FOURIER ORTHOGONAL SERIES ON A PARABOLOID

By

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Abstract. We study the orthogonal structure and Fourier orthogonal series on the surface of a paraboloid

$$\mathbb{V}_0^{d+1} = \{(x,t) : \|x\| = \sqrt{t},\ x \in \mathbb{R}^d,\ 0 \le t < 1\}.$$

The reproducing kernels of the orthogonal polynomials with respect to $t^\beta(1-t)^\gamma$ on $\mathbb{V}_0^{d+1}$ are related to the reproducing kernels of the Jacobi polynomials on the parabolic domain $\{(x_1,x_2) : x_1^2 \le x_2 \le 1\}$ in $\mathbb{R}^2$. This connection serves as an essential tool for our study of the Fourier orthogonal series on the surface of the paraboloid, which allow us, in particular, to study the convergence of the Cesàro means on the surface. Analogous results are also established for the solid paraboloid bounded by $\mathbb{V}_0^{d+1}$ and the hyperplane $t = 1$.

1 Introduction

The Laplace series, so named the generalized Fourier series in spherical harmonics on the unit sphere, has been extensively studied. One essential ingredient for understanding these series is the addition formula for spherical harmonics, which states that the reproducing kernel of the orthogonal projection operator from $L^2(\mathbb{S}^{d-1})$ onto the space of spherical harmonics of degree n can be written as $Z_n(\langle \xi, \eta \rangle)$, where Z_n is a Gegenbauer polynomial of degree n in one variable and $\langle \xi, \eta \rangle$ is the Euclidean inner product of $\xi, \eta \in \mathbb{S}^{d-1}$. Because of this closed formula of the reproducing kernels, much of the study of the Laplace series can be reduced to the study of the Fourier–Gegenbauer series of one variable [1, 5, 14].

The above narrative turns out to be the prototype for Fourier orthogonal series domains in higher dimensions. In the past two decades, starting from the addition formula for classical orthogonal polynomials on the unit ball [21], closed form formulas for reproducing kernels of orthogonal polynomials have been discovered

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for several regular domains, including the unit ball, regular simplex, cylinder, as well as the unit sphere with inner product defined by weighted integrals, which makes study of the Fourier orthogonal series on these domain feasible; see, for example, [4, 3, 5, 6, 7, 10, 11, 13, 18, 19, 20, 21, 22] and their references. For unbounded classical domains, we refer to [16] as well as to [2, 17] for references on more recent works, which however require techniques beyond our narrative.

To step beyond the regular domains, we recently started to analyze orthogonal structure on quadratic surfaces of revolutions other than the unit sphere as well as on domains bounded by such quadratic surfaces. Let $\mathbb{V}_0^{d+1}$ be a quadratic surface in $\mathbb{R}^{d+1}$, parametrized in (x,t), $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$, that is a surface of revolution around the t axis. We consider orthogonal structure defined by the inner product

$$\langle f, g\rangle = \int_{\mathbb{V}_0^{d+1}} f(x,t)g(x,t)\varpi(t)\mathrm{d}\sigma(x,t),$$

where $\varpi(t)$ is a weight function and $\mathrm{d}\sigma$ is the Lebesgue measure of $\mathbb{V}_0^{d+1}$. Taking the cue of spherical harmonics, we look for families of orthogonal polynomials that share two characteristic properties of spherical harmonics, one is an addition formula and the other is the existence of a second order partial differential operator that has orthogonal polynomials as eigenfunctions with eigenvalues depending only on the degree of the polynomials, which is an analogue of the Laplace–Beltrami operator on the unit sphere.

In [23] we studied orthogonal polynomials on the surface of the cone

$$\mathbb{V}_0^{d+1} = \{(x,t) : \|x\| = t, 0 \le t \le b, x \in \mathbb{R}^d\},$$

where $b = 1$ or $b = \infty$. Two families of orthogonal polynomials are identified as eigenfunctions of a differential operator, the Laguerre polynomials on the cone with $b = +\infty$ and the Jacobi polynomials on the cone with $b = 1$. The Jacobi family is shown to possess an addition formula, which is utilized to carry out a preliminary study of the Fourier orthogonal series on the surface of the cone. Moreover, analogous results are also established on the domain bounded by the surface of the cone, together with the hyperplane $t = 1$ when $b = 1$. In [24], we considered the orthogonal structure on the surface of the hyperboloid

$$\mathbb{V}_0^{d+1} = \{(x,t) : \|x\|^2 = c^2(t^2 - \varrho^2), x \in \mathbb{R}^d, \varrho \le |t| \le b\},$$

where $\varrho \ge 0$ and $b = 1+\varrho$ or ∞, which degenerates to the double cone when $\varrho = 0$. In this case the weight function ϖ is an even function. We again identified two families of orthogonal polynomials, the Hermite polynomials on the hyperboloid with $b = \infty$ and the Gegenbauer polynomials on the hyperboloid with $b = 1 + \rho$.

However, for these two families, only those polynomials that are even in t are eigenfunctions of a differential operator. Furthermore, the addition formula holds for the Gegenbauer polynomials on the hyperboloid that are even in t. These results are used to carry out a study of the Fourier orthogonal series for functions that are even in t over the hyperboloid in [24], and analogous results are also established on the domain bounded by the surface of the hyperboloid, together with the hyperplane $t = 1$ when $b = 1 + \varrho$.

In the present paper, we study the orthogonal structure on a paraboloid of revolution, which turns out to be very different from those on the cone and on the hyperboloid. We shall consider the surface of a paraboloid, defined by

$$\mathbb{V}_0^{d+1} = \{(x,t) : \|x\| = \sqrt{t},\ x \in \mathbb{R}^d,\ 0 \le t \le b\},$$

as well as the solid paraboloid $\mathbb{V}^{d+1}$ bounded by $\mathbb{V}_0^{d+1}$ and the hyperplane $t = b$. For our study, we shall consider only $b = 1$, or the compact case, since there is not as much that makes the case $b = \infty$ standing out for paraboloids.

On the surface of the paraboloid, we consider a family of orthogonal polynomials with respect to the weight function $\varpi(t) = t^\beta(1-t)^\gamma$, which shall be called the Jacobi polynomials on the paraboloid. These polynomials will be shown to be eigenfunctions of a second order differential operator but with the eigenvalues depending on two indices, the degree of the polynomials and another index that depends on the particular orthogonal basis, in contrast to the cone and the hyperboloid for which the corresponding eigenvalue property is independent of the choice of orthogonal bases. The Jacobi polynomials on the paraboloid also do not possess an explicit addition formula. What they do have is a connection to an orthogonal structure on the parabolic domain $\mathbb{U} = \{(x_1, x_2) : x_1^2 \le x_2 \le 1\}$ in $\mathbb{R}^2$, bounded by the parabola $x_2 = x_1^2$ and the line $x_2 = 1$. A family of orthogonal polynomials on $\mathbb{U}$, called the Jacobi polynomials on the parabolic domain, was first considered in [8] and shown to satisfy a product formula in [9]. The latter was used to study the Fourier orthogonal series in [3], where the essential ingredient is an addition formula that holds when one argument of the reproducing kernel is at the corner $(1, 1)$ of the domain. Our essential realization is that the reproducing kernels of the orthogonal polynomials on the paraboloid can be expressed in terms of the reproducing kernel on the parabola domain, which provides the tool for studying the Fourier orthogonal series on the paraboloid. In particular, it allows us to study the convergence of the Cesàro means of the series. We shall also show that the connection to the structure on $\mathbb{U}$ also extends to the solid paraboloid, which allows us to carry out our study of the Fourier orthogonal series on the solid paraboloid.

The paper is organized as follows. In the next section, we review the orthogonal structure on the parabolic domain, where enough details will be provided to prepare for their usage in the latter sections. The orthogonal structure and the Fourier series on the surface of the paraboloid will be discussed in Section 3, and analogous results on the solid paraboloid will be discussed in Section 4.

2 Orthogonal polynomials on a parabolic domain

As mentioned in the introduction, our development on $\mathbb{V}^{d+1}$ depends heavily on what is known on the parabolic domain

$$\mathbb{U} = \{x \in \mathbb{R}^2 : x_1^2 \le x_2 \le 1\},$$

bounded by the parabola $x_2 = x_1^2$ and the line $x_2 = 1$, which we review in this section. For $a > -1$ and $b > -\frac{1}{2}$, we define the weight function

$$U_{a,b}(x) = (1-x_2)^a (x_2 - x_1^2)^{b-\frac{1}{2}}$$

on $\mathbb{U}$ and consider orthogonal polynomials with respect to the inner product

$$\langle f, g\rangle_{\mathbb{U}} := \mathbf{d}_{a,b} \int_{\mathbb{U}} f(x) g(x) U_{a,b}(x) \mathrm{d}x, \tag{2.1}$$

where the normalization constant $\mathbf{d}_{a,b}$ is chosen so that $\langle 1, 1\rangle_{a,b} = 1$ and its value can be verified by writing the integral over $\mathbb{U}$ as

$$\int_{\mathbb{U}} f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 = \int_0^1 \int_{-1}^1 f(u\sqrt{x_2}, x_2)\sqrt{x_2} \mathrm{d}u \mathrm{d}x_2. \tag{2.2}$$

An orthogonal basis for this inner product was defined in [8] in terms of the Jacobi polynomials. The Jacobi polynomials $P_n^{(\alpha,\beta)}$ are defined by

$$P_n^{(\alpha,\beta)}(t) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-t}{2}\right),$$

for $\alpha, \beta > -1$. They are also orthogonal with respect to the weight function $w_{\alpha,\beta}(x) := (1-x)^\alpha (1+x)^\beta$,

$$c'_{\alpha,\beta} \int_{-1}^1 P_n^{(\alpha,\beta)}(t) P_m^{(\alpha,\beta)}(t) w_{\alpha,\beta}(t) \mathrm{d}t = h_n^{(\alpha,\beta)} \delta_{n,m},$$

where $c'_{\alpha,\beta} = 2^{-\alpha-\beta-1} c_{\alpha,\beta}$ with

$$c_{\alpha,\beta} := \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \quad \text{and} \quad h_n^{(\alpha,\beta)} := \frac{(\alpha+1)_n (\beta+1)_n (\alpha+\beta+n+1)}{n!(\alpha+\beta+2)_n (\alpha+\beta+2n+1)}. \tag{2.3}$$

In terms of $c_{\alpha,\beta}$, the constant $\mathbf{d}_{a,b}$ in (2.1) is given by $\mathbf{d}_{a,b} = c_{b-\frac{1}{2}, b-\frac{1}{2}} c_{b,a}$.

Let $\mathcal{V}_n(\mathbb{U}, U_{a,b})$, $n = 0, 1, \ldots$, be the space of orthogonal polynomials of degree n with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathbb{U}}$ on the parabolic domain. Then $\dim \mathcal{V}_n(\mathbb{U}, U_{a,b}) = n + 1$. The orthogonal basis for $\mathcal{V}_n(\mathbb{U}, U_{a,b})$ given in [8] consists of polynomials

$$P_{k,n}^{a,b}(x_1, x_2) = P_{n-k}^{(b+k,a)}(1 - 2x_2) x_2^{\frac{k}{2}} P_k^{(b-\frac{1}{2}, b-\frac{1}{2})}\left(\frac{x_1}{\sqrt{x_2}}\right), \quad 0 \le k \le n. \tag{2.4}$$

Their orthogonality can be verified by using (2.2) and so are their L^2 norm. In particular, in terms of the quantities in (2.3),

$$h_{k,n}^{a,b} = \mathbf{d}_{a,b} \int_{\Omega} |P_{k,n}^{a,b}(x)|^2 U_{a,b}(x) \mathrm{d}x = \frac{c_{b,a}}{c_{b+k,a}} h_{n-k}^{(b+k,a)} h_k^{(b-\frac{1}{2}, b-\frac{1}{2})}. \tag{2.5}$$

The polynomials $P_{k,n}^{(\alpha,\beta)}$ satisfy a product formula due to Koornwinder and Schwartz [9]. The formula is rather complicated and takes the following form:

Theorem 2.1. *Let $a \ge b \ge 0$. For $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{U}$,*

$$\begin{aligned} & P_{n,k}^{a,b}(x_1, x_2) P_{n,k}^{a,b}(y_1, y_2) \\ & \quad = P_{n,k}^{a,b}(1, 1) \int_{[0,1]\times[0,\pi]^3} P_{n,k}^{a,b}(\xi_1(x, y; r, \psi_1), \xi_2(x, y; r, \psi)) \mathrm{d}m_{a,b}(r, \psi), \end{aligned} \tag{2.6}$$

where $\psi = (\psi_1, \psi_2, \psi_3)$ and $\mathrm{d}m_{a,b}(r, \psi)$ is a probability measure given by

$$\begin{aligned} & \mathrm{d}m_{a,b}(r, \psi) \\ & \quad = c_{a,b}(1 - r^2)^{a-b-1} r^{2b+1} (\sin \psi_3)^{2b-1} (\sin \psi_2)^{2b-1} (\sin \psi_1)^{2b} \mathrm{d}r \mathrm{d}\psi_1 \, \mathrm{d}\psi_2 \, \mathrm{d}\psi_3. \end{aligned}$$

The complication of the product formula lies in the functions $\xi_1(x, y; r, \psi_1)$ and $\xi_2(x, y; r, \psi)$, which are explicitly given by fairly involved formulas, and they satisfy $\xi = (\xi_1, \xi_2) \in \mathbb{U}$. Since we do not need their explicit formulas, we will not state them here but refer to [9].

