UNIFORM ENVELOPING SEMIGROUPOIDS FOR GROUPOID ACTIONS

By

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Abstract. We establish new characterizations for (pseudo)isometric extensions of topological dynamical systems. For such extensions, we also extend results about relatively invariant measures and Fourier analysis that were previously only known in the minimal case to a significantly larger class, including all transitive systems. To bypass the reliance on minimality of the classical approaches to isometric extensions via the Ellis semigroup, we show that extensions of topological dynamical systems can be described as groupoid actions and then adapt the concept of enveloping semigroups to construct a uniform enveloping semigroupoid for groupoid actions. This approach allows to deal with the more complex orbit structures of nonminimal systems.

We study uniform enveloping semigroupoids of general groupoid actions and translate the results back to the special case of extensions of dynamical systems. In particular, we show that, under appropriate assumptions, a groupoid action is (pseudo)isometric if and only if the uniform enveloping semigroupoid is actually a compact groupoid. We also provide an operator theoretic characterization based on an abstract Peter–Weyl-type theorem for representations of compact, transitive groupoids on Banach bundles which is of independent interest.

Introduction

Given a topological dynamical system (K, φ) consisting of a compact space K and a homeomorphism $\varphi\colon K \to K$, its enveloping Ellis semigroup $\mathrm{E}(K, \varphi)$ introduced by R. Ellis in [Ell60] is the pointwise closure

$$\mathrm{E}(K, \varphi) := \overline{\{\varphi^n \mid n \in \mathbb{Z}\}} \subseteq K^K.$$

It is an important tool in topological dynamics capturing the long-term behavior of a dynamical system. Moreover, it allows to study the system (K, φ) via algebraic properties of $\mathrm{E}(K, \varphi)$. In particular, the elegant theory of compact, right-topological semigroups has been used to describe and study properties of topological dynamical systems. We refer to [Aus88, Chapters 3 and 6] and [Gla07] for the general theory of the Ellis semigroup and to [ABKL15], [Sta19], or [GGY18, Section 4] for some recent applications.

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In the special case of an equicontinuous system, $\mathrm{E}(K, \varphi)$ is actually a compact topological group which agrees with the uniform enveloping semigroup

$$\mathrm{E}_{\mathrm{u}}(K, \varphi) := \overline{\{\varphi^n \mid n \in \mathbb{Z}\}} \subseteq \mathrm{C}(K, K)$$

where the closure is taken with respect to the the compact-open topology on $\mathrm{C}(K,K)$, i.e., the topology of uniform convergence. In this case one can use the Peter–Weyl theorem to understand the representation of the compact group $\mathrm{E}(K, \varphi)$ on $\mathrm{C}(K)$. In particular, one can prove the following characterizations of equicontinuous systems (K, φ) involving the Koopman operator $T_\varphi\colon \mathrm{C}(K) \to \mathrm{C}(K), f \mapsto f \circ \varphi$.

Theorem. *For a topological dynamical system* (K, φ) *the following assertions are equiavalent:*

(a) (K, φ) *is equicontinuous.*
(b) $\mathrm{E}_{\mathrm{u}}(K, \varphi)$ *is a compact group.*
(c) *The Koopman operator* T_φ *has* **discrete spectrum**, *i.e., the union of its eigenspaces is total in* $\mathrm{C}(K)$.

The main goals of this article is to develop the techniques to prove an anlogous statement for "structured" extensions

$$q\colon (K, \varphi) \to (L, \psi)$$

of dynamical systems. The importance of these extensions is in particular due to the famous structure theorem for distal minimal flows proved by H. Furstenberg. It states that any distal minimal flow can be constructed via a **Furstenberg tower** consisting of equicontinuous (equivalently: pseudoisometric) extensions (see, e.g., [dV93, Section V.3]). Beyond this result, such extensions (especially the case of compact group extensions) have continued to play an important role in the structure theory of dynamical systems (see [HK18], [Zie07]), the construction of new dynamical systems (see, e.g., [Dol02] or [GHSY20, Section 6]), and applications to number theory (see [FKPLM16]).

First steps towards a characterization of pseudoisometric extensions were made by A. W. Knapp in [Kna67], though his results use the Ellis semigroup and only cover minimal distal systems, making essential use of minimality. We suggest to work around this built-in dependence on recurrence and propose a new, more general approach to structured extensions $q\colon (K, \varphi) \to (L, \psi)$. Instead of looking at an extension as a morphism between two group actions, we show that an extension can be equivalently regarded as a single system defined by a groupoid action. In analogy to enveloping semigroups, we introduce and study an enveloping semigroupoid $\mathcal{E}_{\mathrm{u}}(q)$ to describe the structuredness of q. This leads in particular to the following characterization of open pseudoisometric extensions. Recall here that in case of minimal systems every pseudoisometric extension is necessarily open (see, e.g., [Aus88, Theorem 7.3]), and therefore this is a natural framework to work with.

Theorem. *For an open extension* $q\colon (K,\varphi)\to(L,\psi)$ *of topological dynamical systems such that* $\dim \mathrm{fix}(T_\psi)=1$*, the following assertions are equivalent:*

(a) *q is pseudoisometric.*
(b) *The uniform enveloping semigroupoid* $\mathcal{E}_{\mathrm{u}}(q)$ *is a compact groupoid.*
(c) *The union of all finitely generated, projective, closed* T_φ*-invariant* $\mathrm{C}(L)$*-submodules of* $\mathrm{C}(K)$ *is dense in* $\mathrm{C}(K)$*.*

Note that previously merely the implication (a) $\Longrightarrow$ (c) was somewhat known and only in the minimal distal case (see [Kna67, Theorem 1.2]). The condition that T_ψ has no nonconstant fixed functions covers a significantly larger family than minimal systems, including all transitive systems but also many dynamical systems with more complex orbit structures.

Groupoids are generalizations of groups that allow to capture local symmetry. They play an important role in noncommutative geometry by providing a framework for studying operator algebras, index theory, and foliations (see [Con95] or [MS06]). In ergodic theory, G. W. Mackey used groupoids for his theory of virtual groups in order "to bring to light and exploit certain apparently far reaching analogies between group theory and ergodic theory" ([Mac66, p. 187 and Section 11]). And in topological dynamics, they have long been used as a bridge between dynamics and C^*-algebras in order to study questions around orbit equivalence; see [Tom87], [GPS95], and more recently [MM14].

It is the goal of this article to demonstrate that groupoids also provide a natural approach to extensions of topological dynamical systems and that the systematic analysis of the occurring groupoid structures allows to simplify and generalize existing results on (pseudo)isometric and equicontinuous extensions. In the process, we investigate the representation theory of compact, transitive groupoids and prove theorems in Section 4 that are of independent interest, including a general Peter–Weyl-type theorem in Theorem 4.8. Beyond the above-mentioned characterization, we also apply our abstract results to prove the existence of relatively invariant measures for certain pseudoisometric extensions (see Theorem 7.3) and to show that, much as in the case of equicontinuous systems, pseudoisometric extensions admit Fourier-analytic decompositions; see Theorem 7.4.

Organization of the article. Since all results on extensions of dynamical systems proved in this article depend only on groupoid structure, most results are formulated in the more general framework of groupoids and their actions. For the reader's convenience, however, the main results (the applications to extensions of topological dynamical systems) are gathered in Section 7.

In Section 1, we recall the concepts of (semi)groupoids and their actions and show in Examples 1.12 and 1.13 how an extension of dynamical systems can be described as a groupoid action. We then begin generalizing concepts for extensions to the context of general groupoid actions (see Definition 1.16). Section 2 is devoted to a generalization of the compact-open topology in Definition 2.8 which we will need to define the uniform enveloping semigroupoid. In particular, we prove a characterization of convergence for nets of mappings defined on distinct fibers of a bundle (see Proposition 2.4). This plays a key role throughout the article. After these preparations, Section 3 then introduces the uniform enveloping semigroupoid of a groupoid action as a generalization of the uniform enveloping semigroup for group actions; see Definition 3.3. We use the generalized Arzelà–Ascoli Theorem 3.13 to show in Theorem 3.27 that—under an assumption of topological ergodicity—a groupoid action is pseudoisometric if and only if its uniform enveloping semigroupoid is a compact groupoid.

To explain what this compactness means on the operator-theoretic level, Section 4 collects results about representations of compact groupoids on Banach bundles. We first prove a Peter–Weyl-type theorem for representations of compact transitive groupoids on Banach bundles in Theorem 4.8. This is then applied to the uniform enveloping (semi)groupoid of pseudoisometric groupoid actions to derive the desired operator-theoretic characterizations of structuredness, one of our main results Theorem 4.14.

In preparation of Section 7, Section 5 investigates the existence and uniqueness of relatively invariant measures for certain (pseudo)isometric groupoid actions; see Theorem 5.12. In Section 7, we then prove Fourier analytic results for transitive actions of compact groupoids, generalizing the Fourier analysis of compact groups and their actions (see Theorem 6.7). Via the uniform enveloping (semi)groupoids this can be used to obtain a better understanding of pseudoisometric groupoid actions. Finally, Section 7 restates all our main results in the case of extensions of topological dynamical systems.

Terminology and notation. All compact spaces are assumed to be Hausdorff though we may occasionally specify the Hausdorff property for emphasis. The neighborhood filter of a point $x \in X$ in a topological space X is denoted by $\mathcal{U}_X(x)$

or simply $\mathcal{U}(x)$ when there is no room for ambiguity. If X is a uniform space, we write $\mathcal{U}_X$ for the uniform structure of X.

At several points in the paper we consider **bundles**, i.e., continuous surjections $p\colon E \to L$ for some topological **total space** E (usually with some additional structure) to a topological (usually compact) **base space** L. For $l \in L$, we write $E_l := p^{-1}(l)$ for the **fiber over** l of such a bundle. Moreover, if $p_1\colon E_1 \to L$ and $p_2\colon E_2 \to L$ are two bundles over the same base space L, we define

$$E_1 \times_{p_1,p_2} E_2 := \{(x, y) \in E_1 \times E_2 \mid p_1(x) = p_2(y)\} \subseteq E_1 \times E_2$$

and equip this set with the subspace topology induced by the product topology on $E_1 \times E_2$. We also write $E_1 \times_L E_2 := E_1 \times_{p_1,p_2} E_2$ if the mappings p_1 and p_2 are clear.

We use the letters S and G for semigroups and groups and the letters $\mathcal{S}$ and $\mathcal{G}$ for semigroupoids and groupoids, respectively. By a **topological dynamical system** we mean a triple (K, G, φ) consisting of a non-empty compact space K, a Hausdorff topological group G, and a continuous action $\varphi\colon G \times K \to K$ of G on K. For $g \in G$, we denote the map $\varphi(g, \cdot)\colon K \to K$ by φ_g. We omit φ from the notation if there is no room for confusion and if $G = \mathbb{Z}$, we abbreviate (K, G, φ) by (K, φ) and identify φ with the map $\varphi(1, \cdot)\colon K \to K$ that completely determines the action.

As usual, a **morphism** $q\colon (K, G, \varphi) \to (L, G, \psi)$ between dynamical systems (K, G, φ) and (L, G, ψ) is a continuous mapping $q\colon K \to L$ such that the diagram

$$\begin{tikzcd} K \arrow[r, "\varphi_g"] \arrow[d, "q"'] & K \arrow[d, "q"] \\ L \arrow[r, "\psi_g"'] & L \end{tikzcd}$$

commutes for all $g \in G$. A surjective morphism $q\colon (K, G, \varphi) \to (L, G, \psi)$ is an **extension** (of topological dynamical systems).

