta^\perp \rangle}{4h(\zeta)^2} :$$

since they are harmonic, by computing their normal derivatives we show that they satisfy

$$(36) \quad (-\Delta)^{\frac{1}{2}} \mathcal{Z}_i = h(\zeta) e^V \mathcal{Z}_i \quad \text{in } \mathbb{S}^1, \quad i = 1, 2.$$

Thus, we can state the following linear invertibility result.

**Theorem 3.1** (see Theorem 3.3 in [1]). *Fix  $p > 1$  and  $\mathfrak{C}' \Subset \mathfrak{C}$ . For any  $\zeta \in \mathfrak{C}'$  and  $\zeta \in L^p(\mathbb{S}^1)$  such that*

$$\int_{\mathbb{S}^1} \zeta = \int_{\mathbb{S}^1} \zeta \mathcal{Z}_i = 0, \quad i = 1, 2,$$

*there exists a unique solution  $\phi \in H^1(\mathbb{D})$  to the problem*

$$\begin{cases} (-\Delta)^{\frac{1}{2}} \phi = h(\zeta) e^{V_\varepsilon} \phi + \zeta & \text{in } \mathbb{S}^1, \\ \int_{\mathbb{S}^1} \phi = \int_{\mathbb{S}^1} \phi \mathcal{Z}_i = 0, & i = 1, 2. \end{cases}$$

*Furthermore*

$$\|\phi\| \leq C_p \|\zeta\|_{L^p},$$

*where the constant  $C_p$  only depends on  $p$  and the compact set  $\mathfrak{C}'$ .*

In order to find a solution of (21), we will solve first the associated projected problem

$$(37) \quad \mathcal{L}_0 \phi = \mathcal{E} + \mathcal{L} \phi + \mathcal{N}(\phi) + c_0 + h(\zeta) e^{\frac{V}{2}} (c_1 \mathcal{Z}_1 + c_2 \mathcal{Z}_2) \quad \text{in } \mathbb{S}^1,$$

with  $\mathcal{L}_0, \mathcal{E}, \mathcal{L}, \mathcal{N}$  defined in (22),  $\mathcal{Z}_1, \mathcal{Z}_2$  given by (35) and  $c_0, c_1, c_2 \in \mathbb{R}$ .

**Lemma 3.2.** *Let  $\phi \in \mathbf{H}$ . Then, for any  $1 < p < \frac{4}{3}$ ,*

$$\|\mathcal{L} \phi\|_{L^p} = O((\delta^{\frac{5}{4}} + \delta |\eta| + \delta^{\frac{1}{4}} \varepsilon + |\eta| \varepsilon + |\tau|) \|\phi\|).$$

**Proof.** By (22) and (32) we can write  $\mathcal{L} = \mathcal{E}_1 + \mathcal{E}_2$  and then

$$\begin{aligned} \|\mathcal{L} \phi\|_{L^p} &= O(\|\mathcal{E}_1 \phi\|_{L^p} + \|\mathcal{E}_2 \phi\|_{L^p}) \\ &= O(\|\mathcal{E}_1\|_{L^{\frac{3}{2}p}} \|\phi\|_{L^{3p}} + \|\mathcal{E}_2\|_{L^{\frac{3}{2}p}} \|\phi\|_{L^{3p}}) \\ &= O((\delta^{\frac{5}{4}} + \delta |\eta| + \delta^{\frac{1}{4}} \varepsilon + |\eta| \varepsilon + |\tau|) \|\phi\|), \end{aligned}$$

where in the last step we have used (33) and (34). □

**Lemma 3.3.** *Let  $\phi, \phi' \in \mathbf{H}$ . For any  $p > 1$ ,*

$$\|\mathcal{N}(\phi) - \mathcal{N}(\phi')\|_{L^p} = O(\|\phi - \phi'\| (\|\phi\| + \|\phi'\|) e^{O(\|\phi\|^2 + \|\phi'\|^2)}).$$

*In particular*

$$\|\mathcal{N}(\phi)\|_{L^p} = O(\|\phi\|^2 e^{O(\|\phi\|^2)}).$$

**Proof.** Using the estimate

$$e^t - t - e^s + s = O(|s - t|(|s| + |t|)(1 + e^{s+t})),$$

Lemma 2.3 and the Moser-Trudinger type inequality in [1, Lemma 3.2] we get

$$\begin{aligned} & \|N(\phi) - N(\phi')\|_{L^p} \\ &= \|h_\varepsilon(f(z))e^{V+W+\tau}(e^\phi - \phi - e^{\phi'} + \phi')\|_{L^p} \\ &= O(\|e^{W+\tau}(|\phi - \phi'|(|\phi| + |\phi'|))(1 + e^{\phi+\phi'})\|_{L^p}) \\ &= O(\|e^{W+\tau}\|_{L^{4p}}\|\phi - \phi'\|_{L^{4p}}(\|\phi\|_{L^{4p}} + \|\phi'\|_{L^{4p}})\|1 + e^{\phi+\phi'}\|_{L^{4p}}) \\ &= O(\|\phi - \phi'\|_{L^{4p}}(\|\phi\|_{L^{4p}} + \|\phi'\|_{L^{4p}})e^{O(\|\phi+\phi'\|^2)}) \\ &= O(\|\phi - \phi'\|(\|\phi\| + \|\phi'\|))e^{O(\|\phi+\phi'\|^2)}. \end{aligned}$$