For $f \in L^2(\mathbb{U}; U_{a,b})$, its Fourier orthogonal series is defined by

$$f = \sum_{n=0}^{\infty} \operatorname{proj}_n(U_{a,b}; f) \quad \text{with } \operatorname{proj}_n(U_{a,b}; f) = \sum_{k=0}^{n} \frac{\langle f, P_{k,n}^{a,b} \rangle_{\mathbb{U}}}{h_{k,n}^{a,b}} P_{k,n}^{a,b}.$$

The operator $\operatorname{proj}_n(U_{a,b}) : L^2(\mathbb{U}; U_{a,b}) \mapsto \mathcal{V}_n(\mathbb{U}, U_{a,b})$ is the orthogonal projection operator, which can be written as an integral

$$\operatorname{proj}_n(U_{a,b}; f, x) = \mathbf{d}_{a,b} \int_{\mathbb{U}} f(y) \mathbf{P}_n(U_{a,b}; x, y) U_{a,b}(y) \mathrm{d}y$$

in terms of the reproducing kernel $\mathbf{P}_n(U_{a,b})$ of $\mathcal{V}_n(\mathbb{U}, U_{a,b})$. The kernel is uniquely determined and it satisfies, in terms of the orthogonal basis (2.4),

$$\mathbf{P}_n(U_{a,b}; x, y) = \sum_{k=0}^{n} \frac{P_{k,n}^{a,b}(x) P_{k,n}^{a,b}(y)}{h_{k,n}^{a,b}}.$$

The product formula (2.6) leads to a convolution structure, which can be defined as follows. Let $\xi_1(x, y; r, \psi_1)$ and $\xi_2(x, y; r, \psi)$ be given in (2.6). For $g \in C(\mathbb{U})$, define

$$(\mathcal{T}_x g)(y) := \int_{[0,1]\times[0,\pi]^3} g(\xi_1(x, y; r, \psi_1), \xi_2(x, y; r, \psi))\mathrm{d}m_{a,b}(r, \psi).$$

The operator $\mathcal{T}_x$ is an analogue of the translation operator, since the product formula can be written as

$$P_{k,n}^{a,b}(x)P_{k,n}^{a,b}(y) = P_{k,n}^{a,b}(\mathbf{1})(\mathcal{T}_x P_{k,n}^{a,b})(y), \quad \mathbf{1} = (1, 1). \tag{2.7}$$

This generalized translation operator is bounded in the space $L^p(\mathbb{U}; U_{a,b})$.

Lemma 2.2. *Let $g \in L^p(\mathbb{U}; U_{a,b})$, $1 \le p < \infty$, or $g \in C(\mathbb{U})$, $p = \infty$. Then, for $x \in \mathbb{U}$,*

$$\|\mathcal{T}_x g\|_{L^p(\mathbb{U};U_{a,b})} \le \|g\|_{L^p(\mathbb{U};U_{a,b})}, \quad 1 \le p \le \infty, \tag{2.8}$$

where the norm is taken as the uniform norm on $\mathbb{U}$ when $p = \infty$.

This is stated and used in [3]. Its proof follows from the product formula. Indeed, for $p = 1$, we use $|\mathcal{T}_x g(y)| \le \mathcal{T}_x(|g|)(y)$ and expand $|g|$ in its Fourier orthogonal series over $\mathbb{U}$, then we use the product formula on the Fourier series of $\mathcal{T}_x(|g|)$ and integrate it to obtain, by orthogonality, the identity

$$\|\mathcal{T}_x|g|\|_{L^1(\mathbb{U};U_{a,b})} = \|g\|_{L^1(\mathbb{U};U_{a,b})}.$$

Thus, (2.8) holds for $p = 1$. The inequality is also trivial for $p = \infty$. The case $1 < p < \infty$ then follows from the Riesz–Thorin theorem.

The boundedness of $\mathcal{T}_x g$ can be used to study the convergence of the Fourier orthogonal expansion on $\mathbb{U}$. For $p \neq 2$, we need to consider a summability method. We choose the Cesàro (C, δ) means, which can be given in terms of $\operatorname{proj}_n(U_{a,b}; f)$ or the partial sum operator $S_n(U_{a,b}; f) = \sum_{k=0}^n \operatorname{proj}_k(U_{a,b}; f)$. For the reason that will become clear later, we choose the latter one. The operator $S_n(U_{a,b}; f)$ has the kernel

$$\mathbf{K}_n(U_{a,b}; x, y) = \sum_{k=0}^{n} \mathbf{P}_k(U_{a,b}; x, y).$$

For $\delta > 0$, let $\mathbf{K}_n^\delta(U_{a,b})$ denote the kernel for the Cesàro (C, δ) means, which can be written in terms of $\mathbf{P}_k(U_{a,b})$ or $\mathbf{K}_m(U_{a,b})$. In particular,

$$\mathbf{K}_n^\delta(U_{a,b}; x, y) = \frac{1}{\binom{n+\delta}{n}} \sum_{m=0}^{n} \binom{n - m + \delta - 1}{n - m} \mathbf{K}_m(U_{a,b}; x, y). \tag{2.9}$$

Then the (C, δ) means $S_n^\delta(U_{a,b};f)$ of the Fourier orthogonal series satisfy

$$S_n^\delta(U_{a,b};f) = \mathbf{d}_{a,b} \int_{\mathbb{U}} f(y)\mathbf{K}_n^\delta(U_{a,b};x,y)U_{a,b}(y)\mathrm{d}y.$$

Since this is a linear integration operator, a standard argument shows that $S_n^\delta f$ converges to f in $L^1(\mathbb{U}, U_{a,b})$ norm or in the uniform norm if and only if

$$\max_{x\in\mathbb{U}} \|\mathbf{K}_n^\delta(U_{a,b};x,\cdot)\|_{L^1(\mathbb{U};U_{a,b})} = \max_{x\in\mathbb{U}} \mathbf{d}_{a,b} \int_{\mathbb{U}} |\mathbf{K}_n^\delta(U_{a,b};x,y)|U_{a,b}(y)\mathrm{d}y < \infty \tag{2.10}$$

uniformly in n. Now, by the definition of $\mathcal{T}_x$, we have

$$\mathbf{K}_n^\delta(U_{a,b};x,y) = \mathcal{T}_x\mathbf{K}_n^\delta(U_{a,b};\mathbf{1},\cdot)(y),$$

so that it follows from the inequality (2.8) that

$$\max_{x\in\mathbb{U}} \|\mathbf{K}_n^\delta(U_{a,b};x,\cdot)\|_{L^1(\mathbb{U};U_{a,b})} \le \|\mathbf{K}_n^\delta(U_{a,b};\mathbf{1},\cdot)\|_{L^1(\mathbb{U};U_{a,b})}. \tag{2.11}$$

In particular, this shows that the convergence of $S_n^\delta(U_{a,b};f)$ follows from the convergence at the point $x = \mathbf{1} = (1,1)$.

More generally, we could define a convolution structure for $f, g \in L^1(\mathbb{U};U_{a,b})$ by

$$f *_U g(x) := \int_{\mathbb{U}} f(y)(\mathcal{T}_x g)(y)U_{a,b}(y)\mathrm{d}y, \quad x \in \mathbb{U}.$$

Since $\mathcal{T}_x g(y)$ is symmetric in x and y by (2.7), it is not difficult to see that this convolution is associative and commutative, as can be seen by first considering polynomials. Furthermore, using (2.8), it follows readily that, for $f \in L^p(\mathbb{U};U_{a,b})$ and $g \in L^1(\mathbb{U};U_{a,b})$,

$$\|f *_U g\|_{L^p(\mathbb{U};U_{a,b})} \le \|f\|_{L^p(\mathbb{U};Ua,b)} \|g\|_{L^1(\mathbb{U};U_{a,b})}, \quad 1 \le p \le \infty. \tag{2.12}$$

The projection operators $\mathrm{proj}_n(U_{a,b};f)$ and the Cesàro means can be written as

$$\mathrm{proj}_n(U_{a,b};f) = f *_U \mathbf{P}_n(U_{a,b};\mathbf{1},\cdot), \quad S_n^\delta(U_{a,b};f) = f *_U \mathbf{K}_n^\delta(U_{a,b};\mathbf{1},\cdot), \tag{2.13}$$

so that it again follows, by (2.12), that the convergence of $S_n^\delta(U_{a,b};f)$ reduces to the boundedness of $\|\mathbf{K}_n^\delta(U_{a,b};\mathbf{1},\cdot)\|_{L^1(\mathbb{U};U_{a,b})}$ of a single point.

It turns out that the kernel $\mathbf{K}_n(U_{a,b};\mathbf{1},\cdot)$ satisfies a closed formula [3].

Theorem 2.3. *For $a > -1$, $b > -\frac{1}{2}$ and $x \in \mathbb{U}$,*

$$\mathbf{K}_n(U_{a,b};\mathbf{1},x) = \frac{P_n^{(a+b+1,b)}(1)}{h_n^{(a+b+1,b)}} c_{\alpha+b+1,b} \int_{-1}^{1} P_n^{(a+b+1,b)}(z(x,t))w_{a+b+1,b}(t)\mathrm{d}t, \tag{2.14}$$

where

$$z(x,t) = 1 - (1-t^2)(1-x_1) - \frac{1}{2}(1-t)^2(1-x_2).$$

In particular, let us denote by $k_n^\delta(w_{\alpha,\beta};\cdot,\cdot)$ the kernels of the Cesàro means of the Jacobi polynomials that are given by

$$k_n^\delta(w_{\alpha,\beta};s,t)=\frac{1}{\binom{n+\delta}{n}}\sum_{k=0}^{n}\binom{n-k+\delta}{n-k}\frac{P_k^{(\alpha,\beta)}(s)P_k^{(\alpha,\beta)}(t)}{h_k^{(\alpha,\beta)}},$$

which are the kernels of the Cesàro (C,δ) means of the Fourier–Jacobi series. Then (2.14) leads to

$$\mathbf{K}_n^\delta(U_{a,b};\mathbf{1},x)=\frac{\delta}{n+\delta}c_{a+b+1,b}\int_{-1}^{1}k_n^{\delta-1}(w_{a+b+1,b};1,z(x,t))w_{a+b+1,b}(t)\mathrm{d}t. \tag{2.15}$$

The identity (2.15) allows us to bound the L^1 norm of $\mathbf{K}_n^\delta(U_{a,b};\mathbf{1},x)$ and, as a result, obtain the convergence of the (C,δ) means $S_n^\delta(U_{a,b};f)$. The result is the following theorem established in [3].

Theorem 2.4. *Let $a\ge b\ge 0$. Then the Cesàro means of the Fourier orthogonal series with respect to $U_{a,b}$ satisfy:*

(1) *if $\delta\ge a+2b+4$, then $S_n^\delta(U_{a,b};f)$ is nonnegative iff f is nonnegative;*

(2) *if $\delta> a+b+\frac{3}{2}$, then $S_n^\delta(U_{a,b};f)$ converge to f in $L^p(\mathbb{U};U_{a,b})$, $1\le p<\infty$, and in $C(\mathbb{U})$.*

Remark 2.1. The proof in [3] shows the boundedness of the L^1 norm of $\mathbf{K}_n^\delta(U_{a,b};\mathbf{1},x)$ for $a>-1$ and $b>-\frac{1}{2}$, so that the convergence of $S_n^\delta(U_{a,b};f)$ at $x=\mathbf{1}$ holds for $\delta>a+b+\frac{3}{2}$ without the restriction $a\ge b\ge 0$. The latter condition is imposed because of the product formula.

The method that we outlined above also applies to other summability methods of the Fourier orthogonal series on $\mathbb{U}$. In particular, it shows that the convergence can often be reduced to that at the point $x=\mathbf{1}$.

3 Orthogonality and Fourier orthogonal series on the surface of paraboloid

We consider orthogonal structure on the surface of the paraboloid of revolution

$$\mathbb{V}_0^{d+1}:=\{(x,t):\|x\|^2=t,\ 0\le t\le 1,\ x\in\mathbb{R}^d\},$$

which is compact since its t direction is bounded by $0\le t\le 1$. For $x=\sqrt{t}\xi$, $\xi\in\mathbb{S}^{d-1}$, we define the measure $\mathrm{d}\sigma(x,t)$ on the surface $\mathbb{V}_0^{d+1}$ by

$$\mathrm{d}\sigma(x,t)=t^{\frac{d-1}{2}}\mathrm{d}\sigma_{\mathbb{S}}(\xi)\mathrm{d}t,$$

where $d\sigma_{\mathbb{S}}$ denotes the surface measure of the unit sphere $\mathbb{S}^{d-1}$. For $d \ge 2$, $\beta > -\frac{d+1}{2}$ and $\gamma > -1$, we define an inner product on the surface $\mathbb{V}_0^{d+1}$

$$\langle f, g\rangle_{\beta,\gamma} = \mathsf{b}_{\beta,\gamma} \int_{\mathbb{V}_0^{d+1}} f(x,t)g(x,t)\varpi_{\beta,\gamma}(t)\mathrm{d}\sigma(x,t),$$

where the weight function is defined by

$$\varpi_{\beta,\gamma}(t) := t^{\beta}(1-t)^{\gamma}, \quad 0 \le t \le 1,$$

and $\mathsf{b}_{\beta,\gamma}$ is the normalization constant given by

$$\mathsf{b}_{\beta,\gamma} = \frac{1}{\omega_d}\frac{1}{\int_0^1 t^{\frac{d-1}{2}}\varpi_{\beta,\gamma}(t)dt} = \frac{1}{\omega_d} c_{\beta+\frac{d-1}{2},\gamma},$$

where the constant $c_{a,b}$ is defined in (2.3) and $\omega_d = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ is the surface area of $\mathbb{S}^{d-1}$. The value of $\mathsf{b}_{\beta,\gamma}$ can be verified by the decomposition of the integral on the surface of the paraboloid

$$\int_{\mathbb{V}_0^{d+1}} f(x,t)\mathrm{d}\sigma(x,t) = \int_0^1 \int_{\|x\|^2=t} f(x,t)\mathrm{d}\sigma(x,t) = \int_0^1 t^{\frac{d-1}{2}} \int_{\mathbb{S}^{d-1}} f(\sqrt{t}\xi, t)\mathrm{d}\sigma_{\mathbb{S}}(\xi)\mathrm{d}t.$$

