THE CAUCHY PROBLEM FOR OPERATORS WITH TRIPLE EFFECTIVELY HYPERBOLIC CHARACTERISTICS: IVRII'S CONJECTURE

By

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Abstract. Ivrii's conjecture asserts that the Cauchy problem is C^∞ well-posed for any lower order term if every singular point of the characteristic variety is effectively hyperbolic. An effectively hyperbolic singular point is at most a triple characteristic. If every characteristic is at most double, this conjecture has been proved in the 1980's. In this paper we prove the conjecture for the remaining cases, that is for operators with triple effectively hyperbolic characteristics.

1 Introduction

This paper is devoted to the Cauchy problem

$$\begin{cases} Pu = D_t^m u + \sum_{j=0}^{m-1} \sum_{|\alpha|+j\le m} a_{j,\alpha}(t,x) D_x^\alpha D_t^j u = 0, \\ D_t^j u(0,x) = u_j(x), \quad j = 0, \dots, m-1, \end{cases} \tag{1.1}$$

where $t \ge 0$, $x \in \mathbb{R}^d$ and the coefficients $a_{j,\alpha}(t,x)$ are C^∞ functions in a neighborhood of the origin of $\mathbb{R}^{1+d}$ and $D_x = (D_{x_1}, \dots, D_{x_d}) = D$, $D_{x_j} = (1/i)(\partial/\partial x_j)$ and $D_t = (1/i)(\partial/\partial t)$. The Cauchy problem (1.1) is C^∞ well-posed at the origin for $t \ge 0$ if one can find a $\delta > 0$ and a neighborhood U of the origin of $\mathbb{R}^d$ such that (1.1) has a unique solution $u \in C^\infty([0,\delta) \times U)$ for any $u_j(x) \in C^\infty(\mathbb{R}^d)$. We assume that the principal symbol of P

$$p(t,x,\tau,\xi) = \tau^m + \sum_{j=0}^{m-1} \sum_{|\alpha|+j=m} a_{j,\alpha}(t,x)\xi^\alpha \tau^j$$

is hyperbolic for $t \ge 0$, that is there exist $\delta' > 0$ and a neighborhood U' of the origin such that

$$p = 0 \text{ has only real roots in } \tau \text{ for } (t,x) \in [0,\delta') \times U' \text{ and } \xi \in \mathbb{R}^d \tag{1.2}$$

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which is indeed necessary in order that the Cauchy problem (1.1) is C^∞ well-posed near the origin for $t \geq 0$ ([19], [22]).

In [12], Ivrii and Petkov proved that if the Cauchy problem (1.1) is C^∞ well-posed for any lower order term near the origin for $t \geq 0$ (such an operator P is called strongly hyperbolic), then the Hamilton map F_p has a pair of non-zero real eigenvalues at every singular point of $p = 0$ located in $[0, \delta'') \times U'' \times (\mathbb{R}^{d+1} \setminus 0)$ ([12, Theorem 3]) with some $\delta'' > 0$ and a neighborhood U'' of $x = 0$. With $X = (t, x)$, $\Xi = (\tau, \xi)$ the Hamilton map F_p is defined by

$$F_p(X, \Xi) = \begin{bmatrix} \frac{\partial^2 p}{\partial X \partial \Xi} & \frac{\partial^2 p}{\partial \Xi \partial \Xi} \\ -\frac{\partial^2 p}{\partial X \partial X} & -\frac{\partial^2 p}{\partial \Xi \partial X} \end{bmatrix}.$$

A singular point of $p = 0$ where the Hamilton map F_p has a pair of non-zero real eigenvalues is called **effectively hyperbolic** ([6], [11]). Ivrii has conjectured that the converse would be also true, that is if every singular point of $p = 0$ is effectively hyperbolic then the Cauchy problem is C^∞ well-posed for any lower order term. If a singular point (t, x, τ, ξ) is effectively hyperbolic, then τ is a characteristic root of multiplicity at most 2 if $t > 0$ and at most 3 if $t = 0$ ([12, Lemma 8.1]). When every multiple characteristic root is at most double, the conjecture has been proved for some special class in [9], [21] and for the general case in [13, 14, 15], [23, 26].

For the case when we have a triple effectively hyperbolic characteristic, Ivrii has also proved in [9] that the conjecture is true if p admits a factorization $p = q_1 q_2$ near singular points with real smooth symbols q_i, transforming the original P, by means of operator powers of evolution generators, to an operator for which a parametrix can be constructed. In this case a singular point is effectively hyperbolic if and only if the Poisson bracket $\{q_1, q_2\}$ does not vanish there. If $m = 3$ it is clear that, for such a factorization to exist, it is necessary that the equation $p = 0$ has a C^∞ real root $\tau = \tau(t, x, \xi)$ near a conic neighborhood of singular points. A typical example is

$$p = q_1 q_2, \quad q_1 = \tau^2 - t|\xi|^2, \; q_2 = \tau$$

where q_1 is the Tricomi operator (symbol). Note that p has a complex characteristic root if $t < 0$. This is a common feature. In fact if $p(0, 0, \tau, \xi) = 0$ has a triple characteristic root and $F_p(0, 0, \tau, \xi) \neq O$, then p has necessarily non-real characteristic roots in the $t < 0$ side near $(0, 0, \xi)$ ([12, Lemma 8.1]).

When $m = 3$, without restrictions we can assume that p has the form

$$p = \tau^3 - a(t, x, \xi)|\xi|^2 \tau + b(t, x, \xi)|\xi|^3,$$

hence the condition (1.2) is reduced to $\Delta = 4\,a^3 - 27\,b^3 \geq 0$ for $(t, x) \in [0, \delta') \times U'$, $|\xi| = 1$. Also note that $(0, 0, 0, \bar{\xi})$ is a triple characteristic, then $(0, 0, 0, \bar{\xi})$ is effectively hyperbolic if and only if $\partial_t a(0,0,\bar{\xi}) \neq 0$, hence $a(t,x,\xi) > 0$ microlocally for small $t > 0$. In [2], Bernardi, Bove and Petkov investigated the case that p has a triple effectively hyperbolic characteristic but p may not be factorized. They studied P with the principal symbol

$$p = \tau^3 - (ta_2(t,x,\xi) + \alpha(x,\xi))|\xi|^2\tau + t^2 b_3(t,x,\xi)|\xi|^3$$

where $a_2(t,x,\xi) \geq c' > 0$, $\alpha(x,\xi) \geq 0$ and proved the conjecture for such P, deriving weighted energy estimates by a separating (multiplier) operator method. Note that $\Delta \geq c\,a^3$ holds with some $c > 0$ for this p. They also proved that if $b_3(0,0,\bar{\xi}) \neq 0$, then a smooth factorization $p = q_1 q_2$ is possible only if $\alpha(x,\bar{\xi}) = 0$ for all x near $x = 0$. This result was extended in [30, 31] such that the conjecture is true if $\Delta \geq c\,ta^2$ or if $\Delta \geq c\,t^2 a$ with some additional conditions, where after reducing the original equation to a first order 3×3 system, a symmetrizer S is constructed and used to get weighted energy estimates. These results are concerned with the case that p is strictly hyperbolic in $t > 0$, while in a general case, double effectively hyperbolic characteristics in $t > 0$ (where $\Delta = 0$) approaching a triple effectively hyperbolic characteristic on $t = 0$ might exist and we must handle them. Moreover, the following example ([30])

$$p(t,x,\tau,\xi) = \tau^3 - (t + \alpha(x))\xi^2\tau + (t^m/2 - t)\sqrt{\alpha(x)}\xi^3, \quad x, \xi \in \mathbb{R}, \tag{1.3}$$

where $\alpha(x) \geq 0$ and $\sqrt{\alpha(x)}$ is smooth, suggests that it is not enough just to study the zeros themselves of Δ. Indeed since

$$\Delta = (t - 2\alpha)^2(4t + \alpha) + 27t^{m+1}\alpha(1 - t^{m-1}/4)$$

so that $\Delta > 0$ for small $t > 0$, while

$$\Delta = 27 \cdot 2^{m+1}\alpha^{m+2}(1 - 2^{m-3}\alpha^{m-1})$$

if $t = 2\alpha$, hence one has no estimate such as $\Delta \geq c\,t^k(t + \alpha)^q$ with $c > 0$ for small $\alpha > 0$ if $m > k + q - 2$.

In [29] we employed a new idea which is to diagonalize the symmetrizer S mentioned above so that the system is transformed to a system with a diagonal symmetrizer. We see that three diagonal entries (the eigenvalues of S) are bounded from below by Δ/a, a, 1 respectively and we recognize here a close relation between the diagonal symmetrizer and two discriminants Δ of $p = 0$ and $\Delta'(= a)$ of $\partial_\tau p = 0$. In example (1.3) we see $\Delta/a \geq (t - 2\alpha)^2$ which looks like $\tau^2 - (\Delta/a)|\xi|^2$ has double effectively hyperbolic characteristics on $t = 2\alpha$ though $\Delta \neq 0$ there (see [26, 27]). When the coefficients of p depend only on t, the behavior of $\Delta(t,\xi)/a(t,\xi)$ can be analyzed relatively easily. Writing $\Delta(t,\xi) = e\prod(t - \nu_j(\xi))$ and dividing $[0, T]$ into subintervals with the end points $\mathrm{Re}\,\nu_j(\xi)$ we can obtain suitable estimates of Δ/a from below in each subinterval. In particular, in this way, we have proved the conjecture for the t dependent case [29, Theorem 4.1]. In this paper we extend this idea and apply it to the general case of which an outline is given in the next section.

For this class of operators we have always a loss of regularity, so the way to obtain microlocal energy estimates for operators of order m from that of order 2 or 3 and the way to prove local C^∞ well-posed results from such obtained microlocal energy estimates with loss of regularity is not so straightforward.

Finally we note that if there is a triple characteristic which is not effectively hyperbolic, the Cauchy problem is not well-posed in any Gevrey class of order $s > 2$ in general, even though the subprincipal symbol vanishes identically ([3]).

In this paper we prove

Theorem 1.1. *Assume* (1.2). *If every singular point* $(0, 0, \tau, \xi)$, $|(\tau, \xi)| \neq 0$ *of* $p = 0$ *is effectively hyperbolic, then for any* $a_{j,\alpha}(t, x)$ *with* $j + |\alpha| \leq m - 1$, *which are* C^∞ *in a neighborhood of* $(0, 0)$, *there exist* $\delta > 0$, *a neighborhood* U *of the origin and* $n > 0$, $\ell > 0$ *such that for any* $s \in \mathbb{R}$ *and any* f *with* $t^{-n+1/2}\langle D\rangle^s f \in L^2((0, \delta) \times \mathbb{R}^d)$ *there exists* u *with* $t^{-n-1/2}\langle D\rangle^{-\ell+s+m-j} D_t^j u \in L^2((0, \delta) \times \mathbb{R}^d)$, $j = 0, 1, \ldots, m - 1$, *satisfying* $Pu = f$ *in* $(0, \delta) \times U$.

Here $\langle D\rangle$ stands for $\sqrt{1 + |D|^2}$ and n, ℓ are given by

$$n = 12\sqrt{2}\, \sup \frac{|P_{\rm sub}(0, 0, \tau, \xi)|}{e(0, 0, \tau, \xi)} + \bar{C}^*, \qquad \ell = k\,(n + 2),$$

where $P_{\rm sub}$ denotes the subprincipal symbol and $e(0, 0, \tau, \xi)$ is the positive real eigenvalue of $F_p(0, 0, \tau, \xi)$, and the supremum is taken over all singular points $(0, 0, \tau, \xi)$ with $|(\tau, \xi)| = 1$ of $p = 0$ and $\bar{C}^*$ is a constant depending only on the principal symbol p. Here k is the maximal number of singular points $(0, 0, \tau, \xi)$ of $p = 0$ with $|(\tau, \xi)| = 1$, hence $k \leq [m/2]$. For a more detailed estimate of $\bar{C}^*$ see (3.14) and (10.2) below. The constant $12\sqrt{2}$ may not be the best.

Theorem 1.2. *Under the same assumption as in Theorem 1.1, for any* $a_{j,\alpha}(t, x)$ *with* $j + |\alpha| \leq m - 1$, *which are* C^∞ *in a neighborhood of* $(0, 0)$, *there exist* $\delta > 0$ *and a neighborhood* U *of the origin such that for any* $u_j(x) \in C_0^\infty(\mathbb{R}^d)$, $j = 0, 1, \ldots, m-1$, *there exists* $u(t, x) \in C^\infty([0, \delta) \times U)$ *satisfying* (1.1) *in* $[0, \delta) \times U$. *If* $u(t, x) \in C^\infty([0, \delta) \times U)$ *with* $\partial_t^j u(0, x) = 0$, $j = 0, 1, \ldots, m - 1$, *satisfies* $Pu = 0$ *in* $[0, \delta) \times U$, *then* $u = 0$ *in a neighborhood of* $(0, 0)$.

Proof. Compute

$$u_j(x) = D_t^j u(0, x)$$

for $j = m, m+1, \dots$ from $u_j(x), j = 0, 1, \dots, m-1$ and the equation $Pu = 0$. By a Borel's lemma there is $w(t, x) \in C_0^\infty(\mathbb{R}^{1+d})$ such that

$$D_t^j w(0, x) = u_j(x) \quad \text{for all } j \in \mathbb{N}.$$

Since

$$(D_t^j P w)(0, x) = 0 \quad \text{for all } j \in \mathbb{N}$$

it is clear that $t^{-n+1/2}\langle D\rangle^s P w \in L^2((0, \delta) \times \mathbb{R}^d)$ for any s. Thanks to Theorem 1.1 there exists v with $t^{-n-1/2}\langle D\rangle^{-\ell+s+m-j} D_t^j v \in L^2((0, \delta) \times \mathbb{R}^d)$, $j = 0, 1, \dots, m-1$ satisfying $Pv = -Pw$ in $(0, \delta) \times U$. Since $D_t^k v \in L^2((0, \delta) \times \mathbb{R}^d)$ for any k, hence $D_t^j v(0, x) = 0, j = 0, 1, \dots, m-1$, we conclude that $u = v + w$ is a desired solution. Local uniqueness follows from Theorem 13.4 below because $\partial_t^k u(0, x) = 0$ for any $k \in \mathbb{N}$ by $Pu = 0$. □

2 Outline of the proof of Theorem 1.1

As noted in the Introduction, if a singular point (t, x, τ, ξ) of $p = 0$ is effectively hyperbolic, then τ is a characteristic root of multiplicity at most 3. This implies that it is essential to study the third order operator P:

$$P = D_t^3 + \sum_{j=1}^{3} a_j(t, x, D)\langle D\rangle^j D_t^{3-j} \tag{2.1}$$

which is a differential operator in t with coefficients $a_j \in S^0$, classical pseudodifferential operators of order 0, where $\langle D\rangle = \mathrm{op}((1 + |\xi|^2)^{1/2})$. One can reduce P to the case with $a_1(t, x, D) = 0$ and hence the principal symbol is

$$p(t, x, \tau, \xi) = \tau^3 - a(t, x, \xi)\langle\xi\rangle^2\tau - b(t, x, \xi)\langle\xi\rangle^3. \tag{2.2}$$

All characteristic roots being real for $t \geq 0$ implies that

$$\Delta = 4\, a(t, x, \xi)^3 - 27\, b(t, x, \xi)^2 \geq 0, \quad (t, x, \xi) \in [0, T) \times U \times \mathbb{R}^d. \tag{2.3}$$

Assume that $p(0, 0, \tau, \bar{\xi}) = 0$ has a triple characteristic root $\tau = \bar{\tau}$, necessarily $\bar{\tau} = 0$. The singular point $(0, 0, \bar{\tau}, \bar{\xi})$ is effectively hyperbolic if and only if

$$\partial_t a(0, 0, \bar{\xi}) \neq 0, \tag{2.4}$$

hence one can write $a = e(t + \alpha(x, \xi))$ where $e > 0$ and $\alpha \geq 0$. From conditions (2.3) and (2.4) the discriminant Δ is essentially a third order polynomial in t. In Section 3, for regularized $a_\epsilon = e(t+\alpha+\epsilon^2)$ and the corresponding discriminant Δ_ϵ we construct a smooth ψ_ϵ such that

$$\Delta_\epsilon \geq c\min\{t^2, (t-\psi_\epsilon)^2\}(t+\rho_\epsilon), \quad t \geq 0 \tag{2.5}$$

where $\rho_\epsilon = \alpha + \epsilon^2$. In Section 4, introducing a large parameter M, we localize the coefficients near reference point $(0, \bar{\xi})$ replacing (x, ξ) by localized coordinates $\chi(M^2x)x$, $\chi(M^2(\xi/\langle\xi\rangle - \bar{\xi}))(\xi - \bar{\xi}\langle\xi\rangle) + \bar{\xi}\langle\xi\rangle$ and taking $\epsilon = M^{1/2}\langle\xi\rangle^{-1/2}$ where $\chi \in C_0^\infty$ is 1 near 0. At this point related symbols are localized near $(0, \bar{\xi})$ (defined in $\mathbb{R}^d \times \mathbb{R}^d$) and (2.5) yields

$$\Delta/a \geq c\min\{t^2, (t-\psi)^2 + M\rho\langle\xi\rangle^{-1}\}, \quad t \geq 0. \tag{2.6}$$

We also estimate such localized symbols in terms of the localized ρ. In particular, we show that

$$|\partial_x^\alpha \partial_\xi^\beta \psi(x,\xi)| \precsim \rho(x,\xi)^{1-|\alpha+\beta|/2}\langle\xi\rangle^{-|\beta|}$$

where, from now on, $A \precsim B$ means that A is bounded by a constant, independent of parameters, times B.

One of the main arguments in the paper is to reduce the original equation to a first order 3×3 system with diagonal symmetrizer. With $U = {}^t(D_t^2u, \langle D\rangle D_tu, \langle D\rangle^2u)$ the equation $Pu = f$ is reduced to $D_tU = A(t,x,D)\langle D\rangle U + B(t,x,D)U + F$ where $A, B \in S^0$, $F = {}^t(f, 0, 0)$ and

$$A(t,x,\xi) = \begin{bmatrix} 0 & a & b \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let S be the Bézout matrix of p and $\partial p/\partial\tau$, that is

$$S(t,x,\xi) = \begin{bmatrix} 3 & 0 & -a \\ 0 & 2a & 3b \\ -a & 3b & a^2 \end{bmatrix};$$

then S is positive semidefinite and symmetrizes A, that is SA is symmetric. We now diagonalize S by an orthogonal matrix T so that $T^{-1}ST = \Lambda$. Then with $V = \mathrm{op}(T^{-1})U$ the system is reduced to, roughly,

$$D_tV = A^T(t,x,D)\langle D\rangle V + \tilde{B}(t,x,D)V \tag{2.7}$$

where Λ is diagonal and symmetrizes $A^T(t,x,\xi) = T^{-1}AT$. This reduction is carried out in Section 9 after examining $T(t,x,\xi)$ carefully in Section 5.

Note that $\Lambda = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3)$ where $0 < \lambda_1 < \lambda_2 < \lambda_3$ are the eigenvalues of S. As mentioned in the Introduction, a significant feature of λ_j is the following:

$$\Delta/a \precsim \lambda_1 \precsim a^2, \quad \lambda_2 \simeq a, \quad \lambda_3 \simeq 1, \quad t \geq 0. \tag{2.8}$$

Section 5 is devoted mainly to estimate derivatives of λ_j and we prove

$$|\partial_x^\alpha \partial_\xi^\beta \lambda_j| \precsim a^{3-j-|\alpha+\beta|/2} \langle\xi\rangle^{-|\beta|}, \quad t \geq 0,\ j = 1, 2, 3$$

which also gives detailed information on the derivatives of T. Since (2.7) is a system with diagonal symmetrizer Λ, a natural energy would be

$$(\mathrm{op}(\Lambda)V, V) = \sum_{j=1}^{3} (\mathrm{op}(\lambda_j)V_j, V_j)$$

and (2.6) and (2.8) suggest that a weighted energy with a scalar pseudodifferential weight $\mathrm{op}(t^{-n}\phi^{-n})$,

$$\phi = \omega + t - \psi, \quad \omega = \sqrt{(t-\psi)^2 + M\rho\langle\xi\rangle^{-1}},$$

would work, where $\mathrm{op}(\phi^{-n})$ is chosen after the weight employed for studying double effectively hyperbolic characteristics in [26] (see also [27]), and satisfies

$$\partial_t(t\phi) = \kappa(t\phi), \quad \kappa = 1/t + 1/\omega.$$

In Section 6, to treat these weight functions, we introduce a σ temperate (uniformly in M) metric

$$g = M^{-1}(\langle\xi\rangle|dx|^2 + \langle\xi\rangle^{-1}|dx|^2)$$

and prove that $\omega^s \in S(\omega^s, g)$, $\phi^s \in S(\phi^s, g)$ with $s \in \mathbb{R}$, uniformly in $t \geq 0$, estimating derivatives of ω, ϕ. In Section 7 we prove that ω, ϕ and λ_j are σ, g temperate uniformly in $t \in [0, M^{-4}]$ (in this paper such functions are called admissible weights for g, while σ is reserved for denoting a certain function). This fact enables us to apply the Weyl calculus of pseudodifferential operators (see [7, Chapter 18]) to $\mathrm{op}(\phi^s)$, $\mathrm{op}(\omega^s)$ and $\mathrm{op}(\lambda_j^s)$ with $s \in \mathbb{R}$, for example we have $\mathrm{op}(\phi^{s_1})\mathrm{op}(\phi^{s_2}) = \mathrm{op}(\phi^{s_1}\#\phi^{s_2})$ where $\phi^{s_1}\#\phi^{s_2} \in S(\phi^{s_1+s_2}, g)$. In Section 8 we prove some basic facts on inverses and L^2 bounds of pseudodifferential operators associated to the metric g which enables us, for example, to write

$$\mathrm{op}(\phi^{s_1})\mathrm{op}(\phi^{s_2}) = \mathrm{op}(1+r)\mathrm{op}(\phi^{s_1+s_2}) \quad \text{with } r \in S(M^{-1}, g).$$

We also give lower bounds of $\mathrm{op}(\lambda_j)$ here.

In Section 10, applying the Weyl calculus of pseudodifferential operators we estimate the weighted energy

$$\mathrm{Re}\, e^{-\theta t}(\mathrm{op}(\Lambda)\mathrm{op}(t^{-n}\phi^{-n})V, \mathrm{op}(t^{-n}\phi^{-n})V)$$

and derive energy estimates for any lower order term (including the term $M\langle D\rangle D_t$ because we have added this to the original operator at the beginning) (Proposition 10.1). In Section 11, using energy estimates for the system coming from the adjoint operator of P, which is obtained repeating exactly the same arguments, we prove an existence result of the Cauchy problem for $P_{\bar{\xi}}$, which is the localized operator near $(0, \bar{\xi})$ of the original P (Theorem11.1). In Section 12, in order to sum up such obtained solutions (which might be considered as a microlocal solution to the Cauchy problem near $(0, \bar{\xi})$), we prove that the wave front sets of such solutions propagate with finite speed (Proposition 12.4). A more precise picture of the propagation of a wave front set of solutions is also proved applying the same arguments (Theorem 12.1). Finally, in Section 13, using the propagation results in Section 12 we complete the proof of Theorem 1.1.

3 Construction of $\psi(x, \xi)$

We study third order operators P of the form (2.1) with $a_1(t, x, D) = 0$, hence the principal symbol has the form (2.2) where $a(t, x, \xi)$ and $b(t, x, \xi)$ are homogeneous of degree 0 in ξ and assumed to satisfy (2.3) with some $T > 0$ and some neighborhood U of the origin of $\mathbb{R}^d$. Assume that $p(t, x, \tau, \xi)$ has a triple characteristic root $\tau = 0$ at $(0, 0, \bar{\xi})$, $|\bar{\xi}| = 1$ and $(0, 0, 0, \bar{\xi})$ is effectively hyperbolic. It is clear that $a(0, 0, \bar{\xi}) = 0$ and $b(0, 0, \bar{\xi}) = 0$. Since $\partial_x^\alpha \partial_\xi^\beta a(0, 0, \bar{\xi}) = 0$ for $|\alpha + \beta| = 1$ and $\partial_x^\alpha \partial_\xi^\beta b(0, 0, \bar{\xi}) = 0$ for $|\alpha + \beta| \le 2$, by (2.3) (see Lemma 4.2 below) it is easy to see that

$$\det(\lambda - F_p(0, 0, 0, \bar{\xi})) = \lambda^{2d}(\lambda^2 - \{\partial_t a(0, 0, \bar{\xi})\}^2), \tag{3.1}$$

hence $(0, 0, 0, \bar{\xi})$ is effectively hyperbolic if and only if $\partial_t a(0, 0, \bar{\xi}) \neq 0$. Then there is a neighborhood $\mathcal{U}$ of $(0, 0, \bar{\xi})$ in which one can write

$$a(t, x, \xi) = e(t, x, \xi)(t + \alpha(x, \xi))$$

where $e > 0$ in $\mathcal{U}$. Note that $\alpha(x, \xi) \ge 0$ near $\bar{\xi}$ because $a(t, x, \xi) \ge 0$ in $[0, T) \times U \times \mathbb{R}^d$.

3.1 A perturbed discriminant. Introducing a small parameter ϵ we consider

$$\tau^3 - e(t,x,\xi)(t+\alpha(x,\xi)+\epsilon^2)|\xi|^2\tau - b(t,x,\xi)\,|\xi|^3 \\ = \tau^3 - a(t,x,\xi,\epsilon)|\xi|^2 - b(t,x,\xi)|\xi|^3. \tag{3.2}$$

From now on we write $b(X)$ or $a(X,\epsilon)$ and so on to make clearer that these symbols are defined in some conic (in ξ) neighborhood of $\bar{X} = (0,\bar{\xi})$ or $(\bar{X},0)$. Consider the discriminant of (3.2); $\Delta(t,X,\epsilon) = 4\,a^3(t,X,\epsilon) - 27\,b^2(t,X)$.

Lemma 3.1. *One can write*

$$\Delta = \tilde{e}(t,X,\epsilon)(t^3 + a_1(X,\epsilon)t^2 + a_2(X,\epsilon)t + a_3(X,\epsilon))$$

in a neighborhood of $(0,\bar{X},0)$ *where* $a_j(\bar{X},0) = 0$, $j = 1,2,3$ *and* $\tilde{e} > 0$.

Proof. It is clear that $\partial_t^k a^3(0,\bar{X},0) = 0$ for $k = 0,1,2$ and $\partial_t^3 a^3(0,\bar{X},0) \neq 0$. We show that $\partial_t b(0,\bar{X},0) = 0$. Suppose the contrary and hence

$$b(t,\bar{X},0) = t(b_1 + tb_2(t)) \quad \text{with } b_1 \neq 0.$$

Since $a(t,\bar{X},0) = c\,t$ with $c > 0$, then $\Delta(t,\bar{X},0) = 4\,c^3\,t^3 - 27\,b(t,\bar{X},0)^2 \geq 0$ leads to a contradiction. Thus $\partial_t^k\Delta(0,\bar{X},0) = 0$ for $k = 0,1,2$ and $\partial_t^3\Delta(0,\bar{X},0) \neq 0$. Then from the Malgrange preparation theorem (e.g., [8, Theorem 7.5.5]) one can conclude the assertion. □

Introducing

$$\rho(X,\epsilon) = \alpha(X) + \epsilon^2 \tag{3.3}$$

one can also write

$$\Delta = 4e^3(t+\rho)^3 - 27b^2 = 4e^3\{(t+\rho)^3 - 27b^2/(4e^3)\} = 4e^3\{(t+\rho)^3 - \hat{b}^2\}$$

with $\hat{b} = 3\sqrt{3}\,b/2e^{3/2}$. Denoting

$$\hat{b}(t,X) = \sum_{j=0}^{2} \hat{b}_j(X)t^j + \hat{b}_3(t,X)t^3,$$

where $\hat{b}_0(\bar{X}) = \hat{b}_1(\bar{X}) = 0$ which is clear from the proof of Lemma 3.1, one can write

$$\Delta/\tilde{e} = \bar{\Delta} = t^3 + a_1(X,\epsilon)t^2 + a_2(X,\epsilon)t + a_3(X,\epsilon) \\ = E\left\{(t+\rho)^3 - \Big(\sum_{j=0}^{2}\hat{b}_j(X)t^j + \hat{b}_3(t,X)t^3\Big)^2\right\} \tag{3.4}$$

with $E(t,X,\epsilon) = 4e^3/\tilde{e}$. Here note that $E(0,\bar{X},0) = 1$.

Lemma 3.2. *There is a neighborhood V of $\bar{X}$ such that $|\hat{b}_1(X)| \le 4\,\alpha^{1/2}(X)$ for $X \in V$.*

Proof. It is clear that $|\hat{b}_0(X)| \le \alpha^{3/2}(X)$. If $\alpha(X) = 0$ then the assertion is obvious. Assume $\alpha(X) \neq 0$. Since

$$(t + \alpha(X))^3 \ge \Big(\sum_{j=0}^{2} \hat{b}_j(X) t^j + \hat{b}_3(t, X) t^3 \Big)^2, \quad 0 \le t \le T, \tag{3.5}$$

choosing $t = 3\alpha(X) \le T$ and writing $\alpha = \alpha(X)$ it follows from (3.5) that

$$8\alpha^{3/2} \ge |\hat{b}_0(X) + 3\hat{b}_1(X)\alpha| - C\alpha^2, \ge 3|\hat{b}_1(X)|\alpha - C\alpha^2 - \alpha^{3/2},$$

hence the assertion is clear because $\alpha(\bar{X}) = 0$. □

Lemma 3.3. *In a neighborhood of $(\bar{X}, 0)$ we have $a_j(X, \epsilon) = O(\rho(X, \epsilon)^j)$ for $j = 1, 2, 3$. More precisely,*

$$\begin{aligned} a_1(X, \epsilon) &= E(0, X, \epsilon)(3\rho(X, \epsilon) - \hat{b}_1^2(X)) + O(\rho^{3/2}), \\ a_2(X, \epsilon) &= E(0, X, \epsilon)(3\rho^2(X, \epsilon) - 2\hat{b}_0(X)\hat{b}_1(X)) + O(\rho^{3/2}), \\ a_3(X, \epsilon) &= E(0, X, \epsilon)(\rho^3(X, \epsilon) - \hat{b}_0^2(X)). \end{aligned}$$

Proof. Since $\bar{\Delta}(0, X, \epsilon) \ge 0$ it follows from (3.4) that

$$a_3(X, \epsilon) = E(0, X, \epsilon)(\rho(X, \epsilon)^3 - \hat{b}_0(X)^2) \ge 0$$

hence $\hat{b}_0 = O(\rho^{3/2})$ and consequently $a_3(X, \epsilon) = O(\rho^3)$. From (3.4)

$$a_2(X, \epsilon) = \partial_t E(0, X, \epsilon) a_3(X, \epsilon) + E(0, X, \epsilon)(3\rho^2(X, \epsilon) - 2\hat{b}_0(X)\hat{b}_1(X)).$$

Since $\hat{b}_0(X)\hat{b}_1(X) = O(\rho^2)$ by Lemma 3.2 hence the above equality shows the assertion for $a_2(X, \epsilon)$. Finally from (3.4) again

$$\begin{aligned} 2a_1(X, \epsilon) = \partial_t^2 E(0, X, \epsilon) a_3(X, \epsilon) &+ 2\partial_t E(0, X, \epsilon)(3\rho^2(X, \epsilon) - 2\hat{b}_0(X)\hat{b}_1(X)) \\ &+ 2E(0, X, \epsilon)(3\rho(X, \epsilon) - \hat{b}_1(X)^2 - 2\hat{b}_0(X)\hat{b}_2(X)) \end{aligned}$$

and from Lemma 3.2 one concludes the assertion for $a_1(X, \epsilon)$. □

3.2 Lower bound of a perturbed discriminant.

Denote

$$\nu(X, \epsilon) = \inf\{t \mid \bar{\Delta}(t, X, \epsilon) > 0\} \tag{3.6}$$

and hence $\bar{\Delta}(\nu, X, \epsilon) = 0$. First check that $\nu(X, \epsilon) \le 0$. Suppose the contrary $\nu(X, \epsilon) = \nu > 0$. Since $\bar{\Delta}(t, X, \epsilon) \ge 0$ for $t \ge 0$ one can write $\bar{\Delta}(t) = (t - \nu)^2(t - \tilde{\nu})$ with a real $\tilde{\nu}$ where $\tilde{\nu} \neq \nu$ and $\tilde{\nu} \le 0$. Therefore we have $\bar{\Delta}(t) > 0$ in $\tilde{\nu} < t < \nu$ which is incompatible with the definition of ν. Write

$$\bar{\Delta}(t, X, \epsilon) = (t - \nu(X, \epsilon))(t^2 + A_1(X, \epsilon)t + A_2(X, \epsilon))$$

where $A_1 = \nu + a_1$. Here we prepare following lemma.