Finally, if K is a compact space, we write $\mathrm{C}(K)$ for the Banach space of all continuous complex-valued functions on K. We identify its dual space $\mathrm{C}(K)$ with the space of all complex regular Borel measures on K and write $\mathrm{P}(K) \subseteq \mathrm{C}(K)'$ for the space of all probability measures in $\mathrm{C}(K)'$. The Dirac measure defined by a point $x \in K$ is denoted by δ_x. If $\vartheta\colon K \to L$ is a continuous mapping between compact spaces K and L, we write $\vartheta_*\mu$ for the pushforward of a measure $\mu \in \mathrm{C}(K)'$, i.e.,

$$\int_L f \,\mathrm{d}\vartheta_*\mu = \int_K f \circ \vartheta \,\mathrm{d}\mu \quad \text{for } f \in \mathrm{C}(L).$$

Moreover, we define the **Koopman operator** $T_\vartheta \in \mathscr{L}(\mathrm{C}(L), \mathrm{C}(K))$ of ϑ by $T_\vartheta f := f \circ \vartheta$ for $f \in \mathrm{C}(L)$. For a topological dynamical system (K, G, φ), the mapping

$$T_\varphi \colon G \to \mathscr{L}(\mathrm{C}(K)), \quad g \mapsto T_{\varphi_{g^{-1}}}$$

is the **Koopman representation** of (K, G, φ).

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1 Groupoids and groupoid actions

Following [MMMM13, Definitions 2.1, 2.2, and 2.17], we recall the definition of groupoids and semigroupoids which generalize the concepts of groups and semigroups, respectively, in the sense that the multiplication is only partially defined. The reader is also referred to [IR19] or [Hig71] as other general introductions to groupoids.

Definition 1.1. A **semigroupoid** consists of a set $\mathcal{S}$, a set $\mathcal{S}^{(2)} \subseteq \mathcal{S} \times \mathcal{S}$ of composable pairs, and a product map $\cdot\colon \mathcal{S}^{(2)} \to \mathcal{S}$ that is associative in the sense that

(i) if $(g_1, g_2), (g_2, g_3) \in \mathcal{S}^{(2)}$, then $(g_1 \cdot g_2, g_3), (g_1, g_2 \cdot g_3) \in \mathcal{S}^{(2)}$ and

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3).$$

We usually abbreviate $g \cdot h$ by gh if there is no room for confusion. We call a semigroupoid $\mathcal{G}$ a **groupoid** if there is an **inverse map** $^{-1}\colon \mathcal{G} \to \mathcal{G}$ such that, additionally, for each $g \in \mathcal{G}$

(ii) $(g^{-1}, g) \in \mathcal{G}^{(2)}$ and if $(g, h) \in \mathcal{G}^{(2)}$, then $g^{-1}(gh) = h$,

(iii) $(g, g^{-1}) \in \mathcal{G}^{(2)}$ and if $(h, g) \in \mathcal{G}^{(2)}$, then $(hg)g^{-1} = h$.

If $\mathcal{G}$ is a groupoid,

$$\mathcal{G}^{(0)} := \{g^{-1}g \mid g \in \mathcal{G}\}$$

is called the **unit space** of $\mathcal{G}$ and the maps

$$s\colon \mathcal{G} \to \mathcal{G}^{(0)}, \quad g \mapsto g^{-1}g,$$
$$r\colon \mathcal{G} \to \mathcal{G}^{(0)}, \quad g \mapsto gg^{-1}$$

are called the **source** and **range maps** of $\mathcal{G}$. For $u, v \in \mathcal{G}^{(0)}$, we write $\mathcal{G}_u := s^{-1}(u)$, $\mathcal{G}^v := r^{-1}(v)$, and $\mathcal{G}_u^v := \mathcal{G}_u \cap \mathcal{G}^v$. A groupoid is **transitive** if $\mathcal{G}_u^v \neq \emptyset$ for all $u, v \in \mathcal{G}^{(0)}$ and a **group bundle** if $\mathcal{G}_u^v = \emptyset$ for all $u, v \in \mathcal{G}^{(0)}$ with $u \neq v$. If $\mathcal{G}$ is a group bundle, we write $p := r = s$. **Subsemigroupoids** and **subgroupoids** of a given semigroupoid or groupoid are defined in a straightforward way.

A **topological semigroupoid** is a semigroupoid $(\mathcal{S}, \mathcal{S}^{(2)}, \cdot)$ with a Hausdorff topology on $\mathcal{S}$ such that the product map is continuous. We define **topological groupoids** analogously by demanding that the inverse map be continuous, too.

Remark 1.2. One can deduce several calculation rules from the groupoid axioms which we will tacitly use throughout the article. For example,

- $(g^{-1})^{-1} = g$ for every $g \in \mathcal{G}$, and
- $(h^{-1}, g^{-1}) \in \mathcal{G}^{(2)}$ with $h^{-1}g^{-1} = (gh)^{-1}$ whenever $(g, h) \in \mathcal{G}^{(2)}$.

We refer to [MMMM13, Section 2, in particular Lemma 2.19 and Proposition 2.21] for the statement and proof of these and other useful rules.

Below, we collect examples of semigroupoids and groupoids which play an important role throughout the article (see also [Ren80, Examples 1.2] for some of these and other examples).

Example 1.3. Let K be a compact space. Then K is a compact groupoid with

$$K^{(2)} = \{(x, x) \mid x \in K\}$$

and multiplication and inversion trivially defined. We call such a groupoid a **trivial groupoid**.

Example 1.4. Given a groupoid $\mathcal{G}$, the subgroupoid

$$\mathrm{Iso}(\mathcal{G}) := \{g \in \mathcal{G} \mid s(g) = r(g)\}$$

of $\mathcal{G}$ is a group bundle called the **isotropy bundle** of $\mathcal{G}$.

Example 1.5. Let K be a set. Then $\mathcal{G}_K := K \times K$ is a groupoid with the set of composable pairs

$$\mathcal{G}_K^{(2)} := \{((x, y), (y, z)) \mid x, y, z \in K\},$$

a product map defined by $(x, y) \cdot (y, z) := (x, z)$, and the inverse map $(x, y) \mapsto (y, x)$. The groupoid $\mathcal{G}_K$ is called the **pair groupoid** of K. It the property that the equivalence relations on K can be identified with full subgroupoids of $\mathcal{G}_K$. Here a subgroupoid is called **full** if it has the same unit space as its ambient groupoid. Note that a subgroupoid of $K \times K$ is transitive if and only if it equals $K \times K$.

Example 1.6. For a topological space X, consider the space

$$\mathrm{P}(X) := \mathrm{C}([0,1], X)$$

of paths in X, and define $\pi_1(X)$ to be the quotient of $\mathrm{P}(X)$ modulo homotopy with fixed end points. Then $\pi_1(X)$ is a groupoid with respect to the product map given by concatenation of paths, called the **fundamental groupoid** of X (cf., [Bro06, Chapter 6]). The source and range maps send an equivalence class $[\gamma]$ to the starting and end points $\gamma(0)$ and $\gamma(1)$, respectively. Moreover, the units in $\pi_1(X)$ are the equivalence classes of constant paths and so the unit space $\pi_1(X)^{(0)}$ may be identified with X. The isotropy groups $\pi_1(X)_x^x$ for $x \in X$ are precisely the usual fundamental groups $\pi_1(X, x)$.

A fundamental groupoid $\pi_1(X)$ is transitive if and only if X is path-connected and such fundamental groupoids are archetypal examples for transitive groupoids. If $\pi_1(X)$ is transitive, all isotropy groups $\pi_1(X)_x^x$, $\pi_1(X)_y^y$ are isomorphic via conjugation by a path η from x to y,

$$c_\eta\colon \pi_1(X)_y^y \to \pi_1(X)_x^x, \quad [\gamma] \mapsto [\eta]^{-1}[\gamma][\eta].$$

In the same way one sees that, in general, all isotropy groups of a transitive groupoid are isomorphic. This explains the heuristic that transitive groupoids behave similarly to groups, which we will make repeated use of. However, this does not mean that the study of a transitive groupoid can always be replaced with the study of a single isotropy group, as it is frequently done for the fundamental groupoid. Isotropy groups contain only part of the picture and, as we will see, the groupoid perspective emerges as the natural conceptual generalization of existing approaches for groups.

The following standard construction allows to completely encode the dynamics of a group action within a groupoid and motivates part of the terminology around groupoids.

Example 1.7. Let (K, G) be a topological dynamical system. Then the **action groupoid** or **transformation groupoid** $G \ltimes K$ is the set $G \times K$ with

$$(G \ltimes K)^{(2)} := \{((g_2, g_1x), (g_1, x)) \mid g_1, g_2 \in G, x \in K\}$$

and

$$\begin{aligned} \cdot : (G \ltimes K)^{(2)} \to G \ltimes K, &\quad ((g_2, g_1x), (g_1, x)) \mapsto (g_2g_1, x), \\ {}^{-1} : G \ltimes K \to G \ltimes K, &\quad (g, x) \mapsto (g^{-1}, gx). \end{aligned}$$

We identify its unit space

$$(G \ltimes K)^{(0)} = \{(1, x) \mid x \in K\}$$

with K.

The "structure-preserving maps" between semigroupoids are the following.

Definition 1.8. A **homomorphism** $\Phi\colon \mathcal{S} \to \mathcal{T}$ of semigroupoids (or groupoids) is a mapping $\Phi\colon \mathcal{S} \to \mathcal{T}$ satisfying

$$(\Phi(g_1), \Phi(g_2)) \in \mathcal{T}^{(2)} \quad \text{and} \quad \Phi(g_1 g_2) = \Phi(g_1)\Phi(g_2)$$

for all $(g_1, g_2) \in \mathcal{S}^{(2)}$. We write $\Phi^{(0)}$ for the induced map $\mathcal{S}^{(0)} \to \mathcal{T}^{(0)}$. Moreover, we call Φ a **factor map** and $\mathcal{T}$ a **factor** of $\mathcal{S}$ if Φ is surjective. Homomorphisms and factors of topological semigroupoids are defined by additionally requiring Φ to be continuous.

Example 1.9. Let $\mathcal{G}$ be a groupoid.

(a) The inclusions of the unit space $\mathcal{G}^{(0)}$ and the isotropy bundle $\mathrm{Iso}(\mathcal{G})$ are groupoid homomorphisms.

(b) The map

$$(r, s)\colon \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}, \quad g \mapsto (r(g), s(g))$$

is a groupoid morphism between $\mathcal{G}$ and the pair groupoid $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. In particular, its image $R_{\mathcal{G}}$ is an equivalence relation, called the **orbit relation** on $\mathcal{G}^{(0)}$.

We now consider groupoid actions (or $\mathcal{G}$-spaces, see, e.g., [ADR00, Section 2.1]). For their definition we remind the reader of the notation for fibers and fiber products of surjective maps fixed in the introduction. Moreover, recall that given a groupoid $\mathcal{G}$ we write $\mathcal{G}_u$ and $\mathcal{G}^u$ for the elements with source and range $u \in \mathcal{G}^{(0)}$, respectively, and set $\mathcal{G}_u^v := \mathcal{G}_u \cap \mathcal{G}^v$ for $u, v \in \mathcal{G}^{(0)}$ (see Definition 1.1).