The second identity follows by just replacing  $\phi' = 0$ . □

**Proposition 3.4.** *Assume  $\delta, |\eta|, |\tau|, \varepsilon \leq \varepsilon_0 \ll 1$ . Then, there exists a unique  $(\phi, c_0, c_1, c_2) \in \mathbf{H} \times \mathbb{R}^3$  such that (37) has a solution, which additionally satisfies*

$$(38) \quad \|\phi\| = O(\delta^{\frac{5}{4}} + \delta|\eta| + \delta^{\frac{1}{4}}\varepsilon + |\eta|\varepsilon + |\tau|).$$

**Proof.** The proof follows replicating the strategy of [1, Proposition 4.3] so we will only highlight the differences. These come from the fact that here the error term  $\mathcal{E}$  has smaller size, which allows us to perform the fixed point argument in a smaller ball.

In particular, following their notation, by Proposition 2.5, Lemmas 3.2 and 3.3 we will have

$$\begin{aligned} (39) \quad \|\mathcal{T}_\xi(\phi)\| &= O(\|\mathcal{E}\|_{L^p} + \|\mathcal{L}\phi\|_{L^p} + \|N(\phi)\|_{L^p}) \\ &= O(\delta^{\frac{5}{4}} + \delta|\eta| + \delta^{\frac{1}{4}}\varepsilon + |\eta|\varepsilon + |\tau| \\ &\quad + (\delta^{\frac{5}{4}} + \delta|\eta| + \delta^{\frac{1}{4}}\varepsilon + |\eta|\varepsilon + |\tau|)\|\phi\| + \|\phi\|^2 e^{O(\|\phi\|^2)}) \\ &= O(\delta^{\frac{5}{4}} + \delta|\eta| + \delta^{\frac{1}{4}}\varepsilon + |\eta|\varepsilon + |\tau| + \|\phi\|^2 e^{O(\|\phi\|^2)}), \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{T}_\xi(\phi) - \mathcal{T}_\xi(\phi')\| &= O(\|\mathcal{L}(\phi - \phi')\|_{L^p} + \|N(\phi) - N(\phi')\|_{L^p}) \\ &= O(\|\phi - \phi'\|(\delta^{\frac{5}{4}} + \delta|\eta| + \delta^{\frac{1}{4}}\varepsilon + |\eta|\varepsilon + |\tau| \\ &\quad + (\|\phi\| + \|\phi'\|)e^{O(\|\phi\|^2 + \|\phi'\|^2)})). \end{aligned}$$

Choosing  $R$  large enough, from (39) we have

$$\|\phi\| \leq R(\delta^{\frac{5}{4}} + \delta|\eta| + \delta^{\frac{1}{4}}\varepsilon + |\eta|\varepsilon + |\tau|) \quad \Rightarrow \quad \|\mathcal{T}_\xi(\phi)\| \leq R(\delta^{\frac{5}{4}} + \delta|\eta| + \delta^{\frac{1}{4}}\varepsilon + |\eta|\varepsilon + |\tau|);$$



proceeding as in [1, Proposition 4.3] we conclude that  $\mathcal{T}_\xi$  is a contraction on a suitable ball and it has a unique fixed point that satisfies (38).  $\square$

### 4 Estimates on the projections

Let  $\phi$  be the solution to the problem (37) provided by Proposition 3.4. Thus, if we prove

$$c_0 = c_1 = c_2 = 0,$$

then  $\phi$  will be a solution of (21). The goal of this section will be to identify the exact expression of these constants.

We begin with multiplying (37) by  $Z_1$  and integrating. Since

$$\int_{\mathbb{S}^1} h(\xi)e^{\frac{\nu}{2}Z_1}Z_2 = \int_{\mathbb{S}^1} h(\xi)e^{\frac{\nu}{2}Z_i}Z_i = 0, \quad i = 1, 2,$$

we deduce that

$$c_1 \int_{\mathbb{S}^1} h(\xi)e^{\frac{\nu}{2}Z_1^2} = \int_{\mathbb{S}^1} \mathcal{L}_0\phi Z_1 - \int_{\mathbb{S}^1} \mathcal{E}Z_1 - \int_{\mathbb{S}^1} \mathcal{L}\phi Z_1 - \int_{\mathbb{S}^1} \mathcal{N}(\phi)Z_1.$$

Integrating by parts and using (36),

$$\int_{\mathbb{S}^1} (\mathcal{L}_0\phi)Z_1 = \int_{\mathbb{S}^1} (\mathcal{L}_0Z_1)\phi = 0,$$

and hence

$$(40) \quad c_1 \int_{\mathbb{S}^1} h(\xi)e^{\frac{\nu}{2}Z_1^2} = - \int_{\mathbb{S}^1} \mathcal{E}Z_1 - \int_{\mathbb{S}^1} \mathcal{L}\phi Z_1 - \int_{\mathbb{S}^1} \mathcal{N}(\phi)Z_1.$$