Lemma 3.4. *One can find a neighborhood $\mathfrak{U}$ of $(\bar{X}, 0)$ such that for any $(X, \epsilon) \in \mathfrak{U}$ there is $j \in \{1, 2, 3\}$ such that $|\nu_j(X, \epsilon)| \ge \rho(X, \epsilon)/9$ where*

$$\bar{\Delta}(t, X, \epsilon) = \prod_{j=1}^{3}(t - \nu_j(X, \epsilon)).$$

Proof. First show that there is $1/3 < \delta < 1/2$ such that

$$\max\{|\rho^3 - \hat{b}_0^2|^{1/3},\ |\rho^2 - 2\hat{b}_0\hat{b}_1/3|^{1/2},\ |\rho - \hat{b}_1^2/3|\} \ge \delta^2\rho. \tag{3.7}$$

In fact denoting $f(\delta) = 2(1 - \delta^6)^{1/2}(1 - \delta^2)^{1/2}/\sqrt{3} - 1 - \delta^4$ it is easy to check that $f(1/3) > 0$ and $f(1/2) < 0$. Take $1/3 < \delta < 1/2$ such that $f(\delta) = 0$. If $|\rho^3 - \hat{b}_0^2|^{1/3} < \delta^2\rho$ and $|\rho - \hat{b}_1^2/3| < \delta^2\rho$, then

$$|\hat{b}_0| \ge (1 - \delta^6)^{1/2}\rho^{3/2} \quad \text{and} \quad |\hat{b}_1| \ge \sqrt{3}(1 - \delta^2)^{1/2}\rho^{1/2},$$

hence

$$|\rho^2 - 2\hat{b}_0\hat{b}_1/3| \ge 2|\hat{b}_0\hat{b}_1|/3 - \rho^2 \ge (f(\delta) + \delta^4)\rho^2 = \delta^4\rho^2.$$

Thus (3.7) is proved. Thanks to Lemma 3.3, taking $E(0, \bar{X}, 0) = 1$ and $1/3 < \delta$ into account, one can find a neighborhood $\mathfrak{U}$ of $(\bar{X}, 0)$ such that

$$\max\{3|a_1(X, \epsilon)|,\ (3^3|a_2(X, \epsilon)|)^{1/2},\ (3^6|a_3(X, \epsilon)|)^{1/3}\} \ge \rho, \quad (X, \epsilon) \in \mathfrak{U}.$$

Then the assertion follows from the relations between $\{\nu_i\}$ and $\{a_i\}$. □

Lemma 3.5. *Denote ν defined in (3.6) by ν_1 and by ν_j, $j = 2, 3$ the other roots of $\bar{\Delta} = 0$. Then one can find a neighborhood $\mathfrak{U}$ of $(\bar{X}, 0)$ and $c_i > 0$ such that*

$$\text{if} \quad \nu_1 + a_1 < 2c_1\rho, \quad (X, \epsilon) \in \mathfrak{U} \quad \text{then} \quad |\nu_1 - \nu_j| \ge c_2\rho, \quad j = 2, 3. \tag{3.8}$$

In particular, $\nu_1(X, \epsilon)$ is smooth in $\mathfrak{U} \cap \{\nu_1 + a_1 < 2c_1\rho\}$.

Proof. Write

$$\bar{\Delta}(t) = \prod_{j=1}^{3}(t - \nu_j) = (t - \nu_1)((t + A_1/2)^2 - D) \tag{3.9}$$

so that $\mathrm{Re}\,\nu_j = -A_1/2$, $j = 2, 3$ where $A_1 = \nu_1 + a_1$. Take $c_1 = 1/27$ and assume $A_1 < 2c_1\rho$. First note that if $\mathrm{Re}\,\nu_j \geq c_1\rho$, $j = 2, 3$, it is clear that $|\nu_1 - \nu_j| \geq |\nu_1 - \mathrm{Re}\,\nu_j| \geq \mathrm{Re}\,\nu_j \geq c_1\rho$ because $\nu_1 \leq 0$, so we may assume

$$-c_1\rho < \mathrm{Re}\,\nu_j = -A_1/2 < c_1\rho, \quad j = 2, 3. \tag{3.10}$$

If $D > 0$ then one has $-A_1/2 + \sqrt{D} \leq 0$. Otherwise $\bar{\Delta}(t)$ would be negative for some $t > 0$ near $-A_1/2 + \sqrt{D}$ which is a contradiction. Thus $\sqrt{D} \leq A_1/2 \leq c_1\,\rho$ which shows that $|\nu_2|, |\nu_3| \leq |A_1|/2 + \sqrt{D} \leq 2c_1\,\rho < \rho/9$, hence $|\nu_1| \geq \rho/9 = 3c_1\rho$ by Lemma 3.4. Therefore $|\nu_1 - \nu_j| \geq |\nu_1| - |\nu_j| \geq c_1\rho$. Turn to the case $D \leq 0$ such that $\nu_2, \nu_3 = -A_1/2 \pm i\sqrt{|D|}$. Thanks to Lemma 3.4 again either $|\nu_1| \geq 3c_1\rho$ or $|\nu_2| = |\nu_3| \geq 3c_1\rho$. If $|\nu_1| \geq 3c_1\rho$ then it follows from (3.10) that

$$|\nu_1 - \nu_j| \geq |\nu_1 + A_1/2| \geq |\nu_1| - |A_1|/2 \geq 2c_1\rho.$$

If $|\nu_2| = |\nu_3| \geq 3c_1\rho$ so that $|A_1|/2 + \sqrt{|D|} \geq 3c_1\rho$, then

$$\sqrt{|D|} \geq 3c_1\rho - |A_1|/2 \geq 2c_1\rho$$

which proves $|\nu_1 - \nu_j| \geq \sqrt{|D|} \geq 2c_1\rho$, hence the assertion. □

Now define $\psi(X, \epsilon)$ which plays a crucial role in our arguments deriving weighted energy estimates. Choose $\chi(s) \in C^\infty(\mathbb{R})$ such that $0 \leq \chi(s) \leq 1$ with $\chi(s) = 1$ if $s \leq 0$ and $\chi(s) = 0$ for $s \geq 1$. Define

$$\psi(X, \epsilon) = -\chi\Big(\frac{\nu_1 + a_1}{2c_1\rho}\Big)\frac{\nu_1 + a_1}{2}, \quad \epsilon \neq 0.$$

Proposition 3.1. *One can find a neighborhood* $\mathcal{U}$ *of* $(\bar{X}, 0)$ *such that*

$$\bar{\Delta}(t, X, \epsilon) \geq \bar{c}\min\{t^2, (t - \psi(X, \epsilon))^2\}\,(t + \rho(X, \epsilon)) \tag{3.11}$$

holds for $(X, \epsilon) \in \mathcal{U}$, $\epsilon \neq 0$ *and* $t \in [0, T]$ *where* $\bar{c} = 1/32$.

Proof. Set $\delta = 1/9$ in this proof. First check that one can find $c \geq \bar{c}$ such that

$$\bar{\Delta}(t, X, \epsilon) \geq c\,t^2(t + \rho) \quad \text{if } A_1 = \nu_1 + a_1 \geq 0. \tag{3.12}$$

Let $D > 0$ in (3.9). It was seen in the proof of Lemma 3.5 that $-A_1/2 + \sqrt{D} \leq 0$ so that $\nu_2, \nu_3 = -A_1/2 \pm \sqrt{D} \leq 0$. If $|\nu_1| \geq \delta\rho$ then

$$t - \nu_1 = t + |\nu_1| \geq t + \delta\rho,$$

hence $\delta^{-1}(t-\nu_1) \geq t+\rho$ and $t-\nu_i = t+|\nu_i| \geq t$, so (3.12) holds with $c=\delta$. Consider the case $D \leq 0$ so that $\nu_2, \nu_3 = -A_1/2 \pm i\sqrt{|D|}$. If $|\nu_1| \geq \delta\rho$ then $\delta^{-1}(t-\nu_1) \geq t+\rho$ as above and $|t-\nu_i| \geq |t+A_1/2| \geq t$, thus (3.12) holds with $c=\delta$. If $|\nu_2| = |\nu_3| \geq \delta\rho$ then $A_1/2+\sqrt{|D|} \geq \delta\rho$. Since

$$(t-\nu_2)(t-\nu_3) \geq (t+A_1/2+\sqrt{|D|})^2/2 \geq (t+\delta\rho)^2/2 \geq \delta t(t+\rho)/2$$

then (3.12) holds with $c=\delta/2$.

Turn to the case $A_1 < 0$. Since $\psi = -(\nu_1+a_1)/2 > 0$, one can write

$$\bar{\Delta}(t) = (t-\nu_1)((t-\psi)^2-D).$$

Note that $D \leq 0$, otherwise $\psi+\sqrt{D} > 0$ would be a positive simple root of $\bar{\Delta}(t)$ and a contradiction. Then $(t-\psi)^2-D = (t-\psi)^2+|D| \geq (t-\psi)^2$. Consider the case $|\nu_1| \geq \delta\rho$. Recalling $t-\nu_1 = t+|\nu_1| \geq \delta(t+\rho)$ we get

$$\bar{\Delta}(t,X,\epsilon) \geq c\,(t-\psi)^2(t+\rho) \tag{3.13}$$

with $c=\delta$. When $|\nu_2| = |\nu_3| = |\psi \pm i\sqrt{|D|}| = \sqrt{\psi^2+|D|} \geq \delta\rho$ one has

$$(t-\nu_2)(t-\nu_3) = (t-\psi)^2+|D| \geq (|t-\psi|+\sqrt{|D|})^2/2.$$

Assume $\psi \geq \sqrt{|D|}$ so that $\sqrt{2}\psi \geq \delta\rho$. For $0 \leq t \leq \psi/2$ we have $t \leq |t-\psi|$ and $\psi/2 \leq |t-\psi|$; one has

$$(1-\gamma)|t-\psi|+\gamma|t-\psi| \geq (1-\gamma)t+\gamma\psi/2 \geq \delta(2\sqrt{2}+\delta)^{-1}(t+\rho)$$

with $\gamma = 2\sqrt{2}/(2\sqrt{2}+\delta)$. Since $|t-\psi|+\sqrt{|D|} \geq |t-\psi| \geq t$ and $|t-\nu_1| = t+|\nu_1| \geq t$ it is clear that (3.12) holds with $c=\delta/(2\sqrt{2}+\delta)$. For $\psi/2 \leq t$ such that $|t-\psi| \leq t$ one sees that

$$t-\nu_1 \geq t = (1-\gamma)t+\gamma t \geq (1-\gamma)t+\gamma\psi/2 \geq \delta(2\sqrt{2}+\delta)^{-1}(t+\rho)$$

and hence

$$(t-\nu_1)((t-\psi)^2+|D|) \geq c\,(t+\rho)(t-\psi)^2$$

which is (3.13) with $c=\delta/(2\sqrt{2}+\delta)$. Next assume $\sqrt{|D|} \geq \psi$ so that $\sqrt{2}\sqrt{|D|} \geq \delta\rho$. For $0 \leq t \leq \psi/2$ one has $|t-\psi| \geq t$ and hence

$$|t-\psi|+\sqrt{|D|} \geq t+\delta\rho/\sqrt{2} \geq (\delta/\sqrt{2})(t+\rho).$$

Noting that $|t-\nu_1| = t+|\nu_1| \geq t$, it is clear that (3.12) holds with $c=\delta/2\sqrt{2}$. For $\psi/2 \leq t$ we see that

$$|t-\psi|+\sqrt{|D|} \geq t-|\psi|+\sqrt{|D|} \geq t, \quad |t-\psi|+\sqrt{|D|} \geq \sqrt{|D|} \geq \delta\rho/\sqrt{2}$$

which shows that $|t-\psi|+\sqrt{|D|} \geq \delta(\sqrt{2}+\delta)^{-1}(t+\rho)$. Recalling $|t-\nu_1| = t+|\nu_1| \geq t$, again one has (3.12) with $c = \delta/2(\sqrt{2}+\delta)$. Thus by choosing

$$\bar{c} = 1/32 < 1/(18\sqrt{2}+2) = \delta/2(\sqrt{2}+\delta)$$

the proof is complete. □

Lemma 3.6. *One can find a neighborhood* $\mathfrak{U}$ *of* $(\bar{X}, 0)$ *and* $C^* > 0$ *such that*

$$\frac{|\partial_t \Delta(t,X,\epsilon)|}{\Delta(t,X,\epsilon)} \leq C^*\Big(\frac{1}{t}+\frac{1}{|t-\psi|+\sqrt{a}\epsilon}\Big), \quad (X,\epsilon)\in\mathfrak{U}, \quad \epsilon>0 \tag{3.14}$$

holds for $t \in (0, T]$.

Proof. It will suffice to show (3.14) for $\Delta(t,X,\sqrt{2}\epsilon)$ which we denote by $\tilde{\Delta}(t,X,\epsilon)$. It is clear that

$$\tilde{\Delta} = \Delta + 4e^3(3(t+\rho)^2\epsilon^2 + 3(t+\rho)\epsilon^4+\epsilon^6) = \Delta + \Delta_r.$$

Writing $\tilde{\Delta} = \tilde{e}(\bar{\Delta}+\bar{\Delta}_r)$ with $\Delta_r = \tilde{e}\bar{\Delta}_r$ it suffices to show the assertion for $\bar{\Delta}+\bar{\Delta}_r$ instead of $\tilde{\Delta}$. Note that

$$|\partial_t\bar{\Delta}_r|/\bar{\Delta}_r \leq C(1+1/(t+\rho)) \leq C'/t \tag{3.15}$$

always holds. Write $\bar{\Delta} = \prod_{j=1}^3(t-\nu_j)$ and note that $\partial_t\bar{\Delta}/\bar{\Delta} = \sum_{j=1}^3(t-\nu_j)^{-1}$. When $A_1 \geq 0$ we see from the proof of Proposition 3.1 that $|t-\nu_j| \geq t$ hence $|\partial_t\bar{\Delta}/\bar{\Delta}| \leq 3/t$. Therefore one has

$$\begin{aligned}|\partial_t\tilde{\Delta}|/\tilde{\Delta} &\leq |\partial_t\bar{\Delta}|/(\bar{\Delta}+\bar{\Delta}_r) + |\partial_t\bar{\Delta}_r|/(\bar{\Delta}+\bar{\Delta}_r)\\ &\leq |\partial_t\bar{\Delta}|/\bar{\Delta} + |\partial_t\bar{\Delta}_r|/\bar{\Delta}_r\end{aligned}$$

which proves the assertion. Let $A_1 < 0$. Then $\bar{\Delta} = (t-\nu_1)((t-\psi)^2-D)$ where $\psi > 0$ and $D \leq 0$ as seen in the proof of Proposition 3.1. If $|D| \geq a\epsilon^2$,

$$|t-\psi|(|t-\psi|+\sqrt{a}\epsilon) \leq \sqrt{2}((t-\psi)^2+|D|)$$

which shows the assertion since $|t-\nu_1| = t+|\nu_1| \geq t$. Similarly if $|t-\psi| \geq \sqrt{a}\epsilon$, one has

$$|t-\psi|(|t-\psi|+\sqrt{a}\epsilon) \leq 2(t-\psi)^2 \leq 2((t-\psi)^2+|D|),$$

hence the assertion. If $|D| < a\epsilon^2$ and $|t-\psi| < \sqrt{a}\,\epsilon$ it follows that

$$|\partial_t\bar{\Delta}| \leq (t-\psi)^2+|D|+2|t-\nu_1||t-\psi| \leq 2a\epsilon^2 + Ca^{3/2}\epsilon$$

because $|t-\nu_1| \leq Ca$. In view of $C\bar{\Delta}_r \geq a^2\epsilon^2$ one concludes that

$$\begin{aligned}|\partial_t\bar{\Delta}|/(\bar{\Delta}+\bar{\Delta}_r) &\leq |\partial_t\bar{\Delta}|/\bar{\Delta}_r \leq C(2a\epsilon^2+Ca^{3/2}\epsilon)/(a^2\epsilon^2)\\ &\leq C\Big(\frac{1}{a}+\frac{1}{\sqrt{a}\,\epsilon}\Big) \leq C'\Big(\frac{1}{t}+\frac{1}{|t-\psi|+\sqrt{a}\,\epsilon}\Big)\end{aligned}$$

which together with (3.15) proves the assertion. □

4 Localized symbols

In the preceding Sections 3.1 and 3.2 all symbols we have studied are defined in some conic (in ξ) neighborhood of $(X, \epsilon) = (\bar{X}, 0)$ or $X = \bar{X}$. In this section we define symbols on $\mathbb{R}^d \times \mathbb{R}^d$ which localizes such symbols around $(X, \epsilon) = (\bar{X}, 0)$ or $X = \bar{X}$ with a parameter M.

4.1 Localization via localized coordinates functions. Let $\bar{X} = (0, \bar{\xi})$ with $|\bar{\xi}| = 1$. Let $\chi(s) \in C^\infty(\mathbb{R})$ be equal to 1 in $|s| \le 1$, and vanishes in $|s| \ge 2$ such that $0 \le \chi(s) \le 1$. Define $y(x)$ and $\eta(\xi)$ by

$$y_j(x) = \chi(M^2 x_j)x_j, \quad \eta_j(\xi) = \chi(M^2(\xi_j\langle\xi\rangle_\gamma^{-1} - \bar{\xi}_j))(\xi_j - \bar{\xi}_j\langle\xi\rangle_\gamma) + \bar{\xi}_j\langle\xi\rangle_\gamma$$

for $j = 1, 2, \dots, d$ with $\langle\xi\rangle_\gamma = (\gamma^2 + |\xi|^2)^{1/2}$, where M and γ are large positive parameters constrained by

$$\gamma \ge M^5. \tag{4.1}$$

It is easy to see that $(1 - CM^{-2})\langle\xi\rangle_\gamma \le |\eta| \le (1 + CM^{-2})\langle\xi\rangle_\gamma$ and

$$|y| \le CM^{-2}, \quad |\eta/|\eta| - \bar{\xi}| \le CM^{-2} \tag{4.2}$$

with some $C > 0$ so that (y, η) is contained in a conic neighborhood of $(0, \bar{\xi})$, shrinking with M. Note that $(y, \eta) = (x, \xi)$ on the conic neighborhood of $(0, \bar{\xi})$,

$$W_M = \{(x, \xi) \mid |x| \le M^{-2}, \quad |\xi_j/|\xi| - \bar{\xi}_j| \le M^{-2}/2, \quad |\xi| \ge \gamma M\}, \tag{4.3}$$

since

$$|\xi_j/\langle\xi\rangle_\gamma - \bar{\xi}_j| \le \big|\xi_j/\langle\xi\rangle_\gamma - \xi_j/|\xi|\big| + \big|\xi_j/|\xi| - \bar{\xi}_j\big| \le M^{-2}$$

if $(x, \xi) \in W_M$ where δ_{ij} is the Kronecker's delta. Let $f(X, \epsilon)$, $h(X)$ be smooth functions in a conic neighborhood of $(\bar{X}, 0)$, $\bar{X}$ respectively which are homogeneous of degree 0 in ξ. Then we define localized symbols of $f(x, \xi)$, $h(x, \xi)$ of $f(X, \epsilon)$, $h(X)$ by

$$f(x, \xi) = f(y(x), \eta(\xi), \epsilon(\xi)), \quad h(x, \xi) = h(y(x), \eta(\xi))$$

with $\epsilon(\xi) = M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$ or $\epsilon(\xi) = \sqrt{2}M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$. In view of (4.2) such extended symbols are defined on $\mathbb{R}^d \times \mathbb{R}^d$, taking M large if necessary. Let

$$G = M^4(|dx|^2 + \langle\xi\rangle_\gamma^{-2}|d\xi|^2).$$

Then it is easy to see that

$$y_j \in S(M^{-2}, G), \quad \eta_j - \bar{\xi}_j\langle\xi\rangle_\gamma \in S(M^{-2}\langle\xi\rangle_\gamma, G), \quad \epsilon(\xi) \in S(M^{-2}, G) \tag{4.4}$$

for $j = 1, \ldots, d$. To avoid confusion we denote $|\eta(\xi)|$ by $[\xi]$, hence

$$[\xi] \in S(\langle\xi\rangle_\gamma, G), \quad [\xi]\langle\xi\rangle_\gamma^{-1} - 1 \in S(M^{-2}, G). \tag{4.5}$$

Lemma 4.1. *Let $f(X, \epsilon)$ be a smooth function in a conic neighborhood of $(\bar{X}, 0)$ which is homogeneous of degree 0 in ξ. If $\partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^k f(\bar{X}, 0) = 0$ for $0 \le |\alpha+\beta|+k < r$, then $f(x, \xi) = f(y(x), \eta(\xi), \epsilon(\xi)) \in S(M^{-2r}, G)$. Let $h(X)$ be a smooth function in a conic neighborhood of $\bar{X}$ which is homogeneous of degree 0 in ξ. Then*

$$h(x, \xi) - h(0, \bar{\xi}) \in S(M^{-2}, G).$$

Proof. We prove the first assertion. By the Taylor formula one can write

$$\begin{aligned} f(y, \eta, \epsilon) = &\sum_{|\alpha+\beta|+k=r} \frac{1}{\alpha!\beta!k!} y^\alpha (\eta - \bar{\xi}\langle\xi\rangle_\gamma)^\beta \epsilon^k \partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^k f(0, \bar{\xi}\langle\xi\rangle_\gamma, 0) \\ &+ (r+1) \sum_{|\alpha+\beta|+k=r+1} \Big[\frac{1}{\alpha!\beta!k!} y^\alpha (\eta - \bar{\xi}\langle\xi\rangle_\gamma)^\beta \epsilon^k \\ &\times \int_0^1 (1-\theta)^r \partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^k f(\theta y, \theta(\eta - \bar{\xi}\langle\xi\rangle_\gamma) + \bar{\xi}\langle\xi\rangle_\gamma, \theta\epsilon) d\theta\Big]. \end{aligned}$$

It is clear that

$$y^\alpha (\eta - \bar{\xi}\langle\xi\rangle_\gamma)^\beta \epsilon^k \partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^k f(0, \bar{\xi}, 0)\langle\xi\rangle_\gamma^{-|\beta|} \in S(M^{-2r}, G)$$

for $|\alpha + \beta| + k = r$ in view of (4.4). Since

$$\langle\xi\rangle_\gamma / C \le |\theta(\eta - \bar{\xi}\langle\xi\rangle_\gamma) + \bar{\xi}\langle\xi\rangle_\gamma| \le C\langle\xi\rangle_\gamma$$

the integral belongs to $S(\langle\xi\rangle_\gamma^{-|\beta|}, G)$, hence the second term on the right-hand side is in $S(M^{-2r-2}, G)$, thus the assertion. □

4.2 Estimate of localized symbols. From now on it is assumed that all constants are independent of M and γ. As explained before we write $A \precsim B$ if A is bounded by a constant, independent of parameters M and γ, times B. Let $\rho(x, \xi)$ be the localized symbol of $\rho(X, \epsilon)$ with $\epsilon = M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$ so that

$$\rho(x, \xi) = \alpha(x, \xi) + M\langle\xi\rangle_\gamma^{-1}.$$

From Lemma 4.1 we see that $\rho \in S(M^{-4}, G)$, hence

$$|\partial_x^\alpha \partial_\xi^\beta \rho| \precsim \langle\xi\rangle_\gamma^{-|\beta|} \quad \text{for } |\alpha + \beta| = 2.$$

Since $\rho \ge 0$ it follows from the Glaeser's inequality that

$$|\partial_x^\alpha \partial_\xi^\beta \rho| \precsim \sqrt{\rho}\,\langle\xi\rangle_\gamma^{-|\beta|}, \quad |\alpha + \beta| = 1. \tag{4.6}$$

Lemma 4.2. *Assume that $f(X,\epsilon)$ is smooth and homogeneous of degree* 0 *in* ξ *in a conic neighborhood of* $(\bar{X},0)$ *and satisfies* $|f(X,\epsilon)|\leq C\rho(X,\epsilon)^n$ *with some* $n>0$ *there. For the localized symbol* $f(x,\xi)$ *there is* $C_{\alpha\beta}>0$ *such that*

$$|\partial_x^\alpha\partial_\xi^\beta f(x,\xi)|\leq C_{\alpha\beta}\rho(x,\xi)^{n-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}. \tag{4.7}$$

Proof. From the assumption it follows that

$$\partial_x^\alpha\partial_\xi^\beta\partial_\epsilon^k f(0,\bar{\xi},0)=0\quad\text{for } |\alpha+\beta|+k<2n,$$

hence Lemma 4.1 shows that $f(x,\xi)\in S(M^{-4n},G)$. Therefore for $|\alpha+\beta|\geq 2n$ one sees that

$$|\langle\xi\rangle_\gamma^{|\beta|}\partial_x^\alpha\partial_\xi^\beta f(x,\xi)|\leq CM^{2|\alpha+\beta|-4n}\leq C(C_0\rho^{-1})^{|\alpha+\beta|/2-n}$$

because $M^4\leq C_0\rho^{-1}$. Hence (4.7) holds for $|\alpha+\beta|\geq 2n$. Assume $|\alpha+\beta|\leq 2n-1$. Write $X=(x,\xi)$, $Y=(y,\eta\langle\xi\rangle_\gamma)$ and apply the Taylor formula to obtain

$$\begin{aligned}|f(X+sY)|&=\Big|\sum_{j=0}^{2n-1}\frac{s^j}{j!}d^jf(X;Y)+\frac{s^{2n}}{(2n)!}d^{2n}f(X+s\theta Y;Y)\Big|\\&\leq C\Big(\sum_{j=0}^{2n-1}\frac{s^j}{j!}d^j\rho(X;Y)+\frac{s^{2n}}{(2n)!}d^{2n}\rho(X+s\theta' Y;Y)\Big)^n\end{aligned} \tag{4.8}$$

where

$$d^jf(X;Y)=\sum_{|\alpha+\beta|=j}(j!/\alpha!\beta!)\partial_x^\alpha\partial_\xi^\beta a(x,\xi)y^\alpha\eta^\beta\langle\xi\rangle_\gamma^{|\beta|},$$

and $0<\theta,\theta'<1$. If $\rho(x,\xi)=0$, then $\partial_x^\alpha\partial_\xi^\beta\rho(x,\xi)=0$ for $|\alpha+\beta|=1$ because $\rho\geq 0$ and then it follows from (4.8) that $\partial_x^\alpha\partial_\xi^\beta f(x,\xi)=0$ for $|\alpha+\beta|\leq 2n-1$, hence (4.7). We fix a small $s_0>0$. If $\rho(x,\xi)\geq s_0$, then one has

$$|\partial_x^\alpha\partial_\xi^\beta f(x,\xi)\langle\xi\rangle_\gamma^{|\beta|}|\leq C_{\alpha\beta}\leq C_{\alpha\beta}s_0^{-n+|\alpha+\beta|/2}\rho^{n-|\alpha+\beta|/2}$$

for $|\alpha+\beta|\leq 2n-1$ which proves (4.7). Assume $0<\rho(x,\xi)<s_0$. Note that

$$|d^{2n}f(X+s\theta Y;Y)|\leq C,\quad d^{2n}\rho(X+s\theta' Y;Y)\leq C\rho(X)^{1-n}$$

for any $|(y,\eta)|\leq 1/2$. Indeed the first relation is clear from $f(x,\xi)\in S(M^{-4n},G)$. To check the second one it is enough to note that $\rho(x,\xi)\in S(M^{-4},G)$ and

$$M^{-4+2j}\leq(C_0\rho^{-1})^{j/2-1},\ \ j\geq 2,\quad \sqrt{2}\langle\xi+\theta\langle\xi\rangle_\gamma\eta\rangle_\gamma\geq\langle\xi\rangle_\gamma/2 \tag{4.9}$$

for $|\eta|\leq 1/2$ and $|\theta|<1$. Take $s=\rho(X)^{1/2}$ in (4.8) to get

$$\Big|\sum_{j=0}^{2n-1}\frac{1}{j!}d^jf(X;Y)\rho(X)^{j/2}\Big|\leq C\Big(\sum_{j=0}^{2n-1}\frac{1}{j!}d^j\rho(X;Y)\rho(X)^{j/2}\Big)^n+C\rho(X)^n$$

where the right-hand side is bounded by $C\rho(X)^n$ for $|d\rho(X;Y)| \le C'\rho(X)^{1/2}$ in view of (4.6) and (4.9) for $j \ge 3$. Replacing (y,η) by $s(y,\eta)$, $|(y,\eta)| = 1/2$, $0 < |s| < 1$ one obtains

$$\Big|\sum_{j=1}^{2n-1} \frac{s^j}{j!} d^j f(X;Y) \frac{\rho(X)^{j/2}}{\rho(X)^n}\Big| \le C_1.$$

Since two norms $\sup_{|s|\le 1} |p(s)|$ and $\max\{|c_j|\}$ on the vector space consisting of all polynomials $p(s) = \sum_{j=0}^{2n-1} c_j s^j$ are equivalent, one obtains

$$|d^j f(X;Y)| \le B'\rho(X)^{n-j/2}.$$

Since $|(y,\eta)| = 1/2$ is arbitrary one obtains (4.7). □

Lemma 4.3. *Let* $s \in \mathbb{R}$. *Then* $|\partial_x^\alpha \partial_\xi^\beta \rho^s| \precsim \rho^{s-|\alpha+\beta|/2} \langle\xi\rangle_\gamma^{-|\beta|}$.

Proof. When $s = 1$ the assertion follows from Lemma 4.2. Since

$$\partial_x^\alpha \partial_\xi^\beta \rho^s = \sum C_{\alpha^{(j)}\beta^{(j)}} \rho^s (\partial_x^{\alpha^{(1)}} \partial_\xi^{\beta^{(1)}} \rho/\rho) \cdots (\partial_x^{\alpha^{(k)}} \partial_\xi^{\beta^{(k)}} \rho/\rho)$$

the proof for the case $s \in \mathbb{R}$ is clear. □

Lemma 4.4. *Let* $a_j(x,\xi)$ *be the localized symbol of* $a_j(X,\epsilon)$. *Then*

$$|\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)| \precsim \rho(x,\xi)^{j-|\alpha+\beta|/2} \langle\xi\rangle_\gamma^{-|\beta|}, \quad j = 1,2,3.$$

Proof. The assertion follows from Lemmas 3.3 and 4.2. □

For the localized symbol $\psi(x,\xi)$ of $\psi(X,\epsilon)$ with $\epsilon = M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$ we have

Lemma 4.5. *One has* $|\partial_x^\alpha \partial_\xi^\beta \psi(x,\xi)| \precsim \rho(x,\xi)^{1-|\alpha+\beta|/2} \langle\xi\rangle_\gamma^{-|\beta|}$.