3.1 Spherical harmonics. In order to understand the orthogonal structure on the surface of the paraboloid, we first review orthogonal polynomials on the unit sphere, which are spherical harmonics. A harmonic polynomial of degree n is a homogeneous polynomial of degree n that satisfies $\Delta Y = 0$, where Δ is the Laplace operator. Its restriction on the unit sphere is called a spherical harmonic. Let $\mathcal{H}_n^d$ denote the space of spherical harmonics of degree n in d variables. It is known that

$$\dim \mathcal{H}_n^d = \binom{n+d-1}{n} - \binom{n+d-3}{n-2}.$$

Spherical harmonics of different degrees are orthogonal on the sphere. For $n \in \mathbb{N}_0$ let $\{Y_\ell^n : 1 \le \ell \le \dim \mathcal{H}_n^d\}$ be an orthonormal basis of $\mathcal{H}_n^d$ in this subsection; then

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} Y_\ell^n(\xi)Y_{\ell'}^m(\xi)\mathrm{d}\sigma_{\mathbb{S}}(\xi) = \delta_{\ell,\ell'}\delta_{m,n}.$$

A fundamental property of the spherical harmonics is that they are eigenfunctions of the Laplace–Beltrami operator Δ_0 [5, (1.4.9)],

$$\Delta_0 Y = -n(n+d-2)Y, \quad Y \in \mathcal{H}_n^d, \tag{3.1}$$

where Δ_0 is the restriction of Δ on the unit sphere. Another important property is that they satisfy an addition formula [5, (1.2.3) and (1.2.7)]: for $\xi \in \mathbb{S}^{d-1}$ and $\eta \in \mathbb{S}^{d-1}$,

$$\sum_{\ell=1}^{\dim \mathcal{H}_n^d} Y_\ell^n(\xi) Y_\ell^n(\eta) = Z_n^{\frac{d-2}{2}}(\langle \xi, \eta \rangle), \quad Z_n^\lambda(t) = \frac{n+\lambda}{\lambda} C_n^\lambda(t), \tag{3.2}$$

where C_n^λ is the Gegenbauer polynomial of degree n that satisfies

$$c_\lambda \int_{-1}^1 C_n^\lambda(t) C_m^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}} \mathrm{d}t = \frac{\lambda}{n+\lambda} C_n^\lambda(1) \delta_{m,n},$$

where $C_n^\lambda(1) = (2\lambda)_n/n!$ and c_λ is the constant determined by

$$c_\lambda = \left(\int_{-1}^1 (1-t^2)^{\lambda-\frac{1}{2}} \mathrm{d}t \right)^{-1} = \frac{\Gamma(\lambda+1)}{\Gamma(\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}. \tag{3.3}$$

The left-hand side of (3.2) is the reproducing kernel of $\mathcal{H}_n^d$ and the kernel of the projection operator $\operatorname{proj}_n : L^2(\mathbb{S}^{d-1}) \to \mathcal{H}_n^d$:

$$\operatorname{proj}_n f(\xi) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) \mathsf{P}_n(\xi, \eta) \mathrm{d}\sigma_{\mathbb{S}}(\eta), \quad \mathsf{P}_n(\xi, \eta) = \sum_{\ell=1}^{\dim \mathcal{H}_n^d} Y_\ell^n(\xi) Y_\ell^n(\eta).$$

Thus, the product formula shows that $\operatorname{proj}_n f$, hence the Fourier series on the sphere defined by

$$L^2(\mathbb{S}^{d-1}) = \bigoplus_{n=0}^\infty \mathcal{H}_n^d : \quad f = \sum_{n=0}^\infty \operatorname{proj}_n f,$$

has a one-dimensional structure that can be used to reduce a large portion of the Fourier analysis on the sphere to that of the Fourier–Gegenbauer series (e.g., [5]).

3.2 Orthogonal structure on the surface of the paraboloid. The space of polynomials on $\mathbb{V}_0^{d+1}$ is the restriction of polynomials in (x, t) variables on the surface $\mathbb{V}_0^{d+1}$, determined by replacing every presence of $\|x\|^2$ with t. In particular, the space of polynomials of degree at most n on $\mathbb{V}_0^{d+1}$ is the direct sum of $t^j \mathcal{H}_k^d$ for $0 \le j+k \le n$, where $\mathcal{H}_n^d$ is the space of solid spherical harmonics in x variables.

For $n = 0, 1, 2, \ldots,$ let $\mathcal{V}_n(\mathbb{V}_0^{d+1}, \varpi_{\beta,\gamma})$ be the space of orthogonal polynomials of degree n with respect to the inner product $\langle \cdot, \cdot \rangle_{\beta,\gamma}$ on the surface $\mathbb{V}_0^{d+1}$. Since $\mathbb{V}_0^{d+1}$ is a quadratic surface, the dimension of $\mathcal{V}_n(\mathbb{V}_0^{d+1}, \varpi_{\beta,\gamma})$ is equal to that of $\mathcal{H}_n^{d+1}$ as established in [12, Cor. 4.2], that is,

$$\dim \mathcal{V}_n(\mathbb{V}_0^{d+1}, \varpi_{\beta,\gamma}) = \binom{n+d}{n} - \binom{n+d-2}{n-2}.$$

An orthogonal basis of $\mathcal{V}_n(\mathbb{V}_0^{d+1}, \varpi_{\beta,\gamma})$ can be given in terms of the Jacobi polynomials and spherical harmonics [12]. We also need norms of the elements in this basis.

Proposition 3.1. *Let $\beta > -\frac{d+1}{2}$ and $\gamma > -1$. Let $\{Y_\ell^m : 1 \le \ell \le \dim \mathcal{H}_m^d\}$ be an orthonormal basis of $\mathcal{H}_m^d$. For $0 \le m \le n$, define*

$$\mathsf{Q}_{m,\ell}^n(x,t) = P_{n-m}^{(\beta+m+\frac{d-1}{2},\gamma)}(1-2t)t^{\frac{m}{2}} Y_\ell^m\Big(\frac{x}{\sqrt{t}}\Big). \tag{3.4}$$

Then $\{\mathsf{Q}_{m,\ell}^n : 0 \le m \le n, 1 \le \ell \le \dim \mathcal{H}_m^d\}$ is an orthogonal basis of $\mathcal{V}_n(\mathbb{V}_0^{d+1}, \varpi_{\beta,\gamma})$. Moreover, the norm square of $\mathsf{Q}_{m,\ell}^n$ is given by

$$\mathsf{h}_{m,n}^{\beta,\gamma} = \langle \mathsf{Q}_{m,\ell}^n, \mathsf{Q}_{m,\ell}^n \rangle_{\beta,\gamma} = \frac{c_{\beta+\frac{d-1}{2},\gamma}}{c_{m+\beta+\frac{d-1}{2},\gamma}} h_{n-m}^{(m+\beta+\frac{d-1}{2},\gamma)}. \tag{3.5}$$

Proof. A simple combinatorial identity shows that the cardinality of

$$\{(m,\ell) : 0 \le m \le n,\ 1 \le \ell \le \dim \mathcal{H}_m^d\}$$

is equal to $\dim \mathcal{V}_n(\mathbb{V}_0^{d+1}, \varpi_{\beta,\gamma})$, so we only need to verify the orthogonality. Since Y_ℓ^m is homogeneous of degree m, the polynomial $\mathsf{Q}_{m,\ell}^n$ is of total degree n in (x,t) variables. Let $\alpha = \beta + \frac{d-1}{2}$. Setting $\xi = x/\sqrt{t}$, we obtain

$$\begin{aligned} \mathsf{b}_{\beta,\gamma} & \int_{\mathbb{V}_0^{d+1}} \mathsf{Q}_{m,\ell}^n(x,t) \mathsf{Q}_{m',\ell'}^{n'}(x,t) \varpi_{\beta,\gamma}(t) \mathrm{d}\sigma(x,t) \\ &= \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} Y_\ell^m(\xi) Y_{\ell'}^{m'}(\xi) \mathrm{d}\sigma_{\mathbb{S}}(\xi) \\ &\quad \times c_{\alpha,\gamma} \int_0^1 P_{n-m}^{(m+\alpha,\gamma)}(1-2t) P_{n'-m'}^{(m'+\alpha,\gamma)}(1-2t) t^{\frac{m+m'}{2}+\alpha}(1-t)^\gamma \mathrm{d}t. \end{aligned}$$

Since Y_ℓ^m are orthonormal, the orthogonality of $\mathsf{Q}_{m,\ell}^n$ follows from that of the Jacobi polynomials in the identity, and so is the formula for $\mathsf{h}_{m,n}^{\beta,\gamma}$. □

In an analog of spherical harmonics, we have shown in [23, 24] that on the surface of the cone and the hyperboloid, orthogonal polynomials of degree n or those that are even in the t variable and of degree n are eigenfunctions of a second order differential operator with eigenvalues depending only on n. This important property, however, does not hold for the paraboloid. The best we can do is the following proposition.

Proposition 3.2. *Let $\beta = -\frac{1}{2}$ and $\gamma > -1$. Then $\mathsf{Q}_{m,\ell}^n$ in (3.4) satisfies the differential equation*

$$\begin{aligned} &\Big[t(1-t)\partial_t^2 + \Big(\frac{d}{2} - \Big(\gamma + \frac{d}{2} + 1\Big)t\Big)\partial_t + \frac{1-t}{4t}\Delta_0^{(\xi)}\Big]y \\ &\qquad = -\Big(n\Big(n+\gamma+\frac{d}{2}\Big) - m\Big(n + \frac{\gamma+d-1}{2}\Big)\Big)y, \end{aligned} \tag{3.6}$$

where $\Delta_0^{(\xi)}$ is the Laplace–Beltrami operator acting on $\xi = x/\sqrt{t} \in \mathbb{S}^{d-1}$.

Proof. Let $f_{n,m}(t) = P_{n-m}^{(m+\alpha,\gamma)}(1-2t)t^{\frac{m}{2}}$ and $\alpha = \beta + \frac{d-1}{2}$, so that

$$\mathsf{Q}_{m,\ell}^n(x,t) = f_{n,m}(t) Y_\ell^m(\xi),$$

where $\xi = x/\sqrt{t}$. Since the Jacobi polynomial $P_n^{(\alpha,\gamma)}(1-2t)$ satisfies the differential equation

$$t(1-t)y'' + (1+\alpha-(\alpha+\gamma+2)t)y' + n(n+\alpha+\gamma+1)y = 0, \tag{3.7}$$

a straightforward computation shows that $f_{n,m}$ satisfies

$$\begin{aligned} t(1-t)f_{n,m}''(t) + (1+\alpha-(2+\alpha+\gamma)t)f_{n,m}'(t) - m(m+2\alpha)\frac{1-t}{4t}f_{n,m}(t) \\ = -\Big(n(n+\alpha+\gamma+1) - \frac{1}{2}m(2n+2\alpha+\gamma)\Big)f_{n,m}(t). \end{aligned}$$

For $\beta = -\frac{1}{2}$, we have $2\alpha = d-2$, so that $-m(m+2\alpha)$ is the eigenvalue of the Laplace–Beltrami operator $\Delta_0^{(\xi)}$. Hence, multiplying the above identity by $Y_\ell^m(\xi)$ and replacing $-m(m+2\alpha)Y_\ell^m$ by $\Delta_0^{(\xi)}Y_\ell^m$, we have proved (3.6). □

Remark 3.1. The right-hand side of (3.6) depends on m so that $\mathsf{Q}_{m,\ell}^n$ is the eigenfunction of the operator in the left-hand side of (3.6) where the eigenvalue depends on both m and n. This is in sharp contrast with the orthogonal structure on the cone and on the hyperboloid, for which the eigenvalue depends only on the degree of the orthogonal polynomials.

The reproducing kernel of $\mathcal{V}_n(\mathbb{V}_0^{d+1}, \varpi_{\beta,\gamma})$ is defined by

$$\mathsf{P}_n(\varpi_{\beta,\gamma};(x,t),(y,s)) = \sum_{m=0}^n \sum_{\ell=1}^{\dim \mathcal{H}_m^d} \frac{\mathsf{Q}_{m,\ell}^n(x,t)\mathsf{Q}_{m,\ell}^n(y,s)}{\mathsf{h}_{m,n}^{\beta,\gamma}},$$

which is the kernel of the orthogonal projection operator $\operatorname{proj}_n(\varpi_{\beta,\gamma})$ from the space $L^2(\mathbb{V}_0^{d+1};\varpi_{\beta,\gamma})$ onto $\mathcal{V}_n(\mathbb{V}_0^{d+1};\varpi_{\beta,\gamma})$. In contrast to the cone and to the hyperboloid, this kernel does not satisfy an addition formula that is of a closed formula one-dimensional in essence. Instead, however, we can express this kernel in terms of the reproducing kernel $\mathbf{P}_n(U_{a,b};\cdot,\cdot)$ on the parabola domain $\mathbb{U}$.

Theorem 3.3. *Let $d \ge 2$ and $\gamma > -1$. Let $(x,t) = (\sqrt{t}\xi, t) \in \mathbb{V}_0^{d+1}$ and $(y,s) = (\sqrt{s}\eta, s) \in \mathbb{V}_0^{d+1}$ with $\xi,\eta \in \mathbb{S}^{d-1}$. Then for $\beta > -\frac{1}{2}$,*

$$\begin{aligned} &\mathsf{P}_n(\varpi_{\beta,\gamma};(x,t),(y,s)) \\ &\quad = c_{\frac{d-2}{2},\beta-\frac{1}{2}} c_{\beta+\frac{1}{2}} \\ &\quad \times \int_{-1}^1 \int_{-1}^1 \mathbf{P}_n\Big(U_{\gamma,\beta+\frac{d-1}{2}};(\sqrt{t},t),\Big(\sqrt{s}\Big(\frac{1-z_1}{2}\langle \xi,\eta\rangle + \frac{1+z_1}{2}z_2\Big),s\Big)\Big) \\ &\quad \times (1-z_1)^{\frac{d-2}{2}}(1+z_1)^{\beta-\frac{1}{2}}(1-z_2^2)^\beta \mathrm{d}z_1\mathrm{d}z_2, \end{aligned} \tag{3.8}$$

and for $\beta = -\frac{1}{2}$,

$$\mathsf{P}_n(\varpi_{-\frac{1}{2},\gamma}; (x,t),(y,s)) = \mathbf{P}_n(U_{\gamma,\frac{d-2}{2}}; (\sqrt{t}, t), (\langle \xi, y\rangle, s)). \tag{3.9}$$

Proof. First we note that the right-hand side of (3.8) is well defined. Indeed, since $|\langle \xi, \eta\rangle| \le 1$ and $-1 \le z_1, z_2 \le 1$, we see that $|\sqrt{s}(\frac{1-z_1}{2}\langle \xi,\eta\rangle + \frac{1+z_1}{2} z_2)| \le \sqrt{s}$, so both variables in $\mathbf{P}_n(U_{\gamma,\beta+\frac{d-1}{2}})$ are elements of $\mathbb{U}$.