Definition 1.10. Let $\mathcal{G}$ be a topological groupoid. A **groupoid action** of $\mathcal{G}$ on a compact space K is a tuple $(K, q, \mathcal{G}, \varphi)$ with a continuous, open surjection $q\colon K \to \mathcal{G}^{(0)}$ and a continuous mapping

$$\varphi\colon \mathcal{G} \times_{s,q} K \to K, \quad (g, x) \mapsto \varphi_g(x) =: gx$$

such that

(i) $q(gx) = r(g)$ for all $(g, x) \in \mathcal{G} \times_{s,q} K$,

(ii) $(g_1 g_2)x = g_1(g_2 x)$ for all $(g_1, g_2) \in \mathcal{G}^{(2)}$ and $x \in K_{s(g_2)}$,

(iii) $ux = x$ for all $u \in \mathcal{G}^{(0)}$ and $x \in K_u$.

If $x \in K$, the **orbit** of x under $\mathcal{G}$ is defined as $\mathcal{G}x := \{gx \mid g \in \mathcal{G}_{q(x)}\}$. A groupoid action is called

- **transitive** if $\mathcal{G}x = K$ for some/every $x \in K$,
- **fiberwise transitive** if the fiber groups $\mathcal{G}_u^u$ act transitively on K_u for every $u \in \mathcal{G}^{(0)}$.

A **morphism** $(p, \Phi)\colon (K_1, q_1, \mathcal{G}_1) \to (K_2, q_2, \mathcal{G}_2)$ of groupoid actions consists of a groupoid morphism $\Phi\colon \mathcal{G}_1 \to \mathcal{G}_2$ and an open continuous map $p\colon K_1 \to K_2$ such that

(i) $q_2 \circ p = \Phi^{(0)} \circ q_1$,
(ii) $p(gx) = \Phi(g)p(x)$ for all $(g, x) \in \mathcal{G} \times_{s,q_1} K_1$.

If $\mathcal{G}_1 = \mathcal{G}_2$ and Φ is the identity, we abbreviate (p, Φ) as p. A morphism (p, Φ) is called a **factor map** or an **extension** if p and Φ are surjective. In this case, $(K_1, q_1, \mathcal{G}_1)$ is called an **extension** of $(K_2, q_2, \mathcal{G}_2)$ and $(K_2, q_2, \mathcal{G}_2)$ a **factor** of $(K_1, q_1, \mathcal{G}_1)$.

Remark 1.11. As in the case of group actions, we usually omit φ and just write $(K, q, \mathcal{G})$ for a groupoid action $(K, q, \mathcal{G}, \varphi)$. We emphasize that for a groupoid action $(K, q, \mathcal{G})$ we always require that K (and consequently also the unit space $\mathcal{G}^{(0)}$) is compact and q to be open. The latter assumption is quite common (see [ADR00, Section 2.1]), but, as explained in the introduction, is also natural for the purpose of examining pseudoisometric extensions. Likewise, we demand that morphisms between groupoid actions are open. Finally, note also that a groupoid action $(K, q, \mathcal{G})$ is transitive if and only if $(K, q, \mathcal{G})$ is fiberwise transitive and $\mathcal{G}$ is transitive.

Example 1.12. Let (K, G, φ) be a topological dynamical system. Then (K, q, G, φ) is a groupoid action where $q\colon K \to \{1\} = G^{(0)}$, $x \mapsto 1$.

Next, we consider one of this article's key examples which motivates our systematic study of groupoid actions.

Example 1.13. Let $q\colon (K, G, \varphi) \to (L, G, \psi)$ be an open extension of topological dynamical systems. Then the action groupoid $G \ltimes L$ defines a groupoid action $(K, q, G \ltimes L, \eta_\varphi)$ via

$$\eta_\varphi\colon (G \ltimes L) \times_{s,q} K \to K, \quad ((g, l), x) \mapsto \varphi_g(x).$$

Conversely, let (L, G, ψ) be a topological dynamical system and $(K, q, G \ltimes L, \eta)$ be an action of $G \ltimes L$ on K. Then it is not hard to check that

$$\varphi_\eta\colon G \times K \to K, \quad (g, x) \mapsto \eta_{(g,q(x))}(x)$$

defines a continuous action (K, G, φ_η) of G on K such that

$$q\colon (K, G, \varphi_\eta) \to (L, G, \psi)$$

is an extension of topological dynamical systems. Since these constructions are mutually inverse, an extension $q\colon (K, G, \varphi) \to (L, G, \psi)$ can be equivalently regarded as a groupoid action $(K, q, G \ltimes L, \eta)$ of the action groupoid $G \ltimes L$ on K. In what follows, the reader should always have this example in mind when thinking about groupoid actions.

Remark 1.14. There is also another way to obtain a groupoid action $(K, q, \mathcal{G})$ from an extension $q\colon (K, G, \varphi) \to (L, G, \psi)$ of dynamical systems: Let

$$\mathcal{S}(q) := \{\varphi_g|_{K_l}\colon K_l \to K_{gl} \mid g \in G, l \in L\}.$$

Then one can also study the groupoid action $(K, q, \mathcal{S}(q))$. Note, however, that there is a loss of information because the action of G on K can no longer be reconstructed from $\mathcal{S}(q)$ as opposed to the action of $G \ltimes L$ which, in addition, keeps track of which element of G acts. This difference is often immaterial, so there is no harm in thinking directly about the transition groupoid $\mathcal{S}(q)$ instead of the action groupoid $G \ltimes L$.

Example 1.15. Every topological groupoid (with compact unit space) acts canonically on its unit space via conjugation. Indeed, if $\mathcal{G}$ is a groupoid with compact unit space, then $(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G}, \varphi)$ with

$$\varphi\colon \mathcal{G} \times_{s,\mathrm{id}_{\mathcal{G}^{(0)}}} \mathcal{G}^{(0)} \to \mathcal{G}^{(0)}, \quad (g, u) \to gug^{-1}$$

is a groupoid action mapping $s(g)$ to $r(g)$ for every $g \in \mathcal{G}$. It is the smallest action of $\mathcal{G}$ in the sense that if $(K, q, \mathcal{G})$ is any groupoid action, then $q\colon K \to \mathcal{G}^{(0)}$ defines an extension $q\colon (\mathcal{G}, q, K) \to (\mathcal{G}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G}^{(0)})$ of groupoid actions. Moreover, $(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ is always fiberwise transitive, and it is transitive if and only if $\mathcal{G}$ is a transitive groupoid.

As noted in Example 1.13, an extension of topological dynamical systems can be equivalently regarded as a certain groupoid action, suggesting that properties of extensions may be rephrased in terms of groupoid actions. The following definition carries out this straightforward translation to general groupoid actions for the standard notions of structuredness of extensions (cf. [dV93, Sections V.2 and V.5] and [Aus13]).

Definition 1.16. A groupoid action $(K, q, \mathcal{G})$ is called

(i) **weakly equicontinuous** or **stable** if for each $u \in \mathcal{G}^{(0)}$ and each entourage $U \in \mathcal{U}_K$ there is an entourage $V \in \mathcal{U}_K$ such that $(gx, gy) \in U$ for all $g \in \mathcal{G}_u$ and $x, y \in K_u$ with $(x, y) \in V$;

(ii) **equicontinuous** if for each entourage $U \in \mathcal{U}_K$ there is an entourage $V \in \mathcal{U}_K$ such that for each $u \in \mathcal{G}^{(0)}$ one has $(gx, gy) \in U$ for all $g \in \mathcal{G}_u$ and $x, y \in K_u$ with $(x, y) \in V$;

(iii) **pseudoisometric** if there is a set P of continuous mappings

$$p\colon K \times_q K \to [0, \infty)$$

such that

- $p_u = p|_{K_u \times K_u}$ is a pseudometric on K_u for every $u \in \mathcal{G}^{(0)}$,
- the pseudometrics p_u for $p \in P$ generate the topology of K_u for every $u \in \mathcal{G}^{(0)}$,
- $p(gx, gy) = p(x, y)$ for all $x, y \in K_{s(g)}$ and $g \in \mathcal{G}$;

(iv) **isometric** if it is pseudoisometric and the set P can be chosen to consist of a single map which is (necessarily) a metric on each fiber.

Remark 1.17.

(a) It is immediate from Definition 1.16 that if $q\colon (K, G) \to (L, G)$ is an extension of topological dynamical systems, the extension is weakly equicontinuous, equicontinuous, ... if and only if the corresponding groupoid action $(K, q, G \ltimes L)$ is.

(b) We show in Porposition 1.18 below that a pseudoisometric groupoid action is equicontinuous. Hence, (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). In general, none of the converse implications hold: For (iii) and (iv) this is obvious, for (ii) and (iii) see Example 3.15 below, and for the relation between and (i) and (ii) we refer to [Aus13].

(c) Recall that if (K, G) and (L, G) are **minimal** group actions, q is equicontinuous if and only if it is pseudoisometric if and only if it is weakly equicontinuous and open, see [dV93, Corollary 5.10] and [Bro79, Theorem 3.13.17].

Proposition 1.18. *Let $(K, q, \mathcal{G})$ be a pseudoisometric groupoid action. Then $(K, q, \mathcal{G})$ is equicontinuous.*

Proof. Pick a set P as in Definition 1.16 iii. For each finite subset $F \subseteq P$ and $\varepsilon > 0$, set

$$U_{F,\varepsilon} := \{(x, y) \in K \times_q K \mid \forall p \in F\colon p(x, y) < \varepsilon\}$$

and note that

$$\bigcap_{\substack{F \subseteq P \text{ finite} \\ \varepsilon > 0}} U_{F,\varepsilon} = \Delta_K.$$

We claim that for every $U \in \mathcal{U}_K$, there are a finite set $F \subseteq P$ and an $\varepsilon > 0$ such that $U_{F,\varepsilon} \subseteq U$ which would yield the claim since $U_{F,\varepsilon}$ is $\mathcal{G}$-invariant. In order to prove the claim, first recall that $\mathcal{U}_K = \mathcal{U}_{K\times K}(\Delta_K)$ is just the neighborhood filter of the diagonal. The claim then follows from the fact that if $(M_\alpha)_{\alpha\in A}$ is a decreasing family of sets in a compact space X and U is an open neighborhood of

$$\bigcap_{\alpha\in A} \overline{M_\alpha},$$

then there is an $\alpha_0 \in A$ such that $M_{\alpha_0} \subseteq U$ (use the finite intersection property). □

In analogy to group actions, we can associate an action groupoid to a groupoid action (cf. [ADR00, Section 2.1]). We note this construction for later reference since it will allow to investigate the orbit structure of groupoid actions.

Definition 1.19. For $(K, q, \mathcal{G})$ a groupoid action we define the **action groupoid** $\mathcal{G} \ltimes K$ **of** $(K, q, \mathcal{G})$ as the fiber product $\mathcal{G} \times_{s,q} K$ with

$$(\mathcal{G} \ltimes K)^{(2)} := \{((h, gx), (g, x)) \mid x \in K, g \in \mathcal{G}_{q(x)}, h \in \mathcal{G}_{r(g)}\}$$

and the operations

$$\begin{aligned} \cdot\colon (\mathcal{G} \ltimes K)^{(2)} \to \mathcal{G} \ltimes K, &\quad ((h, gx), (g, x)) \mapsto (hg, x),\\ ^{-1}\colon\ \mathcal{G} \ltimes K \to \mathcal{G} \ltimes K, &\quad (g, x) \mapsto (g^{-1}, gx). \end{aligned}$$

We identify its unit space

$$(\mathcal{G} \ltimes K)^{(0)} = \{(q(x), x) \mid x \in K\}$$

with K.