Proof. Since Lemma 4.2 is not available for $\psi(X,\epsilon)$ because it is not defined for $\epsilon = 0$, we show the assertion directly. Let $\nu_1(x,\xi)$, $a_1(x,\xi)$ and $\overline{\Delta}(t,x,\xi)$ be localized symbols of $\nu_1(X,\epsilon)$, $a_1(X,\epsilon)$ and $\overline{\Delta}(t,X,\epsilon)$ with $\epsilon = M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$ and hence $\overline{\Delta}(\nu_1(x,\xi),x,\xi) = 0$. Note that $|\partial_t \overline{\Delta}(\nu_1,x,\xi)| \ge 4\, c_2^2\, \rho^2(x,\xi)$ if

$$\nu_1(x,\xi) + a_1(x,\xi) < 2c_1\rho(x,\xi)$$

thanks to Lemma 3.5. Starting with

$$\partial_t \overline{\Delta}(\nu_1,x,\xi) \partial_x^\alpha \partial_\xi^\beta \nu_1 + \partial_x^\alpha \partial_\xi^\beta \overline{\Delta}(\nu_1,x,\xi) = 0, \quad |\alpha+\beta| = 1$$

a repetition of the same argument in Lemma 5.3 below together with Lemma 4.4 shows that

$$|\partial_x^\alpha \partial_\xi^\beta \nu_1| \precsim \rho^{1-|\alpha+\beta|/2} \langle\xi\rangle_\gamma^{-|\beta|}, \quad \nu_1 + a_1 < 2c_1\rho. \tag{4.10}$$

Here we have used $|\nu_1| \precsim \rho$ which also follows from Lemma 4.4. Using (4.10) and Lemmas 4.3 and 4.4 the assertion follows easily. □

4.3 Estimate of a discriminant. Let $\alpha(x,\xi)$, $a(t,x,\xi)$, $b(t,x,\xi)$, $e(t,x,\xi)$ be localized symbols of $\alpha(X)$, $a(t,X)$, $b(t,X)$, $e(t,X)$ so that

$$\tau^3 - e(t,x,\xi)(t+\alpha(x,\xi))[\xi]^2\tau + b(t,x,\xi)[\xi]^3$$

is now defined on $\mathbb{R}^d \times \mathbb{R}^d$ and coincides with the original p in a conic neighborhood W_M of $(0,\bar{\xi})$. We add a term $2Me(t,x,\xi)\langle\xi\rangle_\gamma^{-1}[\xi]^2$ to this:

$$\hat{p} = \tau^3 - e(t+\alpha+2M\langle\xi\rangle_\gamma^{-1})[\xi]^2\tau - b\,[\xi]^3 \tag{4.11}$$

where we denote

$$a_M(t,x,\xi) = e(t,x,\xi)(t+\alpha(x,\xi)+2M\langle\xi\rangle_\gamma^{-1}) \tag{4.12}$$

which is the localized symbol of $a(t,X,\epsilon) = a(t,X)+\epsilon^2$ with $\epsilon = \sqrt{2}M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$. Consider the discriminant

$$\begin{aligned}\Delta_M(t,x,\xi) &= 4\,e^3(t+\alpha+2M\langle\xi\rangle_\gamma^{-1})^3 - 27\,b^2\\ &= 4\,e^3(t+\alpha+M\langle\xi\rangle_\gamma^{-1})^3 - 27\,b^2 + \Delta_r(t,x,\xi)\end{aligned} \tag{4.13}$$

where, recalling $\alpha(x,\xi)+M\langle\xi\rangle_\gamma^{-1} = \rho(x,\xi)$, we have

$$\begin{aligned}\Delta_r &= 4e^3(3(t+\rho)^2M\langle\xi\rangle_\gamma^{-1} + 3(t+\rho)M^2\langle\xi\rangle_\gamma^{-2} + M^3\langle\xi\rangle_\gamma^{-3})\\ &= 12e^3(c_1(x,\xi)t^2 + c_2(x,\xi)t + c_3(x,\xi)) \geq 12e^3M(t+\rho)^2\langle\xi\rangle_\gamma^{-1}.\end{aligned}$$

It is clear that $c_j(x,\xi)$ verifies $|\partial_x^\alpha\partial_\xi^\beta c_j| \precsim \rho^{j-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$. Let $\Delta(t,x,\xi)$, $\bar{\Delta}(t,x,\xi)$ be localized symbols of $\Delta(t,X,\epsilon)$, $\bar{\Delta}(t,X,\epsilon)$ with $\epsilon = M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$. Thanks to Proposition 3.1 one has

$$\bar{\Delta}(t,x,\xi) \geq \bar{c}\,\min\{t^2,(t-\psi)^2\}(t+\rho).$$

Noting that $\Delta(t,x,\xi) = \tilde{e}\,\bar{\Delta}$ we see that

$$\begin{aligned}\Delta(t,x,\xi) &= \tilde{e}\,\bar{\Delta} \geq \tilde{e}\,\bar{c}\,\min\{t^2,(t-\psi)^2\}(t+\rho)\\ &\geq (\tilde{e}/e)\bar{c}\,\min\{t^2,(t-\psi)^2\}e(t+\rho).\end{aligned} \tag{4.14}$$

Therefore choosing a constant $\bar{\nu} > 0$ such that $12\,e^2 \geq (\tilde{e}/e)\bar{c}\,\bar{\nu}$ one obtains from (4.13), (4.14) that

$$\begin{aligned}\Delta_M &\geq (\tilde{e}/e)\bar{c}\,\min\{t^2,(t-\psi)^2\}e(t+\rho) + 12\,e^3(t+\rho)^2M\langle\xi\rangle_\gamma^{-1}\\ &\geq (\tilde{e}/e)\bar{c}(\min\{t^2,(t-\psi)^2\} + \bar{\nu}(t+\rho)M\langle\xi\rangle_\gamma^{-1})e(t+\rho)\\ &\geq (\tilde{e}/e)\bar{c}\,\min\{t^2,(t-\psi)^2 + \bar{\nu}M\rho\langle\xi\rangle_\gamma^{-1}\}e(t+\rho), \quad t\geq 0.\end{aligned} \tag{4.15}$$

Proposition 4.1. *One can write*

$$\Delta_M = e(t^3 + a_1(x,\xi)t^2 + a_2(x,\xi)t + a_3(x,\xi))$$

where $0 < e \in S(1,G)$ *uniformly in* t *and* $|\partial_x^\alpha \partial_\xi^\beta a_j| \precsim \rho^{j-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$. *Moreover, there exist* $\bar{\nu} > 0$ *and* $c > 0$ *such that*

$$\frac{\Delta_M}{a_M} \ge \frac{\tilde{e}}{2e}\bar{c}\min\{t^2, (t-\psi)^2 + \bar{\nu}M\rho\langle\xi\rangle_\gamma^{-1}\}, \qquad \frac{\Delta_M}{a_M} \ge c\,M\langle\xi\rangle_\gamma^{-1}\,a_M \tag{4.16}$$

for $0 \le t \le T$ *where* ψ *and* ρ *satisfy*

$$|\partial_x^\alpha\partial_\xi^\beta\psi|, \quad |\partial_x^\alpha\partial_\xi^\beta\rho| \precsim \rho^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}.$$

Proof. Choosing $\epsilon = \sqrt{2}M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$ and applying Lemma 3.1 one can write Δ_M as a third order polynomial in t, up to a non-zero factor, and can estimate the coefficients thanks to Lemmas 3.3 and 4.2 in terms of $\alpha + 2M\langle\xi\rangle_\gamma^{-1}$. Noting that

$$\rho(x,\xi) \le \alpha(x,\xi) + 2M\langle\xi\rangle_\gamma^{-1} \le 2\rho(x,\xi)$$

we have the desired estimates for a_j. The assertion (4.16) follows from (4.15) for

$$a_M = e(t + \rho + M\langle\xi\rangle_\gamma^{-1/2}) \le 2\,e(t+\rho).$$

The estimates for ψ and ρ are nothing but Lemmas 4.3 and 4.5 with $\epsilon = M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$. □

Remark 4.1. Denoting $\bar{e} = e(0,0,\bar{\xi}) = \partial_t a(0,0,\bar{\xi})$ it is clear from (3.1) that

$$\bar{e} \text{ is the non-zero positive real eigenvalue of } F_p(0,0,0,\bar{\xi})$$

and the coefficient of the right-hand side of (4.16) is

$$\tilde{e}\bar{c}/(2e) = 2\bar{e}^2\bar{c}(1 + O(M^{-2})).$$

On the other hand, denoting the subprincipal symbol of P by P_{sub} and $b_3(0,0,\bar{\xi})$ by $\bar{b}_3$, it is easy to see that

$$P_{\text{sub}}(0,0,0,\bar{\xi}) = \bar{e}/(2i) + \bar{b}_3. \tag{4.17}$$

Lemma 4.6. *With* $\bar{e} = e(0,0,\bar{\xi})$ *we have*

$$|\partial_t b| \le (1 + CM^{-2})(2\sqrt{2/3})\bar{e}\sqrt{a_M}, \quad 0 \le t \le M^{-2}.$$

Proof. Write $b = \beta_0(x,\xi) + t\beta_1(x,\xi) + t^2\beta_3(t,x,\xi)$. Setting $t = 0$ in $27b^2 \le 4a^3$ it is clear that $|\beta_0| \le (2/3\sqrt{3})\bar{e}^{3/2}(1 + CM^{-2})\alpha^{3/2}$. We first check that

$$|\beta_1| \le (1 + CM^{-2})(2/\sqrt{3})\bar{e}^{3/2}\sqrt{\alpha}. \tag{4.18}$$

If $\alpha(x,\xi)=0$ then $\beta_1(x,\xi)=0$, hence (4.18) is clear. If $\alpha(x,\xi)>0$, taking $t=3\alpha$ and noting that $e(3\alpha,x,\xi)\le(1+CM^{-2})\bar{e}+C\alpha$ it follows that

$$\begin{aligned}3\alpha|\beta_1| &\le 2(4^{3/2}(1+CM^{-2})\bar{e}^{3/2}/3\sqrt{3})\alpha^{3/2}+|\beta_0|+C\alpha^2\\ &\le (6/\sqrt{3})(1+CM^{-2})\bar{e}^{3/2}\alpha^{3/2}+C\alpha^2\le(6/\sqrt{3})(1+CM^{-2})\bar{e}^{3/2}\alpha^{3/2}\end{aligned}$$

because $\alpha\le CM^{-4}$ which proves (4.18). Since $|\partial_t b|\le|\beta_1|+Ct$ we see that $|\partial_t b|\le(1+CM^{-2})(2/\sqrt{3})\bar{e}^{3/2}\sqrt{\alpha}+CM^{-2}\sqrt{t}$, thus the proof is immediate. □

5 The Bézout matrix and diagonal symmetrizer

Add $-2M\mathrm{op}(e(t,x,\xi)\langle\xi\rangle_\gamma^{-1})[D]^2D_t$ to the principal part and subtract the same from the lower order part so that the operator is left invariant;

$$\begin{aligned}\hat{P}=D_t^3-a_M(t,x,D)[D]^2D_t-b(t,x,D)\,[D]^3+b_1(t,x,D)D_t^2\\ +(b_2(t,x,D)+d_M(t,x,D))[D]D_t+b_3(t,x,D)[D]^2\end{aligned}$$

where $b_j(t,x,\xi)\in S(1,G)$ and $d_M(t,x,\xi)=2M(e\langle\xi\rangle_\gamma^{-1})\#[\xi]\in S(M,G)$. It follows from Lemma 4.1 and (4.5)

(5.1) $$d_M(t,x,\xi)-2M\bar{e}\in S(M^{-1},g).$$

With $U={}^t(D_t^2u,[D]D_tu,[D]^2u)$ the equation $\hat{P}u=f$ is transformed to

(5.2) $$D_tU=A(t,x,D)[D]U+B(t,x,D)U+F$$

where $F={}^t(f,0,0)$ and

$$A(t,x,\xi)=\begin{bmatrix}0&a_M&b\\1&0&0\\0&1&0\end{bmatrix},\quad B(t,x,\xi)=\begin{bmatrix}b_1&b_2+d_M&b_3\\0&0&0\\0&0&0\end{bmatrix}.$$

Let S be the Bézout matrix of $\hat{p}$ and $\partial\hat{p}/\partial\tau$, that is

$$S(t,x,\xi)=\begin{bmatrix}3&0&-a_M\\0&2a_M&3b\\-a_M&3b&a_M^2\end{bmatrix}.$$

Then S is positive semidefinite and symmetrizes S, namely, SA is symmetric and is easily examined directly, though this is a special case of a general fact (see [16], [28]).

5.1 Eigenvalues of Bézout matrix. To simplify notation denote

$$\sigma(t,x,\xi)=t+\alpha(x,\xi)+2M\langle\xi\rangle_\gamma^{-1}=t+\rho(x,\xi)+M\langle\xi\rangle_\gamma^{-1},$$

hence $a_M(t,x,\xi)=e(t,x,\xi)\sigma(t,x,\xi)$ and $(1-CM^{-2})\bar{e}\sigma\le a_M\le(1+CM^{-2})\bar{e}\sigma$. In what follows we assume that t varies in the interval

$$0\le t\le M^{-4}.$$

Since $\rho\in S(M^{-4},G)$ it is clear that $\sigma(t,x,\xi)\in S(M^{-4},G)$.

Lemma 5.1. *We have* $|\partial_x^\alpha\partial_\xi^\beta\sigma|\precsim\sigma^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$. *In particular* $\sigma\in S(\sigma,g)$.

Proof. It is clear from (4.6) that $|\partial_x^\alpha\partial_\xi^\beta\sigma|\precsim\sqrt{\sigma}\,\langle\xi\rangle_\gamma^{-|\beta|}$ for $|\alpha+\beta|=1$. For $|\alpha+\beta|\ge 2$ we have

$$|\partial_x^\alpha\partial_\xi^\beta\sigma|\precsim M^{2|\alpha+\beta|-4}\langle\xi\rangle_\gamma^{-|\beta|}\precsim\sigma^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$$

from $\rho\in S(M^{-4},G)$ since $C\sigma^{-1}\ge M^4$.

The second assertion is clear from $\sigma^{-1}\le M^{-1}\langle\xi\rangle_\gamma$. □

Corollary 5.1. *Let* $s\in\mathbb{R}$. *Then* $|\partial_x^\alpha\partial_\xi^\beta\sigma^s|\precsim\sigma^{s-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$. *In particular* $\sigma^s\in S(\sigma^s,g)$.

Definition 5.1. To simplify notation we denote by $\mathcal{C}(\sigma^s)$ the set of symbols $r(t,x,\xi)$ satisfying $|\partial_x^\alpha\partial_\xi^\beta r|\precsim\sigma^{s-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$.

It is clear that $\mathcal{C}(\sigma^s)\subset S(\sigma^s,g)$ because $\sigma^{-|\alpha+\beta|/2}\le M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{|\alpha+\beta|/2}$. It is also clear that if $p\in\mathcal{C}(\sigma^s)$ with $s>0$ then $(1+p)^{-1}-1\in\mathcal{C}(\sigma^s)$.

Lemma 5.2. *One has* $a_M^s\in\mathcal{C}(\sigma^s)$ *for* $s\in\mathbb{R}$, $b\in\mathcal{C}(\sigma^{3/2})$, $\partial_t a_M\in\mathcal{C}(1)$ *and* $\partial_t b\in\mathcal{C}(\sqrt{\sigma})$.

Proof. The first assertion is clear from Corollary 5.1 because $a_M=e\sigma$ and $e\in S(1,G)$, $1/C\le e\le C$. To show the second assertion, recalling that $b(t,x,\xi)$ is the localized symbol of $b(t,X)$, write

$$\begin{aligned}b(t,x,\xi)&=b(0,y(x),\eta(\xi))+\partial_t b(0,y(x),\eta(\xi))t\\&\quad+\int_0^1(1-\theta)\partial_t^2 b(\theta t,y(x),\eta(\xi))d\theta\cdot t^2.\end{aligned}\tag{5.3}$$

Since $\partial_x^\alpha\partial_\xi^\beta b(0,0,\bar{\xi})=0$ for $|\alpha+\beta|\le 2$ and $\partial_t b(0,0,\bar{\xi})=0$, then thanks to Lemma 4.1 one has $b(0,y(x),\eta(\xi))\in S(M^{-6},G)$ and $\partial_t b(0,y(x),\eta(\xi))\in S(M^{-2},G)$. Since $0\le t\le M^{-4}$ we conclude that $b(t,x,\xi)\in S(M^{-6},G)$. Since $|b|\le C\sigma^{3/2}$

and $\sigma \in S(M^{-4}, G)$, a repetition of the same arguments in proving Lemma 4.2 shows the second assertion. The third assertion is clear because $\partial_t a_M = e + (\partial_t e)\sigma$. As for the last assertion, recall Lemma 4.6 that $|\partial_t b| \le C a_M^{1/2} \le C' \sigma^{1/2}$. Noting that $\partial_t b \in S(M^{-2}, G)$ which results from (5.3), one sees that

$$|\langle\xi\rangle_\gamma^{|\beta|} \partial_x^\alpha \partial_\xi^\beta \partial_t b| \precsim M^{2|\alpha+\beta|-2} \precsim \sigma^{1/2-|\alpha+\beta|/2}$$

for $|\alpha+\beta| \ge 1$, hence the assertion. □

Recall [29, Proposition 2.1]

Proposition 5.1. *Let* $0 \le \lambda_1(t,x,\xi) \le \lambda_2(t,x,\xi) \le \lambda_3(t,x,\xi)$ *be the eigenvalues of* $S(t,x,\xi)$. *There exist* M_0 *and* $K > 0$ *such that one has for* $M \ge M_0$

$$\begin{aligned} \Delta_M/(6a_M + 2a_M^2 + 2a_M^3) \le & \lambda_1 \le (2/3 + Ka_M)\, a_M^2, \\ (2 - Ka_M)\, a_M \le & \lambda_2 \le (2 + Ka_M)\, a_M, \\ 3 \le & \lambda_3 \le 3 + Ka_M^2. \end{aligned}$$

Proof. Since $a_M = e\,\sigma$ and $\sigma \in S(M^{-4}, G)$, then for any $\bar{\epsilon} > 0$ there is M_0 such that $e\,M_0^{-4} \le \bar{\varepsilon}$. Then the assertion follows from [29, Proposition 2.1]. □

Corollary 5.2. *The eigenvalues* $\lambda_i(t,x,\xi)$ *are smooth in* $(0, M^{-4}] \times \mathbb{R}^d \times \mathbb{R}^d$.

5.2 Estimates of eigenvalues. First we prove

Lemma 5.3. *One has* $\lambda_j \in \mathcal{C}(\sigma^{3-j})$ *for* $j = 1, 2, 3$.

Denote $q(\lambda) = \det(\lambda I - S)$ so that

$$q(\lambda) = \lambda^3 - (3 + 2a_M + a_M^2)\lambda^2 + (6a_M + 2a_M^2 + 2a_M^3 - 9b^2)\lambda - \Delta_M. \tag{5.4}$$

Note that $\partial_\lambda q(\lambda_i)\partial_x^\alpha \partial_\xi^\beta \lambda_i + \partial_x^\alpha \partial_\xi^\beta q(\lambda_i) = 0$ for $|\alpha+\beta| = 1$. Let us write $\partial_x^\alpha \partial_\xi^\beta = \partial_{x,\xi}^{\alpha,\beta}$ for simplicity. We show by induction on $|\alpha+\beta|$ that

$$\begin{aligned} \partial_\lambda q(\lambda_i)\partial_{x,\xi}^{\alpha,\beta}\lambda_i = & \sum_{2|\mu+\nu|+s \ge 2} C_{\mu,\nu,\gamma^{(i)},\delta^{(j)},s} \partial_{x,\xi}^{\mu,\nu} \partial_\lambda^s q(\lambda_i) \\ & \times (\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i) \cdots (\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i) \end{aligned} \tag{5.5}$$

where $\mu + \sum \gamma^{(i)} = \alpha$, $\nu + \sum \delta^{(j)} = \beta$ and $|\gamma^{(i)} + \delta^{(j)}| \ge 1$. The assertion $|\alpha+\beta| = 1$ is clear. Suppose that (5.5) holds for $|\alpha+\beta| = m$. With $|e+f| = 1$ after operating $\partial_{x,\xi}^{e,f}$ on (5.5) the resulting left-hand side is

$$\partial_\lambda q(\lambda_i)\partial_{x,\xi}^{\alpha+e,\beta+f}\lambda_i - \sum_{2|\mu+\nu|+s \ge 2} C_{\mu,\nu,\gamma^{(i)},\delta^{(j)},s} \partial_{x,\xi}^{\mu,\nu} \partial_\lambda^s q(\lambda_i) (\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i) \cdots (\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i)$$

while the resulting right-hand side is

$$\sum C_{\dots}\partial_{x,\xi}^{\mu+e,\nu+f}\partial_\lambda^s q(\lambda_i)(\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i)\cdots(\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i)$$
$$+\sum C_{\dots}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^{s+1} q(\lambda_i)(\partial_{x,\xi}^{e,f}\lambda_i)(\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i)\cdots(\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i)$$
$$+\sum_{j=1}^{s}\sum C_{\dots}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^{s} q(\lambda_i)(\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i)\cdots(\partial_{x,\xi}^{\gamma^{(j)}+e,\delta^{(j)}+f}\lambda_i)\cdots(\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i)$$

which can be written as

$$\sum_{2|\mu+\nu|+s\geq 2} C_{\mu,\nu,\gamma^{(j)},\delta^{(j)},s}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^s q(\lambda_i)(\partial_{x,\xi}^{\gamma^{(1)},\delta^{(1)}}\lambda_i)\cdots(\partial_{x,\xi}^{\gamma^{(s)},\delta^{(s)}}\lambda_i)$$

where $\mu+\sum\gamma^{(i)}=\alpha+e$, $\nu+\sum\delta^{(i)}=\beta+f$ and $|\gamma^{(j)}+\delta^{(j)}|\geq 1$. Therefore we conclude (5.5). In order to estimate $\partial_{x,\xi}^{\alpha,\beta}\lambda_i$ one needs to estimate $\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^s q(\lambda_i)$.

Lemma 5.4. *For any $s\in\mathbb{N}$ and α,β we have that*

$$|\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^s q(\lambda_j)|\precsim \sigma^{4-j-(3-j)s-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|},\quad j=1,2,$$
$$|\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^s q(\lambda_3)|\precsim \sigma^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}.$$

Proof. From Proposition 5.1 and (5.4) one sees that

$$|q(\lambda_i)|\precsim|\lambda_i|^2+|a_M||\lambda_i|+|a_M|^3,$$
$$|\partial_{x,\xi}^{\alpha,\beta}q(\lambda_i)|\precsim(|\partial_{x,\xi}^{\alpha,\beta}a_M|+|\partial_{x,\xi}^{\alpha,\beta}b^2|)|\lambda_i|+|\partial_{x,\xi}^{\alpha,\beta}a_M^3|+|\partial_{x,\xi}^{\alpha,\beta}b^2|,\quad |\alpha+\beta|\geq 1$$

because $|\Delta_M|\precsim a_M^3$ and $|b|\precsim a_M^{3/2}$. Therefore, thanks to Proposition 5.1 and Lemma 5.2 one obtains the assertions for the case $s=0$. Since

$$|\partial_\lambda q(\lambda_i)|\precsim|\lambda_i|+|a_M|,\qquad |\partial_\lambda^s q(\lambda_i)|\precsim 1,\qquad s\geq 2,$$
$$|\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda q(\lambda_i)|\precsim|\partial_{x,\xi}^{\alpha,\beta}a_M||\lambda_i|+|\partial_{x,\xi}^{\alpha,\beta}a_M|+|\partial_{x,\xi}^{\alpha,\beta}b^2|,\qquad |\alpha+\beta|\geq 1,$$
$$|\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^2 q(\lambda_i)|\precsim|\partial_{x,\xi}^{\alpha,\beta}a_M|,\qquad \partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^s q(\lambda_i)=0,\qquad s\geq 3,\ |\alpha+\beta|\geq 1$$

the assertions for the case $s\geq 1$ are clear by Proposition 5.1 and Lemma 5.2. □

Proof of Lemma 5.3. Since $\partial_\lambda q(\lambda_i)=\prod_{k\neq i}(\lambda_i-\lambda_k)$ it follows from Proposition 5.1 that

$$6a_M(1-Ca_M)\leq|\partial_\lambda q(\lambda_i)|\leq 6a_M(1+Ca_M),\quad i=1,2,\ \partial_\lambda q(\lambda_3)\simeq 1. \tag{5.6}$$

Then for $|\alpha+\beta|=1$ one has

$$|\partial_{x,\xi}^{\alpha,\beta}\lambda_j|\precsim|\partial_{x,\xi}^{\alpha,\beta}q(\lambda_j)/\partial_\lambda q(\lambda_j)|\precsim\sigma^{3-j-1/2}\langle\xi\rangle_\gamma^{-|\beta|},\quad j=1,2,3,$$

by Lemma 5.4 with $s = 0$. Assume that $|\partial_{x,\xi}^{\alpha,\beta}\lambda_j| \precsim \sigma^{3-j-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$, $j = 1, 2, 3$, holds for $|\alpha+\beta| \le m$. Lemma 5.4 and (5.5) show that

$$\begin{aligned}|\partial_\lambda q(\lambda_1)\partial_{x,\xi}^{\alpha,\beta}\lambda_1| &\precsim \sum \sigma^{3-2s-|\mu+\nu|/2}\sigma^{2-|\gamma^{(1)}+\delta^{(1)}|/2}\cdots\sigma^{2-|\gamma^{(s)}+\delta^{(s)}|/2}\langle\xi\rangle_\gamma^{-|\beta|}\\ &\precsim \sum \sigma^{3-|\mu+\nu|/2}\sigma^{-|\gamma^{(1)}+\delta^{(1)}|/2}\cdots\sigma^{-|\gamma^{(s)}+\delta^{(s)}|/2}\langle\xi\rangle_\gamma^{-|\beta|}\\ &\precsim \sigma^{3-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}.\end{aligned}$$

This together with (5.6) proves the estimate for λ_1. The same arguments show the assertion for λ_2. The estimate for λ_3 is clear from (5.5) because of (5.6). Thus we have the assertion for $|\alpha+\beta| = m+1$ and the proof is completed by induction on $|\alpha+\beta|$. □

Lemma 5.5. *One has* $\partial_t\lambda_1 \in \mathcal{C}(\sigma)$, $\partial_t\lambda_2 \in \mathcal{C}(1)$ *and* $\partial_t\lambda_3 \in \mathcal{C}(1)$.

Proof. First examine that $\partial_\lambda q(\lambda_i)\partial_{x,\xi}^{\alpha,\beta}\partial_t\lambda_i$ can be written as

$$\begin{aligned}&\sum_{|\alpha'+\beta'|<|\alpha+\beta|} C_{\dots}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^{s+1}q(\lambda_i)(\partial_{x,\xi}^{\alpha',\beta'}\partial_t\lambda_i)(\partial_{x,\xi}^{\gamma^{(1)}+\delta^{(1)}}\lambda_i)\cdots(\partial_{x,\xi}^{\gamma^{(s)}+\delta^{(s)}}\lambda_i)\\ &\qquad+\sum C_{\dots}\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^{s}\partial_t q(\lambda_i)(\partial_{x,\xi}^{\gamma^{(1)}+\delta^{(1)}}\lambda_i)\cdots(\partial_{x,\xi}^{\gamma^{(s)}+\delta^{(s)}}\lambda_i)\end{aligned} \tag{5.7}$$

where $\alpha'+\mu+\sum\gamma^{(i)} = \alpha$, $\beta'+\nu+\sum\delta^{(i)} = \beta$ and $|\gamma^{(i)}+\delta^{(i)}| \ge 1$. Indeed (5.7) is clear when $|\alpha+\beta| = 0$ from $\partial_\lambda q(\lambda_i)\partial_t\lambda_i + \partial_t q(\lambda_i) = 0$. Differentiating this by $\partial_{x,\xi}^{e,f}$ and repeating the same arguments proving (5.5) one obtains (5.7) by induction. To prove Lemma 5.5 first check that

$$|\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^s\partial_t q(\lambda_j)| \precsim \sigma^{3-j-(3-j)s-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}, \quad j = 1, 2, 3. \tag{5.8}$$

In fact from

$$\partial_t q(\lambda) = -\partial_t(2a_M + a_M^2)\lambda^2 + \partial_t(6a_M + 2a_M^2 + 2a_M^3 - 9b^2)\lambda - \partial_t\Delta_M \tag{5.9}$$

it follows that $|\partial_t q(\lambda_i)| \precsim \lambda_i + \sigma^2$ and $|\partial_{x,\xi}^{\alpha,\beta}\partial_t q(\lambda_i)| \precsim (\lambda_i + \sigma^2)\sigma^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$ for $|\alpha+\beta| \ge 1$ in view of Lemma 5.2 and hence the assertion for $s = 0$. Since $|\partial_{x,\xi}^{\alpha,\beta}\partial_\lambda^s\partial_t q(\lambda_i)| \precsim \sigma^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$ for $s \ge 1$ the assertion can be proved. We now show Lemma 5.5 for λ_1 by induction on $|\alpha+\beta|$. Assume

$$|\partial_{x,\xi}^{\alpha,\beta}\partial_t\lambda_1| \precsim \sigma^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}. \tag{5.10}$$

It is clear from (5.6) and (5.8) that (5.10) holds for $|\alpha+\beta| = 0$. Assume that (5.10) holds for $|\alpha+\beta| \le m$. For $|\alpha+\beta| = m+1$, thanks to the inductive assumption, Lemma 5.4 and Lemma 5.3 it follows that

$$\sum_{|\alpha'+\beta'|<|\alpha+\beta|} |\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^{s+1}q(\lambda_1)(\partial_{x,\xi}^{\alpha',\beta'}\partial_t\lambda_1)(\partial_{x,\xi}^{\gamma^{(1)}+\delta^{(1)}}\lambda_1)\cdots(\partial_{x,\xi}^{\gamma^{(s)}+\delta^{(s)}}\lambda_1)|$$
$$\precsim \sum \sigma^{3-2(s+1)-|\mu+\nu|/2}\sigma^{1-|\alpha'+\beta'|/2}\sigma^{2-|\gamma^{(1)}+\delta^{(1)}|/2}\cdots\sigma^{2-|\gamma^{(s)}+\delta^{(s)}|/2}\langle\xi\rangle_\gamma^{-|\beta|}$$

which is bounded by $\sigma^{2-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$. On the other hand one sees

$$\sum |\partial_{x,\xi}^{\mu,\nu}\partial_\lambda^{s}\partial_t q(\lambda_1)(\partial_{x,\xi}^{\gamma^{(1)}+\delta^{(1)}}\lambda_1)\cdots(\partial_{x,\xi}^{\gamma^{(s)}+\delta^{(s)}}\lambda_1)|$$
$$\preceq \sum \sigma^{2-2s-|\mu+\nu|/2}\sigma^{2-|\gamma^{(1)}+\delta^{(1)}|/2}\cdots\sigma^{2-|\gamma^{(s)}+\delta^{(s)}|/2}\langle\xi\rangle_\gamma^{-|\beta|}$$
$$\precsim \sigma^{2-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$$

in view of (5.8) and Lemma 5.3. This proves that (5.10) holds for $|\alpha+\beta| = m+1$ and hence for all α, β. As for λ_2, λ_3 the proof is similar. □

5.3 Eigenvectors of the Bézout matrix. We sometimes denote by $\mathcal{C}(\sigma^s)$ a function belonging to $\mathcal{C}(\sigma^s)$. If we write n_{ij} for the (i,j)-cofactor of $\lambda_k I - S$ then ${}^t(n_{j1}, n_{j2}, n_{j3})$ is, if non-trivial, an eigenvector of S corresponding to λ_k. We take $k=1, j=3$ and hence

$$\begin{bmatrix} a_M(2\,a_M-\lambda_1)\\ 3\,b(\lambda_1-3)\\ (\lambda_1-3)(\lambda_1-2\,a_M)\end{bmatrix} = \begin{bmatrix}\ell_{11}\\ \ell_{21}\\ \ell_{31}\end{bmatrix}$$

is an eigenvector of S corresponding to λ_1 and therefore

$$\mathbf{t}_1 = \begin{bmatrix}t_{11}\\ t_{21}\\ t_{31}\end{bmatrix} = \frac{1}{d_1}\begin{bmatrix}\ell_{11}\\ \ell_{21}\\ \ell_{31}\end{bmatrix}, \quad d_1 = \sqrt{\ell_{11}^2+\ell_{21}^2+\ell_{31}^2}$$

is a unit eigenvector of S corresponding to λ_1. Thanks to Proposition 5.1 and recalling $b \in \mathcal{C}(\sigma^{3/2})$ it is clear that $d_1 = \sqrt{36\,a_M^2 + \mathcal{C}(\sigma^3)} = 6a_M(1+\mathcal{C}(\sigma))$. Therefore since $\ell_{11} = \mathcal{C}(\sigma^2)$, $\ell_{21} = \mathcal{C}(\sigma^{3/2})$ and $\ell_{31} = 6\,a + \mathcal{C}(\sigma^2)$ we have

$$\mathbf{t}_1 = \begin{bmatrix}t_{11}\\ t_{21}\\ t_{31}\end{bmatrix} = \begin{bmatrix} a_M/3+\mathcal{C}(\sigma^2)\\ -3b/(2a_M)+\mathcal{C}(\sigma)\\ 1+\mathcal{C}(\sigma)\end{bmatrix}.$$

Similarly, choosing $k=2, j=2$ and $k=3, j=1$,

$$\begin{bmatrix} -3a_Mb\\ (\lambda_2-3)(\lambda_2-a_M^2)-a_M^2\\ 3b(\lambda_2-3)\end{bmatrix} = \begin{bmatrix}\ell_{12}\\ \ell_{22}\\ \ell_{32}\end{bmatrix}, \quad \begin{bmatrix}(\lambda_3-2a_M)(\lambda_3-a_M^2)-9b^2\\ -3a_Mb\\ -a_M(\lambda_3-2a_M)\end{bmatrix} = \begin{bmatrix}\ell_{13}\\ \ell_{23}\\ \ell_{33}\end{bmatrix}$$

are eigenvectors of S corresponding to λ_2 and λ_3 respectively and

$$\mathbf{t}_j = \begin{bmatrix} t_{1j} \\ t_{2j} \\ t_{3j} \end{bmatrix} = \frac{1}{d_j} \begin{bmatrix} \ell_{1j} \\ \ell_{2j} \\ \ell_{3j} \end{bmatrix}, \quad d_j = \sqrt{\ell_{1j}^2 + \ell_{2j}^2 + \ell_{3j}^2}$$

is a unit eigenvector of S corresponding to λ_j, $j = 2, 3$. Thanks to Proposition 5.1 it is easy to see that $d_2 = 3\lambda_2(1 + \mathcal{C}(\sigma))$ and $d_3 = \lambda_3^2(1 + \mathcal{C}(\sigma))$. Then repeating the same arguments one concludes that

$$\begin{bmatrix} t_{12} \\ t_{22} \\ t_{32} \end{bmatrix} = \begin{bmatrix} \mathcal{C}(\sigma^{3/2}) \\ -1 + \mathcal{C}(\sigma) \\ -3b/\lambda_2 + \mathcal{C}(\sigma) \end{bmatrix}, \quad \begin{bmatrix} t_{13} \\ t_{23} \\ t_{33} \end{bmatrix} = \begin{bmatrix} 1 + \mathcal{C}(\sigma) \\ \mathcal{C}(\sigma^{5/2}) \\ -a_M/\lambda_3 + \mathcal{C}(\sigma^2) \end{bmatrix}.$$

Now $T = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) = (t_{ij})$ is an orthogonal matrix which diagonalizes S;

$$\Lambda = T^{-1}ST = {}^tTST = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Note that ΛA^T is symmetric. We summarize what we have proved in

Lemma 5.6. *Let T be defined as above. Then there is M_0 such that T has the form*

$$T = \begin{bmatrix} a_M/3 + \mathcal{C}(\sigma^2) & \mathcal{C}(\sigma^{3/2}) & 1 + \mathcal{C}(\sigma) \\ -3b/(2a_M) + \mathcal{C}(\sigma) & -1 + \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{5/2}) \\ 1 + \mathcal{C}(\sigma) & -3b/\lambda_2 + \mathcal{C}(\sigma) & -a_M/\lambda_3 + \mathcal{C}(\sigma^2) \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{3/2}) & 1 + \mathcal{C}(\sigma) \\ \mathcal{C}(\sigma^{1/2}) & -1 + \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{5/2}) \\ 1 + \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{1/2}) & \mathcal{C}(\sigma) \end{bmatrix}, \quad M \geq M_0.$$

In particular T, $T^{-1} \in S(1, g)$.