We proceed analogously with  $Z_2$  to obtain

$$(41) \quad c_2 \int_{\mathbb{S}^1} h(\xi)e^{\frac{\nu}{2}Z_2^2} = - \int_{\mathbb{S}^1} \mathcal{E}Z_2 - \int_{\mathbb{S}^1} \mathcal{L}\phi Z_2 - \int_{\mathbb{S}^1} \mathcal{N}(\phi)Z_2.$$

Notice that, since  $\phi \in \mathbf{H}$ ,

$$\int_{\mathbb{S}^1} \mathcal{L}_0\phi = 0,$$

and integrating (37) gives

$$(42) \quad 2\pi c_0 = - \int_{\mathbb{S}^1} \mathcal{E} - \int_{\mathbb{S}^1} \mathcal{L}\phi - \int_{\mathbb{S}^1} \mathcal{N}(\phi).$$

Let us compute the terms involved.

**Proposition 4.1.**

$$\int_{\mathbb{S}^1} \mathcal{E} \mathcal{Z}_1 = \left( \frac{\pi h''(1)}{2 h(1)^3} - \frac{Q(h)}{2\pi h(1)^4} \right) \delta^2 + \frac{\pi (-\Delta)^{\frac{1}{2}} h'(1)}{2 h(1)^3} \delta \eta + \frac{\pi (-\Delta)^{\frac{1}{2}} k(1)}{2 h(1)^3} \delta \varepsilon + O(\delta^{2+\alpha} + \delta \varepsilon (\delta + |\eta|) + \delta |\eta|^2 + |\tau|^2),$$

with

$$Q(h) := \int_{\mathbb{S}^1 \times \mathbb{S}^1} \log \frac{1}{|z-w|} \frac{h(z) - h(1)}{|z-1|^2} \frac{h(w) - h(1)}{|w-1|^2} dwdz.$$

**Proof.** Proceeding as in [1, Proposition 5.2] we obtain

$$\begin{aligned} \int_{\mathbb{S}^1} \mathcal{E} \mathcal{Z}_1 &= \frac{1}{h(1)} (1 + O(|\eta| + |\tau|)) \int_{\mathbb{S}^1} (h(f(z)) - h(\xi)) \mathcal{Z}_1 \\ &+ \frac{1}{2h(1)} (1 + O(|\eta|)) \int_{\mathbb{S}^1} (h(f(z)) - h(\xi)) W \mathcal{Z}_1 \\ &+ \int_{\mathbb{S}^1} O((|W| + |\tau|)^2 (1 + e^{W+\tau})) \\ &+ \frac{\pi (-\Delta)^{\frac{1}{2}} k(1)}{2 h(1)^3} \delta \varepsilon + O(\delta \varepsilon (\delta + |\eta|)). \end{aligned} \tag{43}$$

To estimate the first term we notice that, due to the assumption  $h \in C^{2,\alpha}(\mathbb{S}^1)$ , the ratio

$$\frac{h(y) - h(\xi) - h'(\xi) \langle y, \xi^\perp \rangle}{|y - \xi|^2}$$

will be of class  $C^{0,\alpha}$  on the whole  $\mathbb{S}^1 \times \mathbb{S}^1$ , hence

$$\begin{aligned} \frac{h(y) - h(\xi) - h'(\xi) \langle y, \xi^\perp \rangle}{|y - \xi|^2} &= \frac{h(y) - h(1) - h'(1) \langle y, \iota \rangle}{|y - 1|^2} + O(|\xi - 1|^\alpha) \\ &= \frac{h(y) - h(1)}{|y - 1|^2} + O(|\eta|^\alpha) \\ &= \frac{h''(1)}{2} + O(|y - 1|^\alpha + |\eta|^\alpha) \end{aligned} \tag{44}$$