2 The compact-open topology for fiber maps

In order to define a uniform enveloping semigroupoid for groupoid actions, it is necessary to find an appropriate topological space in which to carry out the compactification. For the uniform enveloping semigroup of a topological dynamical system (K, G), this is the space $\mathrm{C}(K, K)$ endowed with the compact-open topology, as explained in the introduction. To generalize this, we extend the compact-open topology to “fibered mappings” in this section and then introduce the uniform enveloping semigroup in Section 3.

Definition 2.1. For topological spaces X and Y as well as continuous surjections $p\colon X \to L$ and $q\colon Y \to L'$ onto compact spaces L and L' we set

$$\mathrm{C}_p^q(X,Y)_l^{l'} := \mathrm{C}(X_l, Y_{l'}) \quad \text{for } (l,l') \in L \times L'$$

and define the set of **continuous fiber maps** between $p\colon X \to L$ and $q\colon Y \to L'$ as

$$\mathrm{C}_p^q(X,Y) := \bigcup_{l\in L, l'\in L'} \mathrm{C}_p^q(K,X)_l^{l'}.$$

We define "source" and "range" maps

$$s\colon \mathrm{C}_p^q(X,Y) \to L, \quad r\colon \mathrm{C}_p^q(X,Y) \to L'$$

by setting

$$s(\vartheta) := l \quad \text{and} \quad r(\vartheta) := l' \quad \text{for } \vartheta \in \mathrm{C}_p^q(K,X)_l^{l'}.$$

If Y is a topological space and $q\colon Y \to \mathrm{pt}$ is the unique map onto a one-point space pt, we abbreviate $\mathrm{C}_p(X,Y) := \mathrm{C}_p^q(X,Y)$. Moreover, we write

$$\mathrm{C}_p(X) := \mathrm{C}_p(X, \mathbb{C}).$$

Remark 2.2. If $L = L'$ and $p = q$ in the definition above, the set $\mathrm{C}_q^q(K,K)$ with

$$\mathrm{C}_q^q(K,K)^{(2)} := \{(\vartheta, \eta) \in \mathrm{C}_q^q(K,K) \times \mathrm{C}_q^q(K,K) \mid r(\vartheta) = s(\vartheta)\}$$

is a semigroupoid with composition of mappings as the product map. We call this the **semigroupoid of continuous fiber maps** of q.

The following generalization of the compact-open topology for spaces of fiber maps is taken from [BB78, Section 1] where it is considered on the larger set of all partial maps.

Definition 2.3. Let $p\colon X \to L$ and $q\colon Y \to L'$ be continuous surjections of topological spaces X and Y onto compact spaces L and L'. For a compact subset $C \subseteq X$ and an open subset $O \subseteq Y$, set

$$\mathrm{W}(C,O) := \{\vartheta \in \mathrm{C}_p^q(X,Y) \mid \vartheta(C) \subseteq O\} \subseteq \mathrm{C}_p^q(X,Y)$$

where $\vartheta(C) = \{\vartheta(x) \mid x \in C \text{ with } p(x) = s(\vartheta)\}$. We then define the **compact-open topology** on $\mathrm{C}_p^q(X,Y)$ to be the topology generated by all the sets of the form $\mathrm{W}(C,O)$.

The classical characterization of convergence with respect to the compact-open topology for locally compact spaces readily extends to this more general context.

Proposition 2.4. *Let $p\colon X \to L$, $q\colon Y \to L'$ be continuous surjections of topological spaces X and Y onto compact spaces L and L'. Suppose that X is locally compact. Then for a net $(\vartheta_\alpha)_{\alpha\in A}$ in $\mathrm{C}_p^q(X,Y)$ and a $\vartheta \in \mathrm{C}_p^q(X,Y)$ the following assertions are equivalent:*

(a) $\lim_\alpha \vartheta_\alpha = \vartheta$ *with respect to the compact-open topology.*

(b) *The following two conditions are satisfied.*

- $\lim_\alpha s(\vartheta_\alpha) = s(\vartheta)$.
- *If $(\vartheta_\beta)_{\beta\in B}$ is a subnet of $(\vartheta_\alpha)_{\alpha\in A}$, then*

$$\lim_\beta \vartheta_\beta(x_\beta) = \vartheta(x)$$

for every net $(x_\beta)_{\beta\in B}$ in X that converges to some $x \in X$ and satisfies $q(x_\beta) = s(\vartheta_\beta)$ for every $\beta \in B$.

In particular, the compact-open topology is the coarsest topology on $\mathrm{C}_p^q(X,Y)$ such that the maps

$$\begin{aligned} s\colon \mathrm{C}_p^q(X,Y) &\to L, & \vartheta &\mapsto s(\vartheta) \\ \mathrm{ev}\colon \mathrm{C}_p^q(X,Y) \times_{s,p} X &\to Y, & (\vartheta, x) &\mapsto \vartheta(x) \end{aligned}$$

are continuous.

Proof. First, suppose that $(\vartheta_\alpha)_{\alpha\in A}$ and ϑ satisfy (b) and suppose that $(\vartheta_\alpha)_{\alpha\in A}$ does not converge to ϑ. Then there are a compact set $C \subseteq X$ and an open set $O \subseteq Y$ such that $\vartheta \in \mathrm{W}(C,O)$ and such that $(\vartheta_\alpha)_{\alpha\in A}$ does not eventually lie in $\mathrm{W}(C,O)$. This means that there is a subnet $(\vartheta_\beta)_{\beta\in B}$ of $(\vartheta_\alpha)_{\alpha\in A}$ satisfying $\vartheta_\beta \notin \mathrm{W}(C,O)$ for each $\beta \in B$. After again passing to a subnet, we may thus assume that there is a convergent net $(x_\beta)_{\beta\in B}$ in C such that $x_\beta \in X_{s(\vartheta_\beta)}$ and $\vartheta_\beta(x_\beta) \notin O$ for each $\beta \in B$. However, by (b), $\lim_\beta \vartheta_\beta(x_\beta) = \vartheta(x) \in O$, a contradiction. Hence, (b) $\implies$ (a).

Now, suppose that $(\vartheta_\alpha)_{\alpha\in A}$ is a net in $\mathrm{C}_p^q(X,Y)$ converging to $\vartheta \in \mathrm{C}_p^q(X,Y)$ in the compact-open topology. To see that $\lim_\alpha s(\vartheta_\alpha) = s(\vartheta)$, suppose that $s(\vartheta_\alpha) \not\to s(\vartheta)$. We construct a compact set $C \subseteq X$ such that $\vartheta \in \mathrm{W}(C,\emptyset)$ and a subnet of $(\vartheta_\alpha)_{\alpha\in A}$ which avoids $\mathrm{W}(C,\emptyset)$. To do this, first use the compactness of L to find a subnet $(\vartheta_\beta)_{\beta\in B}$ of $(\vartheta_\alpha)_{\alpha\in A}$ such that $(s(\vartheta_\beta))_{\beta\in B}$ converges to another point than $s(\vartheta)$. Moreover, by again passing to a subnet, one may assume that there is a net $(x_\beta)_{\beta\in B}$ in X converging to some $x \in X$ such that $x_\beta \in X_{s(\vartheta_\beta)}$ for each $\beta \in B$. Since by assumption, $p(x_\beta) = s(\vartheta_\beta)$ converges to a point different from $s(\vartheta)$, $p(x) \neq s(\vartheta)$. Therefore, there is a compact neighborhood $C \in \mathfrak{U}(x)$ with $C \cap X_{s(\vartheta)} = \emptyset$. Now $\vartheta \in \mathrm{W}(C,\emptyset)$ whereas eventually $\vartheta_\beta \notin \mathrm{w}(C,\emptyset)$ since $x_\beta \to x$, a contradiction. Therefore, $\lim_\alpha s(\vartheta_\alpha) = s(\vartheta)$.

To establish the second part of (b), let $(\vartheta_\beta)_{\beta\in B}$ and $(x_\beta)_{\beta\in B}$ be as in (b). Let $O \in \mathfrak{U}(\vartheta(x))$ be a neighborhood of $\vartheta(x)$. Then $V := \vartheta^{-1}(O)$ is a neighborhood of x with respect to the subspace topology on $K_{s(\vartheta)}$. Therefore, there is a compact neighborhood $C \in \mathfrak{U}(x)$ such that $C \cap K_{s(\vartheta)} \subseteq V$. Because $x_\beta \to x$, $(x_\beta)_{\beta\in B}$ eventually lies in C and since $\vartheta_\beta \to \vartheta$ in the compact-open topology, we conclude that $(\vartheta_\beta)_{\beta\in B}$ eventually lies in $\mathrm{W}(C, O)$. Since O was an arbitrary neighborhood of $\vartheta(x)$, this shows that $\lim_\beta \vartheta_\beta(x_\beta) = \vartheta(x)$. □

Remark 2.5. In general, the compact-open topology on $\mathrm{C}_p^q(X, Y)$ is not Hausdorff. In fact, it is not difficult to infer from the characterization in Proposition 2.4 that it is Hausdorff if and only if p is open, as it will henceforth always be the case. To show that the compact-open limit ϑ of a net $(\vartheta_\alpha)_{\alpha\in A}$ is unique, it suffices to show for every $x \in X_{s(\vartheta)}$ that $\vartheta(x)$ is uniquely determined. To see that openness implies this, one can use the observation that a continuous surjection $p\colon X \to L$ between a locally compact space X and a compact space L is open if and only if the following condition is fullfilled: For every convergent net $(l_\alpha)_{\alpha\in A}$ in L with limit $l \in L$ and every $x \in X_l$, there are a subnet $(l_\beta)_{\beta\in B}$ of $(l_\alpha)_{\alpha\in A}$ and a net $(x_\beta)_{\beta\in B}$ in X that converges to x and covers $(l_\beta)_{\beta\in B}$ in the sense that $p(x_\beta) = l_\beta$ for every $\beta \in B$. We will make use of this observation at several more occasions.

In order to prove a generalization of the Arzelà–Ascoli theorem below in Theorem 3.13, we will need an equivalent description of the compact-open topology. To find another natural way to topologize $\mathrm{C}_p^q(X, Y)$, observe that an element $\vartheta \in \mathrm{C}_p^q(X, Y)$ may be identified with its graph $\mathrm{Gr}(\vartheta) \subseteq X \times Y$. Therefore, $\mathrm{C}_p^q(X, Y)$ may be regarded as a subspace of the space $\mathscr{C}(X \times Y)$ of closed subsets of $X \times Y$, on which there exist many topologies, e.g., the Vietoris topology.

Definition 2.6. Let X be a topological space and $\mathscr{C}(X)$ the set of its non-empty closed subsets. The **Vietoris topology** on $\mathscr{C}(X)$ is the topology generated by the sets

$$U^- := \{A \in \mathscr{C}(X) \mid A \cap U \neq \emptyset\},$$
$$U^+ := \{A \in \mathscr{C}(X) \mid A \subseteq U\}$$

for open subsets $U \subseteq X$.

Remark 2.7. It is known that if X is a Hausdorff space, then so is $\mathscr{C}(X)$; see [Mic51, Theorem 4.9]. If X is compact, then $\mathscr{C}(X)$ is also compact; see [Mic51, Theorem 4.9] or [EE14, Proposition 5.A.3]. If, additionally, X is a metric space, the Vietoris topology coincides with the topology induced by the Hausdorff metric; see [Mic51, Theorem 3.4 and Proposition 3.6] or [EE14, Exercise 5.4].