Lemma 5.7. *We have*

$$\partial_t T = \begin{bmatrix} \partial_t(a_M/3) + \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{1/2}) & \mathcal{C}(1) \\ -\partial_t(3b/2a_M) + \mathcal{C}(1) & \mathcal{C}(1) & \mathcal{C}(\sigma^{3/2}) \\ \mathcal{C}(1) & -\partial_t(3b/\lambda_2) + \mathcal{C}(1) & -\partial_t(a_M/\lambda_3) + \mathcal{C}(\sigma) \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{C}(1) & \mathcal{C}(\sigma^{1/2}) & \mathcal{C}(1) \\ \mathcal{C}(\sigma^{-1/2}) & \mathcal{C}(1) & \mathcal{C}(\sigma^{3/2}) \\ \mathcal{C}(1) & \mathcal{C}(\sigma^{-1/2}) & \mathcal{C}(1) \end{bmatrix}, \quad M \geq M_0.$$

Proof. Note that every entry of T is a function of a_M, b and λ_j. Then the assertion is clear from Lemmas 5.2 and 5.5. □

From Lemma 5.6 it follows that

$$\langle\xi\rangle_\gamma^{|\beta|}\partial_x^\alpha\partial_\xi^\beta T = \begin{bmatrix} \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) \\ \mathcal{C}(1) & \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(\sigma^2) \\ \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(1) & \mathcal{C}(\sqrt{\sigma}) \end{bmatrix}, \quad |\alpha+\beta| = 1. \tag{5.11}$$

Lemma 5.8. *There is M_0 such that $A^T = T^{-1}AT$ has the form*

$$A^T = \begin{bmatrix} \mathcal{C}(\sqrt{\sigma}) & -1+\mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) \\ \lambda_1\,\mathcal{C}(\sigma^{-1}) & \mathcal{C}(\sqrt{\sigma}) & -1+\mathcal{C}(\sigma) \\ \lambda_1\,\mathcal{C}(\sqrt{\sigma}) & \lambda_2\,\mathcal{C}(1) & \mathcal{C}(\sigma^{5/2}) \end{bmatrix}, \quad M \ge M_0.$$

Proof. Writing $A^T = (\tilde{a}_{ij})$ it is clear that

$$\tilde{a}_{ij} = t_{1i}\,a_M\,t_{2j} + t_{1i}\,b\,t_{3j} + t_{2i}t_{1j} + t_{3i}t_{2j}$$

from which the assertion for $\tilde{a}_{ij}, j \ge i$ follows easily. Therefore one sees that

$$\Lambda A^T = \begin{bmatrix} \lambda_1\mathcal{C}(\sqrt{\sigma}) & \lambda_1(-1+\mathcal{C}(\sigma)) & \lambda_1\mathcal{C}(\sqrt{\sigma}) \\ \lambda_2\tilde{a}_{21} & \lambda_2\tilde{a}_{22} & \lambda_2(-1+\mathcal{C}(\sigma)) \\ \lambda_3\tilde{a}_{31} & \lambda_3\tilde{a}_{32} & \lambda_3\tilde{a}_{33} \end{bmatrix}.$$

Since ΛA^T is symmetric it follows immediately that

$$\tilde{a}_{31} = \lambda_1\mathcal{C}(\sqrt{\sigma})/\lambda_3,\ \tilde{a}_{32} = \lambda_2(-1+\mathcal{C}(\sigma))/\lambda_3,\ \tilde{a}_{21} = \lambda_1(-1+\mathcal{C}(\sqrt{\sigma}))/\lambda_2$$

which proves the assertion because $1/\lambda_3 \in \mathcal{C}(1)$ and $1/\lambda_2 \in \mathcal{C}(\sigma^{-1})$. □

Corollary 5.3. *There is M_0 such that $A^T = T^{-1}AT$ has the form*

$$A^T = \begin{bmatrix} \mathcal{C}(\sqrt{\sigma}) & -1+\mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) \\ \mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) & -1+\mathcal{C}(\sigma) \\ \mathcal{C}(\sigma^{5/2}) & \mathcal{C}(\sigma) & \mathcal{C}(\sigma^{5/2}) \end{bmatrix}, \quad M \ge M_0.$$

Corollary 5.4. *We have*

$$\langle\xi\rangle_\gamma^{|\beta|}\partial_x^\alpha\partial_\xi^\beta A^T = \begin{bmatrix} \mathcal{C}(1) & \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(1) \\ \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(1) & \mathcal{C}(\sqrt{\sigma}) \\ \mathcal{C}(\sigma^2) & \mathcal{C}(\sqrt{\sigma}) & \mathcal{C}(\sigma^2) \end{bmatrix}, \quad |\alpha+\beta| = 1.$$

Proof. The proof is clear since $\langle\xi\rangle_\gamma^{|\beta|}\partial_x^\alpha\partial_\xi^\beta(-1+\mathcal{C}(\sigma)) = \mathcal{C}(\sqrt{\sigma})$. □

Finally consider $T^{-1}(\partial_t T)$. Note that $\langle \partial_t \mathbf{t}_i, \mathbf{t}_j\rangle + \langle \mathbf{t}_i, \partial_t \mathbf{t}_j\rangle = 0$ so that $(\partial_t T^{-1})T$ is antisymmetric. From Lemmas 5.6 and 5.7 one has

$$T^{-1}(\partial_t T) = \begin{bmatrix} 0 & -\partial_t(3b/2a_M) + \mathcal{C}(1) & \mathcal{C}(1) \\ \partial_t(3b/2a_M) + \mathcal{C}(1) & 0 & \mathcal{C}(\sqrt{\sigma}) \\ \partial_t(a_M/3) + \mathcal{C}(\sigma) & \mathcal{C}(\sqrt{\sigma}) & 0 \end{bmatrix}. \tag{5.12}$$

For later use we estimate the (2, 1)-th and (3, 1)-th entries of $T^{-1}(\partial_t T)$. Recalling $a_M = e(t + \alpha + 2M\langle\xi\rangle_\gamma^{-1})$ and $0 \le t \le M^{-4}$ it is clear that $\partial_t a_M - \bar{e} \in S(M^{-2}, g)$. Taking $|b^2/a_M^3| \le 4/27$ into account, thanks to Lemma 4.6 it follows that

$$\begin{aligned} |\sqrt{a_M}\partial_t(3b/2a_M)| &\le 3(|\partial_t b/\sqrt{a_M}| + |b/a_M^{3/2}||\partial_t a_M|)/2 \\ &\le (1 + CM^{-2})((1 + 3\sqrt{2})/\sqrt{3})\,\bar{e}. \end{aligned} \tag{5.13}$$

6 Metric g and estimates of ω and ϕ

Introduce the metric

$$g = g_{(x,\xi)} = M^{-1}(\langle\xi\rangle_\gamma |dx|^2 + \langle\xi\rangle_\gamma^{-1}|d\xi|^2)$$

which is a basic metric with which we work in this paper. Note that $S(M^s, G) \subset S(M^s, g)$ because

$$M^{s+2|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|} \le M^s M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$$

in view of $\langle\xi\rangle_\gamma \ge \gamma \ge M^5$. The metric g is temperate (see [7, Chapter 18]) uniformly in $\gamma \ge M^5 \ge 1$ and will be checked in Section 7.

Lemma 6.1. *One has*

$$\partial_x^\alpha \partial_\xi^\beta \psi \in S(M^{-(|\alpha+\beta|-1)/2}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, g), \quad |\alpha+\beta| \ge 1.$$

Proof. It is enough to remark that

$$|\partial_x^\alpha \partial_\xi^\beta \psi| \precsim \rho^{1/2}\rho^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{-|\beta|} \precsim \rho^{1/2}(M^{-1}\langle\xi\rangle_\gamma)^{(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{-|\beta|}$$

for $|\alpha+\beta| \ge 1$. □

6.1 Estimate ω by metric g. Taking Proposition 4.1 into account we introduce a preliminary weight

$$\omega(t, x, \xi) = \sqrt{(t - \psi(x,\xi))^2 + \bar{\nu} M\rho\langle\xi\rangle_\gamma^{-1}}.$$

Since the exact value of $\bar{\nu} > 0$ is irrelevant in the following arguments, we assume $\bar{\nu} = 1$ from now on. In what follows we work with symbols depending on t where t varies in some fixed interval $[0, T]$, and it is assumed that all constants are independent of $t \in [0, T]$ and γ, M unless otherwise stated. Now $A \precsim B$ implies that A is bounded by a constant, independent of t, M and γ, times B.

Lemma 6.2. *One has*

$$\partial_x^\alpha \partial_\xi^\beta \omega^s \in S(M^{-(|\alpha+\beta|-1)/2}\omega^s\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2+(|\alpha|-|\beta|)/2}, g), \quad |\alpha+\beta| \ge 1.$$

Proof. Recall that $\omega^2 = (t-\psi)^2 + M\rho\langle\xi\rangle_\gamma^{-1}$. Note that for $|\alpha+\beta| \ge 2$

$$\begin{aligned}|\partial_x^\alpha \partial_\xi^\beta (t-\psi)^2| &\precsim \omega|\partial_x^\alpha\partial_\xi^\beta\psi| + \sum |\partial_x^{\alpha'}\partial_\xi^{\beta'}\psi||\partial^{\alpha''}\partial_\xi^{\beta''}\psi| \\ &\precsim \omega^2\{\omega^{-1}\rho^{1/2}\rho^{-(|\alpha+\beta|-1)/2} + \omega^{-2}\rho\rho^{-(|\alpha+\beta|-2)/2}\}\langle\xi\rangle_\gamma^{-|\beta|} \\ &\precsim \omega^2(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2})M^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\end{aligned}$$

since $\rho \ge M\langle\xi\rangle_\gamma^{-1}$ and $\omega \ge \sqrt{M}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}$. When $|\alpha+\beta| = 1$ it is clear that

$$|\partial_x^\alpha\partial_\xi^\beta(t-\psi)^2| \precsim \omega\rho^{1/2}\langle\xi\rangle_\gamma^{-|\beta|} = \omega^2(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2})\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}.$$

Next, it is easy to see that for $|\alpha+\beta| \ge 1$

$$\begin{aligned}|\partial_x^\alpha\partial_\xi^\beta(M\rho\langle\xi\rangle_\gamma^{-1})| &\precsim M\rho\langle\xi\rangle_\gamma^{-1}\rho^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|} \\ &\precsim \omega^2(M\omega^{-2}\rho^{1/2}\langle\xi\rangle_\gamma^{-1})(M^{-1}\langle\xi\rangle_\gamma)^{(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{-|\beta|} \\ &\precsim \omega^2(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2})M^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\end{aligned}$$

because $\omega \ge \sqrt{M}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2} \ge M\langle\xi\rangle_\gamma^{-1}$. Therefore one concludes that

$$|\partial_x^\alpha\partial_\xi^\beta\omega^2| \precsim \omega^2(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2})M^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$$

which proves the assertion for $s = 2$. For general s, noting that

$$|\partial_x^\alpha\partial_\xi^\beta(\omega^2)^{s/2}| \precsim \sum_{|\alpha^i+\beta^i|\ge 1} |(\omega^2)^{s/2}(\partial_x^{\alpha^1}\partial_\xi^{\beta^1}\omega^2/\omega^2)\cdots(\partial_x^{\alpha^l}\partial_\xi^{\beta^l}\omega^2/\omega^2)|$$

the proof is immediate, since $\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2} \le M^{-1/2} \le 1$. □

Corollary 6.1. *We have $\omega^s \in S(\omega^s, g)$ for $s \in \mathbb{R}$.*

6.2 Estimate ϕ by metric g. Introduce a weight function which plays a crucial role in deriving energy estimates

$$\phi(t, x, \xi) = \omega(t, x, \xi) + t - \psi(x, \xi).$$

If $t-\psi(x,\xi)\ge 0$ then $\phi\ge\omega=\omega^2/\omega\ge M\rho\langle\xi\rangle_\gamma^{-1}/\omega$, and if $t-\psi(x,\xi)\le 0$ we see that $\phi=M\rho\langle\xi\rangle_\gamma^{-1}/(\omega+|t-\psi|)\ge\rho M\langle\xi\rangle_\gamma^{-1}/(2\omega)$, hence

$$\phi(t,x,\xi)\ge M\rho\langle\xi\rangle_\gamma^{-1}/(2\,\omega). \tag{6.1}$$

Lemma 6.3. *There is $C>0$ such that $\phi(t,x,\xi)\ge M\langle\xi\rangle_\gamma^{-1}/C$.*

Proof. When $t-\psi(x,\xi)\ge 0$ then $\phi\ge\omega\ge M^{1/2}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\ge M\langle\xi\rangle_\gamma^{-1}$ is obvious for $\rho\ge M\langle\xi\rangle_\gamma^{-1}$. Assume $t-\psi(x,\xi)<0$; then $0\le t<\psi(x,\xi)\le\delta\rho(x,\xi)$ with some $\delta>0$ by Lemma 4.5. Noticing that $|t-\psi(x,\xi)|=\psi(x,\xi)-t\le\delta\rho(x,\xi)$ we have $\omega^2(t,x,\xi)\le\delta^2\rho^2+M\rho\langle\xi\rangle_\gamma^{-1}\le\delta^2\rho^2+\rho^2=(\delta^2+1)\rho^2$. Now the proof is immediate from (6.1). □

Lemma 6.4. *We have $\phi\in S(\phi,g)$.*

Proof. Let $|\alpha+\beta|=1$ and write

$$\partial_x^\alpha\partial_\xi^\beta\phi=\frac{-\partial_x^\alpha\partial_\xi^\beta\psi}{\omega}\phi+\frac{\partial_x^\alpha\partial_\xi^\beta(M\rho\langle\xi\rangle_\gamma^{-1})}{2\omega}=\phi_{\alpha\beta}\phi+\psi_{\alpha\beta}. \tag{6.2}$$

From Corollary 6.1 and Lemma 4.3 it follows that

$$\begin{aligned}|\partial_x^\mu\partial_\xi^\nu(\psi_{\alpha\beta})|&\precsim\omega^{-1}M\rho\langle\xi\rangle_\gamma^{-1}M^{-|\alpha+\beta+\mu+\nu|/2}\langle\xi\rangle_\gamma^{(|\alpha+\mu|-|\beta+\nu|)/2}\\&\precsim\phi M^{-|\alpha+\beta+\mu+\nu|/2}\langle\xi\rangle_\gamma^{(|\alpha+\mu|-|\beta+\nu|)/2}\end{aligned}$$

in view of (6.1). On the other hand, thanks to Lemma 6.1 and Corollary 6.1 it follows that $|\partial_x^\mu\partial_\xi^\nu\phi_{\alpha\beta}|\precsim M^{-|\alpha+\beta+\mu+\nu|/2}\langle\xi\rangle_\gamma^{(|\alpha+\mu|-|\beta+\nu|)/2}$. Hence using (6.2) the assertion is proved by induction on $|\alpha+\beta|$. □

We refine this lemma.

Lemma 6.5. *One has*

$$\partial_x^\alpha\partial_\xi^\beta\phi\in S(\phi M^{-(|\alpha+\beta|-1)/2}\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2},g),\quad |\alpha+\beta|\ge 1.$$

Proof. From Lemma 6.1 one has $\partial_x^\alpha\partial_\xi^\beta\psi\in S(\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2},g)$ for $|\alpha+\beta|=1$ hence $\phi_{\alpha\beta}\in S(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2},g)$ for $|\alpha+\beta|=1$ by Lemma 6.2. From Lemma 6.2 it follows that

$$|\partial_x^\mu\partial_\xi^\nu(\psi_{\alpha\beta})|\precsim\omega^{-1}\rho^{1/2}M\langle\xi\rangle_\gamma^{-1-|\beta|}M^{-|\mu+\nu|/2}\langle\xi\rangle_\gamma^{(|\mu|-|\nu|)/2}$$

for $|\alpha+\beta|=1$ because $\partial_x^\alpha\partial_\xi^\beta(M\rho\langle\xi\rangle_\gamma^{-1})\in S(M\rho^{1/2}\langle\xi\rangle_\gamma^{-1-|\beta|},g)$. Thanks to Lemma 6.3 one sees that $M\langle\xi\rangle_\gamma^{-1}\le C\phi(t,x,\xi)$ and hence

$$\psi_{\alpha\beta}\in S(\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\phi,g),\quad |\alpha+\beta|=1.$$

Since $\phi\in S(\phi,g)$ by Lemma 6.4 we conclude the assertion from (6.2). □

7 ϕ and λ_j are admissible weights for g

Write $z = (x, \xi)$ and $w = (y, \eta)$. It is clear that

$$g^\sigma_z = M(\langle\xi\rangle_\gamma|dx|^2 + \langle\xi\rangle_\gamma^{-1}|d\xi|^2) = M^2 g_z$$

where $g^\sigma_z(t_1, t_2) = \sup|\langle t_2, s_1\rangle - \langle t_1, s_2\rangle|^2/g_z(s_1, s_2)$ (see [7, Chapter 18]). Note that $|\xi - \eta| \le c\,\langle\xi\rangle_\gamma$ with $0 < c < 1$ implies

$$(1 - c)\langle\xi\rangle_\gamma/\sqrt{2} \le \langle\eta\rangle_\gamma \le \sqrt{2}\,(1 + c)\langle\xi\rangle_\gamma.$$

If $g_z(w) < c$ then $|\xi - \eta|^2 < c\,M\langle\xi\rangle_\gamma = c\,M\langle\xi\rangle_\gamma^{-1}\langle\xi\rangle_\gamma^2 \le c\,\langle\xi\rangle_\gamma^2$, so

$$g_z(X)/C \le g_w(X) \le C g_z(X), \quad X \in \mathbb{R}^d \times \mathbb{R}^d$$

with C independent of $\gamma \ge M^5 \ge 1$, namely g_z is slowly varying uniformly in $\gamma \ge M^5 \ge 1$. Similarly, noting that $|\xi-\eta| \ge (\gamma+|\xi|)/2 \ge \langle\xi\rangle_\gamma/2$ if $\langle\eta\rangle_\gamma \le \langle\xi\rangle_\gamma/2\sqrt{2}$ and $|\xi - \eta| \ge (\gamma + |\eta|)/2 \ge \langle\eta\rangle_\gamma/2$ if $\langle\eta\rangle_\gamma \ge 2\sqrt{2}\langle\xi\rangle_\gamma$, it is clear that

$$\frac{\langle\xi\rangle_\gamma}{\langle\eta\rangle_\gamma} + \frac{\langle\eta\rangle_\gamma}{\langle\xi\rangle_\gamma} \le C(1 + \langle\eta\rangle_\gamma^{-1}|\xi - \eta|^2) \le C(1 + g^\sigma_w(z - w)), \tag{7.1}$$

hence $g_w(X) \le C g_z(X)(1 + g^\sigma_w(z - w))$, namely g is a temperate metric uniformly in $\gamma \ge 0$ and $M \ge 1$ (see [7, Chapter 18]). It is clear from (7.1) that

$$g^\sigma_z(z - w) \le C(1 + g^\sigma_w(z - w))^2. \tag{7.2}$$

7.1 ρ and σ are admissible weights for g.

We adopt the same convention as in Sections 6 and 5 even pertaining to weights for g, so that we omit to say uniformly in $t \in [0, M^{-4}]$.

Lemma 7.1. *ρ is an admissible weight for g.*

Proof. First study $\rho^{1/2}$. Assume

$$g_z(w) = M^{-1}\langle\xi\rangle_\gamma(|y|^2 + \langle\xi\rangle_\gamma^{-2}|\eta|^2) < c\ (< 1/2)$$

so that $M^{-1}\langle\xi\rangle_\gamma^{-1}|\eta|^2 < c$, hence $|\eta| < c\langle\xi\rangle_\gamma$ for $M\langle\xi\rangle_\gamma^{-1} \le 1$ so

$$\langle\xi + s\eta\rangle_\gamma/C \le \langle\xi\rangle_\gamma \le C\langle\xi + s\eta\rangle_\gamma, \tag{7.3}$$

where C is independent of $|s| \le 1$. Lemma 4.3 shows that

$$|\rho^{1/2}(z + w) - \rho^{1/2}(z)| \le C(|y| + \langle\xi + s\eta\rangle_\gamma^{-1}|\eta|) \le CM^{1/2}\langle\xi\rangle_\gamma^{-1/2} g_z^{1/2}(w).$$

Since $\rho(z) \geq M\langle\xi\rangle_\gamma^{-1}$ this yields

$$|\rho^{1/2}(z+w) - \rho^{1/2}(z)| \leq C\rho^{1/2}(z)g_z^{1/2}(w). \tag{7.4}$$

Choosing c such that $C\,c < 1/2$ one has $|\rho(z+w)/\rho(z) - 1| < 1/2$, which implies

$$\rho^{1/2}(z+w)/2 \leq \rho^{1/2}(z) \leq 3\,\rho^{1/2}(z+w)/2,$$

namely $\rho^{1/2}$ is g continuous, hence so is ρ. Note that $M\langle\xi\rangle_\gamma^{-1} \leq \rho(z) \leq CM^{-4} \leq C$. If $|\eta| \geq c\,\langle\xi\rangle_\gamma/2$, then $g_z^\sigma(w) \geq Mc^2\langle\xi\rangle_\gamma/4$ and $g_z^\sigma(w) \geq Mc|\eta|/2$ therefore

$$\rho(z+w) \leq C \leq C\langle\xi\rangle_\gamma\rho(z) \leq C'\rho(z)(1+g_z^\sigma(w)).$$

If $|\eta| \leq c\langle\xi\rangle_\gamma$ then (7.4) gives

$$\rho^{1/2}(z+w) \leq C\rho^{1/2}(z)(1+g_z(w))^{1/2} \leq C\rho^{1/2}(z)(1+g_z^\sigma(w))^{1/2}, \tag{7.5}$$

so in view of (7.2), ρ is an admissible weight. □

Lemma 7.2. *σ is an admissible weight for g and $\sigma \in S(\sigma, g)$.*

Proof. Since $\rho(z) + M\langle\xi\rangle_\gamma^{-1}$ is admissible for g by Lemma 7.1, it is clear that so is $\sigma = t + \rho(z) + M\langle\xi\rangle_\gamma^{-1}$ for $t \geq 0$. The second assertion is clear from $|\partial_x^\alpha\partial_\xi^\beta\sigma| \precsim \sigma^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|} \precsim \sigma(M^{-1}\langle\xi\rangle_\gamma)^{|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}$ for $\sigma \geq M\langle\xi\rangle_\gamma^{-1}$. □

7.2 ω and ϕ are admissible weights for g.

We start by showing

Lemma 7.3. *ω and ϕ are g continuous.*

Proof. Denote $f = t - \psi$ and $h = M^{1/2}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2}$ so that $\omega^2 = f^2 + h^2$. Note that

$$\begin{aligned} |\omega(z+w) - \omega(z)| &= |\omega^2(z+w) - \omega^2(z)|/|\omega(z+w) + \omega(z)| \\ &\leq 2|f(z+w) - f(z)| + 2|h(z+w) - h(z)| \end{aligned} \tag{7.6}$$

because $|f(z+w)+f(z)|/|\omega(z+w)+\omega(z)| \leq 2$, $|h(z+w)+h(z)|/|\omega(z+w)+\omega(z)| \leq 2$. Assume $g_z(w) < c$ $(\leq 1/2)$, hence (7.3). It is assumed that constants C may change from line to line but are independent of $\gamma \geq M^5 \geq 1$. Noting that $|f(z+w) - f(z)| = |\psi(z+w) - \psi(z)|$ it follows from Lemma 6.1 that

$$\begin{aligned} |f(z+w) - f(z)| &\leq C\rho^{1/2}(z+sw)(|y| + \langle\xi+s\eta\rangle_\gamma^{-1}|\eta|) \\ &\leq C\rho^{1/2}(z+sw)(|y| + \langle\xi\rangle_\gamma^{-1}|\eta|) \\ &\leq CM^{1/2}\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}g_z^{1/2}(w) \end{aligned} \tag{7.7}$$

since ρ is g continuous. Noting that $\omega(z) \geq M^{1/2}\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}$ we have

$$|f(z+w) - f(z)| \leq C\omega(z)g_z^{1/2}(w). \tag{7.8}$$

A similar argument shows that

$$|h(z+w) - h(z)| \leq CM^{1/2}\langle\xi\rangle_\gamma^{-1}g_z^{1/2}(w).$$

Since $\omega(z) \geq M\langle\xi\rangle_\gamma^{-1}$ we have $|h(z+w) - h(z)| \leq CM^{-1/2}\omega(z)g_z^{1/2}(w)$. Therefore from (7.6) one has $|\omega(z+w) - \omega(z)| \leq C\omega(z)g_z^{1/2}(w)$. Choosing c such that $C\,c < 1/2$ we conclude that ω is g continuous.

Next consider $\phi = \omega + f$. Write

$$\phi(z+w) - \phi(z) = \frac{(f(z+w) - f(z))(\phi(z+w) + \phi(z)) + h^2(z+w) - h^2(z)}{\omega(z+w) + \omega(z)}. \tag{7.9}$$

Since $\omega(z+w)/C \leq \omega(z) \leq C\,\omega(z+w)$, and decreasing $c > 0$ if necessary, together with (7.8) this gives $|f(z+w) - f(z)|/(\omega(z+w) + \omega(z)) \leq Cg_z^{1/2}(w)$. Recalling $h^2(z) = M\rho(z)\langle\xi\rangle_\gamma^{-1}$ and repeating similar arguments one sees that

$$|h^2(z+w) - h^2(z)| \leq CM^{1/2}\rho(z)\langle\xi\rangle_\gamma^{-1}g_z^{1/2}(w) \tag{7.10}$$

for $\rho^{1/2}(z) \geq M^{1/2}\langle\xi\rangle_\gamma^{-1/2}$. Taking (6.1) into account it follows from (7.10) that

$$|h^2(z+w) - h^2(z)|/(\omega(z+w) + \omega(z)) \leq C\phi(z)g_z^{1/2}(w).$$

Combining these estimates we obtain from (7.9) that

$$|\phi(z+w)/\phi(z) - 1| \leq C|\phi(z+w)/\phi(z) + 1|g_z^{1/2}(w) + Cg_z^{1/2}(w),$$

which gives $\phi(z)/C \leq \phi(z+w) \leq C\,\phi(z)$ choosing $c > 0$ small, showing that ϕ is g continuous. □

Lemma 7.4. *ω and ϕ are admissible weights for g and $\omega \in S(\omega, g)$, $\phi \in S(\phi, g)$.*

Proof. Note that

$$\langle\xi\rangle_\gamma^{-1} \leq M\langle\xi\rangle_\gamma^{-1} \leq \sqrt{M}\sqrt{\rho}\langle\xi\rangle_\gamma^{-1/2} \leq \omega \leq CM^{-4} \leq C.$$

Assume $|\eta| \geq c\,\langle\xi\rangle_\gamma$ hence $g_z^\sigma(w) \geq Mc^2\langle\xi\rangle_\gamma \geq c^2\langle\xi\rangle_\gamma$. Therefore

$$\omega(z+w) \leq C \leq C\langle\xi\rangle_\gamma\omega(z) \leq C'\omega(z)(1 + g_z^\sigma(w)). \tag{7.11}$$

Assume $|\eta| \leq c\,\langle\xi\rangle_\gamma$ and note that (7.5). Then checking the proof of Lemma 7.3 we see that

$$|f(z+w) - f(z)| \leq C\omega(z)(1 + g_z^\sigma(w))$$

and

$$|h(z+w) - h(z)| \le C\omega(z)(1+g_z^\sigma(w))^{1/2}.$$

Then (7.11) follows from (7.6) which proves that ω is admissible for g. Turn to ϕ. From Lemma 6.3 it follows that

$$\langle\xi\rangle_\gamma^{-1}/C \le M\langle\xi\rangle_\gamma^{-1}/C \le \phi(z) = \omega(z) + f(z) \le CM^{-4} \le C.$$

If $|\eta| \ge \langle\xi\rangle_\gamma/2$ then $g_z^\sigma(w) \ge M\langle\xi\rangle_\gamma/4 \ge \langle\xi\rangle_\gamma/4$, hence

$$\phi(z+w) \le C \le C^2\langle\xi\rangle_\gamma\phi(z) \le C\phi(z)(1+g_z^\sigma(w)).$$

Assume $|\eta| \le \langle\xi\rangle_\gamma/2$ so that (7.3) holds. From (7.5) and (7.7) we have that

$$|f(z+w) - f(z)| \le C\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}(1+g_z^\sigma(w)).$$

Recalling (7.5) and $M^2 g_z(w) = g_z^\sigma(w)$, the same arguments used to obtain (7.10) show that $|h^2(z+w) - h^2(z)| \le C\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-3/2}(1+g_z^\sigma(w))$. Taking these into account (7.9) yields

$$\begin{aligned} &|\phi(z+w) - \phi(z)| \\ &\le C\Big(\frac{\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}(\phi(z+w)+\phi(z)) + \frac{\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-3/2}}{\omega(z+w)+\omega(z)}\Big)(1+g_z^\sigma(w)). \end{aligned} \tag{7.12}$$

Applying Lemma 6.3 to (7.12) we obtain

$$|\phi(z+w) - \phi(z)| \le C(\phi(z+w) + 2\phi(z))\frac{\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}(1+g_z^\sigma(w)).$$

If $\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}(1+g_z^\sigma(w))\big/(\omega(z+w)+\omega(z)) < 1/4$, then it follows that

$$|\phi(z+w)/\phi(z) - 1| \le (\phi(z+w)/\phi(z) + 2)/4$$

from which we have $\phi(z+w) \le 2\phi(z) \le 5\phi(z+w)$. If

$$\rho^{1/2}(z)\langle\xi\rangle_\gamma^{-1/2}(1+g_z^\sigma(w))\big/(\omega(z+w)+\omega(z)) \ge 1/4,$$

we have

$$32(1+g_z^\sigma(w))^2 \ge 4\langle\xi\rangle_\gamma\omega(z+w)\omega(z)\big/\rho(z) \ge \phi(z+w)\big/\phi(z)$$

by (6.1) and an obvious inequality $\phi(z+w) \le 2\,\omega(z+w)$. Thus we conclude that ϕ is admissible for g. □

7.3 λ_j are admissible weights for g.

Lemma 7.5. *Assume that $\lambda \in \mathcal{C}(\sigma^2)$ and $\lambda \geq cM\sigma\langle\xi\rangle_\gamma^{-1}$ with some $c > 0$. Then λ is an admissible weight for g.*

Proof. Consider $\sqrt{\lambda}$. Assume $g_z(w) < c$ and hence $\langle\xi + s\eta\rangle_\gamma \approx \langle\xi\rangle_\gamma$. Since $\sqrt{\lambda} \in \mathcal{C}(\sigma)$ it follows that

$$|\sqrt{\lambda(z+w)} - \sqrt{\lambda(z)}| \leq C\sqrt{\sigma(z+sw)}M^{1/2}\langle\xi\rangle_\gamma^{-1/2}g_z^{1/2}(w) \tag{7.13}$$

with $|s| < 1$ which is bounded by $C'\sqrt{\sigma(z)}M^{1/2}\langle\xi\rangle_\gamma^{-1/2}g_z^{1/2}(w)$ since σ is g continuous. By assumption $\lambda(z) \geq cM\sigma(z)\langle\xi\rangle_\gamma^{-1}$, one has

$$|\sqrt{\lambda(z+w)} - \sqrt{\lambda(z)}| \leq C''\sqrt{\lambda(z)}\,g_z^{1/2}(w).$$

Choosing $c > 0$ such that $C''\sqrt{c} < 1$ shows that $\sqrt{\lambda(z)}$ is g continuous and so is $\lambda(z)$. From $cM^2\langle\xi\rangle_\gamma^{-2} \leq cM\sigma\langle\xi\rangle_\gamma^{-1} \leq \lambda \leq C'\sigma^2 \leq C'M^{-4}$ one sees that

$$c_1M\langle\xi\rangle_\gamma^{-1} \leq c_1M^{1/2}\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2} \leq \sqrt{\lambda(z)} \leq C.$$

If $|\eta| \geq \langle\xi\rangle_\gamma/2$, hence $g_z^\sigma(w) \geq M\langle\xi\rangle_\gamma/4$, then

$$\sqrt{\lambda(z+w)} \leq C \leq C(c_1M)^{-1}\langle\xi\rangle_\gamma\sqrt{\lambda(z)} \leq C'\sqrt{\lambda(z)}\,g_z^\sigma(w).$$

If $|\eta| \leq \langle\xi\rangle_\gamma/2$, noting that $\sigma(z+w) \leq C\,\sigma(z)(1+g_z^\sigma(w))$, it follows from (7.13) that

$$|\sqrt{\lambda(z+w)} - \sqrt{\lambda(z)}| \leq C\sqrt{\lambda(z)}(1 + g_z^\sigma(w))$$

which proves that $\sqrt{\lambda}$ is an admissible weight for g and hence so is λ. □

Lemma 7.6. *Assume that $\lambda \in \mathcal{C}(\sigma)$ and $\lambda \geq cM\langle\xi\rangle_\gamma^{-1}$ with some $c > 0$. Then λ is an admissible weight for g. If $\lambda \in \mathcal{C}(1)$ and $\lambda \geq c$ with some $c > 0$ then λ is an admissible weight for g.*

Proof. It is enough to repeat the proof of Lemma 7.5. □

Lemma 7.7. *Assume $\lambda \in \mathcal{C}(\sigma^2)$ and $\lambda \geq cM\sigma\langle\xi\rangle_\gamma^{-1}$ with some $c > 0$. Then*

$$\partial_x^\alpha\partial_\xi^\beta\lambda \in S(\sqrt{\sigma}\sqrt{\lambda}\langle\xi\rangle_\gamma^{-|\beta|}, g), \quad |\alpha+\beta| = 1.$$

In particular $\lambda \in S(\lambda, g)$.