Therefore, by making the change of variable  $y = f(z)$  we get

$$\begin{aligned}
 & \int_{\mathbb{S}^1} (h(f(z)) - h(\xi)) \langle z, \xi \rangle dz \\
 &= \int_{\mathbb{S}^1} (h(y) - h(\xi)) \frac{\delta(2 - \delta)}{\delta^2 + (1 - \delta)|y - \xi|^2} \left( -1 + \frac{\delta^2(1 + \langle w, \xi \rangle)}{\delta^2 + (1 - \delta)|y - \xi|^2} \right) dy \\
 &= 2\delta(1 + O(\delta)) \int_{\mathbb{S}^1} \frac{(h(y) - h(\xi)) - h'(\xi) \langle y, \xi^\perp \rangle}{\delta^2 + (1 - \delta)|y - \xi|^2} \left( -1 + \frac{\delta^2(1 + \langle w, \xi \rangle)}{\delta^2 + (1 - \delta)|y - \xi|^2} \right) dy \\
 &= 2\delta(1 + O(\delta)) \left( - \int_{\mathbb{S}^1} \frac{h(y) - h(\xi) - h'(\xi) \langle y, \xi^\perp \rangle}{(1 - \delta)|y - \xi|^2} dy \right. \\
 &\quad \left. + \int_{\mathbb{S}^1} \left( \frac{(h(y) - h(\xi) - h'(\xi) \langle y, \xi^\perp \rangle) \delta^2}{(1 - \delta)|y - \xi|^2 (\delta^2 + (1 - \delta)|y - \xi|^2)} \right. \right. \\
 &\quad \left. \left. + \frac{(h(y) - h(\xi) - h'(\xi) \langle y, \xi^\perp \rangle) \delta^2 (1 + \langle y, \xi \rangle)}{(\delta^2 + (1 - \delta)|y - \xi|^2)^2} \right) dy \right) \\
 &= 2\delta(1 + O(\delta)) \left( \pi(-\Delta)^{\frac{1}{2}} h(\xi) (1 + O(\delta)) \right. \\
 &\quad \left. + \delta^2 \int_{\mathbb{S}^1} \frac{h(y) - h(\xi) - h'(\xi) \langle y, \xi^\perp \rangle}{|y - \xi|^2} \right. \\
 &\quad \left. \times \left( \frac{1}{\delta^2 + (1 - \delta)|y - \xi|^2} + \frac{|y - \xi|^2 (1 + \langle y, \xi \rangle)}{(\delta^2 + (1 - \delta)|y - \xi|^2)^2} \right) dy \right) \tag{45} \\
 &= 2\delta(1 + O(\delta)) \left( \pi(-\Delta)^{\frac{1}{2}} h'(1) \eta + O((\delta + |\eta|)|\eta|) \right. \\
 &\quad \left. + \delta^2 \int_{\mathbb{S}^1} \left( \frac{h''(1)}{2} + O(|y - 1|^\alpha + |\eta|^\alpha) \right) \frac{\delta^2 + (3 - \delta)|y - \xi|^2}{(\delta^2 + (1 - \delta)|y - \xi|^2)^2} dy \right) \\
 &= 2\delta(1 + O(\delta)) \left( \pi(-\Delta)^{\frac{1}{2}} h'(1) \eta + O((\delta + |\eta|)|\eta|) \right. \\
 &\quad \left. + \delta^2 \left( \frac{h''(1)}{2} + O(|\eta|^\alpha) \right) \int_{\mathbb{S}^1} \left( \frac{\delta^2 + 3|y - \xi|^2}{(\delta^2 + |y - \xi|^2)^2} (1 + O(\delta)) + O\left( \frac{|y - 1|^\alpha}{\delta^2 + |y - \xi|^2} \right) \right) dy \right) \\
 &= 2\delta(1 + O(\delta)) \left( \pi(-\Delta)^{\frac{1}{2}} h'(1) \eta + O((\delta + |\eta|)|\eta|) \right. \\
 &\quad \left. + \delta \left( \frac{h''(1)}{2} + O(|\eta|^\alpha) \right) \int_{t=0}^{\left(\frac{1}{\delta}\right)} \left( \frac{1 + 3t^2}{(1 + t^2)^2} (1 + O(\delta)) + O\left( \frac{\delta^\alpha t^\alpha + |\eta|^\alpha}{1 + t^2} \right) \right) dt \right) \\
 &= 2\delta(1 + O(\delta)) \left( \pi \eta (-\Delta)^{\frac{1}{2}} h'(1) + O((\delta + |\eta|)|\eta|) \right. \\
 &\quad \left. + \delta \left( \frac{h''(1)}{2} + O(|\eta|^\alpha) \right) (2\pi + O(\delta^\alpha + |\eta|^\alpha)) \right) \\
 &= 2\pi(-\Delta)^{\frac{1}{2}} h'(1) \delta \eta + 2\pi h''(1) \delta^2 + O(\delta(\delta^{1+\alpha} + |\eta|^2)).
 \end{aligned}$$

To see the second term we use the expression of  $W$  given by (24). Notice first that,

by Proposition 2.1, the lower term can be estimated as

$$\begin{aligned}
 & \int_{\mathbb{S}^1} (h(f(z)) - h(\xi)) \\
 &= O\left(\delta \log \frac{1}{\delta} (\delta + |\eta|) \left(1 + \log^2 \frac{1}{\delta + |z + \xi|}\right)\right) \mathcal{Z}_1 \\
 (46) \quad &= O\left(\delta \log \frac{1}{\delta} (\delta + |\eta|) \int_{\mathbb{S}^1} \frac{\delta}{\delta + |z + \xi|} \left(1 + \log^2 \frac{1}{\delta + |z + \xi|}\right) dz\right) \\
 &= O\left(\delta \log \frac{1}{\delta} (\delta + |\eta|) \left(\int_{|z+\xi| \leq \delta} \left(1 + \log^2 \frac{1}{\delta}\right) dz \right. \right. \\
 &\quad \left. \left. + \int_{|z+\xi| > \delta} \frac{\delta}{|z + \xi|} \left(1 + \log^2 \frac{1}{|z + \xi|}\right) dz\right)\right) \\
 &= O\left(\delta^2 \log^4 \frac{1}{\delta} (\delta + |\eta|)\right).
 \end{aligned}$$