Definition 2.8. If $p\colon X \to L$ and $q\colon Y \to L'$ are continuous surjections of topological spaces X and Y onto compact spaces L and L', we define the **Vietoris topology** on $\mathrm{C}_p^q(X, Y)$ to be the initial topology with respect to the map

$$\mathrm{Gr}\colon \mathrm{C}_p^q(X, Y) \to \mathscr{C}(X \times Y), \quad \vartheta \mapsto \mathrm{Gr}(\vartheta)$$

where $\mathscr{C}(X \times Y)$ is equipped with the Vietoris topology.

Remark 2.9. In [HZ10], a slightly different version of the compact-open topology on $\mathrm{C}_p^q(X, Y)$ is considered which, additionally, uses non-empty open sets $U \subseteq X$ and adds all the sets of the form

$$[U] := \{f \in \mathrm{C}_p^q(X, Y) \mid f^{-1}(Y) \cap U \neq \emptyset\}$$

to generate the topology. If X is compact and p is open, however, $p^{-1}(p(U))^c \subseteq X$ is compact and $[U] = \mathrm{W}(p^{-1}(p(U))^c, \emptyset)$. Therefore, if X is compact, our compact-open topology coincides with the one considered in [HZ10]. In particular, we can note the following theorem which is formulated more generally in [HZ10, Proposition 2.2] for later use.

Theorem 2.10. *Let $p\colon X \to L$ and $q\colon Y \to L'$ be continuous surjections of topological spaces X and Y onto compact spaces L and L'. If X and Y are compact and p is open, then the Vietoris topology and the compact-open topology on $\mathrm{C}_p^q(X, Y)$ coincide.*

3 Uniform enveloping semigroupoids

We are now ready to introduce the uniform enveloping semigroupoid of a groupoid action. The main result of this section is Theorem 3.27 which states that, under appropriate assumptions, $\mathcal{E}_\mathrm{u}(K, q, \mathcal{G})$ is compact if and only if the groupoid action $(K, q, \mathcal{G})$ is pseudoisometric.

To define uniform enveloping semigroupoids, we regard the semigroupoid $\mathrm{C}_q^q(K, K)$ of fiber maps introduced in Remark 2.2 as a topological semigroupoid with respect to the compact-open topology. This allows to define the uniform enveloping semigroupoid of a set of fiber maps.

Definition 3.1. Let $q\colon K \to L$ be an open, continuous surjection of compact spaces and let $\mathcal{F}$ be a subset of the topological semigroupoid $\mathrm{C}_q^q(K, K)$. Then the **uniform enveloping semigroupoid** $\mathcal{E}_\mathrm{u}(\mathcal{F})$ of $\mathcal{F}$ is defined to be the smallest closed subsemigroupoid of $\mathrm{C}_q^q(K, K)$ containing $\mathcal{F}$.

Remark 3.2. Note that this definition makes sense since the intersection of a family of closed subsemigroupoids of a topological semigroupoid is again a closed subsemigroupoid.

Definition 3.3. Let $(K, q, \mathcal{G}, \varphi)$ be a groupoid action and consider the **transition semigroupoid** $\mathcal{S}(K, q, \mathcal{G}, \varphi)$ given by

$$\mathcal{S}(K, q, \mathcal{G}, \varphi) := \{\varphi_g \colon K_{s(g)} \to K_{r(g)} \mid g \in \mathcal{G}\} \subseteq \mathrm{C}_q^q(K, K).$$

We call

$$\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}, \varphi) := \mathcal{E}_{\mathrm{u}}(\mathcal{S}(K, q, \mathcal{G}, \varphi))$$

the **uniform enveloping semigroupoid of** $(K, q, \mathcal{G}, \varphi)$.

Example 3.4. Let (K, G, φ) be a topological dynamical system and interpret it as a groupoid action (K, q, G, φ) of G (see Example 1.12). Then

$$\mathcal{S}(K, q, G, \varphi) = \{\varphi_g \mid g \in G\} \subseteq \mathrm{C}(K, K)$$

is the transition group of (K, G) and the enveloping semigroupoid

$$\mathcal{E}_{\mathrm{u}}(K, q, G, \varphi) = \mathcal{E}_{\mathrm{u}}(\mathcal{S}(K, q, G, \varphi)) \subseteq \mathrm{C}(K, K)$$

is precisely the uniform enveloping semigroup $\mathrm{E}_{\mathrm{u}}(K, G)$. Therefore, $\mathcal{E}_{\mathrm{u}}$ generalizes the uniform enveloping semigroup to arbitrary groupoid actions.

Example 3.5. Let $q \colon (K, G, \varphi) \to (L, G, \psi)$ be an open extension of topological dynamical systems. As noted in Example 1.13, we can equivalently regard the extension as an action $(K, q, G \ltimes L, \eta_\varphi)$ of the action groupoid $G \ltimes L$. For this groupoid action, the transition groupoid is

$$\mathcal{S}(K, q, G \ltimes L, \eta_\varphi) = \{\varphi_g|_{K_l} \mid g \in G, l \in L\} \subseteq \mathrm{C}_q^q(K, K)$$

and the uniform enveloping semigroupoid is

$$\mathcal{E}_{\mathrm{u}}(K, q, G \ltimes L, \eta_\varphi) = \mathcal{E}_{\mathrm{u}}(\{\varphi_g|_{K_l} \mid g \in G, l \in L\}) \subseteq \mathrm{C}_q^q(K, K).$$

We will use the notations $\mathcal{S}(q)$ and $\mathcal{E}_{\mathrm{u}}(q)$ to abbreviate these (semi)groupoids.

Example 3.6. Consider the rotation on the disc with varying speed of rotation, i.e., the system (K, φ) given by $K := \mathbb{D} = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$ and

$$\varphi \colon K \to K, \quad \varphi(z) = \mathrm{e}^{\mathrm{i}|z|} z.$$

If we set $(L, \psi) := ([0, 1], \mathrm{id}_{[0,1]})$, then

$$q \colon (K, \varphi) \to (L, \psi), \quad z \mapsto |z|$$

defines an isometric extension between the two systems. If, for $l \in [0,1]$ and $\alpha \in \mathbb{T}$, one lets

$$\vartheta_{\alpha,l} \colon K_l \to K_l, \quad z \mapsto \alpha z$$

be the rotation by α on K_l, then it is instructive to verify that

$$\mathcal{E}_{\mathrm{u}}(q) = \{\vartheta_{\alpha,l} \mid \alpha \in \mathbb{T}, l \in L\}.$$

In particular, $\mathcal{E}_{\mathrm{u}}(q)$ is a compact groupoid. This should be contrasted with the much larger Ellis semigroup $\mathrm{E}(K,G)$ of the system which contains a homeomorphic copy of $\beta\mathbb{N}$ since (K,G) is not tame (see, e.g., [Gla06, Theorem 1.2]).

Example 3.7. Let $\alpha \in \mathbb{T}$ and

$$\begin{aligned} \psi_\alpha &\colon \mathbb{T} \to \mathbb{T}, & \psi_\alpha(x) &:= \alpha x, \\ \varphi_\alpha &\colon \mathbb{T}^2 \to \mathbb{T}^2, & \varphi_\alpha(x,y) &:= (\alpha x, xy) \end{aligned}$$

be the rotation by α and the corresponding skew rotation. Then

$$q \colon (\mathbb{T}^2, \psi_\alpha) \to (\mathbb{T}, \psi_\alpha),$$

$q(x,y) = x$ defines an isometric extension between the two systems. If α is rational, then φ_α is periodic with some period $N \in \mathbb{N}$ and so

$$\mathcal{E}_{\mathrm{u}}(q) = \mathcal{S}(q) = \{\varphi_\alpha^n|_{\mathbb{T}^2_l} \mid l \in \mathbb{T}, n = 1, \dots, N\}.$$

Clearly, $\mathcal{E}_{\mathrm{u}}(q)$ is a groupoid. Moreover, for fixed $n \in \mathbb{N}$, it follows from the characterization Proposition 2.4 that $\mathcal{E}_{\mathrm{u}}(q)$ is compact. In case α is irrational, $(\mathbb{T}^2, \varphi_\alpha)$ is minimal. Therefore, if one defines for $(\beta,\gamma) \in \mathbb{T}^2$

$$\vartheta_{(\beta,\gamma)} \colon \mathbb{T}^2 \to \mathbb{T}^2, \quad (x,y) \mapsto (\beta x, \gamma y),$$

then

$$\mathcal{E}_{\mathrm{u}}(q) = \{\vartheta_{(\beta,\gamma)} \mid (\beta,\gamma) \in \mathbb{T}^2\}.$$

In particular $\mathcal{E}_{\mathrm{u}}(q)$ is a compact groupoid.

Example 3.8. Let K be a compact space, consider the pair groupoid $K \times K$ and let $R \subseteq K \times K$ be a full subgroupoid, i.e., an equivalence relation on K (see Example 1.5). Then $\mathcal{E}_{\mathrm{u}}(R)$ is the smallest closed equivalence relation on K that contains R. Many important equivalence relations in topological dynamics, such as the equicontinuous structure relation or the distal structure relation, arise in this way.

As a special case, let $\mathcal{G}$ be a topological groupoid with compact unit space and $(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ be its action on its unit space (see Example 1.15). Then $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ lies in the groupoid

$$\mathrm{C}_{\mathrm{id}_{\mathcal{G}^{(0)}}}^{\mathrm{id}_{\mathcal{G}^{(0)}}}(\mathcal{G}^{(0)}, \mathcal{G}^{(0)})$$

which is canonically isomorphic to the pair groupoid $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. Therefore, we can identify the groupoid $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ with a closed equivalence relation on $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. This equivalence relation is given by $\mathcal{E}_{\mathrm{u}}(R_{\mathcal{G}})$ where $R_{\mathcal{G}} \subseteq \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is the orbit relation $R_{\mathcal{G}} = (r, s)(\mathcal{G}) \subseteq \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ of the action of $\mathcal{G}$ on its unit space; see Example 1.9. Therefore, $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ can be identified with the smallest closed equivalence relation on $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ that contains the orbit relation $R_{\mathcal{G}}$.

Remark 3.9. Note that the definition of the uniform enveloping semigroupoid $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ of a groupoid action $(K, q, \mathcal{G})$ is more intricate than that of the uniform enveloping semigroup $\mathrm{E}(K, G)$ of a group action (K, G): The uniform enveloping semigroup is defined as the closure of a semigroup and it turns out that this closure is automatically again a semigroup. In contrast to this, the following example demonstrates that $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is generally not merely the closure of $\mathcal{S}(K, q, \mathcal{G})$.

Example 3.10. Consider the dynamical systems (L, ψ) defined by $L := [-1, 1]$, $\psi(x) := \mathrm{sign}(x)x^2$ for $x \in L$ and (K, φ) given by $K := [-1, 1] \times \mathbb{Z}_2$,

$$\varphi(x, g) := (\psi(x), g + 1) \quad \text{for } (x, g) \in K.$$

Then the map

$$q\colon (K, \varphi) \to (L, \psi), \quad (x, g) \mapsto x$$

defines an isometric extension. The uniform enveloping semigroupoid of q is given by

$$\mathcal{E}_{\mathrm{u}}(q) = \{\vartheta_{x,y,h} \mid x, y \in L, h \in \mathbb{Z}_2\}$$

where $\vartheta_{x,y,h}$ denotes the function

$$\vartheta_{x,y,h}\colon K_x \to K_y, \quad (x, g) \mapsto (y, g + h).$$

In contrast to this,

$$\begin{aligned}\overline{\mathcal{S}(q)} = \mathcal{S}(q) &\cup \{\vartheta_{x,0,h}, \vartheta_{0,x,h} \mid x \in [-1, 1], h \in \mathbb{Z}_2\}\\ &\cup \{\vartheta_{-1,y,h}, \vartheta_{y,-1,h} \mid y \in [-1, 0], h \in \mathbb{Z}_2\}\\ &\cup \{\vartheta_{1,y,h}, \vartheta_{y,1,h} \mid y \in [0, 1], h \in \mathbb{Z}_2\}.\end{aligned}$$

Thus, the inclusion $\overline{\mathcal{S}(q)} \subseteq \mathcal{E}_{\mathrm{u}}(q)$ is generally strict.