Proof. From $\lambda \in \mathcal{C}(\sigma^2)$ we have $|\langle\xi\rangle_\gamma^{|\beta|}\partial_x^\alpha\partial_\xi^\beta\lambda| \le C\,\sigma$ for $|\alpha+\beta| = 2$. Since $\lambda \ge 0$, thanks to Glaeser's inequality one has $|\partial_x^\alpha\partial_\xi^\beta\lambda| \le C'\sqrt{\sigma}\sqrt{\lambda}\,\langle\xi\rangle_\gamma^{-|\beta|}$ for $|\alpha+\beta| = 1$. For $|\alpha'+\beta'| \ge 1$ note that

$$
\begin{aligned}
|\partial_x^{\alpha'}\partial_{\xi'}^{\beta'}(\partial_x^\alpha\partial_\xi^\beta\lambda)| &\precsim \sigma^{1-(|\alpha'+\beta'|-1)/2}\langle\xi\rangle_\gamma^{-|\beta|}\langle\xi\rangle_\gamma^{-|\beta'|}\\
&\precsim \sigma M^{-|\alpha'+\beta'|/2}M^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}\langle\xi\rangle_\gamma^{-|\beta|}\\
&\precsim \sqrt{\sigma}M^{-|\alpha'+\beta'|/2}\sqrt{\lambda}\,\langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}\langle\xi\rangle_\gamma^{-|\beta|}
\end{aligned}
$$

because $\lambda \ge cM\sigma\langle\xi\rangle_\gamma^{-1}$ and $\sigma \ge M\langle\xi\rangle_\gamma^{-1}$ which proves the first assertion. Noting that $\sqrt{\sigma}\langle\xi\rangle_\gamma^{-|\beta|} \le CM^{-1/2}\sqrt{\lambda}\,\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$ it is clear that $\lambda \in S(\lambda, g)$. □

Lemma 7.8. *Assume that $\lambda \in \mathcal{C}(\sigma)$ and $\lambda \ge cM\langle\xi\rangle_\gamma^{-1}$ with some $c > 0$. Then $\lambda \in S(\lambda, g)$. If $\lambda \in \mathcal{C}(1)$ and $\lambda \ge c$ with some $c > 0$, then $\lambda \in S(\lambda, g)$.*

Proof. It suffices to repeat the proof of Lemma 7.7. □

Corollary 7.1. *For $s \in \mathbb{R}$ we have $\lambda_j^s \in S(\lambda_j^s, g)$, $j = 1, 2, 3$.*

Define

$$\kappa = \frac{1}{t} + \frac{1}{\omega} = \frac{t+\omega}{t\,\omega}, \quad t > 0.$$

Lemma 7.9. *κ is an admissible weight for g and $\kappa^s \in S(\kappa^s, g)$ for $s \in \mathbb{R}$.*

Proof. Since ω^{-1} is admissible for g it is clear that so is $\kappa = t^{-1}+\omega^{-1}$. Noting that $\omega^{-1} \in S(\omega^{-1}, g)$ and $\omega^{-1} \le \kappa$ it is also clear that

$$|\partial_x^\alpha\partial_\xi^\beta\kappa| = |\partial_x^\alpha\partial_\xi^\beta\omega^{-1}| \precsim M^{-|\alpha+\beta|/2}\kappa\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$$

for $|\alpha+\beta| \ge 1$, which proves $\kappa \in S(\kappa, g)$. □

Lemma 7.10. *One has*

$$\partial_x^\alpha\partial_\xi^\beta\kappa^s \in S(M^{-(|\alpha+\beta|-1)/2}\kappa^s\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1/2+(|\alpha|-\beta|)/2}, g), \quad |\alpha+\beta| \ge 1.$$

Proof. Since $\partial_x^\alpha\partial_\xi^\beta\kappa^s = \kappa^{s-1}\partial_x^\alpha\partial_\xi^\beta\kappa$ it is enough to show the case $s = 1$. The proof for the case $s = 1$ follows easily from Lemma 6.2. □

Lemma 7.11. *Denote $\bar{\varepsilon} = \sqrt{3}/\bar{e}\,\bar{c}^{1/2} = 4\sqrt{6}/\bar{e}$. There is $C > 0$ such that*

$$\frac{1}{\kappa\lambda_1} \le \bar{\varepsilon}^2(1+CM^{-4})\kappa, \quad \frac{1}{\sigma^2\kappa} \le \kappa.$$

Proof. In view of Propositions 4.1 and 5.1 one sees that

$$\lambda_1 \geq (1/\bar{\varepsilon}^2)(1 - CM^{-4}) \min\{t^2, \omega^2\}.$$

Denote $c = \bar{\varepsilon}^2(1 - CM^{-4})^{-1}$. If $\omega^2 \geq t^2$ and hence $\lambda_1 \geq t^2/c$, then $1/\lambda_1 \leq c/t^2$, which shows that

$$\frac{1}{\kappa\lambda_1} \leq \frac{c}{\kappa t^2} = \frac{c\,t\,\omega}{(t+\omega)t^2} = \frac{c\,\omega}{(t+\omega)t} \leq \frac{c(t+\omega)}{t\omega} = c\kappa.$$

If $t^2 \geq \omega^2$ and hence $\lambda_1 \geq \omega^2/c$, then $1/\lambda_1 \leq c/\omega^2$ and hence

$$\frac{1}{\kappa\lambda_1} \leq \frac{c}{\kappa\omega^2} = \frac{c\,t\,\omega}{(t+\omega)\omega^2} = \frac{c\,t}{(t+\omega)\omega} \leq \frac{c(t+\omega)}{t\,\omega} = c\kappa,$$

thus the first assertion. To show the second assertion it suffices to note that $\sigma \geq t$ and then $\sigma^2(t+\omega)^2 \geq t^2(t+\omega)^2 \geq t^2\omega^2$. □

8 Lower bounds of $\mathrm{op}(\lambda_j)$

8.1 Some preliminary lemmas.

Introduce a metric

$$\bar{g} = \langle\xi\rangle_\gamma |dx|^2 + \langle\xi\rangle_\gamma^{-1}|d\xi|^2$$

independent of M so that $g = M^{-1}\,\bar{g}$. We start with

Lemma 8.1. *Let m be an admissible weight for g and $p \in S(m,g)$ satisfy $p \geq c\,m$ with some constant $c > 0$. Then $p^{-1} \in S(m^{-1}, g)$ and there exist $k, \tilde{k} \in S(M^{-1}, g)$ such that*

$$p\#p^{-1}\#(1+k) = 1, \quad (1+k)\#p\#p^{-1} = 1, \quad p^{-1}\#(1+k)\#p = 1,$$
$$p^{-1}\#p\#(1+\tilde{k}) = 1, \quad (1+\tilde{k})\#p^{-1}\#p = 1, \quad p\#(1+\tilde{k})\#p^{-1} = 1.$$

Proof. In this proof every constant is independent of $\gamma \geq 1$ and M. It is clear that $p^{-1} \in S(m^{-1}, g)$. Write $p\#p^{-1} = 1 - r$ where $r \in S(M^{-1}, g)$. Since

$$|r|^{(l)}_{S(1,\bar{g})} = \sup_{|\alpha+\beta|\leq l, (x,\xi)\in\mathbb{R}^{2d}} |\langle\xi\rangle_\gamma^{(|\beta|-|\alpha|)/2}\partial_x^\alpha\partial_\xi^\beta r| \leq C_l M^{-1},$$

from the L^2-boundedness theorem (see [7, Theorem 18.6.3]), we have

$$\|\mathrm{op}(r)\| \leq CM^{-1}.$$

Therefore for large M there exists the inverse $(1 - \mathrm{op}(r))^{-1}$ which is given by

$$1 + \sum_{\ell=1}^{\infty} r^{\#\ell} \in S(1, \bar{g})$$

(see [1], [20]). Denote $k = \sum_{\ell=1}^{\infty} r^{\#\ell} \in S(1, \bar{g})$ and prove $k \in S(M^{-1}, g)$. It can be seen from the proof (e.g., [20]) that for any $l \in \mathbb{N}$ one can find $C_l > 0$, independent of γ, such that

$$|k|^{(l)}_{S(1,\bar{g})} \le C_l,$$

because $|k|^{(l)}_{S(1,\bar{g})}$ depends only on l, $|r|^{(l')}_{S(1,\bar{g})}$ with some $l' = l'(l)$ and structure constants of $\bar{g}$ which is independent of γ. Note that k satisfies $(1 - r)\#(1 + k) = 1$, that is

$$k = r + r\#k. \tag{8.1}$$

Since $r \in S(M^{-1}, g)$ it follows from (8.1) that $|k|^{(l)}_{S(1,\bar{g})} \le C_l M^{-1}$. Assume that

$$\sup |\langle\xi\rangle_\gamma^{(|\beta|-|\alpha|)/2} \partial_x^\alpha \partial_\xi^\beta k| \le C_{\alpha,\beta,\nu} M^{-1-l/2}, \quad |\alpha + \beta| \ge l \tag{8.2}$$

for $0 \le l \le \nu$. Let $|\alpha + \beta| \ge \nu + 1$ and note that

$$\partial_x^\alpha \partial_\xi^\beta k = \partial_x^\alpha \partial_\xi^\beta r + \sum C_{\ldots} (\partial_x^{\alpha''} \partial_\xi^{\beta''} r)\#(\partial_x^{\alpha'} \partial_\xi^{\beta'} k)$$

where

$$\alpha' + \alpha'' = \alpha \quad \text{and} \quad \beta' + \beta'' = \beta.$$

From the assumption (8.2) we have $\partial_x^{\alpha'} \partial_\xi^{\beta'} k \in S(M^{-1-|\alpha'+\beta'|/2} \langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}, \bar{g})$ if $|\alpha' + \beta'| \le \nu$ and $\partial_x^{\alpha'} \partial_\xi^{\beta'} k \in S(M^{-1-\nu/2} \langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}, \bar{g})$ if $|\alpha' + \beta'| \ge \nu + 1$. Since $r \in S(M^{-1}, g)$ one has

$$(\partial_x^{\alpha''} \partial_\xi^{\beta''} r)\#(\partial_x^{\alpha'} \partial_\xi^{\beta'} k) \in S(M^{-1-(\nu+2)/2} \langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, \bar{g}),$$

which implies that (8.2) holds for $0 \le l \le \nu + 1$ and hence for all ν by induction on ν. This proves that $k \in S(M^{-1}, g)$. The proof of the assertions for $\tilde{k}$ is similar. □

Here recall [27, Lemmas 3.1.6, 3.1.7].

Lemma 8.2. *Let $q \in S(1, g)$ satisfy $q \ge c$ with a constant c independent of M. Then there is $C > 0$ such that*

$$(\mathrm{op}(q)u, u) \ge (c - CM^{-1/2})\|u\|^2.$$

Proof. One can assume that $c = 0$. We see that $q(x, \xi) + M^{-1/2}$ is an admissible weight for $\bar{g}$ and $(q + M^{-1/2})^{1/2} \in S((q + M^{-1/2})^{1/2}, \bar{g})$. Moreover, $\partial_x^\alpha \partial_\xi^\beta (q + M^{-1/2})^{1/2} \in S(M^{-1/2} \langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, \bar{g})$ for $|\alpha + \beta| = 1$. Therefore

$$q + M^{-1/2} = (q + M^{-1/2})^{1/2}\#(q + M^{-1/2})^{1/2} + r, \quad r \in S(M^{-1}, \bar{g})$$

which proves the assertion. □

Lemma 8.3. *Let $q \in S(1, g)$. Then there is $C > 0$ such that*

$$\|\mathrm{op}(q)u\| \le (\sup|q| + CM^{-1/2})\|u\|.$$

Lemma 8.4. *Let $m > 0$ be an admissible weight for g and $m \in S(m, g)$. Then there is $C > 0$ such that*

$$(\mathrm{op}(m)u, u) \ge (1 - CM^{-2})\|\mathrm{op}(\sqrt{m})u\|^2.$$

If $q \in S(m, g)$ then there is $C > 0$ such that

$$|(\mathrm{op}(q)u, u)| \le (\sup(|q|/m) + CM^{-1/2})\|\mathrm{op}(\sqrt{m})u\|^2.$$

Proof. First note that $m^{\pm 1/2}$ are admissible weights and $m^{\pm 1/2} \in S(m^{\pm 1/2}, g)$. Write

$$\tilde{q} = (1 + k)\# m^{-1/2}\# q\# m^{-1/2}\#(1 + \tilde{k}) \in S(1, g),$$

where $m^{1/2}\#(1 + k)\# m^{-1/2} = 1$ and $m^{-1/2}\#(1 + \tilde{k})\# m^{1/2} = 1$ such that

$$m^{1/2}\#\tilde{q}\# m^{1/2} = q.$$

Since $k, \tilde{k} \in S(M^{-1}, g)$ one sees that $\tilde{q} = qm^{-1} + r$ with $r \in S(M^{-1}, g)$. Thanks to Lemma 8.3 we have

$$\|\mathrm{op}(qm^{-1})v\| \le (\sup(|q|/m) + CM^{-1/2})\|v\|$$

hence $|(\mathrm{op}(q)u, u)|$ is bounded by

$$|(\mathrm{op}(qm^{-1})\mathrm{op}(m^{1/2})u, \mathrm{op}(m^{1/2})u)| + CM^{-1}\|\mathrm{op}(m^{1/2})u\|^2$$

which proves the second assertion. The first assertion follows from the second since $m = m^{1/2}\# m^{1/2} + r$ with $r \in S(M^{-2}m, g)$. □

Lemma 8.5. *Let $m_i > 0$ be two admissible weights for g and assume that $m_i \in S(m_i, g)$ and $m_2 \le C\, m_1$ with $C > 0$. Then there is $C' > 0$ such that*

$$\|\mathrm{op}(m_2)u\| \le C'\|\mathrm{op}(m_1)u\|.$$

Proof. Write $\tilde{m}_2 = m_2\# m_1^{-1}\#(1 + k) \in S(1, g)$ such that $m_2 = \tilde{m}_2\# m_1$ with $k \in S(M^{-1}, g)$. Then

$$\|\mathrm{op}(m_2)u\| = \|\mathrm{op}(\tilde{m}_2)\mathrm{op}(m_1)u\| \le C'\|\mathrm{op}(m_1)u\|$$

proves the assertion. □

8.2 Lower bounds of $\mathrm{op}(\lambda_j)$.

Lemma 8.6. *There exist $C > 0$ and M_0 such that*

$$\mathsf{Re}\,(\mathrm{op}(\lambda_j\#\kappa)u, u) \geq (1 - CM^{-2})\|\mathrm{op}(\kappa^{1/2}\lambda_j^{1/2})u\|^2, \quad M \geq M_0.$$

Proof. Since

$$\lambda_j\#\kappa = \kappa\lambda_j + r_{j1} + r_{j2}$$

where r_{j1} is pure imaginary and $r_{j2} \in S(M^{-2}\kappa\lambda_j, g)$, the assertion follows from Lemma 8.4. □

Lemma 8.7. *There exist $c > 0$ and M_0 such that*

$$(\mathrm{op}(\lambda_1)u, u) \geq c\,\|\mathrm{op}(\lambda_1^{1/2})u\|^2 + cM^2\|\langle D\rangle_\gamma^{-1}u\|^2, \quad M \geq M_0.$$

Proof. From Propositions 4.1 and 5.1 it follows that $\lambda_1 \geq c' M\sigma\langle\xi\rangle_\gamma^{-1}$ with some $c' > 0$. Write

$$\lambda_1 - c\,M\sigma\langle\xi\rangle_\gamma^{-1} = \lambda_1/2 + (\lambda_1/2 - c\,M\sigma\langle\xi\rangle_\gamma^{-1}),$$

where $c > 0$ is chosen so that $\tilde{\lambda}_1 = \lambda_1/2 - c\,M\sigma\langle\xi\rangle_\gamma^{-1} \geq c_1 M\sigma\langle\xi\rangle_\gamma^{-1}$ with $c_1 > 0$. Note that $\tilde{\lambda}_1 \in \mathcal{C}(\sigma^2)$ since $M\sigma\langle\xi\rangle_\gamma^{-1} \in \mathcal{C}(\sigma^2)$. Thanks to Lemmas 7.5 and 7.7 it follows that $\tilde{\lambda}_1 \in S(\tilde{\lambda}_1, g)$ and $\tilde{\lambda}_1$ is an admissible weight for g. Thus we have $(\mathrm{op}(\tilde{\lambda}_1)u, u) \geq (1 - CM^{-2})\|\mathrm{op}(\tilde{\lambda}_1^{1/2})u\|^2 \geq 0$ if $M \geq \sqrt{C}$ by Lemma 8.4. Since $M^2\langle\xi\rangle_\gamma^{-2} \leq M\sigma\langle\xi\rangle_\gamma^{-1}$ it follows from Lemma 8.5 that

$$M\|\langle D\rangle_\gamma^{-1}u\|^2 \leq C\|\mathrm{op}(\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2})u\|^2.$$

Therefore the proof follows from Lemma 8.4. □

Similar arguments prove the following lemma.

Lemma 8.8. *There exist $c > 0$ and M_0 such that*

$$(\mathrm{op}(\lambda_2)u, u) \geq c\,\|\mathrm{op}(\lambda_2^{1/2})u\|^2 + c\,M\|\langle D\rangle_\gamma^{-1/2}u\|^2, \quad M \geq M_0,$$
$$(\mathrm{op}(\lambda_3)u, u) \geq c\|u\|^2, \quad M \geq M_0.$$

We now summarize what we have proved in

Proposition 8.1. *There exist $c > 0$, $C > 0$ and M_0 such that*

$$\mathsf{Re}\,(\mathrm{op}(\Lambda\#\kappa)W, W) \geq (1 - CM^{-2})\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2,$$
$$\mathsf{Re}\,(\mathrm{op}(\Lambda)W, W) \geq c\,(\|\mathrm{op}(\Lambda^{1/2})W\|^2 + \|\mathrm{op}(\mathcal{D})W\|^2),$$

for $M \geq M_0$ where $\mathcal{D} = \mathrm{diag}(M\langle\xi\rangle_\gamma^{-1}, M^{1/2}\langle\xi\rangle_\gamma^{-1/2}, 1)$.

9 System with diagonal symmetrizer

Diagonalizing the Bézout matrix introduced in Section 5 we reduce the system (5.2) to a system with a diagonal symmetrizer.

Lemma 9.1. *Let* $p \in \mathcal{C}(\sigma^k)$. *Then* $\partial_x^\alpha \partial_\xi^\beta p \in S(\sigma^{k-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}, g)$.

Proof. The proof is clear from

$$|\partial_x^{\alpha'}\partial_\xi^{\beta'}(\partial_x^\alpha\partial_\xi^\beta p)| \precsim \sigma^{k-|\alpha'+\beta'+\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta'+\beta|}$$
$$\precsim \sigma^{k-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|}\sigma^{-|\alpha'+\beta'|/2}\langle\xi\rangle_\gamma^{-|\alpha'+\beta'|/2}\langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}$$

for $\sigma \ge \rho \ge M\langle\xi\rangle_\gamma^{-1}$. □

Lemma 9.2. *Let* $p \in \mathcal{C}(\sigma^k)$ *and* $q \in \mathcal{C}(\sigma^\ell)$. *Then*

$$p\#p - p^2 \in S(\sigma^{2k-2}\langle\xi\rangle_\gamma^{-2}, g), \quad p\#q - pq \in S(\sigma^{k+\ell-1}\langle\xi\rangle_\gamma^{-1}, g).$$

Proof. The assertions follows from Lemma 9.1 and the Weyl calculus of pseudodifferential operators. □

In what follows, in order to simplify notation we sometimes abbreviate $S(m, g)$ to $S(m)$ where m is admissible for g. Since $a \in \mathcal{C}(\sigma)$, $b \in \mathcal{C}(\sigma^{3/2})$ one sees that $A\#[\xi] = A(t, x, \xi)[\xi] + R$, with R whose first row is $(0, S(\sigma^{1/2}), S(\sigma))$ for $\partial_\xi^\beta[\xi] \in S(1, g)$, by (4.5). Moving R to B we denote $L = D_t - \mathrm{op}(\tilde{A}) - \mathrm{op}(B)$ where

$$\tilde{A} = \begin{bmatrix} 0 & a & b \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}[\xi], \quad B = \begin{bmatrix} b_1 & b_2 + d_M + S(\sigma^{1/2}) & b_3 + S(\sigma) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9.1}$$

and transform L to another system using T introduced in Section 5.3. Note that $T^{-1}\#T = I - R$ with $R \in S(M^{-1}, g)$. Thanks to Lemma 8.1 there is $K \in S(M^{-1}, g)$ such that $(I - R)\#(I + K) = I = (I + K)\#(I - R)$ and hence

$$T^{-1}\#T\#(I + K) = I, \quad (I + K)\#T^{-1}\#T = I, \quad T\#(I + K)\#T^{-1} = I.$$

Therefore one can write

$$L\,\mathrm{op}(T) = \mathrm{op}(T)\,\tilde{L} \tag{9.2}$$

where $\tilde{L} = D_t - \mathrm{op}((I + K)\#T^{-1}\#(\tilde{A} + B)\#T) + \mathrm{op}((I + K)\#T^{-1}\#(D_t T))$.

Lemma 9.3. *Notations being as above. Then* $K \in S(M^{-1}\langle\xi\rangle_\gamma^{-1}, g)$.

Proof. Write $T = (t_{ij})$. Then $T^{-1}\#T = (\sum_{k=1}^{3} t_{ki}\#t_{kj})$. Denote

$$\sum_{k=1}^{3} t_{ki}\#t_{kj} = \delta_{ij} + r_{ij}.$$

Taking Lemma 5.6 into account, we see that $r_{ii} \in S(\sigma^{-1}\langle\xi\rangle_\gamma^{-2}, g) \subset S(M^{-1}\langle\xi\rangle_\gamma^{-1}, g)$ and $r_{ij} \in S(\sigma^{1/2}\langle\xi\rangle_\gamma^{-1}, g) \subset S(M^{-2}\langle\xi\rangle_\gamma^{-1}, g)$ for $i \neq j$ thanks to Lemma 9.2 hence $R \in S(M^{-1}\langle\xi\rangle_\gamma^{-1}, g)$. Since $K \in S(M^{-1}, g)$ satisfies $K = R + R\#K$ we conclude the assertion. □

Therefore $K\#T^{-1}\#(\tilde{A} + B)\#T \in S(M^{-1}, g)$ is clear. Hence

$$\tilde{L} = D_t - \mathrm{op}(T^{-1}\#(\tilde{A} + B)\#T - T^{-1}\#(D_tT)) + \mathrm{op}(S(M^{-1}, g)).$$

In view of Lemmas 5.6 and 5.7 it follows from Lemma 9.2 that

$$T^{-1}\#(\partial_tT) = T^{-1}\partial_tT + \begin{bmatrix} S(\sigma^{-1}\langle\xi\rangle_\gamma^{-1}) & S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}) & S(\langle\xi\rangle_\gamma^{-1}) \\ S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}) & S(\sigma^{-1}\langle\xi\rangle_\gamma^{-1}) & S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}) \\ S(\langle\xi\rangle_\gamma^{-1}) & S(\sigma^{-1/2}\langle\xi\rangle_\gamma^{-1}) & S(\langle\xi\rangle_\gamma^{-1}) \end{bmatrix} \tag{9.3}$$

hence $T^{-1}\#(\partial_tT) = T^{-1}\partial_tT + S(M^{-1}, g)$ because $\sigma \geq M\langle\xi\rangle_\gamma^{-1}$.

We now study $T^{-1}\#\tilde{A}\#T$. Noting that $\partial_x^\alpha\partial_\xi^\beta a \in S(\sigma^{1/2}\langle\xi\rangle_\gamma^{-|\beta|}, g)$ and $\partial_x^\alpha\partial_\xi^\beta b \in S(\sigma\langle\xi\rangle_\gamma^{-|\beta|}, g)$ for $|\alpha + \beta| = 1$ and $\partial_\xi^\beta[\xi] \in S(1, g)$, $|\beta| = 1$ we have

$$T^{-1}\#\tilde{A} = T^{-1}\tilde{A} + R, \quad R = \begin{bmatrix} S(1) & S(M^{-2}) & S(M^{-6}) \\ S(M^{-2}) & S(1) & S(M^{-8}) \\ S(M^{-8}) & S(M^{-2}) & S(M^{-6}) \end{bmatrix}.$$

Therefore $T^{-1}\#\tilde{A}\#T = (T^{-1}\tilde{A})\#T + R_1$ with

$$R_1 = R\#T = \begin{bmatrix} S(M^{-4}) & S(M^{-2}) & S(1) \\ S(M^{-2}) & S(1) & S(M^{-2}) \\ S(M^{-4}) & S(M^{-2}) & S(M^{-8}) \end{bmatrix}.$$

Note that

$$T^{-1}\tilde{A} = \begin{bmatrix} C(\sigma^{1/2}) & 1 + C(\sigma) & C(\sigma^{5/2}) \\ -1 + C(\sigma) & C(\sigma^{1/2}) & C(\sigma^3) \\ C(\sigma^{5/2}) & C(\sigma) & C(\sigma^{3/2}) \end{bmatrix} [\xi]$$

and hence

$$\langle\xi\rangle_\gamma^{|\beta|}\partial_x^\alpha\partial_\xi^\beta(T^{-1}\tilde{A}) = \begin{bmatrix} S(1) & S(1) & S(M^{-8}) \\ S(1) & S(1) & S(M^{-10}) \\ S(M^{-8}) & S(M^{-2}) & S(M^{-4}) \end{bmatrix}, \quad |\alpha + \beta| = 1.$$

Then thanks to (5.11) one sees that

$$(T^{-1}\tilde{A})\#T = T^{-1}\tilde{A}T + R_2, \quad R_2 = \begin{bmatrix} S(1) & S(M^{-2}) & S(M^{-2}) \\ S(1) & S(M^{-2}) & S(M^{-2}) \\ S(M^{-2}) & S(M^{-4}) & S(M^{-6}) \end{bmatrix}.$$

Thus we obtain $T^{-1}\#\tilde{A}\#T = T^{-1}\tilde{A}T + R_1 + R_2$ where

$$R_1 + R_2 = \begin{bmatrix} S(1) & S(M^{-2}) & S(M^{-2}) \\ S(1) & S(M^{-2}) & S(M^{-2}) \\ S(M^{-2}) & S(M^{-4}) & S(M^{-6}) \end{bmatrix}.$$

Recall B given by (9.1). Since $d_M \in S(M, g)$ one sees by Lemma 5.6 that

$$T^{-1}\#B = \begin{bmatrix} S(\sigma) & S(M\sigma) & S(\sigma) \\ S(\sigma^{3/2}) & S(M\sigma^{3/2}) & S(\sigma^{3/2}) \\ b_1 + S(\sigma) & b_2 + d_M + S(\sigma^{1/2}) & b_3 + S(\sigma) \end{bmatrix}$$

because $\sigma \le CM^{-4}$. Thus we conclude that $T^{-1}\#B\#T$ is written as

$$\begin{bmatrix} S(\sigma) & S(M\sigma) & S(\sigma) \\ S(\sigma^{3/2}) & S(M\sigma^{3/2}) & S(\sigma^{3/2}) \\ b_3 + S(M\sigma^{1/2}) & -b_2 - d_M + S(\sigma^{1/2}) & b_1 + S(\sigma) \end{bmatrix}. \tag{9.4}$$

Noting that $b_3(t, x, \xi) - \bar{b}_3 \in S(M^{-2}, g)$ we can summarize what we have proved in

Proposition 9.1. *One can write* $L \cdot \mathrm{op}(T) = \mathrm{op}(T) \cdot \tilde{L}$ *where*

$$\tilde{L} = D_t - \mathrm{op}(\mathcal{A} + \mathcal{B}), \quad \mathcal{A} = (T^{-1}AT)[\xi], \quad \mathcal{B} = \mathcal{B}_1 - T^{-1}D_tT$$

$$\mathcal{B}_1 = T^{-1}\#B\#T = \begin{bmatrix} S(1) & S(1) & S(1) \\ S(1) & S(1) & S(1) \\ \bar{b}_3 + S(M^{-1}) & -2M\bar{e} + S(M^{-1}) & S(1) \end{bmatrix}.$$

10 Weighted energy estimates

10.1 Energy form. Let $w = t\phi(t, x, \xi)$ and consider the energy with scalar weight $\mathrm{op}(w^{-n})$:

$$\mathcal{E}(V) = e^{-\theta t}(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V),$$

where $\theta > 0$ is a large positive parameter and n is fixed such that

$$n > (4\sqrt{2})\,|3\,\bar{b}_3 + i\bar{e}|/\bar{e} + C^* + 2 + 8(1 + 3\sqrt{2}) \tag{10.1}$$

where C^* is given by (3.14). It is clear from (4.17) that (10.1) follows from

$$n > 12\sqrt{2}\,\frac{|P_{\rm sub}(0,0,0,\bar{\xi})|}{\bar{e}} + \bar{C}^*, \quad \bar{C}^* = C^* + 10 + 32\sqrt{2}. \tag{10.2}$$

Note that $\partial_t\phi = \omega^{-1}\phi$ and hence

$$\partial_t w^{-n} = -n\Big(\frac{1}{t}+\frac{1}{\omega}\Big)w^{-n} = -n\kappa\, w^{-n}.$$

Recall that V satisfies

$$\partial_t V = {\rm op}(i\mathcal{A} + i\mathcal{B})V + F, \quad \mathcal{B} = \mathcal{B}_1 - T^{-1}D_tT. \tag{10.3}$$

Noting that Λ is real and diagonal hence ${\rm op}(\Lambda)^* = {\rm op}(\Lambda)$ one has

$$\begin{aligned}\frac{d}{dt}\mathcal{E} = &-\theta e^{-\theta t}({\rm op}(\Lambda){\rm op}(w^{-n})V, {\rm op}(w^{-n})V)\\ &-2n{\rm Re}\, e^{-\theta t}({\rm op}(\Lambda){\rm op}(\kappa w^{-n})V, {\rm op}(w^{-n})V)\\ &+e^{-\theta t}({\rm op}(\partial_t\Lambda){\rm op}(w^{-n})V, {\rm op}(w^{-n})V)\\ &+2{\rm Re}\, e^{-\theta t}({\rm op}(\Lambda){\rm op}(w^{-n})({\rm op}(i\mathcal{A}+i\mathcal{B})V+F), {\rm op}(w^{-n})V).\end{aligned} \tag{10.4}$$

Consider ${\rm op}(\phi^{-n}){\rm op}(\Lambda){\rm op}(\kappa\phi^{-n}) = {\rm op}(\phi^{-n}\#\Lambda\#(\kappa\phi^{-n}))$. Since κ and ϕ^{-n} are admissible weights for g one has $\kappa\#\phi^{-n} = \kappa\phi^{-n} - r$ with $r \in S(M^{-1}\kappa\phi^{-n}, g)$. Let $\tilde{r} = r\#\phi^n\#(1+k) \in S(M^{-1}\kappa, g)$ such that $r = \tilde{r}\#\phi^{-n}$ and hence $\kappa\phi^{-n} = (\kappa+\tilde{r})\#\phi^{-n}$. Thus we have

$$\begin{aligned}{\rm Re}\,({\rm op}(\Lambda){\rm op}(\kappa w^{-n})V, {\rm op}(w^{-n})V) \geq {\rm Re}\,&({\rm op}(\Lambda\#\kappa){\rm op}(w^{-n})V, {\rm op}(w^{-n})V)\\ &-|({\rm op}(\Lambda\#\tilde{r}){\rm op}(w^{-n})V, {\rm op}(w^{-n})V)|.\end{aligned}$$

Since $\lambda_j\#\tilde{r} \in S(M^{-1}\kappa\lambda_j, g)$ thanks to Lemma 8.4 the second term on the right-hand side is bounded by $CM^{-1}\|{\rm op}(\kappa^{1/2}\Lambda^{1/2}){\rm op}(w^{-n})V\|$. Applying Proposition 8.1, and denoting $W_j = {\rm op}(w^{-n})V_j$, one can conclude that

$$\begin{aligned}{\rm Re}\,({\rm op}(\Lambda){\rm op}(\kappa w^{-n})V, {\rm op}(w^{-n})V) &\geq (1-CM^{-1})\|{\rm op}(\kappa^{1/2}\Lambda^{1/2})W\|^2,\\ {\rm Re}\,({\rm op}(\Lambda){\rm op}(w^{-n})V, {\rm op}(w^{-n})V) &\geq c(\|{\rm op}(\Lambda^{1/2})W\|^2 + \|{\rm op}(\mathcal{D})W\|^2),\end{aligned}$$

for $M \geq M_0$.