On the other hand, we can approximate  $\langle z, \xi \rangle$  with  $-1$  since

$$\begin{aligned}
 \int_{\mathbb{S}^1} (h(f(z)) - h(\xi)) W(z) (\langle z, \xi \rangle + 1) dz &= O\left(\|W\|_{L^2} \left\| \frac{\delta}{\delta + |z + \xi|} |z + \xi|^2 \right\|_{L^2}\right) \\
 &= O(\delta \|W\|_{L^2}) \\
 &= O(\delta^{\frac{5}{2}} + \delta^2 |\eta|);
 \end{aligned}$$

therefore the main order term will be given by

$$\begin{aligned}
 & \int_{\mathbb{S}^1} (h(f(z)) - h(\xi)) \left( \frac{2}{\pi h(\xi)} \delta \int_{\mathbb{S}^1} \log \frac{|v - \xi|}{|v - f(z)|} \frac{h(v) - h(\xi) - h'(\xi) \langle v, \xi^\perp \rangle}{|v - \xi|^2} dv \right) dz \\
 &= \frac{2}{\pi h(\xi)} \delta \int_{\mathbb{S}^1 \times \mathbb{S}^1} (h(y) - h(\xi) - h'(\xi) \langle y, \xi^\perp \rangle + O(|\eta| |y - \xi|)) \frac{\delta(2 - \delta)}{\delta^2 + (1 - \delta) |y - \xi|^2} \\
 &\quad \times \log \frac{|v - \xi|}{|v - y|} \frac{h(v) - h(\xi) - h'(\xi) \langle v, \xi^\perp \rangle}{|v - \xi|^2} dv dy \\
 &= \frac{4}{\pi h(\xi)} \delta^2 \int_{\mathbb{S}^1 \times \mathbb{S}^1} \log \frac{|v - \xi|}{|v - y|} \left( \frac{h(y) - h(\xi) - h'(\xi) \langle y, \xi^\perp \rangle}{|y - \xi|^2} + O\left(\frac{|\eta| |y - \xi|}{\delta^2 + |y - \xi|^2}\right) \right. \\
 &\quad \left. + O\left(\frac{\delta}{\delta + |y + \xi|}\right) \right) \cdot \frac{h(v) - h(\xi) - h'(\xi) \langle v, \xi^\perp \rangle}{|v - \xi|^2} dv dy \\
 &= \frac{4}{\pi h(1)} (1 + O(|\eta|)) \delta^2 \\
 &\quad \times \int_{\mathbb{S}^1 \times \mathbb{S}^1} \log \frac{|v - \xi|}{|v - y|} \frac{h(y) - h(1)}{|y - 1|^2} \frac{h(v) - h(1)}{|v - 1|^2} (1 + O(|\eta|^\alpha))^2 dv dy \\
 &\quad + O\left(\delta^2 \int_{\mathbb{S}^1 \times \mathbb{S}^1} \left| \log \frac{|v - \xi|}{|v - y|} \right| \left( |\eta| + \frac{\delta}{\delta + |y + \xi|} + \frac{|\eta| |y - \xi|}{\delta^2 + |y - \xi|^2} \right) \right)
 \end{aligned}$$

and to continue, we have that

$$\begin{aligned}
 & \int_{\mathbb{S}^1} (h(f(z)) - h(\xi)) \left( \frac{2}{\pi h(\xi)} \delta \int_{\mathbb{S}^1} \log \frac{|v - \xi|}{|v - f(z)|} \frac{h(v) - h(\xi) - h'(\xi) \langle v, \xi^\perp \rangle}{|v - \xi|^2} dv \right) dz \\
 &= \frac{4}{\pi h(1)} (1 + O(|\eta|^\alpha)) \delta^2 \int_{\mathbb{S}^1 \times \mathbb{S}^1} \log \frac{|v - \xi|}{|v - y|} \frac{h(v) - h(1) h(y) - h(1)}{|v - 1|^2 |y - 1|^2} dv dy \\
 &\quad + O \left( \delta^2 \int_{\mathbb{S}^1 \times \mathbb{S}^1} \left| \log \frac{|v - \xi|}{|v - y|} \right| \left( |\eta| + \frac{\delta}{\delta + |y + \xi|} + \frac{|\eta| |y - \xi|}{\delta^2 + |y - \xi|^2} \right) \right) \\
 &= \frac{4}{\pi h(1)} (1 + O(|\eta|^\alpha)) \delta^2 \int_{\mathbb{S}^1 \times \mathbb{S}^1} \log \frac{1}{|v - y|} \frac{h(v) - h(1) h(y) - h(1)}{|v - 1|^2 |y - 1|^2} dv dy \\
 &\quad + \frac{4}{\pi h(1)} (1 + O(|\eta|^\alpha)) \delta^2 \int_{\mathbb{S}^1} \log |v - \xi| \frac{h(v) - h(1)}{|v - 1|^2} dv \int_{\mathbb{S}^1} \frac{h(y) - h(1)}{|y - 1|^2} dy \\
 (47) \quad &+ O \left( \delta^2 \int_{\mathbb{S}^1 \times \mathbb{S}^1} \left( 2 + \log \frac{1}{|v - \xi|} + \log \frac{1}{|v - y|} \right) \right. \\
 &\quad \left. \times \left( |\eta| + \frac{\delta}{\delta + |y + \xi|} + \frac{|\eta| |y - \xi|}{\delta^2 + |y - \xi|^2} \right) \right) \\
 &= \frac{4}{\pi h(1)} (1 + O(|\eta|^\alpha)) \delta^2 \int_{\mathbb{S}^1 \times \mathbb{S}^1} \log \frac{1}{|v - y|} \frac{h(v) - h(1) h(y) - h(1)}{|v - 1|^2 |y - 1|^2} dv dy \\
 &\quad + O \left( \delta^2 \left( |\eta| + |\eta| \log \frac{1}{\delta} + \int_{\mathbb{S}^1 \times \mathbb{S}^1} \left( 1 + \log \frac{1}{|u - \xi|} \right) \frac{\delta}{\delta + |y + \xi|} dudv \right) \right) \\
 &= \frac{4}{\pi h(1)} \delta^2 \int_{\mathbb{S}^1 \times \mathbb{S}^1} \log \frac{1}{|v - y|} \frac{h(v) - h(1) h(y) - h(1)}{|v - 1|^2 |y - 1|^2} dv dy \\
 &\quad + O \left( \delta^2 |\eta|^\alpha + \delta^2 |\eta| \log \frac{1}{\delta} + \delta^3 \log \frac{1}{\delta} \right),
 \end{aligned}$$