We first need to specify the reproducing kernel $\mathbf{P}_n(U_{a,b})$ on $\mathbb{U}$ when one of its variables is on the curved boundary of $\mathbb{U}$. Using the well-known identity [15, (4.7.1)]

$$\frac{P_m^{(b-\frac{1}{2},b-\frac{1}{2})}(1) P_m^{(b-\frac{1}{2},b-\frac{1}{2})}(\rho)}{h_m^{(b-\frac{1}{2},b-\frac{1}{2})}} = \frac{C_m^b(1) C_m^b(\rho)}{h_m^b} = Z_m^b(\rho),$$

it follows from (2.4) and (2.5) that

$$\begin{aligned} &\mathbf{P}_n(U_{a,b}; (\sqrt{x_2}, x_2), (y_1,y_2)) \\ &\quad = \sum_{m=0}^n \frac{P_{m,n}^{a,b}(\sqrt{x_2}, x_2) P_{m,n}^{a,b}(y_1,y_2)}{h_{m,n}^{a,b}} \\ &\quad = \sum_{m=0}^n \frac{c_{b+m,a}}{c_{b,a}} \frac{P_{n-m}^{(b+m,a)}(1-2x_2) P_{n-m}^{(b+m,a)}(1-2y_2)}{h_{n-m}^{(b+m,a)}} x_2^{\frac{m}{2}} y_2^{\frac{m}{2}} Z_m^b\Big(\frac{y_1}{\sqrt{y_2}}\Big). \end{aligned} \tag{3.10}$$

In terms of the orthogonal basis (3.4) and using the addition formula (3.2) for the spherical harmonics, we obtain, with $\alpha = \beta + \frac{d-1}{2}$,

$$\mathsf{P}_n(\varpi_{\beta,\gamma}; (x,t),(y,s)) = \sum_{m=0}^n \frac{P_{n-m}^{(\alpha+m,\gamma)}(1-2t) P_{n-m}^{(\alpha+m,\gamma)}(1-2s)}{\mathsf{h}_{m,n}^{\beta,\gamma}} t^{\frac{m}{2}} s^{\frac{m}{2}} Z_m^{\frac{d-2}{2}}(\langle \xi,\eta\rangle),$$

where $\xi = \frac{x}{\sqrt{t}} \in \mathbb{S}^{d-1}$ and $\eta = \frac{y}{\sqrt{s}} \in \mathbb{S}^{d-1}$. For $\beta = -\frac{1}{2}$, the sum in the right-hand side can be identified, using (3.5), with $\mathbf{P}_n(U_{\gamma,\frac{d-2}{2}})$ in (3.10) by setting $x_2 = t$, $y_2 = s$ and $y_1 = \sqrt{s}\langle \xi,\eta\rangle$, which proves (3.9).

For $\beta > -\frac{1}{2}$, we need to increase the value of the index in the zonal harmonic by using the following identity, proved recently in [22]:

$$\begin{aligned} Z_m^\lambda(t) = c_{\lambda,\sigma-1} c_\sigma \int_{-1}^1 \int_{-1}^1 & Z_m^{\lambda+\sigma}\Big(\frac{1-z_1}{2} t + \frac{1+z_1}{2} z_2\Big) \\ & \times (1-z_1)^\lambda (1+z_1)^{\sigma-1} (1-z_2^2)^{\sigma-\frac{1}{2}} \mathrm{d}z_1 \mathrm{d}z_2 \end{aligned} \tag{3.11}$$

with $\lambda = \frac{d-2}{2}$ and $\sigma = \beta + \frac{1}{2}$. This shows that

$$\begin{aligned} &\mathsf{P}_n(\varpi_{\beta,\gamma}; (x,t),(y,s)) \\ &\quad = c_{\frac{d-2}{2},\beta-\frac{1}{2}} c_{\beta+\frac{1}{2}} \int_{-1}^1 \int_{-1}^1 \sum_{m=0}^n \frac{P_{n-m}^{(\alpha+m,\gamma)}(1-2t) P_{n-m}^{(\alpha+m,\gamma)}(1-2s)}{\mathsf{h}_{m,n}^{\beta,\gamma}} \\ &\quad \times t^{\frac{m}{2}} s^{\frac{m}{2}} Z_m^\alpha\Big(\frac{1-z_1}{2}\langle \xi,\eta\rangle + \frac{1+z_1}{2} z_2\Big)(1-z_1)^{\frac{d-2}{2}} (1+z_1)^{\beta-\frac{1}{2}} (1-z_2^2)^\beta \mathrm{d}z_1 \mathrm{d}z_2. \end{aligned}$$

The sum in the right-hand side can be identified, using (3.5), with the reproducing kernel $\mathbf{P}_n(U_{\gamma,\beta+\frac{d-1}{2}})$ in (3.10), which gives (3.8). □

The identity (3.9) for $\beta = -\frac{1}{2}$ can be regarded as the limit of (3.8) as $\beta \to -\frac{1}{2}$ by using the limit relation (4.3).

The kernel $\mathsf{P}_n(\varpi_{\beta,\gamma})$, however, does not satisfy a closed formula in general. In the case that (x,t) is on the boundary $t = 1$ of the paraboloid $\mathbb{V}_0^{d+1}$, however, we could derive a closed form formula for the kernel of the partial sum operator

$$\mathsf{K}_n(\varpi_{\beta,\gamma},(x,t),(y,s)) = \sum_{m=0}^{n} \mathsf{P}_m(\varpi_{\beta,\gamma};(x,t),(y,s))$$

on the paraboloid by using the closed formula of $\mathbf{K}_n(U_{a,b};\mathbf{1},\cdot)$ in Theorem 2.3. We state the result for $\beta = -\frac{1}{2}$ as an example.

Corollary 3.4. *Let $d \ge 2$ and $\gamma > -1$. Then, for $\xi \in \mathbb{S}^{d-1}$,*

$$\begin{aligned}\mathsf{K}_n(\varpi_{-\frac{1}{2},\gamma};(\xi,1),(y,s)) \\ = \frac{P_n^{(\gamma+\frac{d}{2},\frac{d-2}{2})}(1)}{h_n^{(\gamma+\frac{d}{2},\frac{d-2}{2})}} c_{\gamma+\frac{d}{2},\frac{d-2}{2}} \int_{-1}^{1} P_n^{(\gamma+\frac{d}{2},\frac{d-2}{2})}(z'(\xi,y,v))\, w_{\gamma+\frac{d}{2},\frac{d-2}{2}}(v)\,\mathrm{d}v,\end{aligned}$$

where, with $y = \sqrt{s}\eta$,

$$z'(\xi,y,v) = z((\langle\xi,\eta\rangle,s),v) = 1-(1-v^{2})(1-\langle\xi,y\rangle) - \frac{1}{2}(1-v)^2(1-s).$$

This is a corollary of (3.9) and (2.14). Similarly, using (3.8) and (2.14), we can derive an explicit formula for $\beta > -\frac{1}{2}$, which is however more involved.

3.3 Summability of Fourier orthogonal series. Let $\mathrm{proj}_n(\varpi_{\beta,\gamma})$ be the orthogonal projection operator

$$\mathrm{proj}_n(\varpi_{\beta,\gamma}) : L^2(\mathbb{V}_0^{d+1};\varpi_{\beta,\gamma}) \mapsto \mathcal{V}_n(\mathbb{V}_0^{d+1},\varpi_{\beta,\gamma}).$$

In terms of the reproducing kernel $\mathsf{P}_n(\varpi_{\beta,\gamma})$ of $\mathcal{V}_n(\mathbb{V}_0^{d+1},\varpi_{\beta,\gamma})$, we have

$$\mathrm{proj}_n(\varpi_{\beta,\gamma};f) = \mathsf{b}_{\beta,\gamma}\int_{\mathbb{V}_0^{d+1}} f(y,s)\mathsf{P}_n(\varpi_{\beta,\gamma};(x,t),(y,s))\varpi_{\beta,\gamma}(s)\mathrm{d}\sigma(y,s).$$

For $f \in L^2(\mathbb{V}_0^{d+1};\varpi_{\beta,\gamma})$, the Fourier orthogonal series of f on $\mathbb{V}_0^{d+1}$ is defined by

$$f = \sum_{n=0}^{\infty} \mathrm{proj}_n(\varpi_{\beta,\gamma};f).$$

Below we study the summability of this Fourier orthogonal series. We start with a definition. Recall that $\mathbb{U}$ denotes the domain bounded by the parabolic $x_2 = x_1^2$ and $x_2 = 1$ in $\mathbb{R}^2$. We further denote by $\mathbb{U}_0$ the curved portion of the boundary of $\mathbb{U}$,

$$\mathbb{U}_0 := \{(x_1, x_2) \in \mathbb{U} : x_2 = x_1^2\}.$$

Definition 3.5. Let $\beta \geq -\frac{1}{2}$ and $\gamma > -1$. Let $g : \mathbb{U}_0 \times \mathbb{U} \mapsto \mathbb{R}$ such that, for each $t \in [0, 1]$, the function $x \mapsto g((\sqrt{t}, t); x)$ is in $L^1(\mathbb{U}; U_{\gamma,\alpha})$. For $(x, t) = (\sqrt{t}\xi, t) \in \mathbb{V}_0^{d+1}$ and $(y, s) = (\sqrt{s}\eta, s) \in \mathbb{V}_0^{d+1}$, define for $\beta > -\frac{1}{2}$

$$\begin{aligned} &\mathsf{T}_{\beta,\gamma} g((x,t),(y,s)) \\ &:= c_{\frac{d-2}{2},\beta-\frac{1}{2}} c_{\beta+\frac{1}{2}} \int_{-1}^{1}\int_{-1}^{1} g\Big((\sqrt{t},t),\Big(\sqrt{s}\Big(\frac{1-z_1}{2}\langle\xi,\eta\rangle+\frac{1+z_1}{2}z_2\Big),s\Big)\Big) \\ &\qquad\qquad \times (1-z_1)^{\frac{d-2}{2}}(1+z_1)^{\beta-\frac{1}{2}}(1-z_2^2)^{\beta}\mathrm{d}z \end{aligned}$$

and, furthermore, define for $\beta = -\frac{1}{2}$

$$\mathsf{T}_{-\frac{1}{2},\gamma} g((x,t),(y,s)) := g((\sqrt{t},t),(\langle\xi,y\rangle,s)).$$

The definition of $\mathsf{T}_{\beta,\gamma}$ is motivated by the relation, by (3.8) and (3.9),

$$\mathsf{P}_n(\varpi_{\beta,\gamma};(x,t),(y,s)) = \mathsf{T}_{\beta,\gamma}\mathbf{P}_n(U_{\gamma,\beta+\frac{d-1}{2}})((x,t),(y,s)). \tag{3.12}$$

For each fixed $(x, t) \in \mathbb{V}_0^{d+1}$, this is a bounded operator as seen below.

Proposition 3.6. *Let $\beta \geq -\frac{1}{2}$ and $\gamma > -1$. Let $g : \mathbb{U}_0 \times \mathbb{U} \mapsto \mathbb{R}$ such that, for each $t \in [0, 1]$, the function $g((\sqrt{t}, t); \cdot)$ is in $L^1(\mathbb{U}; U_{\gamma,\alpha})$ with $\alpha = \beta + \frac{d-1}{2}$. Then, for $(x, t) \in \mathbb{V}_0^{d+1}$,*

$$\int_{\mathbb{V}_0^{d+1}} |\mathsf{T}_{\beta,\gamma} g((x,t),(y,s))|\varpi_{\beta,\gamma}(s)\mathrm{d}\sigma(y,s) \leq c\int_{\mathbb{U}} |g((\sqrt{t},t),z)|U_{\gamma,\alpha}(z)\mathrm{d}z. \tag{3.13}$$

Proof. Let $G(z) = g((t, \sqrt{t}), z)$ for $z \in \mathbb{U}$ in this proof. We first consider the case $\beta = -\frac{1}{2}$. Using the well-known integral relation

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(\langle\xi,\eta\rangle)\mathrm{d}\sigma(\xi) = c_{\frac{d-2}{2}}\int_{-1}^{1} f(u)(1-u^2)^{\frac{d-3}{2}}\mathrm{d}u \tag{3.14}$$

we obtain, setting $y = \sqrt{s}\,\eta$ with $\eta \in \mathbb{S}^{d-1}$, that

$$
\begin{aligned}
\mathsf{b}_{-\frac12,\gamma} & \int_{\mathbb{V}^{d+1}} |\mathsf{T}_{-\frac12,\gamma} g((x,t),(y,s))| \varpi_{-\frac12,\gamma}(s) \mathrm{d}\sigma(y,s) \\
&= \mathsf{b}_{-\frac12,\gamma} \int_0^1 s^{\frac{d-1}{2}} \int_{\mathbb{S}^{d-1}} |G(\sqrt{s}\,\langle \xi, \eta\rangle, s)| \varpi_{-\frac12,\gamma}(s) \mathrm{d}\sigma(\xi) \mathrm{d}s \\
&= \mathsf{b}_{-\frac12,\gamma} c_{\frac{d-2}{2}} \omega_d \int_0^1 \int_{-1}^1 |G(\sqrt{s}\,u, s)| (1-u^2)^{\frac{d-3}{2}} \mathrm{d}u\, s^{\frac{d-2}{2}} (1-s)^\gamma \mathrm{d}s \\
&= \mathbf{d}_{\gamma,\frac{d-2}{2}} \int_{\mathbb{U}} |G(z)| (z_2 - z_1^2)^{\frac{d-3}{2}} (1-z_2)^\gamma \mathrm{d}z,
\end{aligned}
$$

where we have used (2.2) in the last step, and the constant can be verified simply by the fact that if $g = 1$, then $\mathsf{T}_{\beta,\gamma} g = 1$. In particular, this shows that the inequality (3.13) is in fact an identity for $\beta = -\frac12$.