Definition 10.1. To simplify notation we denote

$$\begin{aligned}\mathcal{E}_1(V) &= \|{\rm op}(\kappa^{1/2}\Lambda^{1/2}){\rm op}(w^{-n})V\|^2 = t^{-2n}\|{\rm op}(\kappa^{1/2}\Lambda^{1/2}){\rm op}(\phi^{-n})V\|^2,\\ \mathcal{E}_2(V) &= \|{\rm op}(\Lambda^{1/2}){\rm op}(w^{-n})V\|^2 = t^{-2n}\|{\rm op}(\Lambda^{1/2}){\rm op}(\phi^{-n})V\|^2.\end{aligned}$$

Now we summarize:

Lemma 10.1. *One can find $C>0$, $c>0$ and M_0 such that*

$$n\mathrm{Re}\,(\mathrm{op}(\Lambda)\mathrm{op}(\kappa w^{-n})V, \mathrm{op}(w^{-n})V)+\theta\mathrm{Re}\,(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V)$$
$$\geq n(1-CM^{-1})\mathcal{E}_1(V)+c\,\theta\mathcal{E}_2(V), \quad M\geq M_0.$$

10.2 The term $(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(\mathcal{B})V, \mathrm{op}(w^{-n})V)$**.** Recall that $\lambda_i\in S(\lambda_i, g)$ and $\lambda_1\leq C\sigma\lambda_2\leq C\sigma^2\lambda_3$ with some $C>0$. We show

Lemma 10.2. *Let $W=\mathrm{op}(\phi^{-n})V$. Then we have*

$$|(\mathrm{op}(\lambda_i)\mathrm{op}(b)W_j, W_i)|\leq CM^{-2}\mathcal{E}_1(V)+CM^2\mathcal{E}_2(V), \qquad b\in S(\sigma^{-1/2}, g),\ j\geq i,$$
$$|(\mathrm{op}(\lambda_3)\mathrm{op}(b)W_2, W_3)|\leq CM^{-2}\mathcal{E}_1(V)+CM^{2+2l}\mathcal{E}_2(V), \qquad b\in S(M^l, g),$$
$$|(\mathrm{op}(\lambda_3)\mathrm{op}(b)W_1, W_3)|\leq (\sqrt{3}\,\bar{\varepsilon}\|\mathrm{op}(b)\|+CM^{-1/2})\mathcal{E}_1(V), \quad b\in S(1, g),$$
$$|(\mathrm{op}(\lambda_2)\mathrm{op}(b)W_1, W_2)|\leq (\bar{\varepsilon}\|\mathrm{op}(\lambda_2^{1/2}b)\|+CM^{-1/2})\mathcal{E}_1(V), \quad b\in S(\sigma^{-1/2}, g).$$

Proof. Let $b\in S(\sigma^{-1/2}, g)$. Noting that $\lambda_i^{1/2}\lambda_j^{-1/2}\in S(\sigma^{(j-i)/2}, g)$ one can write

$$r=(1+k)\#(\kappa^{-1/2}\lambda_i^{-1/2})\#(\lambda_i\#b)\#\lambda_j^{-1/2}\#(1+\tilde{k})\in S(\sigma^{(j-i)/2}, g), \quad j\geq i$$

for $\sigma\kappa\geq 1$, such that $(\kappa^{1/2}\lambda_i^{1/2})\#r\#\lambda_j^{1/2}=\lambda_i\#b$. Then we have

$$|(\mathrm{op}(\lambda_i)\mathrm{op}(b)W_j, W_i)|\leq M^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2+CM^2\|\mathrm{op}(\Lambda^{1/2})W\|^2$$

for $j\geq i$. Let $b\in S(M^l, g)$ and denote

$$r=(1+k)\#(\kappa^{-1/2}\lambda_2^{-1/2})\#(\lambda_3\#b)\#\lambda_3^{-1/2}\#(1+\tilde{k})$$

such that $(\kappa^{1/2}\lambda_2^{1/2})\#r\#\lambda_3^{1/2}=\lambda_3\#b$. Since $r\in S(\kappa^{-1/2}\lambda_3^{1/2}\lambda_2^{-1/2}, g)\subset S(1, g)$ in view of Lemma 7.11 then $|(\mathrm{op}(\lambda_3)\mathrm{op}(b)W_2, W_3)|$ is bounded by

$$CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2+CM^{2+2l}\|\mathrm{op}(\Lambda^{1/2})W\|^2.$$

We check $(\mathrm{op}(\lambda_3)\mathrm{op}(b)W_1, W_3)$ for $b\in S(1, g)$. Noting that $\kappa^{-1}\lambda_1^{-1/2}\in S(1, g)$, by Lemma 7.11, write

$$r=(1+k)\#(\kappa^{-1/2}\lambda_1^{-1/2})\#(\lambda_3\#b)\#(\kappa^{-1/2}\lambda_3^{-1/2})\#(1+\tilde{k})\in S(1, g)$$

such that $(\kappa^{1/2}\lambda_1^{1/2})\#r\#(\kappa^{1/2}\lambda_3^{1/2})=\lambda_3\#b$. Since k, $\tilde{k}\in S(M^{-1}, g)$ it is easy to see that $r=(\lambda_3^{1/2}\lambda_1^{-1/2}\kappa^{-1})\#b+\tilde{r}$ with $\tilde{r}\in S(M^{-1/2}, g)$. By Proposition 5.1 and Lemma 7.11 one sees that $|\lambda_3^{1/2}\lambda_1^{-1/2}\kappa^{-1}|\leq\sqrt{3}\,\bar{\varepsilon}+CM^{-4}$, hence

$$|(\mathrm{op}(\lambda_3)\mathrm{op}(b)W_1, W_3)|=|(\mathrm{op}(r)\mathrm{op}(\kappa^{1/2}\lambda_1^{1/2})W_1, \mathrm{op}(\kappa^{1/2}\lambda_3^{1/2})W_3)|$$
$$\leq(\sqrt{3}\,\bar{\varepsilon}\|\mathrm{op}(b)\|+CM^{-1/2})\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2.$$

Next consider $(\mathrm{op}(\lambda_2)\mathrm{op}(b)W_1, W_2)$ for $b \in S(\sigma^{-1/2}, g) = S(\lambda_2^{-1/2}, g)$. Denote

$$r = (1 + k)\#(\kappa^{-1/2}\lambda_2^{-1/2})\#(\lambda_2\#b)\#(\lambda_1^{-1/2}\kappa^{-1/2})\#(1 + \tilde{k}) \in S(1, g)$$

such that $(\kappa^{1/2}\lambda_2^{1/2})\#r\#(\lambda_1^{1/2}\kappa^{1/2}) = \lambda_2\#b$. Write $r = (\kappa^{-1}\lambda_1^{-1/2})\#(\lambda_2^{1/2}b) + \tilde{r}$ with $\tilde{r} \in S(M^{-1}, g)$. Thus repeating the same arguments as above one concludes the last assertion. □

In particular, this lemma implies

Corollary 10.1. *Let $B = (b_{ij}) \in S(1, g)$. Then with $W = \mathrm{op}(w^{-n})V$*

$$|(\mathrm{op}(\Lambda)\mathrm{op}(B)W, W)| \le (\sqrt{3}\,\bar{\varepsilon}\|\mathrm{op}(b_{31})\| + CM^{-1/2})\mathcal{E}_1(V) + C\mathcal{E}_2(V).$$

From Proposition 9.1 there results $\phi^{-n}\#\mathcal{B}_1 - \mathcal{B}_1\#\phi^{-n} \in S(M^{-1}\phi^{-n}, g)$, so one concludes by Corollary 10.1 that

$$|(\mathrm{op}(\Lambda)[\mathrm{op}(w^{-n}), \mathrm{op}(\mathcal{B}_1)]V, W)| \le CM^{-1}\mathcal{E}_1(V) + C\mathcal{E}_2(V) \tag{10.5}$$

where $W = \mathrm{op}(w^{-n})V$ again. Write $T^{-1}\partial_t T = (\tilde{t}_{ij})$ and recall (5.12) and note that $\tilde{t}_{12} = -\tilde{t}_{21} \in \mathcal{C}(\sigma^{-1/2})$ and $\tilde{t}_{31} \in S(1, g)$. Then thanks to Lemma 6.5 one sees that $\lambda_j\#(\phi^{-n}\#\tilde{t}_{j1} - \tilde{t}_{j1}\#\phi^{-n})\#\phi^n$ is in

$$S(\sigma^{1-j/2}\omega^{-1}\rho^{1/2}\langle\xi\rangle_\gamma^{-1}, g) \subset S(M^{-1}\sqrt{\kappa\lambda_1}\sqrt{\kappa\lambda_j}, g), \quad j = 2, 3,$$

because $C\lambda_1 \ge M\rho\langle\xi\rangle_\gamma^{-1}$, $C\lambda_2 \ge M\langle\xi\rangle_\gamma^{-1}$ and $\omega^{-1} \le \kappa$. Therefore repeating similar arguments one concludes that

$$|(\mathrm{op}(\Lambda)[\mathrm{op}(w^{-n}), \mathrm{op}(T^{-1}\partial_t T)]V, W)| \le CM^{-1}\mathcal{E}_1(V). \tag{10.6}$$

Recalling $\mathcal{B} = \mathcal{B}_1 - T^{-1}D_t T$ it follows from (10.5) and (10.6) that

$$|(\mathrm{op}(\Lambda)[\mathrm{op}(w^{-n}), \mathrm{op}(\mathcal{B})]V, W)| \le CM^{-1}\mathcal{E}_1(V) + C\mathcal{E}_2(V). \tag{10.7}$$

With $\mathcal{B} = (q_{ij})$ we see that $q_{ij} \in S(\sigma^{-1/2}, g)$ for $j \ge i$ and

$$q_{21} = i\partial_t(3b/2a_M) + S(1), \quad q_{31} = \bar{b}_3 + i\bar{e}/3 + S(M^{-1}), \quad q_{32} = -2M\bar{e} + S(M^{-1})$$

by Proposition 9.1. Applying Lemma 10.2, we have from (5.13), recalling Proposition 5.1 and $\bar{\varepsilon} = \sqrt{3}/\bar{e}\bar{c}^{1/2}$, that

$$\begin{aligned}&|(\mathrm{op}(\Lambda)\mathrm{op}(\mathcal{B})\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V)| \\ &\quad \le (|3\bar{b}_3 + i\bar{e}|/\bar{e}\bar{c}^{1/2} + (6 + \sqrt{2})/\bar{c}^{1/2} + CM^{-1/2})\mathcal{E}_1(V) + CM^4\mathcal{E}_2(V).\end{aligned} \tag{10.8}$$

Combining the estimates (10.8) and (10.7) we obtain

Lemma 10.3. *The term $|(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(\mathcal{B})V, \mathrm{op}(w^{-n})V)|$ is bounded by the right-hand side of* (10.8).

10.3 The term $(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(i\mathcal{A})V, \mathrm{op}(w^{-n})V)$**.** Note that

$$\phi^{-n}\#([\xi]r) - ([\xi]r)\#\phi^{-n} \in S(\phi^{-n}\sigma^{s-1/2}\omega^{-1}\rho^{1/2}, g) \quad \text{for } r \in \mathcal{C}(\sigma^s)$$

by Lemma 6.5. Recalling Corollary 5.3 and $\mathcal{A} = A^T[\xi]$, then denoting

$$\phi^{-n}\#\mathcal{A} - \mathcal{A}\#\phi^{-n} = (r_{ij}),$$

we see that $r_{ij} \in S(\phi^{-n}\omega^{-1}\rho^{1/2}, g) \subset S(M^{-2}\kappa\phi^{-n}, g)$ for $j \geq i$ because $\omega^{-1} \leq \kappa$. Writing $\lambda_i\#r_{ij} = \lambda_i\#\tilde{r}_{ij}\#\phi^{-n}$ with $\tilde{r}_{ij} \in S(M^{-2}\kappa, g)$ one obtains

$$|(\mathrm{op}(\lambda_i)\mathrm{op}(r_{ij})V_j, W_i)| \leq CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2 \quad \text{for } j \geq i$$

since $\lambda_i\#\tilde{r}_{ij} \in S(M^{-2}\kappa\lambda_i, g)$. From Lemma 5.8 one has $\tilde{a}_{21} = \lambda_1\,\mathcal{C}(\sigma^{-1})$, hence thanks to Lemmas 6.5 and 7.7

$$\phi^{-n}\#(\tilde{a}_{21}[\xi]) - \phi^{-n}\tilde{a}_{21}[\xi] \in S(\sigma^{-1/2}\lambda_1^{1/2}\omega^{-1}\rho^{1/2}\phi^{-n}, g) \subset S(\lambda_1^{1/2}\kappa\phi^{-n}, g)$$

for $\omega^{-1} \leq \kappa$ again. Thus we have

$$|(\mathrm{op}(\lambda_2)\mathrm{op}(r_{21})V_1, W_2)| \leq CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2$$

since $\lambda_2^{1/2} \leq CM^{-2}$. Similarly from $\tilde{a}_{31} = \lambda_1\,\mathcal{C}(\sigma^{1/2})$, $\tilde{a}_{32} = \lambda_2\,\mathcal{C}(1)$ and Lemma 7.7 it follows that $r_{3j} \in S(\sigma^{2-j}\lambda_j^{1/2}\omega^{-1}\rho^{1/2}\phi^{-n}, g) \subset S(M^{-2}\lambda_j^{1/2}\kappa\phi^{-n}, g)$ for $j = 1, 2$. Here we have used $\partial_x^\alpha\partial_\xi^\beta\lambda_2 \in S(\lambda_2^{1/2}\langle\xi\rangle_\gamma^{-|\beta|}, g)$ for $|\alpha + \beta| = 1$ which follows from $\lambda_2 \in \mathcal{C}(\sigma)$ easily. Then one obtains

$$|(\mathrm{op}(\lambda_3)\mathrm{op}(r_{3j})V_j, W_3)| \leq CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2, \quad j = 1, 2.$$

Therefore $(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(\mathcal{A})V, \mathrm{op}(w^{-n})V) - (\mathrm{op}(\Lambda)\mathrm{op}(\mathcal{A})W, W)$ is bounded by a constant times $M^{-2}\mathcal{E}_1(V)$.

Next we study $\Lambda\#\mathcal{A} - \Lambda\mathcal{A} = (q_{ij})$. From Corollary 5.3 and Lemma 7.7 we have

$$\lambda_1\#(\tilde{a}_{1j}[\xi]) - \lambda_1\tilde{a}_{1j}[\xi] \in S(\sigma^{1/2}\lambda_1^{1/2}, g), \ \ \lambda_2\#(\tilde{a}_{2j}[\xi]) - \lambda_2\tilde{a}_{2j}[\xi] \in S(\lambda_2^{1/2}, g)$$

for $j \geq 1$ and $j \geq 2$ respectively. Then noting that $C\lambda_1^{1/2}\kappa \geq 1$ and $C\lambda_2\kappa \geq 1$

$$|(\mathrm{op}(q_{ij})W_j, W_i)| \leq CM^{-2}\|\mathrm{op}(\kappa^{1/2}\Lambda^{1/2})W\|^2 + CM^2\|\mathrm{op}(\Lambda^{1/2})W\|^2, \ \ j \geq i.$$

Repeating similar arguments one has $\lambda_i\#(\tilde{a}_{ij}[\xi]) - \lambda_i\tilde{a}_{ij}[\xi] \in S(M^{-2}\kappa\lambda_i^{1/2}\lambda_j^{1/2}, g)$ and hence

$$|(\mathrm{op}(q_{ij})W_i, W_j)| \leq CM^{-2}\|\mathrm{op}(\Lambda^{1/2}\kappa^{1/2})W\|^2 \quad \text{for } i > j.$$

Thus we conclude that

$$\begin{aligned} &|(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(\mathcal{A})V, \mathrm{op}(w^{-n})V) - (\mathrm{op}(\Lambda\mathcal{A})W, W)| \\ &\qquad\qquad \leq CM^{-2}\mathcal{E}_1(V) + CM^2\mathcal{E}_2(V). \end{aligned} \tag{10.9}$$

Since $\Lambda\mathcal{A} = \mathcal{A}^*\Lambda$ we have

Lemma 10.4. *One can find $C > 0$ such that*

$$|\mathsf{Re}\,(\mathrm{op}(\varLambda)\mathrm{op}(w^{-n})\mathrm{op}(i\mathcal{A})V, \mathrm{op}(w^{-n})V)| \le CM^{-2}\mathcal{E}_1(V) + CM^2\mathcal{E}_2(V).$$

10.4 The term $(\mathrm{op}(\partial_t \varLambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V)$. We start with

Lemma 10.5. *We have $\partial_t\lambda_j \in S(\kappa\lambda_j, g)$, $j = 1, 2$.*

Proof. Note that Lemma 3.6 with $\epsilon = \sqrt{2}M\langle\xi\rangle_\gamma^{-1}$ implies

$$|\partial_t \Delta_M| \le C^*(1/t + 1/\omega)\Delta_M = C^*\kappa\Delta_M.$$

Recalling $\partial_t\lambda_1 = -\partial_t q(\lambda_1)/\partial_\lambda q(\lambda_1)$ it follows from (5.6) and (5.9) that

$$|\partial_t\lambda_1| \le (1 + CM^{-2})(|\partial_t a_M/a_M|\lambda_1 + |\partial_t\Delta_M|/6a_M).$$

Since $(1+CM^{-2})\lambda_1 \ge \Delta_M/6a_M$ by Proposition 5.1 and $1/a_M \le \kappa/e$ by Lemma 7.11 one concludes that

$$|\partial_t\lambda_1| \le (1 + CM^{-2})(C^* + 1)\kappa\lambda_1. \tag{10.10}$$

Note that

$$|\partial_x^\alpha\partial_\xi^\beta\partial_t\lambda_1| \le C\sigma^{1-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{-|\beta|} \le C\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \quad \text{for } \partial_t\lambda_1 \in \mathcal{C}(\sigma).$$

From Lemma 7.11 and $C\lambda_1 \ge M\sigma\langle\xi\rangle_\gamma^{-1}$ it follows that

$$\kappa\lambda_1 \ge \kappa\sqrt{\lambda_1}M^{1/2}\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2}/C \ge M^{1/2}\sigma^{1/2}\langle\xi\rangle_\gamma^{-1/2}/C',$$

which proves that $|\partial_x^\alpha\partial_\xi^\beta\partial_t\lambda_1| \le CM^{-1/2}\kappa\lambda_1\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$ for $|\alpha+\beta| = 1$. For $|\alpha+\beta| \ge 2$ it follows that

$$\begin{aligned}|\partial_x^\alpha\partial_\xi^\beta\partial_t\lambda_1| &\precsim \sigma^{-(|\alpha+\beta|-2)/2}\langle\xi\rangle_\gamma^{-|\beta|} \precsim M\langle\xi\rangle_\gamma^{-1}M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\\ &\le \sigma^{-1}M\sigma\langle\xi\rangle_\gamma^{-1}M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \le C\kappa\lambda_1M^{-|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\end{aligned}$$

because $\kappa\sigma \ge 1$. Thus $\partial_t\lambda_1 \in S(\kappa\lambda_1, g)$. On the other hand, $\partial_t\lambda_j \in S(\kappa\lambda_j, g)$, $j = 2, 3$ is clear since $\partial_t\lambda_j \in \mathcal{C}(1) \subset S(1, g) \subset S(\kappa\lambda_2, g)$ for $C\lambda_2\kappa \ge 1$. This completes the proof. □

Note that from (5.6), (5.9) and $|\partial_t\Delta_M| \precsim a_M^2$ we see that

$$|\partial_t\lambda_2| \le (1 + CM^{-2})|\partial_t a_M|\lambda_2/a_M + Ca_M \le (1 + CM^{-2})\kappa\lambda_2 \tag{10.11}$$

for $C\kappa\lambda_2 \ge 1$. Since $|(\mathrm{op}(\partial_t\lambda_3)W_3, W_3)| \le C\|\mathrm{op}(\lambda_3)W_3\|^2$ is evident, by applying Lemma 8.4 one obtains from (10.10) and (10.11)

Lemma 10.6. *We have*

$$|(\mathrm{op}(\partial_t \varLambda)\mathrm{op}(w^{-n})V, \mathrm{op}(w^{-n})V)| \le (C^* + 2 + CM^{-1/2})\mathcal{E}_1(V) + C\mathcal{E}_2(V).$$

10.5 Conclusion. In what follows we denote $\|u\|_s = \|\langle D\rangle_\gamma^s u\|$, and by $H^s = H^s(\mathbb{R}^d)$ the set of tempered distributions u on $\mathbb{R}^d$ such that $\|u\|_s < +\infty$.

Definition 10.2. Denote by $\mathcal{H}_{-n,s}(0,\delta)$ the set of f such that

$$t^{-n}\langle D\rangle_\gamma^s f(t,\cdot) \in L^2((0,\delta)\times\mathbb{R}^d).$$

Consider the term $\mathsf{Re}\,(\mathrm{op}(\varLambda)\mathrm{op}(\omega^{-n})F, \mathrm{op}(\omega^{-n})V)$ where $F = {}^t(F_1, F_2, F_3)$. Write $\varLambda = (\kappa^{1/2}\varLambda^{1/2})\#R\#(\kappa^{-1/2}\varLambda^{1/2})$ with $R \in S(1,g)$. Because of the choice of n it follows from (10.4) and Lemmas 10.1, 10.3, 10.4 and 10.6 that one can find $c_i > 0$ and M_0, γ_0, θ_0 such that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(V) \leq &-c_1 e^{-\theta t}\mathcal{E}_1(V) - c_2\theta e^{-\theta t}\mathcal{E}_2(V)\\ &+|\mathsf{Re}\,(\mathrm{op}(\varLambda)\mathrm{op}(\omega^{-n})F, \mathrm{op}(\omega^{-n})V)|\\ &\qquad\text{for } M \geq M_0,\ \gamma \geq \gamma_0 \text{ and } \theta \geq \theta_0. \end{aligned}\tag{10.12}$$

Thanks to Lemma 6.3 one has $\kappa^{-1/2}\phi^{-n}\lambda_j^{1/2} \in S(M^{-n}\sqrt{t}\,\langle\xi\rangle_\gamma^n, g)$, so we see easily that

$$|\mathsf{Re}\,(\mathrm{op}(\varLambda)\mathrm{op}(\omega^{-n})F, \mathrm{op}(\omega^{-n})V)| \leq CM^{-1}\mathcal{E}_1(V) + CM^{-2n+1}t^{2n+1}\|F\|_n^2.$$

Since $M\langle\xi\rangle_\gamma^{-1}/C \leq \phi \leq CM^{-4}$ and $t^{-1/2} \leq \kappa^{1/2} \leq t^{-1/2} + \omega^{-1/2} \leq t^{-1/2} + M^{-1}\langle\xi\rangle_\gamma$ and $\langle\xi\rangle_\gamma^{-3/2+j/2} \leq C\lambda_j^{1/2}$ for $1 \leq j \leq 3$, then

$$\begin{aligned} M^{8n}t^{-1}\|V\|_{-1}^2/C \leq t^{2n}\mathcal{E}_1(V) \leq CM^{-2n}(t^{-1}\|V\|_n^2 + \|V\|_{n+1}^2),\\ M^{8n}\|V\|_{-1}^2/C \leq t^{2n}\mathcal{E}_2(V) \leq CM^{-2n}\|V\|_n^2. \end{aligned}\tag{10.13}$$

Assume $D_t^j V \in \mathcal{H}_{-n-1/2+j/2, n+1-j}(0, M^{-4})$, $j = 0, 1$. From this one sees that $\lim_{t\to+0}\|V(t)\|_n$ exists and is 0. Using this, we see that $\lim_{t\to+0} t^{-n}\|V(t)\|_n = 0$. Noting that $\mathcal{E}(V) \leq CM^{-n}t^{-2n}\|V(t)\|_n^2$ and integrating (10.12) over t we obtain

Proposition 10.1. *There exist $c_i > 0$, $C > 0$ and M_0, γ_0, θ_0 such that for any V with $D_t^j V \in \mathcal{H}_{-n-1/2+j/2, n+1-j}(0, M^{-4})$, $j = 0, 1$, one has*

$$\begin{aligned} c_1 t^{-2n}e^{-\theta t}\|V(t)\|_{-1}^2 + c_2\int_0^t e^{-\theta s}s^{-2n-1}\|V(s)\|_{-1}^2 ds + c_3\theta\int_0^t e^{-\theta s}s^{-2n}\|V(s)\|_{-1}^2 ds\\ \leq CM^{1-10n}\int_0^t e^{-\theta s}s^{-2n+1}\|\tilde{L}V(s)\|_n^2 ds \end{aligned}$$

for $0 \leq t \leq M^{-4}$, $M \geq M_0$, $\gamma \geq \gamma_0$, $\theta \geq \theta_0$.

Corollary 10.2. *For any V with $D_t^j V \in \mathcal{H}_{-n-1/2+j/2, n+1-j}(0, M^{-4})$, $j = 0, 1$*

$$\int_0^t s^{-2n-1}\|V(s)\|_{-1}^2 ds \leq C\int_0^t s^{-2n+1}\|\tilde{L}V(s)\|_n^2 ds, \quad 0 \leq t \leq M^{-4}.$$

Consider the adjoint operator $\hat{P}^*$ of $\hat{P}$. Noting that $a_M \in \mathcal{C}(\sigma)$, $b \in \mathcal{C}(\sigma^{3/2})$ and (4.5), (5.1), we see that

$$\hat{P}^* = D_t^3 - a_M(t,x,D)[D]^2 D_t - b(t,x,D)\,[D]^3 \\ + b_1 D_t^2 + (\tilde{b}_2 + d_M)[D]D_t + \tilde{b}_3[D]^2 + \tilde{c}_1 D_t + \tilde{c}_2[D]$$

with $\tilde{b}_j \in S(1,g)$ and $\tilde{c}_j \in S(M^2,g)$, hence $\tilde{c}_j[D]^{-1} \in S(M^{-3},g)$ where it is not difficult to check that $\tilde{b}_3 - (b_3 + ie) \in S(M^{-3},g)$. Denote by $\tilde{L}^*$ the corresponding first order system (which is not the adjoint of $\tilde{L}$). Since the power n of the weight ϕ^{-n} depends only on a, b and b_3 (see (10.1)), then we can choose the same n for $\hat{P}^*$ as for $\hat{P}$. Now employing the weighted energy

$$\mathcal{E}^*(V) = e^{\theta t}(\mathrm{op}(\Lambda)\mathrm{op}(t^n\phi^n)V, \mathrm{op}(t^n\phi^n)V)$$

and repeating the same arguments for $\mathcal{E}(V)$ and carrying out the integration

$$-\int_t^\delta \frac{d}{dt}\mathcal{E}^*\,dt, \quad 0 < t < \delta = M^{-4},$$

we have

Proposition 10.2. *There exist $c_i > 0$, $C > 0$ and M_0, γ_0, θ_0 such that for any V with $D_t^j V \in \mathcal{H}_{n-1/2+j/2,1-j}(0, M^{-4})$, $j = 0, 1$, one has*

$$c_1 t^{2n} e^{\theta t}\|V(t)\|^2_{-n-1} + c_2\int_t^\delta e^{\theta\tau}\tau^{2n-1}\|V(\tau)\|^2_{-n-1}d\tau \\ + c_3\theta\int_t^\delta e^{-\theta\tau}\tau^{2n}\|V(\tau)\|^2_{-n-1}d\tau \\ \le CM^{-10n}\delta^{2n}e^{\theta\delta}\|V(\delta)\|^2 + CM^{1-10n}\int_t^\delta e^{\theta\tau}\tau^{2n+1}\|\tilde{L}^*V(\tau)\|^2 d\tau, \quad 0 \le t \le \delta,$$

for $M \ge M_0$, $\gamma \ge \gamma_0$, $\theta \ge \theta_0$ where

$$\tilde{L}^*V = \mathrm{op}(T)^t(\hat{P}^*u, 0, 0) \quad \textit{and} \quad \mathrm{op}(T)V = {}^t(D_t^2 u, [D]D_t u, [D]^2 u).$$

Remark 10.1. It is clear from the proof that for any $n' \ge n$, Propositions 10.1 and 10.2 hold.

11 Preliminary existence result

Let $s \in \mathbb{R}$ and we estimate $\langle D\rangle^s V$. In what follows we fix M and γ (actually it is enough to choose $\gamma = M^5$, see (4.1)) such that Propositions 10.1 and 10.2 hold, therefore from now on we can assume $\langle D\rangle_\gamma = \langle D\rangle = [D]$ without restrictions while θ remains free. From (10.3) one has

$$\partial_t(\langle D\rangle^s V) = (\mathrm{op}(i\mathcal{A} + i\mathcal{B}) + i[\langle D\rangle^s, \mathrm{op}(\mathcal{A}+\mathcal{B})]\langle D\rangle^{-s})\langle D\rangle^s V + \langle D\rangle^s F.$$

Lemma 11.1. *For any $s \in \mathbb{R}$ there is $C > 0$ such that*

$$|([\langle D\rangle^s, \mathrm{op}(\mathcal{A})]V, \mathrm{op}(\Lambda)\langle D\rangle^s V)| \le C\mathcal{E}_2(\langle D\rangle^s V).$$

Proof. Denoting $T^{-1}AT = (\tilde a_{ij})$ thanks to Lemmas 5.8 and 7.7 we see that

$$\begin{gathered}((\tilde a_{3j}[\xi])\#\langle\xi\rangle^s - \langle\xi\rangle^s\#(\tilde a_{3j}[\xi]))\#\langle\xi\rangle^{-s} \in S(\sigma^{2-j}\sqrt{\lambda_j}, g), \quad j = 1, 2,\\ ((\tilde a_{21}[\xi])\#\langle\xi\rangle^s - \langle\xi\rangle^s_\gamma\#(\tilde a_{21}[\xi]))\#\langle\xi\rangle^{-s} \in S(\sigma^{-1/2}\sqrt{\lambda_1}, g)\end{gathered} \tag{11.1}$$

where $S(\sigma^{-1/2}\sqrt{\lambda_1}, g) = S(\lambda_2^{-1/2}\sqrt{\lambda_1}, g)$. From Corollary 5.3 it is easy to see that $((\tilde a_{ij}[\xi])\#\langle\xi\rangle^s - \langle\xi\rangle^s\#(\tilde a_{ij}[\xi]))\#\langle\xi\rangle^{-s} \in S(1, g)$ for $j \ge i$. Then together with (11.1) the proof follows from a repetition of similar arguments. □

Lemma 11.2. *For any $s \in \mathbb{R}$ and any $\epsilon > 0$ there is $C > 0$ such that*

$$|([\langle D\rangle^s, \mathrm{op}(\mathcal{B})]V, \mathrm{op}(\Lambda)\langle D\rangle^s V)| \le \epsilon\,\mathcal{E}_1(\langle D\rangle^s V) + C\mathcal{E}_2(\langle D\rangle^s V).$$

Proof. Write $\mathcal{B}_1 = (\tilde b_{ij})$. Since $\tilde b_{ij} \in S(1, g)$ then $\lambda_i\#(\langle\xi\rangle^s\#\tilde b_{ij} - \tilde b_{ij}\#\langle\xi\rangle^s)\#\langle\xi\rangle^{-s}$ is in $S(\langle\xi\rangle^{-1/2}\lambda_i, g)$. Noting that $C\lambda_1 \ge \sigma\langle\xi\rangle^{-1}$ and $C\lambda_2 \ge \sigma \ge \langle\xi\rangle^{-1}$ it is easy to see that $S(\langle\xi\rangle^{-1/2}\lambda_i, g) \subset S(\lambda_i^{1/2}\lambda_j^{1/2}, g)$ except for $(i, j) = (3, 1)$. For $(i, j) = (3, 1)$ recalling $\tilde b_{31} = b_3 + S(\sigma^{1/2})$ by (9.4) one sees that

$$\lambda_3\#(\langle\xi\rangle^s\#\tilde b_{31} - \tilde b_{31}\#\langle\xi\rangle^s)\#\langle\xi\rangle^{-s} \in S(\sigma^{1/2}\langle\xi\rangle^{-1/2}, g) \subset S(\lambda_1^{1/2}\lambda_3^{1/2}, g)$$

(recall that M is fixed). Therefore one obtains

$$|([\langle D\rangle^s, \mathrm{op}(\mathcal{B}_1)]V, \mathrm{op}(\Lambda)\langle D\rangle^s V)| \le C\mathcal{E}_2(\langle D\rangle^s V). \tag{11.2}$$

Next consider $T^{-1}\partial_t T = (\tilde t_{ij})$. Recalling $\tilde t_{21} \in \mathcal{C}(\sigma^{-1/2})$, $\tilde t_{31} \in \mathcal{C}(1)$ and $\tilde t_{32} \in \mathcal{C}(\sigma^{1/2})$ and noting that $C\kappa\lambda_1 \ge \langle\xi\rangle^{-1}$ and $C\lambda_2 \ge \sigma \ge \langle\xi\rangle^{-1}$, one has

$$\lambda_i\#(\langle\xi\rangle^s\#\tilde t_{ij} - \tilde t_{ij}\#\langle\xi\rangle^s)\#\langle\xi\rangle^{-s} \in S(\sqrt{\kappa\lambda_j}\sqrt{\lambda_i}, g), \quad i > j.$$

Therefore we have

$$\begin{aligned}|([\langle D\rangle^s, \mathrm{op}(T^{-1}\partial_t T)]V, \mathrm{op}(\Lambda)\langle D\rangle^s V)| &\le C\sqrt{\mathcal{E}_1(\langle D\rangle^s V)}\sqrt{\mathcal{E}_2(\langle D\rangle^s V)}\\ &\le \epsilon\mathcal{E}_1(\langle D\rangle^s V) + C^2\epsilon^{-1}\mathcal{E}_2(\langle D\rangle^s V),\end{aligned}$$

which together with (11.2) proves the assertion. □

Choosing $\epsilon > 0$ smaller than c_2 in Proposition 10.1 and θ large, we conclude the following

Proposition 11.1. *For any $s \in \mathbb{R}$ there exists $C > 0$ such that for any V with $D_t^j V \in \mathcal{H}_{-n-1/2+j/2,n+s+1-j}(0,\delta)$, $j = 0, 1$, one has*

$$t^{-2n}\|V(t)\|_{s-1}^2 + \int_0^t \tau^{-2n-1}\|V(\tau)\|_{s-1}^2 d\tau \le C\int_0^t \tau^{-2n+1}\|\tilde{L}V(\tau)\|_{n+s}^2 d\tau$$

for $0 \le t \le \delta$.