where we have used estimate (31) and the fact that  $h'(1) = (-\Delta)^{\frac{1}{2}} h(1) = 0$ . Finally, by Lemma 2.3,

$$\begin{aligned}
 (48) \quad & \int_{\mathbb{S}^1} O((|W| + |\tau|)^2 (1 + e^{W+\tau})) = O(\| |W| + |\tau| \|_{L^2}^2 \| 1 + e^{W+\tau} \|_{L^2}) \\
 &= O(\| W \|_{L^4}^2 + |\tau|^2) \\
 &= O(\delta^{\frac{5}{2}} + \delta^2 |\eta|^2 + |\tau|^2).
 \end{aligned}$$

Substituting (45)–(48) into (43) the result follows. □

**Proposition 4.2.**

$$\begin{aligned}
 \int_{\mathbb{S}^1} \mathcal{E} \mathcal{Z}_2 = & -\frac{\pi}{2} \frac{(-\Delta)^{\frac{1}{2}} h'(1)}{h(1)^3} \delta^2 + \frac{\pi}{2} \frac{h''(1)}{h(1)^3} \delta \eta + \frac{\pi}{2} \frac{k'(1)}{h(1)^3} \delta \varepsilon \\
 & + O(\delta^{\frac{5}{2}} + \delta \varepsilon (\delta + |\eta|) + \delta |\eta|^{1+\alpha} + |\tau|^2).
 \end{aligned}$$

**Proof.** Following [1, Proposition 5.4] we can write the integral as

$$\begin{aligned}
 \int_{\mathbb{S}^1} \mathcal{E} \mathcal{Z}_2 &= \frac{1}{h(1)}(1 + O(|\eta| + |\tau|)) \int_{\mathbb{S}^1} (h(f(z)) - h(\xi)) \mathcal{Z}_2 \\
 (49) \quad &+ \frac{1}{2h(1)}(1 + O(|\eta|)) \int_{\mathbb{S}^1} (h(f(z)) - h(\xi)) W \mathcal{Z}_2 + \frac{\pi}{2} \frac{k'(1)}{h(1)^3} \delta \mathcal{E} \\
 &+ O(\delta^{\frac{5}{2}} + (\delta + |\eta|)\delta \mathcal{E} + |\tau|^2).
 \end{aligned}$$

To estimate the first term we make the change of variable  $y = f(z)$  and we integrate by parts to obtain