3.1 Characterizing compactness. Usually, the uniform enveloping semi-groupoid is neither compact, nor a groupoid. We therefore try to answer the question: When is the uniform enveloping semigroupoid actually a compact groupoid? As a first step to address this problem, we observe that the groupoid property follows automatically once we have ensured compactness.

Proposition 3.11. *Let $(K, q, \mathcal{G})$ be a groupoid action. If $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is compact, then it is a compact groupoid, i.e., every $\vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ has an inverse $\vartheta^{-1} \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ and the mapping ${}^{-1}\colon \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}) \to \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is a homeomorphism.*

Proof. Consider the set M of all elements $\vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ having an inverse ϑ^{-1} in $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$. Then M is certainly closed under compositions and contains $\mathcal{S}(K, q, \mathcal{G})$. To see that $M = \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ it therefore suffices to show that M is closed in $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$. Pick a net $(\vartheta_\alpha)_{\alpha \in A}$ in M converging to $\vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$. Passing to a subnet, we may assume that $(\vartheta_\alpha^{-1})_{\alpha \in A}$ converges to some element $\varrho \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$. Using the characterization from Proposition 2.4 and the openness of q, we conclude that $\varrho = \vartheta^{-1}$. This shows that $M = \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$. Moreover, if $(\vartheta_\alpha)_{\alpha \in A}$ is a net in $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ converging to some $\vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$, then a similar argument shows that ϑ^{-1} is the only cluster point of the net $(\vartheta_\alpha^{-1})_{\alpha \in A}$. □

We now try to characterize the compactness of the uniform enveloping semi-groupoid by investigating when a set is (pre)compact in the compact-open topology. To this end, recall that if K is a compact space and Y is a uniform space, the precompactness of a subset $\mathcal{F} \subseteq \mathrm{C}(K, Y)$ in the compact-open topology is characterized by the classical Arzelà–Ascoli theorem: $\mathcal{F}$ is precompact if and only if $\mathcal{F}$ is equicontinuous and $\mathrm{im}(\mathcal{F}) = \bigcup_{f \in \mathcal{F}} \mathrm{im}(f)$ is precompact in Y. In what follows, we generalize the notion of equicontinuity and the Arzelà–Ascoli theorem to compact bundles.

Definition 3.12. Let $p\colon X \to L$, $q\colon Y \to L'$ be continuous surjections onto compact spaces and X and Y be uniform spaces. A subset $\mathcal{F} \subseteq \mathrm{C}_p^q(X, Y)$ is called (**uniformly**) **equicontinuous** if for each $U \in \mathcal{U}_Y$ there is a $V \in \mathcal{U}_X$ such that $(\vartheta(x_1), \vartheta(x_2)) \in U$ for every $\vartheta \in \mathcal{F}$ and every $(x_1, x_2) \in V \cap X \times_L X$ with $s(\vartheta) = p(x_1) = p(x_2)$.

Theorem 3.13. *Let $p\colon K \to L$, $q\colon Y \to L'$ be continuous surjections onto compact spaces, K be compact, and Y be a Hausdorff uniform space. If p is open, then a subset $\mathcal{F} \subseteq \mathrm{C}_p^q(K, Y)$ is precompact if and only if the following two conditions are fulfilled:*

(i) *$\mathrm{im}(\mathcal{F}) \subseteq Y$ is precompact.*

(ii) *$\mathcal{F}$ is equicontinuous.*

Proof. Suppose that (i) and (ii) hold. We may then assume that Y is compact. In view of Remark 2.7, it suffices to show that the closure $\overline{\mathrm{Gr}(\mathcal{F})}$ in $\mathscr{C}(K \times Y)$ is in fact contained in $\mathrm{Gr}(\mathrm{C}_p^q(K, Y))$. So we pick $C \in \overline{\mathrm{Gr}(\mathcal{F})}$ and show that $C = \mathrm{Gr}(\vartheta)$ for some $\vartheta \in \mathrm{C}_p^q(K, Y)$.

Let $(\vartheta_\alpha)_{\alpha\in A}$ be a net in $\mathcal{F}$ such that $\mathrm{Gr}(\vartheta_\alpha) \to C$ with respect to the Vietoris topology. First, let $(x, y) \in C$ and set $l := p(x), l' := q(x)$. We claim that $C \subseteq K_l \times Y_{l'}$: If $U \in \mathcal{U}_L(l)$ and $V \in \mathcal{U}_{L'}(l')$ are open neighborhoods of l and l', then

$$C \cap p^{-1}(U) \times q^{-1}(V) \neq \emptyset.$$

Thus, there is an $\alpha_0 \in A$ such that for all $\alpha \geqslant \alpha_0$

$$\mathrm{Gr}(\vartheta_\alpha) \cap p^{-1}(U) \times q^{-1}(V) \neq \emptyset.$$

Since $\vartheta_\alpha \in \mathrm{C}_p^q(K, Y)$, it follows that $\mathrm{Gr}(\vartheta_\alpha) \subseteq p^{-1}(U) \times q^{-1}(V)$ for $\alpha \geqslant \alpha_0$ and hence that

$$C \subseteq \overline{p^{-1}(U)} \times \overline{q^{-1}(V)}.$$

Since U and V were arbitrary, $C \subseteq K_l \times Y_{l'}$.

Since p is open, it follows that for every $x \in K_l$ there is a $y \in Y_{l'}$ such that $(x, y) \in C$: Use Remark 2.5 and the compactness of $\mathrm{im}(\mathcal{F})$ to find a subnet $(\mathrm{Gr}(\vartheta_\beta))_{\beta\in B}$ and a net $(x_\beta)_{\beta\in B}$ such that $(x_\beta)_{\beta\in B}$ converges to x,

$$p(x_\beta) = s(\vartheta_\beta)$$

for every $\beta \in B$, and $(\vartheta_\beta(x_\beta))_{\beta\in B}$ converges to some $y \in Y$. Since $(\mathrm{Gr}(\vartheta_\beta))_{\beta\in B}$ converges to C with respect to the Vietoris topology, this then shows that $(x, y) \in C$. In order to see that C is, in fact, the graph of a function $\vartheta\colon K_l \to Y_{l'}$, assume that $(x, y), (x, y') \in C$. Then there are nets $(x_\alpha, \vartheta_\alpha(x_\alpha))_{\alpha\in A}$, $(x'_\alpha, \vartheta_\alpha(x'_\alpha))_{\alpha\in A}$ converging to (x, y) and (x, y'). It then follows from the equicontinuity of $\mathcal{F}$ that the nets $(\vartheta_\alpha(x_\alpha))_{\alpha\in A}$ and $(\vartheta_\alpha(x'_\alpha))_{\alpha\in A}$ have the same limits. This shows that $y = y'$, i.e., there is a function $\vartheta\colon K_l \to Y_{l'}$ with $C = \mathrm{Gr}(\vartheta)$. Since $\vartheta(K_l)$ is contained in the compact space Y, the closed graph theorem shows that ϑ is continuous, i.e., $\vartheta \in \mathrm{C}_p^q(K, Y)$. Hence, $\mathcal{F}$ is precompact.

For the converse implication, we may assume $\mathcal{F}$ to be compact. Using the characterization of convergent nets in the compact-open topology from Proposition 2.4, it is then easy to see that $\mathrm{im}(\mathcal{F})$ is compact. If $\mathcal{F}$ were not equicontinuous, we would find a net $((\vartheta_\alpha, x_\alpha, x'_\alpha))_{\alpha\in A}$ in $\mathcal{F} \times_L K \times_L K$ and a $U \in \mathcal{U}_Y$ such that $\lim_\alpha x_\alpha = \lim_\alpha x'_\alpha$ and $(\vartheta_\alpha(x_\alpha), \vartheta_\alpha(x'_\alpha)) \notin U$ for every $\alpha \in A$ which clearly contradicts the compactness of $\mathcal{F}$. Thus, $\mathcal{F}$ is equicontinuous. $\square$

Corollary 3.14. *For a groupoid action $(K, q, \mathcal{G})$ the following assertions are equivalent:*

(a) *$(K, q, \mathcal{G})$ is equicontinuous.*

(b) *$\mathcal{S}(K, q, \mathcal{G}) \subseteq \mathrm{C}_q^q(K, K)$ is precompact.*

(c) *$\{f|_{K_{r(g)}} \circ \varphi_g \mid g \in \mathcal{G}\} \subseteq \mathrm{C}_q(K)$ is equicontinuous for all $f \in \mathrm{C}(K)$ from one/every subset M of $\mathrm{C}(K)$ that generates $\mathrm{C}(K)$ as a C^*-algebra.*

(d) *$\{f|_{K_{r(g)}} \circ \varphi_g \mid g \in \mathcal{G}\} \subseteq \mathrm{C}_q(K)$ is relatively compact for all $f \in \mathrm{C}(K)$ from one/every subset M of $\mathrm{C}(K)$ that generates $\mathrm{C}(K)$ as a C^*-algebra.*

Proof. Given Theorem 3.13, the equivalence of (a) and (b) is hard not to prove. Similarly, (c) and (d) are equivalent. So suppose $M \subseteq \mathrm{C}(K)$ is as in (c). Straightforward arguments show that the property

$$\{f|_{K_{r(g)}} \circ \varphi_g \mid g \in \mathcal{G}\} \subseteq \mathrm{C}_q(K) \text{ is equicontinuous}$$

is preserved under taking finite linear combinations, products, and conjugates of functions in $\mathrm{C}(K)$. Thus, we may assume that M is dense in $\mathrm{C}(K)$. Now, to verify the equicontinuity of $(K, q, \mathcal{G})$, let $V \in \mathcal{U}_K$ be a given entourage. Since the functions in $\mathrm{C}(K)$ generate the uniformity on K and M is dense, we can find an $\varepsilon > 0$ and an $f \in M$ such that $U_{f,\varepsilon} \subseteq V$ where

$$U_{f,\varepsilon} = \{(x, y) \in K \times K \mid |f(x) - f(y)| < \varepsilon\}.$$

By assumption,

$$\{f|_{K_{r(g)}} \circ \varphi_g \mid g \in \mathcal{G}\}$$

is equicontinuous and so we may find an entourage $U \in \mathcal{U}_K$ such that for all $g \in \mathcal{G}$ and all $(x, y) \in U$ with $q(x) = q(y) = s(g)$ one has

$$|f(gx) - f(gy)| < \varepsilon.$$

In other words, $\mathcal{G}$ maps $K \times_q K \cap U$ into $U_{f,\varepsilon} \subseteq V$, so $\mathcal{G}$ is equicontinuous and (c) implies (a). The converse implication is again easy to verify. □

In particular, if $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is compact, $(K, q, \mathcal{G})$ is necessarily equicontinuous. The following example shows that the converse is generally not true because the inclusion $\overline{\mathcal{S}(K, q, \mathcal{G})} \subseteq \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is generally strict, as noted in Remark 3.9 and Example 3.10.