Since $\tilde{L} = \mathrm{op}(I+K)\mathrm{op}(T^{-1}) \cdot L \cdot \mathrm{op}(T)$ with $T, T^{-1} \in S(1,g)$, then

$$\|\tilde{L}V\|_s \le C_s\|L\cdot\mathrm{op}(T)V\|_s \quad \text{and} \quad \|\mathrm{op}(T)V\|_s \le C_s\|V\|_s$$

with some $C_s > 0$. Thanks to Lemma 5.7 one has $\|\mathrm{op}(\partial_t T)V\|_s \le C_s t^{-1/2}\|V\|_s$, so replacing $\mathrm{op}(T)V$ by U one sees that for any U with $D_t^j U \in \mathcal{H}_{n-1/2+j/2,n+s+1-j}(0,\delta)$, $j = 0, 1$, we have

$$t^{-2n}\|U(t)\|_{s-1}^2 + \int_0^t \tau^{-2n-1}\|U(\tau)\|_{s-1}^2 d\tau \le C\int_0^t \tau^{-2n+1}\|LU(\tau)\|_{n+s}^2 d\tau.$$

Since $U = {}^t(D_t^2 u, [D]D_t u, [D]^2 u)$ and $LU = {}^t(\hat{P}u, 0, 0)$ we have

Proposition 11.2. *For any $s \in \mathbb{R}$ there is $C > 0$ such that for any u with $D_t^j u \in \mathcal{H}_{-n-1/2,n+s+3-j}(0,\delta)$, $0 \le j \le 3$, one has*

$$\begin{aligned}(11.3)\qquad t^{-2n}\sum_{j=0}^{2}\|D_t^j u(t)\|_{s+1-j}^2 &+ \sum_{j=0}^{2}\int_0^t \tau^{-2n-1}\|D_t^j u(\tau)\|_{s+1-j}^2 d\tau \\ &\le C\int_0^t \tau^{-2n+1}\|\hat{P}u(\tau)\|_{n+s}^2 d\tau, \quad 0 \le t \le \delta.\end{aligned}$$

Repeating the same arguments we conclude the following

Proposition 11.3. *For any $s \in \mathbb{R}$ there is $C > 0$ such that for any u with $D_t^j u \in \mathcal{H}_{n-1/2+j/2,s+3-j}(0,\delta)$, $0 \le j \le 3$, we have*

$$\begin{aligned}(11.4)\qquad & t^{2n}\sum_{j=0}^{2}\|D_t^j u(t)\|_{s+1-j}^2 + \sum_{j=0}^{2}\int_t^\delta \tau^{2n-1}\|D_t^j u(\tau)\|_{s+1-j}^2 d\tau \\ &\le C\left(\sum_{j=0}^{2}\|D_t^j u(\delta)\|_{n+s+2-j}^2 + \int_t^\delta \tau^{2n+1}\|\hat{P}^* u(\tau)\|_{n+s}^2 d\tau\right), \quad 0 < t \le \delta.\end{aligned}$$

Since (11.4) holds with $\bar{n} \ge n+3$ as noted in Remark 10.1, then, in the resulting (11.4), replacing s by $-3\bar{n}-s-1$, we have

$$\int_0^\delta t^{2\bar{n}-1}\|u(t)\|_{-3\bar{n}-s}^2 dt \le C\int_0^\delta t^{2\bar{n}+1}\|\hat{P}^* u(t)\|_{-2\bar{n}-s-1}^2 dt$$

for any $u \in C_0^\infty((0,\delta)\times\mathbb{R}^d)$. This implies that

$$\Big|\int_0^\delta (f,v)dt\Big| \le \Big(\int_0^\delta t^{-2\bar{n}+1}\|f\|^2_{3\bar{n}+s}dt\Big)^{1/2}\Big(\int_0^\delta t^{2\bar{n}-1}\|v\|^2_{-3\bar{n}-s}dt\Big)^{1/2}$$
$$\le C\Big(\int_0^\delta t^{-2\bar{n}+1}\|f\|^2_{3\bar{n}+s}dt\Big)^{1/2}\Big(\int_0^\delta t^{2\bar{n}+1}\|\hat{P}^*v\|^2_{-2\bar{n}-s-1}dt\Big)^{1/2}$$

for any $v \in C_0^\infty((0,\delta)\times\mathbb{R}^d)$ and $f \in \mathcal{H}_{-\bar{n}+1/2,3\bar{n}+s}(0,\delta)$. Using the Hahn-Banach theorem to extend the anti-linear form in $\hat{P}^*v$:

$$\hat{P}^*v \;\mapsto \int_0^\delta (f,v)dt, \tag{11.5}$$

we conclude that there is some $u \in \mathcal{H}_{-\bar{n}-1/2,2\bar{n}+s+1}(0,\delta)$ such that

$$\int_0^\delta (f,v)dt = \int_0^\delta (u,\hat{P}^*v)dt, \quad v \in C_0^\infty((0,\delta)\times\mathbb{R}^d).$$

This implies that $\hat{P}u = f$. Since $u \in \mathcal{H}_{0,2\bar{n}+s+1}(0,\delta)$ and $f \in \mathcal{H}_{0,3\bar{n}+s}(0,\delta)$, it follows from [7, Theorem B.2.9] that $D_t^j u \in \mathcal{H}_{0,2\bar{n}+s+1-j}(0,\delta)$ for $j = 0,1,2,\dots$. Thus with $w = \langle D\rangle^{\bar{n}+s}u$ one has $D_t^j w \in L^2((0,\delta)\times\mathbb{R}^d)$ for $j = 0,\dots,\bar{n}+1$ hence $D_t^j w(0)$ exists in $L^2(\mathbb{R}^d)$ which is 0 for $j = 0,\dots,\bar{n}$ for $w \in \mathcal{H}_{-\bar{n}+1/2,0}(0,\delta)$. Thus one can write

$$w(t) = \int_0^t (t-\tau)^{\bar{n}}\partial_t^{\bar{n}+1}w(\tau)d\tau/\bar{n}!.$$

From this, one concludes that $D_t^j u \in \mathcal{H}_{-\bar{n}+j-1/2,\bar{n}+s}(0,\delta)$, hence we have that $D_t^j u \in \mathcal{H}_{-n-1/2,n+s+3-j}(0,\delta)$ for $0 \le j \le 3$ because $\bar{n} \ge n+3$, thus (11.3) holds for this u. Now let $f \in \mathcal{H}_{-n+1/2,n+s}(0,\delta)$. Take a rapidly decreasing function $\rho(\xi)$ with $\rho(0) = 1$; then $f_\epsilon = e^{-\epsilon/t}\rho(\epsilon D)f \in \mathcal{H}_{-\bar{n}+1/2,2\bar{n}+s+1}(0,\delta)$ and $f_\epsilon \to f$ in $\mathcal{H}_{-n+1/2,n+s}(0,\delta)$. As just proved above there is u_ϵ satisfying $\hat{P}u_\epsilon = f_\epsilon$ and (11.3). Therefore choosing a weakly convergent subsequence $\{u_{\epsilon'}\}$ one can conclude the following

Theorem 11.1. *There exists $\delta > 0$ such that for any s in $\mathbb{R}$ and any f in $\mathcal{H}_{-n+1/2,n+s}(0,\delta)$ there exists a unique u with $D_t^j u \in \mathcal{H}_{-n-1/2,1+s-j}(0,\delta)$, $j = 0,1,2$, satisfying $\hat{P}u = f$ and* (11.3).

Instead of (11.5), by considering the anti-linear form in $\hat{P}v$:

$$\hat{P}v \;\mapsto \int_0^\delta (f,v)dt + \sum_{j=0}^1 (w_{2-j}, D_t^j v(\delta,\cdot)) + (w_0, (D_t^2 - [D]^2 a(\delta,x,D))v(\delta,\cdot))$$

for $v \in C_0^\infty((0,\infty)\times\mathbb{R}^d)$ and repeating similar arguments adopting (11.3), we conclude the following

Theorem 11.2. *There exists $\delta > 0$ such that for any $s \in \mathbb{R}$ and any $f \in \mathcal{H}_{n+1/2,n+s}(0,\delta)$ and $w_j \in H^{n+s+2-j}$, $j = 0,1,2$, there is a unique u with $D_t^j u \in \mathcal{H}_{n-1/2,1+s-j}(0,\delta)$ satisfying $\hat{P}^* u = f$, $D_t^j u(\delta,\cdot) = w_j$, $j = 0,1,2$, and* (11.4).

Indeed we first see that there is $u \in \mathcal{H}_{n-1/2,1+s}(0,\delta)$ satisfying $\hat{P}^* u = f$ and $D_t^j u(\delta) = w_j$, $j = 0,1,2$ (e.g., [7, Chapter 23]). Since $f \in \mathcal{H}_{0,n+s}(\varepsilon,\delta)$ and $u \in \mathcal{H}_{0,1+s}(\varepsilon,\delta)$ it follows from [7, Theorem B.2.9] that $D_t^j u \in \mathcal{H}_{0,1+s-j}(\varepsilon,\delta)$, $0 \le j \le 2$, for any $\varepsilon > 0$. Applying (11.4) with $t = \varepsilon$ we conclude that $D_t^j u \in \mathcal{H}_{n-1/2,1+s-j}(0,\delta)$, $j = 1,2$, since $\varepsilon > 0$ is arbitrary.

12 Propagation of the wave front set

In Section 11 we have proved an existence result of the Cauchy problem for $\hat{P}$, which coincides with the original P only in W_M. Following [23], [10] (also [27]) we show that the wave front set of $u(t,\cdot)$, obtained by Theorem 11.1, propagates with finite speed. This fact enables us to solve the Cauchy problem for the original P.

12.1 Estimate of the wave front set. Let $\chi(x) \in C_0^\infty(\mathbb{R}^d)$ be equal to 1 near $x = 0$ and vanish in $|x| \ge 1$. Set

$$
\begin{aligned}
d_\epsilon(x,\xi;y,\eta) &= \{\chi(x-y)|x-y|^2 + |\xi\langle\xi\rangle^{-1} - \eta\langle\eta\rangle^{-1}|^2 + \epsilon^2\}^{1/2},\\
f_\epsilon(t,x,\xi;y,\eta) &= t - T + \nu d_\epsilon(x,\xi;y,\eta), \quad T > 0,
\end{aligned} \tag{12.1}
$$

where $(y,\eta) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ and ν is a positive small parameter. Note that

$$
|\partial_x^\alpha \partial_\xi^\beta d_\epsilon| \le C\langle\xi\rangle^{-|\beta|}, \quad |\alpha+\beta| = 1, \tag{12.2}
$$

where C is independent of $\epsilon > 0$. Define Φ_ϵ by

$$
\Phi_\epsilon(t,x,\xi) = \begin{cases} \exp(1/f_\epsilon(t,x,\xi)) & \text{if } f_\epsilon < 0 \\ 0 & \text{otherwise} \end{cases} \tag{12.3}
$$

and note that $\Phi_\epsilon \in S(1,g_0)$ for any fixed $\epsilon > 0$ where $g_0 = |dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2$. From now on, to simplify notation, we denote

$$
\mathcal{E}_1(\langle D\rangle^s V) + \mathcal{E}_2(\langle D\rangle^s V) = t^{-2n}\mathcal{N}_s(V), \quad \mathcal{N}_0(V) = \mathcal{N}(V).
$$

Lemma 12.1. *There exists $\nu_0 > 0$ such that for any $0 < \nu \le \nu_0$ and any $\epsilon > 0$ there is $C > 0$ such that for any V with $\int_0^\delta t^{-2n}\mathcal{N}_{-1/4}(V)dt < +\infty$ and $\tilde{L}V \in \mathcal{H}_{-n+1/2,l}(0,\delta)$ with some l we have*

$$\begin{aligned}&\mathcal{E}_1(\mathrm{op}(\Phi_\epsilon)V) + \int_0^t \tau^{-2n}\mathcal{N}(\mathrm{op}(\Phi_\epsilon)V)d\tau\\&\quad\le C\int_0^t \tau^{-2n+1}\|\mathrm{op}(\Phi_\epsilon)\tilde{L}V\|_n^2 d\tau + C\int_0^t \tau^{-2n}\mathcal{N}_{-1/4}(V)d\tau, \quad 0 < t \le \delta.\end{aligned}$$

Proof. Denote

$$V^\mu = \langle \mu D\rangle^{-\bar{n}}V$$

with small $\mu > 0$ where we choose $\bar{n} = 2n + \max\{-l, 0\} + 3$. Assume $\tilde{L}V = F$ so that $\tilde{L}V^\mu = F^\mu + R^\mu V^\mu = G^\mu$ where

$$R^\mu = [\langle\mu D\rangle^{-\bar{n}}, \mathrm{op}(\mathcal{A}+\mathcal{B})]\langle\mu D\rangle^{\bar{n}} \quad \text{and} \quad F^\mu = \langle\mu D\rangle^{-\bar{n}}F.$$

Note that $\Phi_{\epsilon 1} = f_\epsilon^{-1}\Phi_\epsilon \in S(1, g_0)$ for any fixed $\epsilon > 0$ and $\Phi_\epsilon - f_\epsilon \# \Phi_{\epsilon 1} \in S(\langle\xi\rangle^{-1}, g_0)$. Since $\partial_t\Phi_\epsilon = -\Phi_{\epsilon 1}/f_\epsilon$ one can write

$$\begin{aligned}\partial_t(\mathrm{op}(\Phi_\epsilon)V^\mu) = &-\mathrm{op}(f_\epsilon^{-1}\Phi_{\epsilon 1})V^\mu + (\mathrm{op}(i\mathcal{A}+i\mathcal{B}))\mathrm{op}(\Phi_\epsilon)V^\mu\\&+[\mathrm{op}(\Phi_\epsilon), \mathrm{op}(i\mathcal{A}+i\mathcal{B})]V^\mu + \mathrm{op}(\Phi_\epsilon)G^\mu.\end{aligned} \tag{12.4}$$

Since $\Phi_\epsilon\#\mathcal{B}_1 - \mathcal{B}_1\#\Phi_\epsilon \in S(\langle\xi\rangle^{-1/2}, g)$ by Proposition 9.1 it is not difficult to see from the proof of Corollary 10.1 that

$$|(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})[\mathrm{op}(\Phi_\epsilon), \mathrm{op}(\mathcal{B}_1)]V^\mu, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_\epsilon)V^\mu)| \le c(\epsilon)\mathcal{N}_{-1/4}(V^\mu).$$

Denote $\Phi_\epsilon\#(T^{-1}\partial_t T) - (T^{-1}\partial_t T)\#\Phi_\epsilon = (\varphi_{ij})$, hence $\varphi_{21} \in S(\sigma^{-1}\langle\xi\rangle^{-1}, g)$ and $\varphi_{31} \in S(\sigma^{-1/2}\langle\xi\rangle^{-1}, g)$ from (5.12). Thus $\lambda_j\#\varphi_{j1} \in S(\langle\xi\rangle^{-1/2}\sqrt{\kappa\lambda_1}\sqrt{\kappa\lambda_j}, g)$ for $j = 2, 3$ because $C\lambda_1 \ge \sigma\langle\xi\rangle^{-1}$, $C\lambda_2 \ge \sigma$ and $\kappa\sigma \ge 1$. A repetition of similar arguments proving (10.6) shows that

$$|(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})[\mathrm{op}(\Phi_\epsilon), \mathrm{op}(T^{-1}\partial_t T)]V^\mu, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_\epsilon)V^\mu)| \le c(\epsilon)\mathcal{N}_{-1/4}(V^\mu).$$

Note that $\Phi_\epsilon\#\mathcal{A} - \mathcal{A}\#\Phi_\epsilon$ can be written

$$\sum_{|\alpha+\beta|=1}\frac{(-1)^{|\alpha|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!}(\partial_x^\alpha\partial_\xi^\beta\Phi_\epsilon\partial_x^\beta\partial_\xi^\alpha\mathcal{A} - \partial_x^\beta\partial_\xi^\alpha\Phi_\epsilon\partial_x^\alpha\partial_\xi^\beta\mathcal{A}) + R_\epsilon = H_\epsilon + R_\epsilon$$

where it follows from Corollary 5.3 that $R_\epsilon \in S(\langle\xi\rangle^{-1/2}, g)$ for $\sigma \ge \langle\xi\rangle^{-1}$. It is not difficult to see from the proof of Corollary 10.1 that

$$|(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(R_\epsilon)V^\mu, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_\epsilon)V^\mu)| \le c(\epsilon)\mathcal{N}_{-1/4}(V^\mu).$$

Note that $H_\epsilon \in S(1,g)$ because $\partial_x^\alpha \partial_\xi^\beta \mathcal{A} \in S(\langle\xi\rangle^{1-|\beta|}, g)$ for $|\alpha+\beta|=1$. Write

$$\Phi_\epsilon = f_\epsilon \# \Phi_{\epsilon 1} + r_\epsilon \quad \text{with } r_\epsilon \in S(\langle\xi\rangle^{-1}, g_0).$$

Noting that $\phi^{-n}\# f_\epsilon - f_\epsilon \# \phi^{-n} \in S(\omega^{-1}\rho^{1/2}\langle\xi\rangle^{-1}\phi^{-n}, g) \subset S(\phi^{-n}\langle\xi\rangle^{-1/2}, g)$ and $f_\epsilon \# \lambda_j - \lambda_j \# f_\epsilon \in S(\lambda_j \langle\xi\rangle^{-1/2}, g)$ we see that

$$\begin{aligned}&|(\mathrm{op}(\varLambda)\mathrm{op}(\phi^{-n})\mathrm{op}(iH_\epsilon)V^\mu, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_\epsilon)V^\mu)\\ &\qquad - (\mathrm{op}(\varLambda)\mathrm{op}(\phi^{-n})\mathrm{op}(f_\epsilon)\mathrm{op}(iH_\epsilon)V^\mu, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^\mu)|\end{aligned}$$

is bounded by $c(\epsilon)\mathcal{N}(\langle D\rangle^{-1/4}V^\mu)$. Here we examine at iH_ϵ more carefully:

$$iH_\epsilon = \Big(\sum_{|\alpha+\beta|=1} \partial_x^\alpha \partial_\xi^\beta (\tilde{a}_{ij}[\xi])(\partial_x^\beta \partial_\xi^\alpha f_\epsilon)\frac{1}{f_\epsilon}\Phi_{1\epsilon}\Big) = (h_{ij}^\epsilon)\frac{1}{f_\epsilon}\Phi_{1\epsilon}.$$

Taking $h_{ij}^\epsilon \in S(1,g)$ and $f_\epsilon^{-1}\Phi_{\epsilon 1}$, $\Phi_{\epsilon 1} \in S(1, g_0)$ into account one can write $f_\epsilon \#(iH_\epsilon) = (h_{ij}^\epsilon)\#\Phi_{\epsilon 1} + R_\epsilon$ with $R_\epsilon \in S(\langle\xi\rangle^{-1/2}, g)$, so denoting $\tilde{H}_\epsilon = (h_{ij}^\epsilon)$,

$$\begin{aligned}&|(\mathrm{op}(\varLambda)\mathrm{op}(\phi^{-n})\mathrm{op}(f_\epsilon)\mathrm{op}(iH_\epsilon)V^\mu, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^\mu)\\ &\qquad - (\mathrm{op}(\varLambda)\mathrm{op}(\phi^{-n})\mathrm{op}(\tilde{H}_\epsilon)\mathrm{op}(\Phi_{\epsilon 1})V^\mu, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^\mu)|\end{aligned}$$

is bounded by $c(\epsilon)\mathcal{N}_{-1/4}(V^\mu)$. From Corollary 5.3 we see that $h_{ij}^\epsilon \in \mathcal{C}(1)$ for $j \ge i$, $h_{21}^\epsilon, h_{32}^\epsilon \in \mathcal{C}(\sigma^{1/2})$ and $h_{31}^\epsilon \in \mathcal{C}(\sigma)$, so in view of Lemma 6.5 one has

$$\lambda_i \#(\phi^{-n}\# h_{ij}^\epsilon - h_{ij}^\epsilon \# \phi^{-n}) \in S(\kappa\lambda_i \langle\xi\rangle^{-1}\phi^{-n}, g) \quad \text{for } j \ge i$$

and

$$\lambda_i \#(\phi^{-n}\# h_{ij}^\epsilon - h_{ij}^\epsilon \# \phi^{-n}) \in S(\kappa\lambda_i\lambda_j^{1/2} \langle\xi\rangle^{-1/2}\phi^{-n}, g) \quad \text{for } i > j.$$

From this we see that

$$\begin{aligned}&|(\mathrm{op}(\varLambda)\mathrm{op}(\phi^{-n})\mathrm{op}(\tilde{H}_\epsilon)\mathrm{op}(\Phi_{\epsilon 1})V^\mu, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^\mu)\\ &\qquad - (\mathrm{op}(\varLambda)\mathrm{op}(\tilde{H}_\epsilon)\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^\mu, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^\mu)|\end{aligned}$$

is bounded by $c(\epsilon)\mathcal{N}_{-1/4}(V^\mu)$.

Lemma 12.2. *One can write*

$$h_{ij}^\epsilon = \sum_{|\alpha+\beta|=1} k_{ij\alpha\beta}^\epsilon \# l_{ij\alpha\beta} + r_{ij}^\epsilon,$$

where $k_{ij\alpha\beta}^\epsilon \in S(1, g_0)$ such that

$$|k_{ij\alpha\beta}^\epsilon| \le C\nu$$

with some $C > 0$ independent of ν and ϵ for any $1 \le i, j \le 3$. As for $l_{ij\alpha\beta}$ and r_{ij}^ϵ one has $l_{ij\alpha\beta} \in S(1,g)$ and $r_{ij}^\epsilon \in S(\sigma^{-1/2}\langle\xi\rangle^{-1}, g)$ for $j \ge i$ and $l_{ij\alpha\beta} \in S(\sigma^{(i-3)/2}\sqrt{\lambda_j}, g)$, $r_{ij}^\epsilon \in S(\sigma^{(i-3)/2}\sqrt{\lambda_j}\langle\xi\rangle^{-1/2}, g)$ for $i > j$.

Proof. Set $k^{\epsilon}_{ij\alpha\beta} = \langle\xi\rangle^{|\alpha|}\partial_x^{\beta}\partial_{\xi}^{\alpha} f_{\epsilon}$ and $l_{ij\alpha\beta} = \langle\xi\rangle^{-|\alpha|}\partial_x^{\alpha}\partial_{\xi}^{\beta}(\tilde{a}_{ij}[\xi])$; then the assertion for $k^{\epsilon}_{ij\alpha\beta}$ is clear from (12.2). The assertions for $l_{ij\alpha\beta}$ follow from Corollary 5.3 and Lemmas 5.8, 7.7. Note that $\partial_x^{\mu}\partial_{\xi}^{\nu} l_{ij\alpha\beta} \in S(\sigma^{-1/2}\langle\xi\rangle^{-|\nu|}, g)$,

$$|\mu+\nu| = 1 \quad \text{for } j \ge i,$$

and $\partial_x^{\mu}\partial_{\xi}^{\nu} l_{21\alpha\beta}$, $\partial_x^{\mu}\partial_{\xi}^{\nu} l_{32\alpha\beta} \in S(\langle\xi\rangle^{-|\nu|}, g)$ and $\partial_x^{\mu}\partial_{\xi}^{\nu} l_{31\alpha\beta} \in S(\sigma^{1/2}\langle\xi\rangle^{-|\nu|}, g)$ for $|\mu+\nu| = 1$, which follows from $\tilde{a}_{21}$, $\tilde{a}_{32} \in \mathcal{C}(\sigma)$ and $\tilde{a}_{31} \in \mathcal{C}(\sigma^{5/2})$. Then since $\sigma \ge \langle\xi\rangle_{\gamma}^{-1}$ and $C\lambda_1 \ge \sigma\langle\xi\rangle_{\gamma}^{-1}$ the assertions for r^{ϵ}_{ij} are checked immediately. □

With $R^{\epsilon} = (r^{\epsilon}_{ij})$ and $W = \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^{\mu}$, recalling $\lambda_1 \le C\sigma\lambda_2 \le C'\sigma^2\lambda_3$, it is easy to see that

$$|(\mathrm{op}(R^{\epsilon})W, \mathrm{op}(\Lambda)W)| \le c(\epsilon)\|\mathrm{op}(\Lambda^{1/2})\langle D\rangle^{-1/4}W\|^2.$$

Consider $|(\mathrm{op}(h^{\epsilon}_{ij})W_j, \mathrm{op}(\lambda_i)W_i)|$. Thanks to Lemma 12.2 this is bounded by

$$C\|\mathrm{op}(\lambda_j^{1/2})W_j\|\,\|\mathrm{op}(k^{\epsilon}_{ij\alpha\beta})\mathrm{op}(\lambda_i^{1/2})W_i\|$$

with C independent of ϵ because $\lambda_i^{1/2}\# l_{ij\alpha\beta} \in S(\lambda_j^{1/2}, g)$ in view of Lemma 12.2. On the other hand, taking Lemma 12.2 into account, it follows from the sharp Gårding inequality (e.g., [7, Theorem 18.1.14]) that

$$\|\mathrm{op}(k^{\epsilon}_{ij\alpha\beta})\mathrm{op}(\lambda_i^{1/2})W_i\| \le C\nu\|\mathrm{op}(\lambda_i^{1/2})W_i\| + C(\nu,\epsilon)\|\mathrm{op}(\lambda_i^{1/2})\langle D\rangle^{-1/2}W_i\|.$$

Therefore applying the above-obtained estimates one can find $C > 0$ independent of ϵ and ν such that

$$\begin{aligned}&|\mathsf{Re}(\mathrm{op}(\Lambda)\mathrm{op}(\tilde{H}_{\epsilon})\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^{\mu}, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^{\mu})|\\ &\le C\nu\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^{\mu}\|^2 + C'(\nu,\epsilon)\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^{-n})\langle D\rangle^{-1/4}V^{\mu}\|^2.\end{aligned}$$

It follows from the same reasoning that

$$\begin{aligned}&|(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(f_{\epsilon}^{-1}\Phi_{\epsilon})V^{\mu}, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon})V^{\mu})|\\ &\qquad - (\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^{\mu}, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^{\mu})| \le c(\epsilon)\mathcal{N}_{-1/4}(V^{\mu}).\end{aligned}$$

We conclude finally that $-\mathsf{Im}(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\tilde{L}(\mathrm{op}(\Phi_{\epsilon})V^{\mu}), \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon})V^{\mu})$ is bounded by

$$\begin{aligned}&-(1-C\nu)\|\mathrm{op}(\Lambda^{1/2})\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon 1})V^{\mu}\|^2 + c(\nu,\epsilon)\mathcal{N}_{-1/4}(V^{\mu})\\ &\qquad + \mathsf{Re}(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon})G^{\mu}, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon})V^{\mu}).\end{aligned} \tag{12.5}$$

We fix ν_0 such that $1 - C\nu_0 \ge 0$. Since $|\partial_{\xi}^{\alpha}\langle\mu\xi\rangle^{-\bar{n}}| \le C_{\alpha}\langle\mu\xi\rangle^{-\bar{n}}\langle\xi\rangle^{-|\alpha|}$ with C_{α} independent of $\mu > 0$ we see that

$$|(\mathrm{op}(\Lambda)\mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon})R^{\mu}V^{\mu}, \mathrm{op}(\phi^{-n})\mathrm{op}(\Phi_{\epsilon})V^{\mu})| \le C\mathcal{E}_1(\mathrm{op}(\Phi_{\epsilon})V^{\mu}).$$

Therefore $|(\mathrm{op}(\Lambda)\mathrm{op}(w^{-n})\mathrm{op}(\Phi_\epsilon)G^\mu, \mathrm{op}(w^{-n})\mathrm{op}(\Phi_\epsilon)V^\mu)|$ is bounded by

$$\epsilon_1 \mathcal{E}_1(\mathrm{op}(\Phi_\epsilon)V^\mu) + C_{\epsilon_1} t^{-2n+1}\|\mathrm{op}(\Phi_\epsilon)F^\mu\|_n^2 + C\mathcal{E}_1(\mathrm{op}(\Phi_\epsilon)V^\mu)$$

for any $\epsilon_1 > 0$. Note that $D_t^j V^\mu \in \mathcal{H}_{0,2n+7/4-j}(0,\delta)$, $j = 0, 1, \ldots$, hence $D_t^j V(0)$ exists in $H^{n+3/4}$ which is 0 for $j = 0, 1, \ldots, n$, thus $\lim_{t\to+0} t^{-n}\|V^\mu(t)\|_n = 0$ for $\mu > 0$. Applying (10.12) to $\mathrm{op}(\Phi_\epsilon)V^\mu$ instead of V, choosing $\epsilon_1 < c_1$ and then letting $\mu \to 0$ one concludes the proof. □

Applying $\langle D\rangle^s$ to (12.4) and repeating similar arguments one obtains

Proposition 12.1. *For any $s \in \mathbb{R}$, any $0 < \nu \le \nu_0$ and any $\epsilon > 0$ one can find $C > 0$ such that for any V with $\int_0^\delta t^{-2n}\mathcal{N}_{s-1/4}(V)dt < +\infty$ and $\tilde{L}V \in \mathcal{H}_{-n+1/2,l}(0,\delta)$ with some l we have*

$$\mathcal{E}_1(\langle D\rangle^s \mathrm{op}(\Phi_\epsilon)V) + \int_0^t \tau^{-2n}\mathcal{N}_s(\mathrm{op}(\Phi_\epsilon)V)d\tau \le C\Big(\int_0^t \tau^{-2n+1}\|\mathrm{op}(\Phi_\epsilon)\tilde{L}V\|_{n+s}^2 d\tau + \int_0^t \tau^{-2n}\mathcal{N}_{s-1/4}(V)d\tau\Big), \quad 0 \le t \le \delta.$$

12.2 The wave front set propagates with finite speed.

Lemma 12.3. *Assume that $V \in \mathcal{H}_{-n-1/2,l_1+1}(0,\delta)$ and $\tilde{L}V \in \mathcal{H}_{-n+1/2,l_2}(0,\delta)$, and that $\mathrm{op}(\Phi_{\epsilon_0})\tilde{L}V \in \mathcal{H}_{-n+1/2,n+s_0}(0,\delta)$ with some $l_1, l_2, s_0 \in \mathbb{R}$, $\epsilon_0 > 0$. Then for every $\epsilon > \epsilon_0$ we have $\mathrm{op}(\Phi_\epsilon)V \in \mathcal{H}_{-n-1/2,s}(0,\delta)$ for all $s \le s_0 - 5/4$. Moreover,*

$$\int_0^t \tau^{-2n-1}\|\mathrm{op}(\Phi_\epsilon)V(\tau)\|_s^2 d\tau \le C\int_0^t (\tau^{-2n-1}\|V(\tau)\|_{l_1+1}^2 + \tau^{-2n+1}\|\tilde{L}V(\tau)\|_{l_2}^2)d\tau + C\int_0^t \tau^{-2n+1}\|\mathrm{op}(\Phi_{\epsilon_0})\tilde{L}V(\tau)\|_{n+s_0}^2 d\tau, \quad 0 < t \le \delta.$$

Proof. We may assume $l_1 \le s_0$ otherwise there is nothing to be proved. Let J be the largest integer such that $l_1 + J/4 \le s_0$. Take $\epsilon_j > 0$ such that

$$\epsilon_0 < \epsilon_1 < \cdots < \epsilon_J = \epsilon.$$

We write $\Phi_{\epsilon_j} = \Phi_j$ and $f_j = f_{\epsilon_j}$ in this proof. Inductively we show that

$$\int_0^t \tau^{-2n}\mathcal{N}_{l_1+j/4}(\mathrm{op}(\Phi_j)V)d\tau \le C\int_0^t \tau^{-2n-1}\|V(\tau)\|_{l_1+1}^2 d\tau + C\int_0^t \tau^{-2n+1}\{\|\tilde{L}V(\tau)\|_{l_2}^2 + \|\mathrm{op}(\Phi_0)\tilde{L}V(\tau)\|_{l_1+n+j/4}^2\}d\tau. \tag{12.6}$$

Choose $\psi_j(x,\xi) \in S(1, g_0)$ so that

$$\operatorname{supp} \psi_j \subset \{f_j < 0\} \quad \text{and} \quad \{f_{j+1} < 0\} \subset \{\psi_j = 1\}.$$

Noting that

$$\mathrm{op}(\Phi_{j+1})\tilde{L}\,\mathrm{op}(\psi_j) = \mathrm{op}(\Phi_{j+1}\#\psi_j)\tilde{L} + \mathrm{op}(\Phi_{j+1})[\tilde{L}, \mathrm{op}(\psi_j)]$$

we apply Proposition 12.1 with $s = l_1 + (j+1)/4$, $\Phi = \Phi_{j+1}$ and $V = \mathrm{op}(\psi_j)V$. Since $\Phi_{j+1}\#\psi_j - \Phi_{j+1} \in S^{-\infty}$, then $\|\mathrm{op}(\Phi_{j+1})\tilde{L}\,\mathrm{op}(\psi_j)V\|^2_{l_1+(j+1)/4+n}$ is bounded by $c\|\mathrm{op}(\Phi_{j+1})\tilde{L}V\|^2_{l_1+(j+1)/4+n} + C(j)\|V\|^2_{l_1+1}$ and hence by

$$C(j)\big\{\|\mathrm{op}(\Phi_0)\tilde{L}V\|^2_{l_1+(j+1)/4+n} + \{\|\tilde{L}V\|^2_{l_2} + \|V\|^2_{l_1+1}\} \tag{12.7}$$

because $\Phi_{j+1} - k_j\#\Phi_0 \in S^{-\infty}$ with some $k_j \in S(1, g_0)$. Since $\psi_j - \tilde{k}_j\#\Phi_j \in S^{-\infty}$ with some $\tilde{k}_j \in S(1, g_0)$ it follows that

$$\mathcal{N}_{l_1+j/4}(\mathrm{op}(\Phi_{j+1})\mathrm{op}(\psi_j)V) \le C\mathcal{N}_{l_1+j/4}(\mathrm{op}(\Phi_j)V) + C\|V\|^2_{l_1+1}.$$

Consider $\mathcal{N}_{l_1+(j+1)/4}(\mathrm{op}(\Phi_{j+1})\mathrm{op}(\psi_j)V)$. Noting that $\Phi_{j+1}\#\psi_j - \Phi_{j+1} \in S^{-\infty}$ the same reasoning shows that

$$\mathcal{N}_{l_1+(j+1)/4}(\mathrm{op}(\Phi_{j+1})V) \le C\mathcal{N}_{l_1+(j+1)/4}(\mathrm{op}(\Phi_{j+1})\mathrm{op}(\psi_j)V) + C\|V\|^2_{l_1+1}. \tag{12.8}$$

Multiply (12.8) and (12.7) by t^{-2n} and t^{-2n+1} respectively and integrate from 0 to t. We conclude from Proposition 12.1 that (12.6) holds for $j+1$ and hence for $j = J$. Since $l_1 + J/4 \le s_0$, $l_1 + J/4 > s_0 - 1/4$ and $\|V\|_{s-1}/C \le \mathcal{N}_s(V)$ the assertion follows. □

Let Γ_i $(i = 1, 2, 3)$ be open conic sets in $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ with relatively compact basis such that $\Gamma_1 \Subset \Gamma_2 \Subset \Gamma_3$. Take $h_i(x,\xi) \in S(1, g_0)$ with $\operatorname{supp} h_1 \subset \Gamma_1$ and $\operatorname{supp} h_2 \subset \Gamma_3 \setminus \Gamma_2$. Consider a solution V with $V \in \mathcal{H}_{-n-1/2,l}(0,\delta)$ to the equation

$$\tilde{L}V = \mathrm{op}(h_1)F, \quad F \in \mathcal{H}_{-n+1/2,s}(0,\delta).$$

Proposition 12.2. *The notation is as above. There exists $\delta' = \delta'(\Gamma_i) > 0$ such that for any $r \in \mathbb{R}$ there is $C > 0$ such that*

$$\begin{aligned}\int_0^t \tau^{-2n-1}\|\mathrm{op}(h_2)V(\tau)\|_r^2 d\tau \\ \le C\int_0^t \{\tau^{-2n+1}\|F(\tau)\|_s^2 + \tau^{-2n-1}\|V(\tau)\|_l^2\} d\tau, \quad 0 < t \le \delta'.\end{aligned} \tag{12.9}$$