$$\begin{aligned}
 &\int_{\mathbb{S}^1} (h(f(z)) - h(\xi)) \langle z, \xi^\perp \rangle dz \\
 &= \int_{\mathbb{S}^1} (h(y) - h(\xi)) \frac{\delta(2 - \delta)}{\delta^2 + (1 - \delta)|y - \xi|^2} \frac{\delta(2 - \delta) \langle y, \xi^\perp \rangle}{\delta^2 + (1 - \delta)|y - \xi|^2} dy \\
 &= 4\delta^2(1 + O(\delta)) \int_{\mathbb{S}^1} (h(y) - h(\xi)) \left( - \frac{1}{2(1 - \delta)(\delta^2 + (1 - \delta)|y - \xi|^2)} \right)' dy \\
 &= -4\delta^2(1 + O(\delta)) \int_{\mathbb{S}^1} h'(y) \left( - \frac{1}{2(1 - \delta)(\delta^2 + (1 - \delta)|y - \xi|^2)} \right) dy \\
 &= 2\delta^2(1 + O(\delta)) \left( \int_{\mathbb{S}^1} \frac{h'(y) - h'(\xi)}{\delta^2 + (1 - \delta)|y - \xi|^2} dy + h'(\xi) \int_{\mathbb{S}^1} \frac{dy}{(\delta^2 + (1 - \delta)|y - \xi|^2)} \right) \\
 &= 2\delta^2(1 + O(\delta)) \left( -\pi(-\Delta)^{\frac{1}{2}} h'(\xi) + O(\delta) + h'(\xi) \int_{\mathbb{S}^1} \frac{dy}{(\delta^2 + |y - \xi|^2)} (1 + O(\delta)) \right) \\
 &= 2\delta^2(1 + O(\delta)) \left( -\pi(-\Delta)^{\frac{1}{2}} h'(1) + O(\delta + |\eta|^\alpha) \right. \\
 &\quad \left. + (\eta h''(1) + O(|\eta|^{1+\alpha})) \int_{\mathbb{S}^1} \frac{dy}{(\delta^2 + |y - \xi|^2)} (1 + O(\delta)) \right) \\
 &= 2\delta^2(1 + O(\delta)) \left( -\pi(-\Delta)^{\frac{1}{2}} h'(1) + O(\delta + |\eta|^\alpha) \right. \\
 &\quad \left. + (\eta h''(1) + O(|\eta|^{1+\alpha})) \frac{1}{\delta} \int_{t=O(\frac{1}{\delta})} \frac{dt}{1+t^2} (1 + O(\delta)) \right) \\
 &= 2\delta^2(1 + O(\delta)) \\
 &\quad \times \left( -\pi(-\Delta)^{\frac{1}{2}} h'(1) + O(\delta + |\eta|^\alpha) + (\eta h''(1) + O(|\eta|^{1+\alpha})) \left( \frac{\pi}{\delta} + O(1) \right) \right) \\
 &= -2\pi(-\Delta)^{\frac{1}{2}} h'(1) \delta^2 + 2\pi h''(1) \delta \eta + O(\delta(\delta^2 + |\eta|^{1+\alpha})),
 \end{aligned}$$

where we applied that  $h \in C^{2,\alpha}(\mathbb{S}^1)$ , and hence  $(-\Delta)^{\frac{1}{2}} h \in C^{1,\alpha}(\mathbb{S}^1)$ .

Using Lemma 2.3 and Proposition 2.1 we can estimate the second integral as

$$\int_{\mathbb{S}^1} (h(f(z)) - h(\xi)) W(z) \langle z, \xi^\perp \rangle dz = O\left( \|W\|_{L^2} \left\| \frac{\delta}{\delta + |z + \xi|} |z + \xi| \right\|_{L^2} \right) = O(\delta^{\frac{5}{2}} + \delta^2 |\eta|),$$

where we have used the fact  $\langle z, \xi^\perp \rangle = O(|z + \xi|)$ .

Replacing in (49) we conclude. □

**Proposition 4.3.**

$$\int_{\mathbb{S}^1} \varepsilon = \tau(\pi h(1) + O(|\eta| + |\tau|)) + O\left(\delta^2 \log^2 \frac{1}{\delta} + \delta|\eta| + \delta \log \frac{1}{\delta} \varepsilon\right).$$

**Proof.** The result follows from [1, Proposition 5.6] by fixing  $K \equiv 0$ . □

We can finally write the exact expressions of the constants  $c_0, c_1$  and  $c_2$ .

**Corollary 4.4.** *The constants  $c_0, c_1, c_2$  in problem (37) satisfy:*

$$\begin{aligned} c_1 &= -16h(1)\delta \left( \underbrace{\delta(h''(1) - \frac{Q(h)}{\pi^2 h(1)})}_{=:a_{11}} + \underbrace{\eta(-\Delta)^{\frac{1}{2}}h'(1)}_{=:a_{12}} + \underbrace{\varepsilon(-\Delta)^{\frac{1}{2}}k(1)}_{=:b_1} \right) \\ &\quad + O(\delta^{2+\alpha} + \delta\varepsilon(\delta + |\eta|) + \delta|\eta|^2 + |\tau|^2), \\ c_2 &= -16h(1)\delta \left( \underbrace{\delta(-(-\Delta)^{\frac{1}{2}}h'(1))}_{=:a_{21}} + \underbrace{\eta h''(1)}_{=:a_{22}} + \underbrace{\varepsilon k'(1)}_{=:b_2} \right) \\ &\quad + O(\delta^{\frac{5}{2}} + \delta\varepsilon(\delta + |\eta|) + \delta|\eta|^{1+\alpha} + |\tau|^2), \\ c_0 &= -h(1)\tau + O\left(|\eta||\tau| + |\tau|^2 + \delta^2 \log^2 \frac{1}{\delta} + \delta|\eta| + \delta \log \frac{1}{\delta} \varepsilon\right) \end{aligned}$$

where  $Q$  is defined in (11).