Example 3.15. Let $L_0 := [0, \infty)$ and

$$\psi_0 \colon L_0 \to L_0, \quad \psi_0(x) := \lfloor x \rfloor + (x - \lfloor x \rfloor)^2$$

as well as $K_0 := L_0 \times \mathbb{Z}_2$ and

$$\varphi_0 \colon K_0 \to K_0, \quad \varphi_0(x, g) := (\psi_0(x), g + 1).$$

Then $q_0 \colon K_0 \to L_0, (x, h) \mapsto x$ is continuous and intertwines φ_0 and ψ_0. Since ψ_0, φ_0, and q are proper, they extend canonically to the one-point compactifications $K := K_0 \cup \{\infty_{K_0}\}$ and $L := L_0 \cup \{\infty_{L_0}\}$ of K_0 and L_0 and thereby yield an extension $q \colon (K, \varphi) \to (L, \psi)$ of topological dynamical systems. It is easy to see that $\overline{\mathcal{S}(q)}$ is compact since

$$\overline{\mathcal{S}(q)} \subseteq \{\vartheta_\infty\} \cup \bigcup_{n \in \mathbb{N}_0} \{\vartheta_{x,y,g} \mid x, y \in [n, n+1], g \in \mathbb{Z}_2\}$$

where for $x, y \in L$ and $g \in \mathbb{Z}_2$, we define $\vartheta_{x,y,g}$ and ϑ_x as

$$\begin{aligned} \vartheta_{x,y,g} \colon K_x \to K_y, &\qquad (x, h) \mapsto (y, g + h), \\ \vartheta_x \colon K_x \to \{\infty_{K_0}\}, &\qquad (x, h) \mapsto \infty_{K_0}. \end{aligned}$$

However,

$$\mathcal{E}_{\mathrm{u}}(q) = \{\vartheta_{x,y,g} \mid x, y \in L_0, g \in \mathbb{Z}_2\} \cup \{\vartheta_x \mid x \in L\}$$

and since ϑ_x is not invertible for $x \neq \infty_{L_0}$, $\mathcal{E}_{\mathrm{u}}(q)$ is neither a groupoid nor compact (use Proposition 3.11).

Thus, in contrast to the case of group actions, in order to characterize the compactness of $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$, a more restrictive property than equicontinuity is needed. The following proposition shows that pseudoisometry is a sufficient condition for the enveloping semigroupoid to be a compact groupoid.

Proposition 3.16. *Let $(K, q, \mathcal{G})$ be a pseudoisometric groupoid action. Then $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is a compact groupoid.*

Proof. Pick a set P as in Definition 1.16 (iii) and consider the set

$$\mathrm{I}(P) := \left\{ \vartheta \in \mathrm{C}_q^q(K, K) \,\middle|\, \begin{array}{l} \vartheta \colon K_{s(\vartheta)} \to K_{r(\vartheta)} \text{ is bijective and for all } p \in P, \\ x, y \in K_{s(\vartheta)} \text{ one has } p(\vartheta(x), \vartheta(y)) = p(x, y) \end{array} \right\}.$$

By Theorem 3.13, $\mathrm{I}(P)$ is a compact (semi)groupoid containing $\mathcal{S}(K, q, \mathcal{G})$ and therefore $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}) \subseteq \mathrm{I}(P)$ is itself a compact semigroupoid. It follows from Proposition 3.11 above that it is in fact a groupoid. □

The following proposition shows that if $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is transitive, then we can actually characterize pseudoisometric exensions via the compactness of the uniform enveloping semigroupoid.

Proposition 3.17. *Let $(K, q, \mathcal{G})$ be a groupoid action such that $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is a compact transitive groupoid. Then $(K, q, \mathcal{G})$ is pseudoisometric.*

Proof. Let P be a family of pseudometrics generating the topology of K. Then for $p \in P$, define

$$p' \colon K \times_{\mathcal{G}^{(0)}} K \to [0, \infty), \quad (x, y) \mapsto \max_{\substack{\vartheta \in \mathcal{E}_{\mathrm{u}}(K,q,\mathcal{G}) \\ s(\vartheta)=q(x)}} p(\vartheta(x), \vartheta(y)).$$

Then the family $P' := \{p' \mid p \in P\}$ generates the topology of K_u for each $u \in \mathcal{G}^{(0)}$ since $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})_u$ is compact. Moreover, since the range and source map of a compact transitive groupoid are open by Proposition 3.18 below, each p' is continuous and one readily verifies the invariance of the p'. □

Proposition 3.18. *Let $\mathcal{G}$ be a compact transitive groupoid. Then (s, r), s, and r are open and so is the restriction p of s and r to* $\mathrm{Iso}(\mathcal{G})$.

Proof. We start with the restrictions to $\mathrm{Iso}(\mathcal{G})$: Pick $g \in \mathrm{Iso}(\mathcal{G})$ and set $u := p(g) \in \mathcal{G}^{(0)}$. Moreover, let $(u_\alpha)_{\alpha \in A}$ be a net in $\mathcal{G}^{(0)}$ converging to u. For each $\alpha \in A$ there is an $h_\alpha \in \mathcal{G}_u^{u_\alpha}$ and by passing to a subnet, we may assume that $\lim_\alpha h_\alpha = h \in \mathcal{G}_u^u$. But then $g = \lim_\alpha h_\alpha(h^{-1}gh)h_\alpha^{-1}$ and so we have found a net $(g_\alpha)_{\alpha \in A}$ in $\mathrm{Iso}(\mathcal{G})$ that converges to g and satisfies $r(g_\alpha) = u_\alpha$ for every $\alpha \in A$. Thus, r is open.

To show that (s, r), s, and r are open, it suffices to show that (s, r) is open, so let $g \in \mathcal{G}$ and $(u_\alpha, v_\alpha)_{\alpha \in A}$ be a net in $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ converging to $(u, v) = (s(g), r(g))$. Since $\mathcal{G}$ is transitive, there is a net $(h_\alpha)_{\alpha \in A}$ in $\mathcal{G}$ with $s(h_\alpha) = u_\alpha$ and $r(h_\alpha) = v_\alpha$ for each $\alpha \in A$. By compactness of $\mathcal{G}$, we may assume that $(h_\alpha)_{\alpha \in A}$ converges to some element $h \in \mathcal{G}$ with $s(h) = s(g)$ and $r(h) = s(g)$. Set $\gamma := gh^{-1} \in \mathrm{Iso}(\mathcal{G})_{r(g)}$ and, using the openness result for the isotropy bundle, find, after possibly passing to a subnet, a net $(\gamma_\alpha)_{\alpha \in A}$ in $\mathrm{Iso}(\mathcal{G})$ with $p(\gamma_\alpha) = v_\alpha$ for each $\alpha \in A$. Then the net $(\gamma_\alpha h_\alpha)_{\alpha \in A}$ converges to g and satisfies

$$(s(\gamma_\alpha h_\alpha), r(\gamma_\alpha h_\alpha)) = (s(h_\alpha), r(h_\alpha)) = (u_\alpha, v_\alpha)$$

for each $\alpha \in A$. Hence, (s, r) is open. □

3.2 Characterizing transitivity.

Proposition 3.17 is unsatisfying in that it is not yet clear when $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is a transitive groupoid. Therefore, we show in this subsection that the transitivity of $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ can be characterized purely in terms of $\mathcal{G}$. To this end, recall from Example 1.15 that a groupoid is transitive if and only if the action on its unit space is transitive. This allows to reduce the question when $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is transitive to a question purely about $\mathcal{G}$ and its action $(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ on its unit space. The following lemma and Corollary 3.20 show that we thus only need to consider the question when $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ is transitive.

Lemma 3.19. *Let $p\colon (K_1, q_1, \mathcal{G}) \to (K_2, q_2, \mathcal{G})$ be an extension of groupoid actions. If $\mathcal{E}_{\mathrm{u}}(K_1, q_1, \mathcal{G})$ is compact, then*

$$\Phi_p\colon \mathcal{E}_{\mathrm{u}}(K_1, q_1, \mathcal{G}) \to \mathcal{E}_{\mathrm{u}}(K_2, q_2, \mathcal{G}), \quad \vartheta \to \Phi_p(\vartheta)$$

is a factor map of topological groupoids where

$$\Phi_p(\vartheta)\colon (K_2)_{s(\vartheta)} \to (K_2)_{r(\vartheta)}, \quad p(x) \mapsto p(\vartheta(x))$$

for $\vartheta \in \mathcal{E}_{\mathrm{u}}(q_1)$. Moreover,

$$\Phi_p^{(0)}\colon \mathcal{E}_{\mathrm{u}}(K_1, q_1, \mathcal{G})^{(0)} \to \mathcal{E}_{\mathrm{u}}(K_2, q_2, \mathcal{G})^{(0)}$$

is bijective.

Proof. We first check that Φ_p is well-defined. Let $\mathcal{S}$ be the set of all elements $\vartheta \in \mathcal{E}_{\mathrm{u}}(K_1, q_1, \mathcal{G})$ with the following property: If $x, y \in (K_1)_{s(\vartheta)}$ with $p(x) = p(y)$, then $p(\vartheta(x)) = p(\vartheta(y))$. Then $\mathcal{S}$ is a semigroupoid containing $\mathcal{S}(K_1, q_1, \mathcal{G})$ and we show that it is closed in $\mathcal{E}_{\mathrm{u}}(K_1, q_1, \mathcal{G})$. Let $(\vartheta_\alpha)_{\alpha \in A}$ be a net in $\mathcal{S}$ converging to $\vartheta \in \mathcal{E}_{\mathrm{u}}(K_1, q_1, \mathcal{G})$ and $x, y \in (K_1)_{s(\vartheta)}$ with $p(x) = p(y)$. Since p and q_2 are open, we find, by passing to a subnet, a $((x_\alpha, y_\alpha))_{\alpha \in A}$ in $K_1 \times K_1$ such that $x = \lim_\alpha x_\alpha$, $y = \lim_\alpha y_\alpha$ and $p(x_\alpha) = p(y_\alpha)$ as well as $q_1(x_\alpha) = q_1(y_\alpha) = s(\vartheta_\alpha)$ for every $\alpha \in A$. But then $p(\vartheta_\alpha(x_\alpha)) = p(\vartheta_\alpha(y_\alpha))$ for every $\alpha \in A$ and therefore

$$p(\vartheta(x)) = \lim_\alpha p(\vartheta_\alpha(x_\alpha)) = \lim_\alpha p(\vartheta_\alpha(y_\alpha)) = p(\vartheta(y)).$$

Thus, $\mathcal{S}$ is closed and therefore $\mathcal{S} = \mathcal{E}_{\mathrm{u}}(K_1, q_1, \mathcal{G})$. It is now clear, that

$$\Phi_p\colon \mathcal{E}_{\mathrm{u}}(K_1, q_1, \mathcal{G}) \to \mathrm{C}_{q_2}^{q_2}(K_2, K_2), \quad \vartheta \mapsto \Phi_p(\vartheta)$$

is a well-defined morphism of semigroupoids and a moment's thought reveals that it is continuous. Since $\mathcal{E}_{\mathrm{u}}(K_1, q_1, \mathcal{G})$ is compact, we obtain that its image is a closed subsemigroupoid of $\mathrm{C}_{q_2}^{q_2}(K_2, K_2)$ containing $\mathcal{S}(q_2)$ and therefore containing $\mathcal{E}_{\mathrm{u}}(K_2, q_2, \mathcal{G})$. On the other hand, $\Phi_p^{-1}(\mathcal{E}_{\mathrm{u}}(K_2, q_2, \mathcal{G}))$ is a closed subsemigroupoid of $\mathcal{E}_{\mathrm{u}}(K_1, q_1, \mathcal{G})$ containing $\mathcal{S}(K_1, q_1, \mathcal{G})$ showing that the image of Φ_p is precisely $\mathcal{E}_{\mathrm{u}}(K_2, q_2, \mathcal{G})$. Moreover, $\Phi_p^{(0)}$ is easily shown to be bijective since p is an extension of groupoid actions, i.e., $q_1 = q_2 \circ p$. □

Corollary 3.20. *Let $(K, q, \mathcal{G})$ be a groupoid action such that its enveloping semigroupoid $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is compact. Then $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is transitive if and only if $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ is transitive.*

Proof. Consider the extension $q\colon (K, q, \mathcal{G}) \to (\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ of groupoid actions. Then Lemma 3.19 shows that there is a surjective groupoid morphism from $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ to $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ which is bijective on the level of unit spaces. Thus, $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is transitive if and only if $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ is. □

The question that now remains is: When is $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ transitive? To understand this, first recall from Example 3.8 that the groupoid $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ is isomorphic to $\mathcal{E}_{\mathrm{u}}(R_{\mathcal{G}})$ where $R_{\mathcal{G}}$ is the orbit relation on $\mathcal{G}^{(0)}$. Therefore, we need to understand when the equivalence relation $\mathcal{E}_{\mathrm{u}}(R_{\mathcal{G}})$ is transitive which, as noted in Example 1.5, amounts to understanding when $\mathcal{E}_{\mathrm{u}}(R_{\mathcal{G}}) = \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. We now consider the following illustrating examples.