Proof. Let $f_\epsilon = t - \nu_0\hat\tau + \nu_0 d_\epsilon(x,\xi;y,\eta)$ with a small $\hat\tau > 0$. It is clear that there is $\hat\epsilon > 0$ such that $\{t \geq 0\} \cap \{f_{\hat\epsilon} \leq 0\} \cap (\mathbb{R} \times \operatorname{supp} h_1) = \emptyset$ for any $(y,\eta) \notin \Gamma_2$. Take $\hat\epsilon < \tilde\epsilon < \hat\tau$. It is also clear that one can find a finite number of $(y_i,\eta_i) \in \Gamma_3 \setminus \Gamma_2$, $i = 1,\dots,M$, such that with $\delta' = \nu_0(\hat\tau - \tilde\epsilon)/2$

$$\Gamma_3 \setminus \Gamma_2 \Subset \left(\bigcup_{i=1}^{M}\{f_{\tilde\epsilon}(\delta', x,\xi;y_i,\eta_i) \leq 0\}\right),$$
$$\{t \geq 0\} \cap \{f_{\hat\epsilon}(t,x,\xi;y_i,\eta_i) \leq 0\} \cap (\mathbb{R} \times \operatorname{supp} h_1) = \emptyset.$$

Now $\Phi_{i\epsilon}$ is defined by (12.3) with $f_\epsilon(t,x,\xi;y_i,\eta_i)$. Then since $\sum \Phi_{i\tilde\epsilon} > 0$ on $[0,\delta'] \times \operatorname{supp} h_2$ there is $k \in S(1,g_0)$ such that

$$h_2 - k\sum \Phi_{i\tilde\epsilon} \in S^{-\infty}.$$

Noting that $\mathrm{op}(\Phi_{i\hat\epsilon})\mathrm{op}(h_1)F \in \mathcal{H}_{-n+1/2,r}(0,\delta)$ for any $r \in \mathbb{R}$ we apply Lemma 12.3 with $\Phi_{\epsilon_0} = \Phi_{\hat\epsilon}$, $\Phi_\epsilon = \Phi_{i\tilde\epsilon}$ and $s_0 = r + 5/4$ to obtain

$$\begin{aligned}\int_0^t &\tau^{-2n-1}\|\mathrm{op}(\Phi_{i\tilde\epsilon})V(\tau)\|_r^2 d\tau\\ &\leq C\int_0^t \tau^{-2n-1}\|V(\tau)\|_l^2 d\tau\\ &\quad + \int_0^t \tau^{-2n+1}(\|\mathrm{op}(\Phi_{i\hat\epsilon})\mathrm{op}(h_1)F(\tau)\|^2_{2n+r+5/4} + \|F(\tau)\|_s)d\tau\end{aligned}$$

for $\|\tilde L V(\tau)\|_s \leq C\|F(\tau)\|_s$. Since $\Phi_{i\hat\epsilon}\# h_1 \in S^{-\infty}$ on summing up the above estimates over $i = 1,\dots,M$ one concludes the desired assertion. □

Lemma 12.4. *The same assertion as Proposition 12.2 holds for L.*

Proof. Assume that $U \in \mathcal{H}_{-n-1/2,l}(0,\delta)$ satisfies $LU = \mathrm{op}(h_1)F$ where $F \in \mathcal{H}_{-n+1/2,s}(0,\delta)$. Choose $\tilde\Gamma_i$ such that $\Gamma_1 \Subset \tilde\Gamma_1 \Subset \tilde\Gamma_2 \Subset \Gamma_2 \Subset \Gamma_3 \Subset \tilde\Gamma_3$ and $\tilde h_i \in S(1,g_0)$ such that $\operatorname{supp}\tilde h_1 \subset \tilde\Gamma_1$, $\operatorname{supp}\tilde h_2 \subset \tilde\Gamma_3 \setminus \tilde\Gamma_2$ and $\tilde h_i = 1$ on the support of h_i. Recall that $L\,\mathrm{op}(T) = \mathrm{op}(T)\,\tilde L$, hence

$$\tilde L V = (I + \mathrm{op}(K))\mathrm{op}(T^{-1})\mathrm{op}(h_1)F$$

with $U = \mathrm{op}(T)V$. Since there is $\tilde T \in S(1,g)$ such that $(I+K)\#T^{-1}\#h_1 - \tilde h_1\tilde T \in S^{-\infty}$, it follows from Proposition 12.2 (or rather its proof) that (12.9) holds with $\tilde h_2$ in place of h_2. Similarly, since there is $\tilde T \in S(1,g)$ such that $h_2\#T - \tilde h_2\tilde T \in S^{-\infty}$ repeating the same arguments we conclude the assertion. □

Returning to $\hat P$ we have

Proposition 12.3. *The notation is as above. Then there exists* $\delta' = \delta'(\Gamma_i) > 0$ *such that for any* $s, r \in \mathbb{R}$ *there is* C *such that for any* u *with* $D_t^j u \in \mathcal{H}_{-n-1/2,l+2-j}(0, \delta')$, $j = 0, 1, 2$, *with some* l *satisfying* $\hat{P}u = \mathrm{op}(h_1)f$ *where* $f \in \mathcal{H}_{-n+1/2,s}(0, \delta')$, *one has*

$$\sum_{j=0}^{2} \int_0^t \tau^{-2n-1} \|\mathrm{op}(h_2) D_t^j u(\tau)\|_{r+2-j}^2 d\tau$$
$$\leq C\Big(\int_0^t \tau^{-2n+1} \|f(\tau)\|_s^2 d\tau + \sum_{j=0}^{2} \int_0^t \tau^{-2n-1} \|D_t^j u(\tau)\|_{l+2-j}^2 d\tau \Big), \quad 0 < t \leq \delta'.$$

Thanks to Theorem 11.1, for any $f \in \mathcal{H}_{-n+1/2,n+s}(0, \delta)$ there is a unique solution $u \in \mathcal{H}_{-n-1/2,s+1}(0, \delta)$ to $\hat{P}u = f$ satisfying (11.3). Denote this map by

$$\hat{G} : \mathcal{H}_{-n+1/2,n+s}(0, \delta) \ni f \mapsto u \in \mathcal{H}_{-n-1/2,s+1}(0, \delta).$$

From Proposition 12.3 and Theorem 11.1 we conclude

Proposition 12.4. *With the notation as above, let* Γ_i $(i = 1, 2, 3)$ *be open conic sets in* $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ *with relatively compact basis such that* $\Gamma_1 \Subset \Gamma_2 \Subset \Gamma_3$ *and* $h_i(x, \xi) \in S(1, g_0)$ *with* $\mathrm{supp}\, h_1 \subset \Gamma_1$ *and* $\mathrm{supp}\, h_2 \subset \Gamma_3 \setminus \Gamma_2$. *Then there exists* $\delta' = \delta'(\Gamma_i) > 0$ *such that for any* r, s *one can find* $C > 0$ *such that*

$$\sum_{j=0}^{2} \int_0^t \tau^{-2n-1} \|\mathrm{op}(h_2) D_t^j \hat{G}\, \mathrm{op}(h_1) f(\tau)\|_{r-j}^2 d\tau \leq C \int_0^t \tau^{-2n+1} \|f(\tau)\|_s^2 d\tau$$

for $0 < t \leq \delta'$ *and for any* $f \in \mathcal{H}_{-n+1/2,s}(0, \delta')$.

Denote by $\mathcal{H}^*_{n,s}(0, \delta]$ the set of all f with $t^n \langle D\rangle^s f(t, \cdot) \in L^2((0, \infty) \times \mathbb{R}^d)$ such that $f = 0$ for $t \geq \delta$. Thanks to Theorem 11.2, for any $f \in \mathcal{H}^*_{n+1/2,n+s}(0, \delta]$ there is a unique solution $u \in \mathcal{H}_{n-1/2,s+1}(0, \delta)$ to $\hat{P}^* u = f$ with $D_t^j u(\delta) = 0$, $j = 0, 1, 2$, satisfying (11.4), hence $u \in \mathcal{H}^*_{n-1/2,s+1}(0, \delta]$. Denote this map by

$$\hat{G}^* : \mathcal{H}^*_{n+1/2,n+s}(0, \delta] \ni f \mapsto u \in \mathcal{H}^*_{n-1/2,s+1}(0, \delta].$$

Repeating similar arguments to those proving Proposition 12.4, one obtains

Proposition 12.5. *The notation is as in Proposition 12.4. There exists* $\delta' = \delta'(\Gamma_i) > 0$ *such that for any* r, s *one can find* $C > 0$ *such that*

$$\sum_{j=0}^{2} \int_t^{\delta'} \tau^{2n-1} \|\mathrm{op}(h_2) D_t^j \hat{G}^*\, \mathrm{op}(h_1) f(\tau)\|_{r-j}^2 d\tau \leq C \int_t^{\delta'} \tau^{2n+1} \|f(\tau)\|_s^2 d\tau$$

for $0 < t \leq \delta'$ *and* $f \in \mathcal{H}^*_{n+1/2,s}(0, \delta']$.

Remark 12.1. As already remarked in Remark 10.1, it is clear from the proof that Theorems 11.1 and 11.2 and Propositions 12.4 and 12.5 hold for any n' greater than n.

12.3 Remarks on propagation of singularities. In this section we give a more precise picture of the propagation of wave front set $WF(u)$ of u applying the same arguments as in Sections 12.1 and 12.2. Denote $X = [0, \delta) \times U$ and $\overset{\circ}{X} = (0, \delta) \times U$ and by Σ, Σ_1 the set of characteristics and simple characteristics of p respectively. As explained in the Introduction every characteristic in $T^*\overset{\circ}{X} \setminus 0$ is at most double and a double characteristic is effectively hyperbolic. Let $\mathcal{U}$ be an open conical set in $\subset T^*\overset{\circ}{X} \setminus 0$. According to [21] we say that a continuous curve $\gamma(t) : (0, a] \to \mathcal{U} \cap \Sigma$, parametrized by t, is a generalized bicharacteristic if $\gamma^{-1}(\Sigma \setminus \Sigma_1) = \{t_i\}$ is discrete in $(0, a]$ and γ is a parametrized smooth bicharacteristic of p on (t_i, t_{i+1}). In the present case a generalized bicharacteristic is described rather easily. Let $\rho \in \mathcal{U}$ be a double characteristic. Then from [27] one can find a conical open set $\mathcal{V} \ni \rho$ and a smooth function $\psi(x, \xi)$, homogeneous of degree 0 in ξ, such that the double characteristics of p in $\mathcal{V}$ are contained in $\{t = \psi\}$ and there are exactly two smooth bicharacteristics enter ρ transversally to $t = \psi$ in the direction of decreasing t (also exactly two in the direction of increasing t) (see, e.g.,[18]). Therefore γ consists of segments of a smooth bicharacteristic of p, the only end points of these segments lying in Σ, and a transition to one of two segments takes place there. Let γ_i, $i = 1, 2$ be two bicharacteristic segments entering ρ in the direction of decreasing t (or two segments in the direction of increasing t). Then from [24, Theorem 2.1], [21, Theorem 1.7] we have $\rho \notin WF(u)$ if $\gamma_i \notin WF(u)$, $i = 1, 2$, and $\rho \notin WF(Pu)$.

Lemma 12.5. *If $\gamma(t)$ is a generalized bicharacteristic, then*

$$\lim_{t \to +0} \gamma(t)$$

exists.

Proof. Write $p = \prod_{j=1}^{m}(\tau - \tau_j(t, x, \xi))$. Then thanks to [4] for any $0 < \delta_1 < \delta$ and $U' \Subset U$ there is $L > 0$ such that

$$|\nabla_x \tau_j(t, x, \xi)|/|\xi|,\ |\nabla_\xi \tau_j(t, x, \xi)| \le L,$$
$$(t, x, \xi) \in [0, \delta_1] \times U' \times \mathbb{R}^d,\ 1 \le j \le m.$$

In each (t_i, t_{i+1}) it is clear that $\gamma(t) = (t, x(t), \tau(t), \xi(t))$ is a bicharacteristic of some $\tau - \tau_j(t, x, \xi)$ and hence $dx/dt = -\nabla_\xi \tau_j(t, x, \xi)$ and $d\xi/dt = \nabla_x \tau_j(t, x, \xi)$. This shows that $x(t)$ and $\xi(t)$ are uniformly Lipschitz continuous in $(0, a]$ with the Lipschitz constant L, though $\tau(t)$ is not Lipschitz continuous in $(0, a]$ in general. Then $\lim_{t \to +0} x(t)$ and $\lim_{t \to +0} \xi(t)$ exist. Since $\tau(t) = \tau_k(t, x(t), \xi(t))$ for some k and $\tau_k(t, x, \xi)$ are continuous in $X \times \mathbb{R}^d$, then $\lim_{t \to +0} \tau(t)$ also exists. □

Denote

$$K^{-}(\rho) = \bigcup_{\gamma} \gamma(t)$$

where γ varies over all generalized bicharacteristics such that $\gamma(a) = \rho$, extended to $t = 0$ according to Lemma 12.5. Note that for any $\varepsilon > 0$ the set $K^{-}(\rho) \cap \{t = \varepsilon\}$ consists of a finite number of characteristic points. Thanks to the propagation results near double effectively hyperbolic characteristics mentioned above, if

$$WF(u) \cap K^{-1}(\rho) \cap \{t = \varepsilon\} = \emptyset \tag{12.10}$$

with some $\varepsilon > 0$ then we have $\rho \notin WF(u)$.

Theorem 12.1. *With the notation as above, let $K_0(\rho) = \pi(K^{-}(\rho) \cap \{t = 0\})$ where $\pi : (t, x, \tau, \xi) \to (x, \xi)$ is the projection. Let $Pu \in C^{\infty}(X)$ and $D_t^j u(0, \cdot) = u_j$. If $K_0(\rho) \cap (\bigcup_{j=0}^{m-1} WF(u_j)) = \emptyset$ then $\rho \notin WF(u)$.*

We give a sketch of the proof. Since characteristics are at most triple we may assume that $m = 3$. It suffices to show that (12.10) holds with some $\varepsilon > 0$. In (12.1) we take $T = 2\nu$. Since $K_0(\rho)$ is compact one can find $0 < \nu < \nu_0$ and finitely many (y_i, η_i), $i = 1, \ldots, k$ such that

$$K_0(\rho) \subset \bigcup_{i=1}^{k} \{f_0(0, x, \xi, y_i, \eta_i) < 0\}$$

and

$$\{f_0(0, x, \xi, y_i, \eta_i) \le 0\} \cap \left(\bigcup_{j=0}^{2} WF(u_j) \right) = \emptyset.$$

Let $\Phi_{i,\epsilon}$ be the symbol defined by (12.3) with $f_\epsilon(t, x, \xi, y_i, \eta_i)$. Then we have

$$\mathrm{op}(\Phi_{i,\epsilon}(0)) u_j \in H^s, \quad 0 \le j \le 2, \quad 1 \le i \le k, \ \forall s \in \mathbb{R} \tag{12.11}$$

for small enough $\epsilon > 0$. Using the same reduction as in the proof of Theorem 1.2 it suffices to study $Pu = f$ where $D_t^j u \in \mathcal{H}_{-n-1/2, l_1 - j}(0, \delta)$, $0 \le j \le 2$ and $f \in \mathcal{H}_{-n-1/2, l_2}(0, \delta)$ with some $l_j \in \mathbb{R}$ such that

$$\mathrm{op}(\Phi_{i,\epsilon}) f \in \mathcal{H}_{-n-1/2, s}(0, \delta), \quad 1 \le i \le k, \ \ \forall s \in \mathbb{R}$$

for small enough $\epsilon > 0$ which follows from (12.11). Repeating the same arguments as in Sections 12.1 and 12.2 we conclude that $\mathrm{op}(\Phi_{i,0}) D_t^j u \in L^2((0, \delta'), H^s)$ for any $1 \le i \le k$, $0 \le j \le 2$ and any $s \in \mathbb{R}$ with some $\delta' > 0$. Finally, using the equation we have

$$\mathrm{op}(\Phi_{i,0}) D_t^j u \in L^2((0, \delta'), H^s), \quad 1 \le i \le k, \quad \forall s \in \mathbb{R}, \ \forall j \in \mathbb{N}.$$

Since $\Phi_{i,0}(t, y_i, \eta_i) < 0$ for $t < \nu$ this proves (12.10) with some $\varepsilon > 0$. □

13 Proof of Theorem 1.1

Applying Proposition 12.4 we prove Theorem 1.1 following [23], [27].

13.1 Solution operator with finite speed of propagation.

Consider

$$P = D_t^m + \sum_{j=1}^{m} a_j(t,x,D)\langle D\rangle^j D_t^{m-j} \tag{13.1}$$

which is a differential operator in t with coefficients $a_j \in S^0$. We say that G is a solution operator for P with finite propagation speed of the wave front set (which we abbreviate to "solution operator with finite speed of propagation" from now on) with loss of (n, l) derivatives if G satisfies the following conditions:

(i) There exists $\delta > 0$ such that for any $s \in \mathbb{R}$ there is $C > 0$ such that for $f \in \mathcal{H}_{-n+1/2,s}(0,\delta)$ we have $PGf = f$ and

$$\sum_{j=0}^{m-1}\int_0^t \tau^{-2n-1}\|D_t^j Gf(\tau)\|^2_{-l+s+m-j}d\tau \le C\int_0^t \tau^{-2n+1}\|f(\tau)\|_s^2 d\tau.$$

(ii) For any $h_j(x,\xi) \in S(1, g_0), j = 1,2$ with $\operatorname{supp} h_2 \Subset (\mathbb{R}^d \times \mathbb{R}^d) \setminus \operatorname{supp} h_1$ there exists $\delta' > 0$ such that for any $r, s \in \mathbb{R}$ there is $C > 0$ such that

$$\sum_{j=0}^{m-1}\int_0^t \tau^{-2n-1}\|\mathrm{op}(h_2)D_t^j G\mathrm{op}(h_1)f(\tau)\|^2_{r-j}d\tau \le C\int_0^t \tau^{-2n+1}\|f(\tau)\|_s^2 d\tau$$

for $f \in \mathcal{H}_{-n+1/2,s}(0,\delta')$ and $0 < t \le \delta'$.

Let P_1 and P_2 be two operators of the form (13.1). We say that

$$P_1 \equiv P_2 \quad \text{at } (\hat{x},\hat{\xi})$$

if there exist $\delta' > 0$ and a conic neighborhood W of $(\hat{x},\hat{\xi})$ such that

$$P_1 - P_2 = \sum_{j=1}^{m} R_j(t,x,D)\langle D\rangle^j D_t^{m-j} \tag{13.2}$$

with $R_j \in S^0$ which are in $S^{-\infty}(W)$ uniformly in $0 \le t \le \delta'$.

Theorem 13.1. *Assume that for any* $(\hat{x},\eta)$, $|\eta| = 1$ *one can find* P_η *of the form* (13.1) *for which there is a solution operator with finite speed of propagation with loss of* $(n, \ell(\eta))$ *derivatives such that* $P \equiv P_\eta$ *at* $(\hat{x},\eta)$. *Then there exist* $\delta > 0$, $\ell = \sup \ell(\eta)$ *and a neighborhood* U *of* $\hat{x}$ *such that for every* $f \in \mathcal{H}_{-n+1/2,s+\ell}(0,\delta)$ *there exists* u *with* $D_t^j u \in \mathcal{H}_{-n,s+m-j}(0,\delta)$, $0 \le j \le m-1$, *satisfying* $Pu = f$ *in* $(0,\delta)\times U$.

Proof. By assumption, P_η has a solution operator G_η with finite speed of propagation with loss of $(n, \ell(\eta))$ derivatives. There are finite open conic neighborhoods W_i of $(\hat{x}, \eta_i)$ such that $\bigcup_i W_i \supset \Omega \times (\mathbb{R}^d \setminus \{0\})$, where Ω is a neighborhood of $\hat{x}$, and $P \equiv P_{\eta_i}$ at $(\hat{x}, \eta)$ with $W = W_i$ in (13.2). Now take another open conic covering $\{V_i\}$ of $\Omega \times (\mathbb{R}^d \setminus \{0\})$ with $V_i \Subset W_i$, and a partition of unity $\{\alpha_i(x, \xi)\}$ subordinate to $\{V_i\}$ so that $\sum_i \alpha_i(x, \xi) = \alpha(x)$ where $\alpha(x) = 1$ in a neighborhood of $\hat{x}$. Define

$$G = \sum_i G_{\eta_i} \alpha_i.$$

Then denoting $P - P_{\eta_i} = R_i$ we have

$$PGf = \sum_i PG_{\eta_i}\alpha_i f = \sum_i P_{\eta_i} G_i \alpha_i f + \sum_i R_i G_i \alpha_i f = \alpha(x) f - Rf$$

where $R = \sum_i R_i G_{\eta_i} \alpha_i$. Then

$$\int_0^t \tau^{-2n-1} \|Rf(\tau)\|_{s+\ell}^2 d\tau \le C \int_0^t \tau^{-2n+1} \|f(\tau)\|_{s+\ell}^2 d\tau$$

for $0 \le t \le \delta''$ with some $\delta'' > 0$ by the condition (ii) where $\ell = \max_i \ell(\eta_i)$. Choosing $0 < \delta_1 \le \delta''$ such that $\delta_1^2 C \le 1/2$ one has

$$\int_0^t \tau^{-2n+1} \|Rf(\tau)\|_{s+\ell}^2 d\tau \le \frac{1}{2} \int_0^t \tau^{-2n+1} \|f(\tau)\|_{s+\ell}^2 d\tau, \quad 0 < t \le \delta_1$$

for $f \in \mathcal{H}_{-n+1/2, s+\ell}(0, \delta)$. With $S = \sum_{k=0}^\infty R^k$ one has $Sf \in \mathcal{H}_{-n+1/2, s+\ell}(0, \delta_1)$ and

$$\int_0^t \tau^{-2n+1} \|Sf(\tau)\|_{s+\ell}^2 d\tau \le 2 \int_0^t \tau^{-2n+1} \|f(\tau)\|_{s+\ell}^2 d\tau.$$

Let $\gamma(x) \in C_0^\infty(\mathbb{R}^d)$ be equal to 1 near $\hat{x}$ such that $\operatorname{supp} \gamma \Subset \{\alpha = 1\}$. Since $\gamma(\alpha - R)S = \gamma(I - R)S = \gamma$ it follows that $\gamma(x) PGSf = \gamma(x) f$, that is

$$P(GSf) = f \quad \text{on } \{\gamma(x) = 1\}.$$

With $u = GSf$ one has

$$\sum_{j=0}^{m-1} \int_0^t \tau^{-2n-1} \|D_t^j u(\tau)\|_{s+m-j}^2 d\tau \le C \int_0^t \tau^{-2n+1} \|Sf(\tau)\|_{s+\ell}^2 d\tau$$

which proves the assertion. □

We define a solution operator with finite speed of propagation for P^* with obvious modifications.

Theorem 13.2. *Assume that for every* $(\hat{x}, \eta)$, $|\eta| = 1$, *one can find* P^*_η *of the form* (13.1) *such that* $P^* \equiv P^*_\eta$ *at* $(\hat{x}, \eta)$ *for which a solution operator with finite speed of propagation with loss of* $(n, \ell(\eta))$ *derivatives exists. Then there exist* $\delta > 0$, $\ell = \sup \ell(\eta)$ *and a neighborhood* U *of* $\hat{x}$ *such that for every* $f \in \mathcal{H}^*_{n+1/2, s+\ell}(0, \delta]$ *there exists* u *with* $D^j_t u \in \mathcal{H}^*_{n-1/2, s+m-j}(0, \delta]$, $0 \le j \le m-1$, *satisfying* $P^* u = f$ *in* $(0, \delta) \times U$.

13.2 Local existence and uniqueness. First consider a third order operator P of the form (2.1). To reduce P to the case $a_1(t, x, D) = 0$ we apply a Fourier integral operator, which is actually the solution operator $S(t', t)$ of the Cauchy problem

$$D_t u + a_1(t, x, D) u = 0, \quad u(t', x) = \phi(x)$$

such that $S(t', t)\phi = u(t)$. Then it is clear that $S(t, 0)(D_t + a_1)S(0, t) = D_t$. Now $\tilde{P} = S(t, 0) P S(0, t)$ has the form (2.1) with $a_1 = 0$ (see, e.g., [5], [33]). Assume that $\tilde{P}$ has a solution operator with finite speed of propagation $\tilde{G}$ with loss of (n, ℓ) derivatives. Then one can show that $G = S(0, t)\tilde{G}S(t, 0)$ is a solution operator for P with finite speed of propagation with loss of (n, ℓ) derivatives.

Let $|\eta| = 1$ be given. Assume that p has a triple characteristic root $\bar{\tau}$ at $(0, 0, \eta)$ and $(0, 0, \bar{\tau}, \eta)$ is effectively hyperbolic. Theorem 11.1 and Proposition 12.4 imply that $\hat{P}$, which coincides with the original P in W_M, given by (4.3), has a solution operator with finite speed of propagation with loss of $(n, n+2)$ derivatives.

Next assume that p has a double characteristic root $\bar{\tau}$ at $(0, 0, \eta)$ such that $(0, 0, \bar{\tau}, \eta)$ is effectively hyperbolic characteristic if it is a singular point. Note that one can write

$$p(t, x, \tau, \xi) = (\tau + b(t, x, \xi))(\tau^2 + a_1(t, x, \xi)\tau + a_2(t, x, \xi)) = p_1 p_2$$

in a conic neighborhood of $(0, 0, \eta)$ where $p_1(0, 0, \bar{\tau}, \eta) \neq 0$. There exist $\hat{P}_i$ such that $P \equiv \hat{P}_1 \cdot \hat{P}_2$ at $(0, \eta)$ where the principal symbol of $\hat{P}_i$ coincides with p_i in a conic neighborhood of $(0, 0, \eta)$. If $\hat{P}_i$ has a solution operator with finite speed of propagation G_i with loss of (n, ℓ_i) derivatives, then one can see that $G_2 G_1$ is a solution operator with finite speed of propagation for $\hat{P}_1 \cdot \hat{P}_2$ with loss of $(n, \ell_1 + \ell_2)$ derivatives. Consider the case that $(0, 0, \bar{\tau}, \eta)$ is a singular point. Then $F_p(0, 0, \bar{\tau}, \eta) = c F_{p_2}(0, 0, \bar{\tau}, \eta)$ with some $c \neq 0$ and $(0, 0, \bar{\tau}, \eta)$ is an effectively hyperbolic characteristic of p_2. Write p_2 as

$$p_2(t, x, \tau, \xi) = \tau^2 - a(t, x, \xi)|\xi|^2 \tag{13.3}$$

such that $\bar{\tau} = 0$ is a double characteristic and $(0, 0, 0, \eta)$ is a singular point. To apply earlier results [26, 27] on operators with double effectively hyperbolic characteristics, we need some modifications because $a(t,x,\xi)$ is assumed to be non-negative only in the $t \geq 0$ side in the present case. One can improve [27, Lemma 1.2.2] to

Proposition 13.1. *Assume that $a(t,x,\xi)$ is smooth in some conic neighborhood of $(0,0,\eta)$, homogeneous of degree 0 in ξ, and non-negative in $t \geq 0$, and $(0,0,0,\eta)$ is an effectively hyperbolic singular point of $p_2 = 0$. Then there exist a smooth function $\psi(x,\xi)$ in a conic neighborhood V of $(0,\eta)$ and constants $0 < \kappa < 1$, $c > 0$ such that*

$$\{\psi, a\}^2 \leq 4\kappa a, \quad a(t,x,\xi) \geq c \min\{t^2, (t-\psi(x,\xi))^2\} \tag{13.4}$$

for $(x,\xi) \in V$, $t \geq 0$ where $\psi(x,\xi)$ satisfies $|\partial_x^\alpha \partial_\xi^\beta \psi| \precsim \langle\xi\rangle^{-|\beta|}$.

Indeed that the same time function given in [25] under the assumption $a(t,x,\xi) \geq 0$ in a full neighborhood in t, denoted by $Y(t,x,\xi)$ there, satisfies (13.4) can be proved by applying [25, Theorem 1.1]. Then repeating similar arguments as in [26, 27] we conclude that there is a solution operator with finite speed of propagation for $\hat{P}_2$. Since $\hat{P}_1$ is a first order operator with real principal symbol p_1, it is easy to see that $\hat{P}_1$ has a solution operator with finite speed of propagation. Therefore P has a solution operator with finite speed of propagation. Consider now the case that $(0,0,0,\eta)$ is not a singular point. It is easy to see that $(0,0,0,\eta)$ is not a singular point implies $\partial_t a(0,0,\eta) > 0$, which is the case that $\hat{P}_2$ is a hyperbolic operator of principal type, and some detailed discussion is found in [7, Chapter 23.4]. It is easily proved that $\hat{P}_2$ has a solution operator with finite speed of propagation, because it suffices to employ the weight t^{-n} (ϕ^{-n} is now absent) in order to obtain weighted energy estimates.

Turn to the general case. Let $|\eta| = 1$ be arbitrarily fixed. Write

$$p(0,0,\tau,\eta) = \prod_{j=1}^{r} (\tau - \tau_j)^{m_j}$$

where $\sum m_j = m$ and τ_j are real and distinct from each other, with $m_j \leq 3$ which follows from the assumption. There exist $T > 0$ and a conic neighborhood U of $(0,\eta)$ such that one can write

$$p(t,x,\tau,\xi) = \prod_{j=1}^{r} p^{(j)}(t,x,\tau,\xi),$$
$$p^{(j)}(t,x,\tau,\xi) = \tau^{m_j} + a_{j,1}(t,x,\xi)\tau^{m_j-1} + \cdots + a_{j,m_j}(t,x,\xi),$$

for $(t,x,\xi) \in (-T,T) \times U$ where $a_{j,k}(t,x,\xi)$ are real valued, homogeneous of degree k in ξ and $p^{(j)}(0,0,\tau,\eta) = (\tau - \tau_j)^{m_j}$ and $p^{(j)}(t,x,\tau,\xi) = 0$ has only real roots in τ for $(t, x, \xi) \in [0, T) \times U$. If $(0, 0, \tau_j, \eta)$ is a singular point of p, and necessarily $m_j \geq 2$, then $(0, 0, \tau_j, \eta)$ is a singular point of $p^{(j)}$ and it is easy to see that $F_p(0, 0, \tau_j, \eta) = c_j F_{p^{(j)}}(0, 0, \tau_j, \eta)$ with some $c_j \neq 0$, and hence $F_{p^{(j)}}(0, 0, \tau_j, \eta)$ has non-zero real eigenvalues if $F_p(0, 0, \tau_j, \eta)$ does, and vice versa. It is well known that one can find $P^{(j)}$ such that

$$P \equiv P^{(1)}P^{(2)} \cdots P^{(r)} \quad \text{at } (0, \eta)$$

where $P^{(j)}$ are operators of the form (13.1) with $m = m_j$ whose principal symbol coincides with $p^{(j)}$ in some conic neighborhood of $(0, 0, \eta)$. Since each $P^{(j)}$ has a solution operator with finite speed of propagation with loss of $(n_j, n_j + 2)$ derivatives, thanks to Theorem 11.1 and Proposition 12.4, hence so does P with loss of $(n, r(n + 2))$ derivatives with $n = \max_j n_j$ noting Remark 12.1. Therefore Theorem 1.1 results from Theorem 13.1.

Repeating parallel arguments to the existence proof for P above, we obtain

Theorem 13.3. *Under the same assumption as in Theorem 1.1 there exist $\delta > 0$, a neighborhood U of the origin and $n > 0$, $\ell > 0$ such that for any $s \in \mathbb{R}$ and any $f \in \mathcal{H}^*_{n+1/2,s}(0, \delta]$ there exists u with $D_t^j u \in \mathcal{H}^*_{n-1/2,-\ell+s+m-j}(0, \delta]$, $0 \leq j \leq m - 1$, satisfying $P^*u = f$ in $(0, \delta) \times U$.*

Now we prove a local uniqueness result for the Cauchy problem for P applying Theorem 13.3. From the assumption one can find a neighborhood W of the origin of $\mathbb{R}^d$ and $T > 0$ such that every multiple characteristic of p on $(t, x, \xi) \in (0, T) \times W$ is at most double, and a double characteristic is effectively hyperbolic. Thanks to [17, Main Theorem] there exists $\hat{c} > 0$ such that for any solution v to $P^*v = f$ vanishing in $t \geq \delta'$ with $f \in C_0^\infty((0, \delta') \times \{ |x| < \varepsilon\})$ $(\delta' \leq T)$ one has

$$\mathrm{supp}_x v(t, \cdot) \subset \{ |x| \leq \varepsilon + \hat{c}\,\delta'\}, \quad 0 < t \leq \delta'.$$

Now assume that u satisfies $Pu = 0$ in $(0, \delta) \times U$ and $\partial_t^k u(0, x) = 0$ for all k. Choose $\varepsilon > 0$ and $\delta' > 0$ such that $\{ |x| \leq \varepsilon + \hat{c}\,\delta'\} \subset U$, $\delta' \leq \delta$. Then we see that

$$0 = \int_0^{\delta'} (Pu, v)dt = \int_0^{\delta'} (u, P^*v)dt = \int_0^{\delta'} (u, f)dt.$$

Since $f \in C_0^\infty((0, \delta') \times \{ |x| < \varepsilon\})$ is arbitrary, we conclude that

$$u(t, x) = 0, \quad (t, x) \in (0, \delta') \times \{ |x| \leq \varepsilon\}.$$

Theorem 13.4. *Assume (1.2) and that every singular point $(0, 0, \tau, \xi)$, $|(\tau, \xi)| \neq 0$ of $p = 0$ is effectively hyperbolic. If $u(t, x) \in C^\infty([0, \delta) \times U)$ satisfies $Pu = 0$ in $[0, \delta) \times U$ and $\partial_t^k u(0, x) = 0$ for all k, then $u = 0$ in a neighborhood of $(0, 0)$.*

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