**Proof.** By [1, Proposition 7.9] with  $K \equiv 0$  we know that

$$(50) \quad \int_{\mathbb{S}^1} h(\xi)e^{\frac{\nu}{2}Z_1^2} = \frac{\pi}{16h(1)^4}(1 + O(|\eta|)).$$

Furthermore, by Lemmas 3.2 and 3.3, for  $1 < p < \frac{4}{3}$ ,

$$\begin{aligned} \int_{\mathbb{S}^1} \mathcal{L}\phi Z_1 + \int_{\mathbb{S}^1} \mathcal{N}(\phi)Z_1 &= O(\|\mathcal{L}\phi\|_{L^p} + \|\mathcal{N}(\phi)\|_{L^p}) \\ &= O((\delta^{\frac{5}{4}} + \delta|\eta| + \delta^{\frac{1}{4}}\varepsilon + |\eta|\varepsilon + |\tau|)\|\phi\| + \|\phi\|^2 e^{O(\|\phi\|^2)}) \\ &= O(\delta^{\frac{5}{2}} + \delta^2|\eta|^2 + \delta^{\frac{1}{2}}\varepsilon^2 + |\eta|^2\varepsilon^2 + |\tau|^2), \end{aligned}$$

where in the last step we have used Proposition 3.4. Applying (50), (51) and Proposition 4.1 in (40) we obtain the expression for  $c_1$ .

The identities for  $c_2$  and  $c_0$  similarly follow from (41) and (42) using Propositions 4.2 and 4.3 respectively. □

## 5 The finite dimensional reduction: Proof of Theorem 1.1

Let  $\delta, |\eta|, |\tau|, \varepsilon$  be small enough so that Proposition 3.4 can be applied to find a solution to (37). If we choose  $\delta, \eta, \tau$  in such a way that  $c_0 = c_1 = c_2 = 0$ , then  $\phi$  also solves (21), and hence we get a solution to the problem (8).

We study first the cases of  $c_1$  and  $c_2$ . From Corollary 4.4 and the estimates on  $\|\phi\|$ , assuming that

$$\eta = s\varepsilon \quad \text{and} \quad \delta = d\varepsilon \quad \text{with } d > 0 \text{ if } \varepsilon > 0 \text{ or } d < 0 \text{ if } \varepsilon < 0,$$

we have that  $c_1 = c_2 = 0$  if

$$\begin{cases} a_{11}d + a_{12}s + b_1 + o_\varepsilon(1) = 0, \\ a_{21}d + a_{22}s + b_2 + o_\varepsilon(1) = 0. \end{cases}$$

This system can be rewritten as  $\tilde{\mathfrak{F}}_\varepsilon(d, s) = \tilde{\mathfrak{F}}_0(d, s) + o_\varepsilon(1) = 0$  where  $\tilde{\mathfrak{F}}_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by

$$\tilde{\mathfrak{F}}_0(d, s) := \mathfrak{A} \begin{pmatrix} d \\ s \end{pmatrix} + \mathfrak{B}, \quad \text{with } \mathfrak{A} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } \mathfrak{B} := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Therefore, if

$$\det \mathfrak{A} = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0, \quad \text{i.e.,} \quad a_{11}a_{22} - a_{21}a_{12} \neq 0 \text{ (see (10)),}$$

there exists a unique  $(d_0, s_0) \in \mathbb{R}^2$  such that  $\tilde{\mathfrak{F}}_0(d_0, s_0) = 0$  with  $d_0 \neq 0$  if

$$\det \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix} \neq 0, \quad \text{i.e.,} \quad a_{22}b_1 - a_{12}b_2 \neq 0 \text{ (see (12)).}$$

Moreover, the Brouwer degree of  $\tilde{\mathfrak{F}}_\varepsilon$  is not zero and since  $\tilde{\mathfrak{F}}_\varepsilon \rightarrow \tilde{\mathfrak{F}}_0$  uniformly on compact sets of  $\mathbb{R} \times \mathbb{R} \setminus \{0\}$ , there exists  $(d_\varepsilon, s_\varepsilon) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$  such that  $\tilde{\mathfrak{F}}_\varepsilon(d_\varepsilon, s_\varepsilon) = 0$  with  $(d_\varepsilon, s_\varepsilon) \rightarrow (d_0, s_0)$  as  $\varepsilon \rightarrow 0$ .

Once  $d_\varepsilon, s_\varepsilon$  are fixed, the existence of a  $\tau = \tau_\varepsilon$  so that

$$-h(1)\tau + O\left(|\eta||\tau| + |\tau|^2 + \delta^2 \log^2 \frac{1}{\delta} + \delta|\eta| + \delta \log \frac{1}{\delta} \varepsilon\right) = 0$$

is immediate, and we conclude that  $c_0 = 0$ . Notice that  $\tau_\varepsilon = o_\varepsilon(\varepsilon)$ . This finishes the proof of the existence of a solution.

Thanks to the estimates on  $\mathcal{E}, \mathcal{L}, \mathcal{N}$  and  $\|\phi\|$ , from (21) we get

$$\|(-\Delta)^{\frac{1}{2}}\phi\|_{L^p} = o_\varepsilon(1) \quad \text{for some } p > 1,$$

and therefore  $\|\phi\|_{L^\infty} = o_\varepsilon(1)$ . Moreover, since  $V(f^{-1}(z))$  and  $W(f^{-1}(z))$  both concentrate at  $\xi = 1$ , we conclude that the solution

$$u = V(f^{-1}(z)) + W(f^{-1}(z)) + \tau + \phi(f^{-1}(z))$$

concentrates at  $\xi = 1$ . This ends the proof of Theorem 1.1.



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