We now consider the case $\beta > -\frac12$. Using (3.14), we obtain

$$
\begin{aligned}
\int_{\mathbb{V}_0^{d+1}} & |\mathsf{T}_{\beta,\gamma} g((x,t),(y,s))| \varpi_{\beta,\gamma}(s) \mathrm{d}\sigma(y,s) \\
&\le c \int_0^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \left| G\Big(\sqrt{s}\Big(\frac{1-z_1}{2} u + \frac{1+z_1}{2} z_2\Big), s\Big) \right| \\
&\quad \times (1-z_1)^{\frac{d-2}{2}} (1+z_1)^{\beta-\frac12} (1-z_2^2)^\beta (1-u^2)^{\frac{d-3}{2}} s^{\beta+\frac{d-1}{2}} (1-s)^\gamma \mathrm{d}z \mathrm{d}u \mathrm{d}s,
\end{aligned}
$$

where $c = \omega_d c_{\frac{d-2}{2}} c_{\frac{d-2}{2},\beta-\frac12} c_{\beta+\frac12}$. Making a change of variables $z_1 \mapsto y$ with

$$
y = \frac{1-z_1}{2} u + \frac{1+z_1}{2} z_2
$$

and dividing the integral over $\mathrm{d}u\mathrm{d}z_2$ on $[-1,1]^2$ into two integrals over the triangles $\{(u,z_2) \in [-1,1]^2 : u \ge z_2\}$ and $\{(u,z_2) \in [-1,1]^2 : u < z_2\}$, respectively, we can write the triple integral against $\mathrm{d}u\mathrm{d}z$ as a sum of two integrals:

$$
\begin{aligned}
& 2^{\beta+\frac{d-1}{2}} \int_{-1}^1 \int_u^1 \int_u^{z_2} |G(\sqrt{s}y, s)| (z_2 - y)^{\frac{d-2}{2}} (y-u)^{\beta-\frac12} dy \frac{(1-z_2^2)^\beta (1-u^2)^{\frac{d-3}{2}}}{(z_2-u)^{\beta+\frac{d-1}{2}}} \mathrm{d}z_2 \mathrm{d}u \\
& + 2^{\beta+\frac{d-1}{2}} \int_{-1}^1 \int_{-1}^u \int_{z_2}^u |G(\sqrt{s}y, s)| (y - z_2)^{\frac{d-2}{2}} (u-y)^{\beta-\frac12} dy \frac{(1-z_2^2)^\beta (1-u^2)^{\frac{d-3}{2}}}{(u-z_2)^{\beta+\frac{d-1}{2}}} \mathrm{d}z_2 \mathrm{d}u.
\end{aligned}
$$

Changing the order of integrals in both terms, we see that this sum is equal to

$$
\begin{aligned}
2^{\beta+\frac{d-1}{2}} \int_{-1}^1 |G(\sqrt{s}y, s)| \bigg[& \int_{-1}^y \int_y^1 (z_2 - y)^{\frac{d-2}{2}} (y-u)^{\beta-\frac12} \frac{(1-z_2^2)^\beta (1-u^2)^{\frac{d-3}{2}}}{(z_2-u)^{\beta+\frac{d-1}{2}}} \mathrm{d}z_2 \mathrm{d}u \\
& + \int_y^1 \int_{-1}^y (y-z_2)^{\frac{d-2}{2}} (u-y)^{\beta-\frac12} \frac{(1-z_2^2)^\beta (1-u^2)^{\frac{d-3}{2}}}{(u-z_2)^{\beta+\frac{d-1}{2}}} \mathrm{d}z_2 \mathrm{d}u \bigg] \mathrm{d}y.
\end{aligned}
$$

Making a change of variables $(u, z_2) \mapsto (v_1, v_2)$ with $v_1 = (z_2 - y)/(1 - y)$ and $v_2 = (y-u)/(1+y)$ in the first double integral in the square bracket, and a change of variables $(u, z_2) \mapsto (v_1, v_2)$ with $v_1 = (y - z_2)/(1 + y)$ and $v_2 = (u - y)/(1 - y)$ in the second double integral in the square bracket, we see that the expression in the above square bracket is equal to

$$(1-y^2)^{\beta+\frac{d-2}{2}}\Bigg[(1-y)\int_0^1\int_0^1 \frac{(1+y+(1-y)v_1)^{\beta}(1-y+(1+y)v_2)^{\frac{d-3}{2}}}{((1-y)v_1+(1+y)v_2)^{\beta+\frac{d-1}{2}}} \times v_1^{\frac{d-2}{2}} v_2^{\beta-\frac{1}{2}}(1-v_1)^{\beta}(1-v_2)^{\frac{d-3}{2}}\mathrm{d}v_1\mathrm{d}v_2 + (1+y)\int_0^1\int_0^1 \frac{(1-y+(1+y)v_1)^{\beta}(1+y+(1-y)v_2)^{\frac{d-3}{2}}}{((1+y)v_1+(1-y)v_2)^{\beta+\frac{d-1}{2}}} \times v_1^{\frac{d-2}{2}} v_2^{\beta-\frac{1}{2}}(1-v_1)^{\beta}(1-v_2)^{\frac{d-3}{2}}\mathrm{d}v_1\mathrm{d}v_2\Bigg]\mathrm{d}y.$$

Since $0 \le v_1, v_2 \le 1$ and $1 \pm y \ge 0$, it follows that $(1-y)v_1 \le (1-y)v_1 + (1+y)v_2$,

$$\frac{v_2(1+y+(1-y)v_1)}{((1-y)v_1+(1+y)v_2)} \le 1 \quad \text{and} \quad \frac{v_1(1-y+(1+y)v_2)}{((1-y)v_1+(1+y)v_2)} \le 1,$$

which implies that the first term in the square bracket is bounded by

$$\int_0^1\int_0^1 v_1^{-\frac{1}{2}} v_2^{-\frac{1}{2}}(1-v_1)^{\beta}(1-v_2)^{\frac{d-3}{2}}\mathrm{d}v_1\mathrm{d}v_2 = \frac{\pi\Gamma(\beta+1)\Gamma(\frac{d-1}{2})}{\Gamma(\beta+\frac{3}{2})\Gamma(\frac{d}{2})}.$$

Similarly, it is easy to see that the same bound holds for the second term in the square bracket. Putting all these estimates together, we conclude that

$$\begin{aligned}\int_{\mathbb{V}_0^{d+1}} &|\mathsf{T}_{\beta,\gamma}g((x,t),(y,s))|\varpi_{\beta,\gamma}(s)\mathrm{d}\sigma(y,s)\\ &\le c\int_0^1\int_{-1}^1 |G(\sqrt{s}y,s)|(1-y^2)^{\beta+\frac{d-2}{2}}s^{\beta+\frac{d-1}{2}}(1-s)^{\gamma}\mathrm{d}y\mathrm{d}s\\ &= c\int_{\mathbb{U}}|G(z_1,z_2)|U_{\gamma,\beta+\frac{d-1}{2}}(z_1,z_2)\mathrm{d}z,\end{aligned}$$

where the last step follows from (2.2). This completes the proof. □

Definition 3.7. Let $\beta \ge -\frac{1}{2}$ and $\gamma > -1$. Let $g : \mathbb{U}_0 \times \mathbb{U} \mapsto \mathbb{R}$ such that, for each $t \in [0, 1]$, the function $x \mapsto g((\sqrt{t}, t); x)$ is in $L^1(\mathbb{U}; U_{\gamma,,\beta+\frac{d-1}{2}})$. For $f \in L^1(\mathbb{V}_0^{d+1}; \varpi_{\beta,\gamma})$ and $(y, s) \in \mathbb{V}_0^{d+1}$, define

$$(f *_{\mathbb{V}_0} g)(y,s) = \mathsf{b}_{\beta,\gamma}\int_{\mathbb{V}_0^{d+1}} f(x,t)\mathsf{T}_{\beta,\gamma}g((x,t),(y,s))\varpi_{\beta,\gamma}(t)\mathrm{d}\sigma(x,t).$$

The operator $\mathsf{T}_{\beta,\gamma}$ plays the role of a translation in the definition of the pseudo convolution $f *_{\mathbb{V}_0} g$. By (3.12), it follows that the project operator onto $\mathcal{V}_n(\mathbb{V}_0^{d+1}, \varpi_{\beta,\gamma})$ satisfies

$$\operatorname{proj}_n(\varpi_{\beta,g}; f) = f *_{\mathbb{V}_0} \mathbf{P}_n(U_{\gamma,\beta+\frac{d-1}{2}}).$$

Proposition 3.8. *Let $\beta \geq -\frac{1}{2}$ and $\gamma > -1$. For $f \in L^p(\mathbb{V}_0^{d+1}; \varpi_{\beta,\gamma})$, $1 \leq p < \infty$, and $f \in C(\mathbb{V}_0^{d+1})$ for $p = \infty$,*

$$\|f *_{\mathbb{V}_0} g\|_{L^p(\mathbb{V}_0^{d+1};\varpi_{\beta,\gamma})} \leq c\|f\|_{L^p(\mathbb{V}_0^{d+1};\varpi_{\beta,\gamma})} \max_{t\in[0,1]} \int_{\mathbb{U}} |g((\sqrt{t}, t), z)| U_{\gamma,\beta+\frac{d-1}{2}}(z) \mathrm{d}z.$$

Proof. By the Minkowski inequality, we obtain

$$\|f *_{\mathbb{V}_0} g\|_{L^p(\mathbb{V}_0^{d+1};\varpi_{\beta,\gamma})} \leq \|f\|_{L^p(\mathbb{V}_0^{d+1};\varpi_{\beta,\gamma})} \mathsf{b}_{\beta,\gamma} \int_{\mathbb{U}} |T_{\beta,\gamma} g((x,t),(y,s))| \varpi_{\beta,\gamma}(s) \mathrm{d}\sigma(y,s).$$

Applying the inequality (3.13) on the integral on the right-hand side, the stated inequality follows readily by taking the maximum over t. □

The boundedness of the pseudo convolution can be used to study the convergence of the Fourier orthogonal series on the surface of the paraboloid. As in the case of the previous section, we consider the Cesàro means. For $\delta > -1$, let $\mathsf{K}_n^\delta(\varpi_{\beta,\gamma}; (x,t), (y,s))$ be the kernel of the Cesàro means $\mathsf{S}_n^\delta(\varpi_{\beta,\gamma}; f)$, which can be written in terms of the reproducing kernel $\mathsf{P}_n(\varpi_{\beta,g})$ analogously to (2.9), and it satisfies, by (3.12), that

$$\mathsf{K}_n^\delta(\varpi_{\beta,\gamma}; (x,t), (y,s)) = \mathsf{T}_{\beta,\gamma} \mathbf{K}_n^\delta(U_{\gamma,\beta+\frac{d-1}{2}})((x,t),(y,s)). \tag{3.15}$$

In terms of the pseudo convolution, we can write

$$\mathsf{S}_n^\delta(\varpi_{\beta,\gamma}; f, (x,t)) = f *_{\mathbb{V}_0} \mathbf{K}_n^\delta(U_{\gamma,\beta+\frac{d-1}{2}})(x,t).$$

Theorem 3.9. *Let $d \geq 2$, $\beta \geq -\frac{1}{2}$ and $\gamma > -1$. If $f \in C(\mathbb{V}_0^{d+1})$, then $\mathsf{S}_n^\delta(\varpi_{\beta,\gamma}; f, (\xi, 1))$ converges to $f(\xi, 1)$ uniformly for $\xi \in \mathbb{S}^{d-1}$ provided*

$$\delta > \beta + \gamma + \frac{d+2}{2}.$$

Proof. The convergence of $\mathsf{S}_n^\delta(\varpi_{\beta,\gamma}; f, (\xi, 1))$ holds if and only if

$$\sup_{\xi\in\mathbb{S}^{d-1}} \int_{\mathbb{V}_0^{d+1}} |\mathsf{K}_n^\delta(\varpi_{\beta,\gamma}; (\xi,1), (y,s))| \varpi_{\beta,\gamma}(s) \mathrm{d}\sigma(y,s)$$

is bounded uniformly in n. By (3.15) and the inequality (3.13), this is bounded by

$$\int_{\mathbb{U}} |\mathbf{K}_n^\delta(U_{\gamma,\beta+\frac{d-1}{2}}; \mathbf{1}, z)| U_{\gamma,\beta+\frac{d-1}{2}}(z) \mathrm{d}z.$$

For $\delta > \gamma + \beta + \frac{d-1}{2} + \frac{3}{2}$, the last integral is bounded uniformly in n by Theorem 2.4 and by Remark 2.1. □

Theorem 3.10. *Let $d \ge 2$, $\gamma \ge \beta + \frac{d-1}{2}$ and $\beta \ge -\frac{1}{2}$. Let $f \in L^p(\mathbb{V}_0^{d+1}, \varpi_{\beta,\gamma})$ for $1 \le p < \infty$ and $f \in C(\mathbb{V}_0^{d+1})$ for $p = \infty$. Then the Cesàro means $\mathsf{S}_n^\delta(\varpi_{\beta,\gamma}; f)$ satisfy:*

(1) *if $\delta \ge 2\beta + \gamma + d + 3$, then $\mathsf{S}_n^\delta(\varpi_{\beta,g}; f)$ is nonnegative if f is nonnegative;*
(2) *if $\delta > \beta + \gamma + \frac{d+2}{2}$, then $\mathsf{S}_n^\delta(\varpi_{\beta,g}; f)$ converges to f in $L^p(\mathbb{V}_0^{d+1}; \varpi_{\beta,\gamma})$, $1 \le p < \infty$, and in $C(\mathbb{V}_0^{d+1})$.*

Proof. The positivity of $\mathsf{S}_n^\delta(\varpi_{\beta,\gamma}; f,)$ follows from the positivity of its kernel. Hence, the first item is a consequence of (3.15) and the positivity of $\mathsf{T}_{\beta,\gamma}\mathbf{K}_n^\delta(U_{\gamma,\beta+\frac{d-1}{2}})$, where the latter follows from the positivity of $\mathbf{K}_n^\delta(U_{\gamma,\beta+\frac{d-1}{2}})$, which in turn follows from the definition of $\mathsf{T}_{\beta,\gamma}$ and Theorem 2.4; the last theorem requires $a \ge b \ge 0$ in $U_{a,b}$, which is satisfied in our case by the assumption $\gamma \ge \beta + \frac{d-1}{2}$ and $\beta \ge -\frac{1}{2}$.

For the second item, it suffices to show that L^p norm of $\mathsf{S}_n^\delta(\varpi_{\beta,g}; f)$ is uniformly bounded. By Proposition 3.8, it is sufficient to show that

$$\max_{t\in[0,1]} \int_{\mathbb{U}} |\mathbf{K}_n^\delta(U_{\gamma,\beta+\frac{d-1}{2}}; (\sqrt{t}, t), z)| U_{\gamma,\beta+\frac{d-1}{2}}(z) \mathrm{d}z$$

is bounded uniformly in n when $\delta > \gamma + \beta + \frac{d-1}{2} + \frac{3}{2}$. This follows immediately from (2.11) and Theorem 2.4. □

4 Orthogonality and Fourier orthogonal series on the solid paraboloid

We consider orthogonal structure on the solid paraboloid of revolution

$$\mathbb{V}^{d+1} := \{(x, t) : \|x\|^2 \le t,\ 0 \le t \le 1,\ x \in \mathbb{R}^d\},$$

which is bounded by the surface $\mathbb{V}_0^{d+1}$ and the hyperplane $t = 1$ of $\mathbb{R}^{d+1}$. The t-section of the domain, $\{x : \|x\| \le \sqrt{t}\}$, is the ball of radius $\sqrt{t}$ in $\mathbb{R}^d$. W review the orthogonal structure on the unit ball first.

4.1 Classical orthogonal polynomials on the unit ball. For $\mu > -\frac{1}{2}$, let ϖ_μ be the weight function

$$\varpi_\mu(x) := (1 - \|x\|^2)^{\mu-\frac{1}{2}}, \quad \|x\| < 1.$$

The classical orthogonal polynomials on the unit ball are orthogonal with respect to the inner product

$$\langle f, g\rangle_\mu = b_\mu \int_{\mathbb{B}^d} f(x)g(x)\varpi_\mu(x)\mathrm{d}x \quad \text{with } b_\mu = \frac{\Gamma(\mu + \frac{d+1}{2})}{\pi^{\frac{d}{2}}\Gamma(\mu + \frac{1}{2})},$$

where b_μ is the normalization constant of ϖ_μ so that $\langle 1, 1\rangle = 1$.

Let $\mathcal{V}_n(\mathbb{B}^d, \varpi_\mu)$ be the space of orthogonal polynomials of degree n with respect to ϖ_μ. Then $\dim \mathcal{V}_n(\mathbb{B}^d, \varpi_\mu) = \binom{n+d-1}{n}$. An orthogonal basis of $\mathcal{V}_n(\mathbb{B}^d, \varpi)$ can be given in terms of the Jacobi polynomials or spherical harmonics, see [6, Chapter 5], which we shall call the **basis with parity** since its elements are polynomials that are even in each of its variables if n is even and odd in each of its variables if n is odd. The orthogonal polynomials of degree n are eigenfunctions of a second order differential operator: for $u \in \mathcal{V}_n(\mathbb{B}^d, \varpi_\mu)$,

$$(\Delta - \langle x, \nabla \rangle^2 - (2\mu + d - 1)\langle x, \nabla \rangle)u = -n(n + 2\mu + d - 1)u. \tag{4.1}$$

Furthermore, these polynomials also satisfy an addition formula. Let $\mathbf{P}_n(\varpi_\mu; \cdot, \cdot)$ be the reproducing kernel of the space $\mathcal{V}_n(\mathbb{B}^d, \varpi_\mu)$. In terms of an orthonormal basis $\{P_{\mathbf{k}}^n : |\mathbf{k}| = n\}$ of $\mathcal{V}_n(\mathbb{B}^d, \varpi_\mu)$, the kernel can be written as

$$\mathbf{P}_n(\varpi_\mu; x, y) = \sum_{|\mathbf{k}|=n} P_{\mathbf{k}}^n(x) P_{\mathbf{k}}^n(y).$$

The addition formula on the unit ball states [21], for $\mu \ge 0$,

$$\begin{aligned} \mathbf{P}_n(\varpi_\mu; x, y) = c_{\mu-\frac{1}{2}} \int_{-1}^1 & Z_n^{\mu+\frac{d-1}{2}}\left(\langle x, y\rangle + t\sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2}\right) \\ & \times (1 - t^2)^{\mu-1} \mathrm{d}t, \end{aligned} \tag{4.2}$$

where the identity holds for $\mu = 0$ under the limit

$$\lim_{\mu \to 0} c_{\mu-\frac{1}{2}} \int_{-1}^1 f(t)(1 - t^2)^{\mu-1} \mathrm{d}t = \frac{f(1) + f(-1)}{2}. \tag{4.3}$$

4.2 Orthogonal structure of the solid paraboloid. For $\beta > -\frac{d+1}{2}$, $\gamma > -1$ and $\mu > -\frac{1}{2}$, we define a weight function $W_{\beta,\gamma,\mu}$ on $\mathbb{V}^{d+1}$,

$$W_{\beta,\gamma,\mu}(x, t) := t^\beta (1 - t)^\gamma (t - \|x\|^2)^{\mu-\frac{1}{2}}, \quad (x, t) \in \mathbb{V}^{d+1}.$$

With respect to this weight function, we define an inner product

$$\langle f, g\rangle_{\beta,\gamma,\mu} = \mathbf{b}_{\beta,\gamma,\mu} \int_{\mathbb{V}^{d+1}} f(x, t) g(x, t) W_{\beta,\gamma,\mu}(x, t) \mathrm{d}x \mathrm{d}t,$$

where $\mathbf{b}_{\beta,\gamma,\mu} = b_\mu c_{\beta+\mu+\frac{d-1}{2},\gamma}$ with b_μ the normalization constant of ϖ_μ on the unit ball and $c_{\alpha,\gamma}$ is defined in (2.3). The weight function $W_{\beta,\gamma,\mu}$ can be written as

$$W_{\beta,\gamma,\mu}(x, t) = t^{\beta+\mu-\frac{1}{2}} (1 - t)^\gamma (1 - \|x'\|^2)^{\mu-\frac{1}{2}}, \quad \text{with } x' = \frac{x}{\sqrt{t}} \in \mathbb{B}^d. \tag{4.4}$$

Hence, the value of the constant $\mathbf{b}_{\beta,\gamma,\mu}$ can be verified by using the identity

$$\int_{\mathbb{V}^{d+1}} f(x, t) \mathrm{d}x \mathrm{d}t = \int_0^1 \int_{\|x\|^2 \le t} f(x, t) \mathrm{d}x \mathrm{d}t = \int_0^1 t^{\frac{d}{2}} \int_{\mathbb{B}^d} f(\sqrt{t} y, t) \mathrm{d}y \mathrm{d}t.$$

For $n = 0, 1, 2, \ldots$, let $\mathcal{V}_n(\mathbb{V}^{d+1}, W_{\beta,\gamma,\mu})$ denote the space of orthogonal polynomials of degree n in (x, t) variables with respect to the inner product $\langle \cdot, \cdot \rangle_{\beta,\gamma,\mu}$ on the paraboloid. Then $\dim \mathcal{V}_n(\mathbb{V}^{d+1}, W_{\beta,\gamma,\mu}) = \binom{n+d}{n}$. An orthogonal basis of this space can be given in terms of the Jacobi polynomials and the classical orthogonal polynomials on the unit ball [12]. We will also need the norms of these orthogonal polynomials.

Proposition 4.1. *Let* $\beta > -\frac{d+1}{2}$ *and* $\gamma > -1$. *Let* $\{P^m_{\mathbf{k}} : |\mathbf{k}| = m, \mathbf{k} \in \mathbb{N}^d_0\}$ *denote an orthonormal basis with parity of* $\mathcal{V}^d_m(\mathbb{B}^d, \varpi_\mu)$. *For* $0 \le m \le n$, *define*

$$\mathbf{Q}^n_{m,\mathbf{k}}(x, t) = P^{(m+\beta+\mu+\frac{d-1}{2},\gamma)}_{n-m}(1 - 2t)t^{\frac{m}{2}} P^m_{\mathbf{k}}\Big(\frac{x}{\sqrt{t}}\Big), \quad |\mathbf{k}| = m,\ 0 \le m \le n. \tag{4.5}$$

Then $\{\mathsf{Q}^n_{m,\mathbf{k}} : |\mathbf{k}| = m, 0 \le m \le n, \mathbf{k} \in \mathbb{N}^d_0\}$ *is an orthogonal basis of* $\mathcal{V}_n(\mathbb{V}^{d+1}, W_{\beta,\gamma,\mu})$. *Moreover, the norm square of* $\mathsf{Q}^n_{m,\ell}$ *is given by*

$$\mathbf{h}^{\beta,\gamma,\mu}_{m,n} = \langle \mathbf{Q}^n_{m,\mathbf{k}}, \mathbf{Q}^n_{m,\mathbf{k}} \rangle_{\beta,\gamma,\mu} = \frac{c_{\beta+\mu+\frac{d-1}{2},\gamma}}{c_{m+\beta+\mu+\frac{d-1}{2},\gamma}} h^{(m+\beta+\mu+\frac{d-1}{2},\gamma)}_{n-m}. \tag{4.6}$$

Proof. Using the parity of $P^m_{\mathbf{k}}$, it is not difficult to see that $\mathbf{Q}^n_{m,\mathbf{k}}$ is a polynomial of degree n in (x, t) variables. Let $\alpha = \beta + \mu + \frac{d-1}{2}$. Setting $y = x/\sqrt{t} \in \mathbb{B}^d$, we obtain

$$\mathbf{b}_{\beta,\gamma,\mu} \int_{\mathbb{V}^{d+1}} \mathbf{Q}^n_{m,\mathbf{k}}(x, t)\mathbf{Q}^{n'}_{m',\mathbf{k}'}(x, t)W_{\beta,\gamma,\mu}(x, t)\mathrm{d}x\mathrm{d}t = b_\mu \int_{\mathbb{B}^d} P^m_{\mathbf{k}}(y)P^{m'}_{\mathbf{k}'}(y)\varpi_\mu(y)\mathrm{d}y$$
$$\times c_{\alpha,\gamma} \int_0^1 P^{(m+\alpha,\gamma)}_{n-m}(1 - 2t)P^{(m'+\alpha,\gamma)}_{n'-m'}(1 - 2t)t^{\frac{m+m'}{2}+\alpha}(1 - t)^\gamma \mathrm{d}t.$$

Since $P^m_{\mathbf{k}}$ are orthonormal, it follows that the second integral in the right-hand side is non-zero only when $m = m'$, from which the orthogonality of $\mathbf{Q}^n_{m,\mathbf{k}}$ and the formula for $\mathbf{h}^{\beta,\gamma,\mu}_{m,n}$ follow from the corresponding properties of the Jacobi polynomials. □

We know that orthogonal polynomials on the solid cone and hyperboloid are eigenfunctions of a second order linear differential operator with the eigenvalues depending only on the degree of the polynomials [23, 24]. In particular, this means that all polynomials of degree n are eigenfunctions independent of the choice of bases. In contrast, the orthogonal polynomials on the solid paraboloid, as those on the surface of the paraboloid, do not possess this property. For polynomials $\mathbf{Q}^n_{m,\mathbf{k}}$, we can find a differential operator for which the eigenvalues depend on n and m but not on $\mathbf{k}$, as seen in the following analog of Proposition 3.2, where we assume $\beta = 0$. The latter assumption is consistent with $\beta = -\frac{1}{2}$ in Proposition 3.2 because $W_{\beta,\gamma,\mu}$ contains the factor $t^{\beta+\mu-\frac{1}{2}}$ when writing in the form (4.4) and, for $\mu = 0$, ϖ_0 is the Chebyshev weight function on the unit ball, or the projection of the surface measure of $\mathbb{S}^d$ onto $\mathbb{B}^d$.

Proposition 4.2. *Let $\beta = 0$, $\gamma > -1$ and $\mu > -\frac{1}{2}$. Then $\mathbf{Q}^n_{m,\mathbf{k}}$ in* (4.5) *satisfies the differential equation*

$$\begin{aligned}\Big[t(1-t)\partial_t^2 + (1-t)\langle x, \nabla_x\rangle \partial_t + \frac{1}{4}(1-t)\Delta_x \\ + \Big(\mu + \frac{d+1}{2}\Big)(1-t)\partial_t - \frac{\gamma+1}{2}(2t\partial_t + \langle x, \nabla_x\rangle)\Big] u \\ = -\Big(n(n+\mu+\gamma+\frac{d+1}{2}) - m\Big(n+\mu+\frac{\gamma+d}{2}\Big)\Big) u.\end{aligned} \tag{4.7}$$

Proof. Let $\alpha = \beta + \mu + \frac{d-1}{2}$. Set $g(t) = P_{n-m}^{(m+\alpha,\gamma)}(1-2t)$ and $H(x,t) = t^{\frac{m}{2}} P^m_{\mathbf{k}}(\frac{x}{\sqrt{t}})$, so that $\mathbf{Q}^n_{m,\mathbf{k}}(x,t) = g(t)H(x,t)$. Since $H(x,s^2)$ is a homogeneous polynomial of degree m in (x,s) and, for $t = s^2$, $2\sqrt{t}\frac{\partial}{\partial t} = \frac{\partial}{\partial s}$, it follows by Euler's formula for homogenous polynomials that

$$\Big(2t\frac{\partial}{\partial t} + \langle x, \nabla_x\rangle\Big) H = mH. \tag{4.8}$$

Furthermore, since $\sqrt{t}\frac{\partial}{\partial t}H(x,t) = \frac{\partial}{\partial s}H(x,s^2)$ is a homogeneous polynomial of degree $m-1$ in (x,s^2), applying (4.8) on $\sqrt{t}\frac{\partial}{\partial t}H$ and simplifying gives

$$2t\frac{\partial^2 H}{\partial t^2} + \langle x, \nabla_x\rangle \frac{\partial H}{\partial t} = (m-2)\frac{\partial H}{\partial t}. \tag{4.9}$$

Let $u = \mathbf{Q}^n_{m,\mathbf{k}}$. Then $u = g(t)H(x,t)$. Taking the derivative by the chain rule, a straightforward computation, using (4.8) once, shows that

$$\begin{aligned}t(1-t)\partial_{tt}u + (1-t)\langle x, \nabla_x\rangle \partial_t u =& (t(1-t)g''(t) + m(1-t)g'(t))H \\ &+ (1-t)g(t)\Big(t\frac{\partial^2 H}{\partial t^2} + \langle x, \nabla_x\rangle \frac{\partial H}{\partial t}\Big).\end{aligned}$$

The Jacobi polynomial satisfies the differential equation (3.7), so that g satisfies (3.7) with α replaced by $\alpha + m$ and n replaced by $n - m$, which leads to

$$\begin{aligned}t(1-t)\partial_{tt}u + (1-t)\langle x, \nabla_x\rangle \partial_t u + (1+\alpha-(\alpha+\gamma+2)t)\partial_t u \\ = -(n-m)(n+\alpha+\gamma+1)u - (\gamma+1)tg(t)\frac{\partial H}{\partial t} \\ + (1-t)g(t)\Big[t\frac{\partial^2 H}{\partial t^2} + \langle x, \nabla_x\rangle \frac{\partial H}{\partial t} + (1+\alpha)\frac{\partial H}{\partial t}\Big].\end{aligned} \tag{4.10}$$

The polynomial $P^m_{\mathbf{k}}$ satisfies a second order differential equation (4.1) with n replaced by m, from which it follows that H satisfies

$$(t\Delta_x - \langle x, \nabla_x\rangle^2 - (2\mu+d-1)\langle x, \nabla_x\rangle)H = -m(m+2\mu+d-1)H.$$

Now, applying (4.8) and (4.9), the square bracket in the right-hand side of (4.10) satisfies

$$\begin{aligned}[\dots] &= \frac{1}{4t}(\langle x, \nabla\rangle + 2\alpha + m)(m - \langle x, \nabla_x\rangle)H \\ &= \frac{1}{4t}(-\langle x, \nabla_x\rangle^2 - 2\alpha\langle x, \nabla_x\rangle + m(m+2\alpha))H = -\frac{1}{4}\Delta_x H,\end{aligned}$$

where in the last step we have used $2\alpha = 2\mu + d - 1$ for $\beta = 0$ and the differential equation satisfied by H. Substituting this into (4.10) and using (4.8) one more time, the resulted identity is simplified to give (4.7). □

Next we consider the reproducing kernel of $\mathcal{V}_n(\mathbb{V}^{d+1}, W_{\beta,\gamma,\mu})$. In terms of the basis (4.5), the kernel is given by

$$\mathbf{P}_n(W_{\beta,\gamma,\mu}; (x,t), (y,s)) = \sum_{m=0}^{n} \sum_{|\mathbf{k}|=m} \frac{\mathbf{Q}^n_{m,\mathbf{k}}(x,t)\mathbf{Q}^n_{m,\mathbf{k}}(y,s)}{\mathbf{h}^{\beta,\gamma,\mu}_{m,n}}.$$

With the help of the addition formula for the orthogonal polynomials on the unit ball, we can express this kernel in terms of the reproducing kernel $\mathbf{P}_n(U_{a,b}; \cdot, \cdot)$ on the parabola domain $\mathbb{U}$.

Theorem 4.3. *Let $d \ge 2$, $\mu \ge 0$ and $\gamma > -1$. Let $\alpha = \beta + \mu + \frac{d-1}{2}$. Then, for $(x,t) \in \mathbb{V}^{d+1}$, $(y,s) \in \mathbb{V}^{d+1}$ and $\beta > 0$,*

$$\begin{aligned}(4.11)\qquad &\mathbf{P}_n(W_{\beta,\gamma,\mu}; (x,t), (y,s)) \\ &= c\int_{[-1,1]^3} \mathbf{P}_n(U_{\gamma,\alpha}; (\sqrt{t}, t), (\sqrt{s}\xi(x,t,y,s;z,u), s)) \\ &\quad \times (1-z_1)^{\mu+\frac{d-1}{2}}(1+z_1)^{\beta-1}(1-z_2^2)^{\beta-\frac{1}{2}}(1-u^2)^{\mu-1}\mathrm{d}z\mathrm{d}u,\end{aligned}$$

where $c = c_{\mu+\frac{d-1}{2},\beta-1}c_\beta c_\mu$ and

$$\begin{aligned}\xi(x,t,y,s;z,u) &= \frac{1-z_1}{2}\xi_0(x,t,y,s;u) + \frac{1+z_1}{2}z_2, \\ \textit{with}\quad \xi_0(x,t,y,s;u) &= \frac{1}{\sqrt{st}}(\langle x,y\rangle + u\sqrt{t-\|x\|^2}\sqrt{s-\|y\|^2});\end{aligned}$$

furthermore, for $\beta = 0$,

$$\begin{aligned}(4.12)\qquad &\mathbf{P}_n(W_{0,\gamma,\mu}; (x,t), (y,s)) \\ &= c_\mu\int_{-1}^{1} \mathbf{P}_n(U_{\gamma,\mu+\frac{d-1}{2}}; (\sqrt{t}, t), (\sqrt{s}\xi_0(x,t,y,s;u), s))(1-u^2)^{\mu-1}\mathrm{d}u.\end{aligned}$$

In both cases, the identity holds for $\mu = 0$ under the limit (4.3).

Proof. Since $\|x\| \le t$ and $\|y\| \le s$, we see that $|\xi_0(x,t,,y,s,u)| \le 1$ by the Cauchy inequality. Consequently, $|\xi(x,t,y,s;z,u)| \le 1$, so that both variables in $\mathbf{P}_n(U_{\gamma,\beta+\frac{d-1}{2}})$ are elements of $\mathbb{U}$.

By (4.5) and the assumption that $P_{\mathbf{k}}^m$ is orthonormal, it follows from the addition formula (4.2) on the unit ball that, with $\alpha = \beta + \mu + \frac{d-1}{2}$,

$$\begin{aligned}
&\mathbf{P}_n(W_{\beta,\gamma,\mu};(x,t),(y,s)) \\
&\quad = c_\mu \int_{-1}^{1} \sum_{m=0}^{n} \frac{P_{n-m}^{(\alpha+m,\gamma)}(1-2t)P_{n-m}^{(\alpha+m,\gamma)}(1-2s)}{\mathbf{h}_{m,n}^{\beta,\gamma,\mu}} t^{\frac{m}{2}} s^{\frac{m}{2}} \\
&\qquad \times Z_m^{\mu+\frac{d-1}{2}}\Big(\frac{\langle x,y\rangle}{\sqrt{st}} + u\sqrt{1-\frac{\|x\|^2}{t}}\sqrt{1-\frac{\|y\|^2}{s}}\Big)(1-u^2)^{\mu-1}\mathrm{d}u.
\end{aligned}$$

If $\beta = 0$, then $\alpha = \mu + \frac{d-1}{2}$, so that the sum under the integral sign can be identified with $\mathbf{P}_n(U_{\gamma,\alpha})$ by (3.10) with $x_2 = t$, $y_2 = s$ and $y_1 = \xi_0(x,t,y,s;u)$. This proves (4.12). For $\beta > 0$, we increase the value of the index of Z_m^b from $\mu + \frac{d-1}{2}$ to $\alpha = \mu + \beta + \frac{d-1}{2}$ by (3.11) with $\lambda = \mu + \frac{d-1}{2}$ and $\sigma = \beta$, so that the sum under the integral sign becomes

$$\begin{aligned}
&c_{\mu+\frac{d-1}{2},\beta-1} c_\beta \int_{-1}^{1}\int_{-1}^{1} \sum_{m=0}^{n} \frac{P_{n-m}^{(\alpha+m,\gamma)}(1-2t)P_{n-m}^{(\alpha+m,\gamma)}(1-2s)}{\mathbf{h}_{m,n}^{\beta,\gamma,\mu}} t^{\frac{m}{2}} s^{\frac{m}{2}} \\
&\times Z_m^{\alpha}\Big(\frac{1-z_1}{2\sqrt{s}}\xi_0(x,t,y,s;u) + \frac{1+z_1}{2}z_2\Big)(1-z_1)^{\mu+\frac{d-1}{2}}(1+z_1)^{\beta-1}(1-z_2^2)^{\beta-\frac{1}{2}}\mathrm{d}z \\
&\quad = c_{\mu+\frac{d-1}{2},\beta-1} c_\beta \int_{-1}^{1}\int_{-1}^{1} \mathbf{P}_n(U_{\gamma,\alpha};(\sqrt{t},t),(\xi(x,t,y,s;z,u),s)) \\
&\qquad\qquad \times (1-z_1)^{\mu+\frac{d-1}{2}}(1+z_1)^{\beta-1}(1-z_2^2)^{\beta-\frac{1}{2}}\mathrm{d}z,
\end{aligned}$$

where the second step follows from (4.6) and (3.10). Putting the two displayed identities together, we have proved (4.11). □

If we allow $d = 1$, then $\mathbb{V}^2$ with $W_{0,\gamma,\mu}(x_1,x_2)$ should just be the domain $\mathbb{U}$ with $U_{\gamma,\mu}(x_1,x_2)$. We know that $\mathbf{P}_n(U_{\alpha,b})$ does not have a closed formula except in the case that one of its variables is $\mathbf{1} = (1,1)$. For $d \ge 2$, we can deduce accordingly a closed formula for $\mathbf{P}_n(W_{\beta,\gamma,\mu})$ on the hyperplane $t = 1$ of $(x,t) \in \mathbb{V}^{d+1}$. We state this formula for the kernel of the partial sum operator

$$\mathbf{K}_n(W_{\beta,\gamma,\mu},(x,t),(y,s)) = \sum_{m=0}^{n} \mathbf{P}_m(W_{\beta,\gamma,\mu};(x,t),(y,s))$$

by using the closed formula of $\mathbf{K}_n(U_{a,b};\mathbf{1},\cdot)$ in Theorem 2.3. We again state the result only for the case $\beta = 0$, for which the formula is relatively simple.

Corollary 4.4. *Let $d \ge 2$, $\gamma > -1$ and $\mu \ge 0$. Let $\tau = \mu + \frac{d-1}{2}$. Then, for $x \in \mathbb{B}^d$,*

$$\mathbf{K}_n(W_{0,\gamma,\mu};(x,1),(y,s)) = \frac{P_n^{(\gamma+\tau+1,\tau)}(1)}{h_n^{(\gamma+\tau+1,\tau)}} c_{\gamma+\tau+1,\tau} c_\mu \times \int_{[-1,1]^2} P_n^{(\gamma+\tau+1,\tau)}(z'(x,y,s,u,v)) w_{\gamma+\tau 1,\tau}(v)(1-u^2)^{\mu-1} \mathrm{d}u\mathrm{d}v,$$

where $z'(x,y,s,u,v) = z((\sqrt{s}\xi_0(x,1,y,s;u),s),v)$ or

$$z'(x,y,s,u,v) = 1-(1-v^{\,2})(1-\sqrt{s}\xi_0(x,1,y,s;u)) - \frac{1}{2}(1-v)^2(1-s).$$

This is a corollary of (4.12) and (2.14). A more involved closed form formula for $\beta > 0$ can be written down using (4.11) and (2.14).

4.3 Summability of Fourier orthogonal series. Denote by $\mathrm{proj}_n(W_{\beta,\gamma,\mu})$ the orthogonal projection operator

$$\mathrm{proj}_n(W_{\beta,\gamma,\mu}) : L^2(\mathbb{V}^{d+1}; W_{\beta,\gamma,\mu}) \mapsto \mathcal{V}_n(\mathbb{V}^{d+1}, W_{\beta,\gamma,\mu}).$$

In terms of the reproducing kernel $\mathbf{P}_n(W_{\beta,\gamma,\mu})$ of $\mathcal{V}_n(\mathbb{V}^{d+1}, W_{\beta,\gamma,\mu})$, we can write

$$\mathrm{proj}_n(W_{\beta,\gamma,\mu};f) = \mathsf{b}_{\beta,\gamma,\mu} \int_{\mathbb{V}_0^{d+1}} f(y,s)\mathbf{P}_n(W_{\beta,\gamma,\mu};(x,t),(y,s))W_{\beta,\gamma,\mu}(x,t)\mathrm{d}y\mathrm{d}s.$$

For $f \in L^2(\mathbb{V}^{d+1}; W_{\beta,\gamma,\mu})$, the Fourier orthogonal series of f on $\mathbb{V}^{d+1}$ is defined by

$$f = \sum_{n=0}^{\infty} \mathrm{proj}_n(W_{\beta,\gamma,\mu};f).$$

Recall that $\mathbb{U}_0$ denotes the curved portion of the boundary of the parabola domain $\mathbb{U}$. Analogously to Definition 3.5, we give the following definition:

Definition 4.5. Let $d \ge 2$, $\mu \ge 0$, $\beta \ge 0$ and $\gamma > -1$. Set $\alpha = \beta + \mu + \frac{d-1}{2}$. Let $g : \mathbb{U}_0 \times \mathbb{U} \mapsto \mathbb{R}$ such that, for each $t \in [0,1]$, the function $x \mapsto g((\sqrt{t},t);x)$ is in $L^1(\mathbb{U}; U_{\gamma,\alpha})$. For $(x,t) \in \mathbb{V}^{d+1}$ and $(y,s) \in \mathbb{V}^{d+1}$, define for $\beta > 0$

$$\mathbf{T}_{\beta,\gamma,\mu} g((x,t),(y,s)) := c \int_{[-1,1]^3} g((\sqrt{t},t),(\sqrt{s}\xi(x,t,y,s;z,u),s)) \times (1-z_1)^{\mu+\frac{d-1}{2}}(1+z_1)^{\beta-1}(1-z_2^2)^{\beta-\frac{1}{2}}(1-u^2)^{\mu-1}\mathrm{d}z\mathrm{d}u,$$

where $c = c_{\mu+\frac{d-1}{2},\beta-1} c_\beta c_\mu$, and define for $\beta = 0$,

$$\mathbf{T}_{0,\gamma,\mu} g((x,t),(y,s)) := c_\mu \int_{-1}^{1} g((\sqrt{t},t),(\sqrt{s}\xi_0(x,t,y,s;u),s))(1-u^2)^{\mu-1}\mathrm{d}u.$$

In both cases the definition holds under the limit (4.3) when $\mu = 0$.

By (4.11) and (4.12), the definition $\mathbf{T}_{\beta,\gamma,\mu}$ is motivated by

$$\mathbf{P}_n(W_{\beta,\gamma,\mu};(x,t),(y,s)) = \mathbf{T}_{\beta,\gamma,\mu}\mathbf{P}_n(U_{\gamma,\beta+\mu+\frac{d-1}{2}})((x,t),(y,s)). \tag{4.13}$$

For each fixed $(x,t) \in \mathbb{V}^{d+1}$, this is a bounded operator as seen below.

Proposition 4.6. *Let $\beta \ge 0$, $\mu \ge 0$ and $\gamma > -1$. Let $g : \mathbb{U}_0 \times \mathbb{U} \mapsto \mathbb{R}$ such that, for each $t \in [0,1]$, the function $g((\sqrt{t},t);\cdot)$ is in $L^1(\mathbb{U};U_{\gamma,\alpha})$ with $\alpha = \beta+\mu+\frac{d-1}{2}$. Then, for $(x,t) \in \mathbb{V}^{d+1}$,*

$$\begin{aligned}\int_{\mathbb{V}^{d+1}} &|\mathbf{T}_{\beta,\gamma,\mu}g((x,t),(y,s))|W_{\beta,\gamma,\mu}(y,s)\mathrm{d}y\mathrm{d}s \\ &\le c\int_{\mathbb{U}}|g((\sqrt{t},t),z)|U_{\gamma,\alpha}(z)\mathrm{d}z.\end{aligned} \tag{4.14}$$

Proof. We follow the approach for the proof of Proposition 3.6. Instead of the integral relation (3.14), we use the following identity for $h : [-1,1] \mapsto \mathbb{R}$ and $v \in \mathbb{B}^d$:

$$\begin{aligned} b_\mu \int_{\mathbb{B}^d}\int_{-1}^{1} & h(\langle u, v\,\rangle \\ &+ \sqrt{1-\|u\|^2}\sqrt{1-\|v\,\|^2}\,r)(1-r^2)^{\mu-1}\mathrm{d}r(1-\|u\|^2)^{\mu-\frac12}\mathrm{d}u \\ &= c_{\mu+\frac{d-1}{2}}\int_{1}^{1} h(t)(1-t^2)^{\mu+\frac{d-2}{2}}\mathrm{d}t. \end{aligned} \tag{4.15}$$

This identity is established in the proof of [21, Theorem 5.3] for a specific function h but the proof clearly holds for all generic h. Let $G(z) = g((t,\sqrt{t}),z)$ for $z \in \mathbb{U}$. Then, in the case of $\beta = 0$, we obtain

$$\begin{aligned} \mathbf{b}_{0,\gamma,\mu} & \int_{\mathbb{V}^{d+1}} |\mathbf{T}_{0,\gamma,\mu}g((x,t),(y,s))|W_{0,\gamma,\mu}(y,s)\mathrm{d}y\mathrm{d}s \\ &= \mathbf{b}_{0,\gamma,\mu}\int_0^1\int_{\mathbb{B}^d}|\mathbf{T}_{0,\gamma,\mu}g((x,t),(\sqrt{s}y',s))|(1-\|y'\|^2)^{\mu-\frac12}\mathrm{d}y's^{\mu+\frac{d-1}{2}}(1-s)^\gamma\mathrm{d}s \\ &\le \mathbf{b}_{0,\gamma,\mu}\int_0^1\int_{\mathbb{B}^d}c_\mu\int_{-1}^1|G(\sqrt{s}\xi_0(x,t,\sqrt{s}y',s;r),s)| \\ &\quad\times(1-r^2)^{\mu-1}\mathrm{d}r(1-\|y'\|^2)^{\mu-\frac12}\mathrm{d}y's^{\mu+\frac{d-1}{2}}(1-s)^\gamma\mathrm{d}s. \end{aligned}$$

Since $\xi_0(x,t,\sqrt{s}y',s;z) = \langle x',y'\rangle + z\sqrt{1-\|x'\|^2}\sqrt{1-\|y'\|^2}$ with $x' = x/\sqrt{t} \in \mathbb{B}^d$ and $y' \in \mathbb{B}^d$, we can apply (4.15) to bound the above inequality by

$$\begin{aligned} c\int_0^1\int_{-1}^1 & |G(\sqrt{s}u,s)|(1-u^2)^{\mu+\frac{d-2}{2}}\mathrm{d}us^{\mu+\frac{d-1}{2}}(1-s)^\gamma\mathrm{d}s \\ &= \mathbf{d}_{\gamma,\mu+\frac{d-1}{2}}\int_{\mathbb{U}}|G(z)|U_{\gamma,\mu+\frac{d-1}{2}}(z)\mathrm{d}z, \end{aligned}$$

which follows from changing variables $z_1 = \sqrt{s}u$ and $z_2 = s$ and the the last constant is determined by setting $G(z) = 1$. Consequently, this establishes (4.14) for $\beta = 0$.

Let now $\beta > 0$. Following the proof in the case of $\beta = 0$ by using (4.15), we obtain

$$\begin{aligned}
&\mathbf{b}_{\beta,\gamma,\mu}\int_{\mathbb{V}^{d+1}} |\mathbf{T}_{0,\gamma,\mu} g((x,t),(y,s))| W_{\beta,\gamma,\mu}(y,s)\mathrm{d}y\mathrm{d}s\\
&\le \mathbf{b}_{\beta,\gamma,\mu}\int_0^1\int_{-1}^1\int_{-1}^1\int_{\mathbb{B}^d} c_\mu \int_{-1}^1 |G(\sqrt{s}\xi(x,t,\sqrt{s}y',s;z,r),s)|(1-r^2)^{\mu-1}\mathrm{d}r\\
&\quad\times(1-\|y'\|^2)^{\mu-\frac12}\mathrm{d}y'(1-z_1)^{\mu+\frac{d-1}{2}}(1+z_1)^{\beta-1}(1-z_2)^{\beta-\frac12}\mathrm{d}z s^{\beta+\mu+\frac{d-1}{2}}(1-s)^\gamma\mathrm{d}s\\
&\le c\int_0^1\int_{-1}^1\int_{-1}^1\int_{-1}^1 \left|G\Big(\sqrt{s}\Big(\frac{1-z_1}{2}u+\frac{1+z_1}{2}z_2\Big),s\Big)\right|(1-u^2)^{\mu+\frac{d-2}{2}}\mathrm{d}u\\
&\quad\times(1-z_1)^{\mu+\frac{d-1}{2}}(1+z_1)^{\beta-1}(1-z_2)^{\beta-\frac12}s^{\beta+\mu+\frac{d-1}{2}}(1-s)^\gamma\mathrm{d}z\mathrm{d}s.
\end{aligned}$$

Besides the difference in their parameters, the last integral is the same as the quadruple integral in the proof of Proposition 3.6. Indeed, if we replace β by $\beta+\frac12$ and $\mu+\frac{d-1}{2}$ by $\frac{d-2}{2}$ in the above integral, then the two quadruple integrals are the same. Hence, we can estimate the above integral as in the proof of Proposition 3.6 to complete the proof of (4.14) for $\beta > 0$. This completes the proof. □

As in the case on the surface $\mathbb{V}_0^{d+1}$, we define a pseudo-convolution on $\mathbb{V}^{d+1}$.

Definition 4.7. Let $\beta \ge 0$, $\mu \ge 0$ and $\gamma > -1$. Let $g : \mathbb{U}_0 \times \mathbb{U} \mapsto \mathbb{R}$ such that, for each $t \in [0,1]$, the function $x \mapsto g((\sqrt{t},t);x)$ is in $L^1(\mathbb{U}; U_{\gamma,,\beta+\mu+\frac{d-1}{2}})$. For $f \in L^1(\mathbb{V}^{d+1}; W_{\beta,\gamma,\mu})$ and $(y,s) \in \mathbb{V}^{d+1}$, define

$$(f *_{\mathbb{V}} g)(y,s) = \mathbf{b}_{\beta,\gamma,\mu}\int_{\mathbb{V}^{d+1}} f(x,t)\mathbf{T}_{\beta,\gamma,\mu} g((x,t),(y,s))W_{\beta,\gamma,\mu}(x,t)\mathrm{d}x\mathrm{d}t.$$

By (4.13), it follows that the project operator onto $\mathcal{V}_n(\mathbb{V}^{d+1}, W_{\beta,\gamma,\mu})$ satisfies

$$\operatorname{proj}_n(W_{\beta,\gamma,\mu};f) = f *_{\mathbb{V}} \mathbf{P}_n(U_{\gamma,\beta+\mu+\frac{d-1}{2}}).$$

Proposition 4.8. *Let $\beta \ge 0$, $\mu \ge 0$ and $\gamma > -1$. Let $\alpha = \beta+\mu+\frac{d-1}{2}$. For $f \in L^p(\mathbb{V}^{d+1}; W_{\beta,\gamma,\mu})$, $1 \le p < \infty$, and $f \in C(\mathbb{V}^{d+1})$ for $p = \infty$,*

$$\|f *_{\mathbb{V}} g\|_{L^p(\mathbb{V}^{d+1};W_{\beta,\gamma,\mu})} \le c\|f\|_{L^p(\mathbb{V}^{d+1};W_{\beta,\gamma,\mu})} \max_{t\in[0,1]}\int_{\mathbb{U}} |g((\sqrt{t},t),z)|U_{\gamma,\alpha}(z)\mathrm{d}z.$$

Proof. Using (4.14), the proof follows exactly as that of Proposition 3.8. □

We now use this boundedness of the pseudo-convolution to study the Cesàro means of the Fourier orthogonal series on the solid paraboloid.

For $\delta > -1$, let $\mathbf{K}_n^\delta((x,t),(y,s))$ be the kernel of the Cesàro (C,δ) means $\mathbf{S}_n^\delta(W_{\beta,\gamma,\mu};f)$. Analogously to (3.15), we derive from (4.13) that

$$\mathbf{K}_n^\delta(W_{\beta,\gamma,\mu};(x,t),(y,s)) = \mathbf{T}_{\beta,\gamma,\mu}\mathbf{K}_n^\delta(U_{\gamma,\beta+\mu+\frac{d-1}{2}})((x,t),(y,s)). \tag{4.16}$$

Furthermore, in terms of the pseudo-convolution, we can write

$$\mathbf{S}_n^\delta(W_{\beta,\gamma,\mu};f,(x,t)) = f *_{\mathbb{V}} \mathbf{K}_n^\delta\left(U_{\gamma,\beta+\mu+\frac{d-1}{2}}\right)(x,t).$$

Theorem 4.9. *Let $d \ge 2$, $\beta \ge 0$, $\mu \ge 0$ and $\gamma > -1$. If $f \in C(\mathbb{V}^{d+1})$, then the Cesàro means $\mathbf{S}_n^\delta(W_{\beta,\gamma,\mu};f,(x,1))$ converge to $f(x,1)$ uniformly for $x \in \mathbb{B}^d$ provided $\delta > \beta+\gamma+\mu+\frac{d+2}{2}$.*

Proof. The convergence of $\mathbf{S}_n^\delta(W_{\beta,\gamma,\mu};f,(x,1))$ holds if and only if

$$\sup_{x\in\mathbb{B}^d}\int_{\mathbb{V}^{d+1}} |\mathbf{K}_n^\delta(W_{\beta,\gamma,\mu};(x,1),(y,s))| W_{\beta,\gamma,\mu}(y,s)\mathrm{d}y\mathrm{d}s$$

is bounded uniformly in n. By (4.16), the fact that $t = 1$ and the inequality (4.14) show that this follows from the boundedness of the L^1 norm of $\mathbf{K}_n^\delta(U_{\gamma,\beta+\mu+\frac{d-1}{2}};\mathbf{1},z)$, which holds for $\delta > \gamma+\beta+\mu+\frac{d-1}{2}+\frac{3}{2}$ by Theorem 2.4 and by Remark 2.1. □

Theorem 4.10. *Let $d \ge 2$, $\beta \ge 0$ and $\mu \ge 0$, $\gamma \ge \beta+\mu+\frac{d-1}{2}$. Let $f \in L^p(\mathbb{V}^{d+1}, W_{\beta,\gamma,\mu})$ for $1 \le p < \infty$ and $f \in C(\mathbb{V}^{d+1})$ for $p = \infty$. Then the Cesàro means $\mathbf{S}_n^\delta(W_{\beta,\gamma,\mu};f)$ satisfy:*

1. *if $\delta \ge 2\beta+2\mu+\gamma+d+3$, then $\mathbf{S}_n^\delta(W_{\beta,\gamma,\mu};f)$ is nonnegative if f is nonnegative;*
2. *if $\delta > \beta+\mu+\gamma+\frac{d+2}{2}$, then $\mathbf{S}_n^\delta(W_{\beta,\gamma,\mu};f)$ converge to f in $L^p(\mathbb{V}^{d+1};W_{\beta,\gamma,\mu})$, $1 \le p < \infty$, and in $C(\mathbb{V}^{d+1})$.*

Proof. Using the identity (4.16), the proof reduces to properties possessed by the kernel $\mathbf{K}_n^\delta(U_{\gamma,\beta+\mu+\frac{d-1}{2}})$ on the parabola domain $\mathbb{U}$. The detail follows exactly as in the proof of Theorem 3.10 and we leave it to interested readers. □

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