Example 3.21. Let (L, G) be a topological dynamical system and let $G \ltimes L$ be the action groupoid of (L, G). In this case, $R_{G \ltimes L} = \{(x, y) \mid \exists g \in G\colon gx = y\}$ is the regular orbit relation on L.

(a) If every orbit in (L, G) equals L, then $R_{G\times L} = L \times L$, $G \ltimes L$ is transitive, and so is $\mathcal{E}_{\mathrm{u}}(R_{G\times L})$. However, this case almost never occurs in topological dynamics.

(b) If (L, G) has a dense orbit, then $R_{G\ltimes L}$ is dense in $L \times L$ and so

$$\mathcal{E}_{\mathrm{u}}(R_{G\ltimes L}) = L \times L,$$

so $\mathcal{E}_{\mathrm{u}}(R_{G\ltimes L})$ is transitive.

(c) Even if (L, G) does not have any transitive point, $\mathcal{E}_{\mathrm{u}}(L, \mathrm{id}_L, G \ltimes L)$ may still be transitive. To see this, revisit the system (L, ψ) considered in Example 3.10: It follows either by direct computation or by observing the transitivity of $\mathcal{E}_{\mathrm{u}}(K, q, \mathbb{Z} \ltimes L)$ that $\mathcal{E}_{\mathrm{u}}(R_{G\ltimes L})$ is transitive. However, $(L, \mathbb{Z})$ itself is not transitive.

(d) Consider the system (L, ψ) on $L := [0, 1] \times \mathbb{Z}_2$ given by the map

$$\psi\colon [0, 1] \times \mathbb{Z}_2 \to [0, 1] \times \mathbb{Z}_2, \quad (x, g) \mapsto (x^2, g).$$

Then

$$\mathcal{E}_{\mathrm{u}}(R_{G\ltimes L}) = \{((x, g), (y, h)) \in [0, 1] \times \mathbb{Z}_2 \mid g = h\} \subsetneq L \times L.$$

In particular, $\mathcal{E}_{\mathrm{u}}(R_{G\ltimes L})$ is not transitive.

Remark 3.22. In light of Example 3.21, it is apparent that the orbit structure of the action of a groupoid $\mathcal{G}$ on its unit space $\mathcal{G}^{(0)}$ plays an essential role for the transitivity of the enveloping groupoid $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ and the equivalence relation $\mathcal{E}_{\mathrm{u}}(R_{\mathcal{G}})$. For topological dynamical systems (L, ψ), it has been characterized when $\mathcal{E}_{\mathrm{u}}(R_{G \ltimes L})$, the smallest closed equivalence relation containing the orbit relation, is all of $L \times L$: It is equivalent to each of the following assertions:

(i) The fixed space $\mathrm{fix}(T_\psi)$ of the Koopman operator $T_\psi\colon \mathrm{C}(L) \to \mathrm{C}(L)$ is one-dimensional.

(ii) The maximal trivial factor of (L, ψ) is a point.

See [Kü21] and [Ede21, Section 1] for more information. In analogy with ergodic measure-preserving systems, we call such systems (L, G) **topologically ergodic**. We now extend this characterization to groupoid actions.

Definition 3.23. A factor $(M, t, \mathcal{H})$ of $(K, q, \mathcal{G})$ with factor map (p, Φ) is called a **trivial factor** if the acting groupoid $\mathcal{H}$ is trivial in the sense of 1.3. It is a **maximal trivial factor**, if for any factor map $(\tilde{p}, \tilde{\Phi})\colon (K, q, \mathcal{G}) \to (\tilde{M}, \tilde{t}, \tilde{\mathcal{H}})$ onto another trivial factor there is a unique factor map (m, Θ) such that the following diagram commutes:

$$\begin{tikzcd}
(K, q, \mathcal{G}) \arrow[rr, "{(p,\Phi)}"] \arrow[dr, "{(\tilde{p},\tilde{\Phi})}"'] & & (M, t, H) \arrow[dl, dashed, "{\exists!\,(m,\Theta)}"] \\
& (\tilde{M}, \tilde{t}, \tilde{\mathcal{H}}) &
\end{tikzcd}$$

We call $(K, q, \mathcal{G})$ **topologically ergodic** if every trivial factor is a point and say that a groupoid $\mathcal{G}$ is **topologically ergodic** if its action $(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ on its unit space is topologically ergodic.

Lemma 3.24. *Let $(K, q, \mathcal{G}, \varphi)$ be a groupoid action. Then the folowing assertions hold:*

(i) *Maximal trivial factors are unique up to isomorphism.*

(ii) *If*

$$R_\varphi = \{(x, y) \in K \times K \mid y \in \mathcal{G}x\}$$

denotes the orbit relation, then the space $K/\mathcal{E}_{\mathrm{u}}(R_\varphi)$ defines a maximal trivial factor of $(K, q, \mathcal{G})$.

Proof. Two trivial factors $(M_1, t_1, \mathcal{H}_1)$ and $(M_2, t_2, \mathcal{H}_2)$ are isomorphic if and only if the associated equivalence relations on K agree. By construction, $\mathcal{E}_{\mathrm{u}}(R_\varphi)$ is the smallest closed equivalence relation on K that contains the equivalence relation R_φ. Thus, $K/\mathcal{E}_{\mathrm{u}}(R_\varphi)$ is a maximal trivial factor of $(K, q, \mathcal{G})$ and every other maximal trivial factor is isomorphic to it. □

In view of Lemma 3.24, we may from now on speak of the maximal trivial factor $\mathrm{fix}(K, q, \mathcal{G})$ of a groupoid action $(K, q, \mathcal{G})$. Summarizing our observations, we obtain the following characterization.

Theorem 3.25. *Let $\mathcal{G}$ be a topological groupoid with compact unit space. Then the following assertions are equivalent.*

(i) *$\mathcal{G}$ is topologically ergodic.*
(ii) *The equivalence relation $\mathcal{E}_{\mathrm{u}}(R_{\mathcal{G}})$ equals $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$.*
(iii) *The enveloping groupoid $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ is transitive.*
(iv) *The enveloping groupoid $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is transitive for every action $(K, q, \mathcal{G})$ of $\mathcal{G}$ such that $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is compact.*

Since we are ultimately interested in groupoid actions that arise from extensions of topological dynamical systems as in Example 1.13, the following corollary provides a simple criterion for topological ergodicity.

Corollary 3.26. *Let (K, G) be a topological dynamical system. Then the action groupoid $G \ltimes K$ is topologically ergodic if and only if the system (K, G) is.*

We are now ready to state the main result of this section.

Theorem 3.27. *Let $(K, q, \mathcal{G})$ be a groupoid action by a topologically ergodic groupoid $\mathcal{G}$. Then the following assertions are equivalent:*

(a) *$(K, q, \mathcal{G})$ is pseudoisometric.*
(b) *$\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is a compact groupoid.*

Proof. The implication (a) $\Longrightarrow$ (b) was established more generally in Porposition 3.16. The converse implication follows from Proposition 3.17 since if $\mathcal{G}$ is topologically ergodic, $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is a compact transitive groupoid by Theorem 3.25. □

Remark 3.28. Let $(K, q, \mathcal{G})$ be a groupoid action by a topologically ergodic groupoid $\mathcal{G}$. If K is metrizable, then Theorem 3.27 combined with the proof of Proposition 3.17 reveal that $(K, q, \mathcal{G})$ is isometric if and only if it is pseudoisometric.

3.3 Maximal trivial factors and Koopman representations. As noted in Remark 3.22, topological ergodicity of a dynamical system (L, ψ) can be characterized in terms of its Koopman operator, allowing for a very convenient criterion. We extend this characterization to groupoids.

Definition 3.29. Let $(K, q, \mathcal{G}, \varphi)$ be a groupoid action. The map

$$T_\varphi\colon \mathcal{G} \to \bigcup_{u,v\in\mathcal{G}^{(0)}} \mathscr{L}(\mathrm{C}(K_u), \mathrm{C}(K_v)), \quad g \mapsto T_g$$

with $T_g f := f \circ \varphi_{g^{-1}}$ for $f \in \mathrm{C}(K_{s(g)})$ is called the **Koopman representation of** $(K, q, \mathcal{G}, \varphi)$. Moreover, the set

$$\mathrm{fix}(T_\varphi) := \{f \in \mathrm{C}(K) \mid \forall g \in \mathcal{G}\colon T_g(f|_{K_{s(g)}}) = f|_{K_{r(g)}}\}$$

is called its **fixed space**. If $\mathcal{G}$ is a topological groupoid with compact unit space, we write $T_\mathcal{G}$ for the Koopman representation of $(\mathcal{G}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G}^{(0)})$ and call this the **Koopman representation of** $\mathcal{G}$.

Remark 3.30. If $(K, q, \mathcal{G}, \varphi)$ is a groupoid action, we can recover its fixed space from the action groupoid $\mathcal{G} \ltimes K$ (see Definition 1.19). Concretely, we obtain, under the usual identification of the unit space, the identity $\mathrm{fix}(T_\varphi) = \mathrm{fix}(T_{\mathcal{G}\ltimes K})$.

The fixed space of the Koopman representation is always a unital commutative C*-algebra and therefore isomorphic to $\mathrm{C}(X)$ where X is its (compact) Gelfand space. Using this observation we obtain the following result characterizing the maximal trivial factor of a groupoid action.

Proposition 3.31. *Let $(K, q, \mathcal{G}, \varphi)$ be a groupoid actions. Then the Gelfand space of the fixed space $\mathrm{fix}(T_\varphi)$ defines a maximal trivial factor of $(K, q, \mathcal{G}, \varphi)$.*

Proof. The Gelfand space of $\mathrm{fix}(T_\varphi)$ is homeomophic to the compact space $M = K/R_{\mathrm{fix}}$ with

$$R_{\mathrm{fix}} := \{(x, y) \in K \times K \mid \forall f \in \mathrm{fix}(T_\varphi)\colon f(x) = f(y)\}.$$

Clearly, R_{fix} is a closed and invariant equivalence relation containg the orbit relation R_φ. We therefore immediately obtain that $\mathcal{E}_\mathrm{u}(R_\varphi) \subseteq R_{\mathrm{fix}}$. On the other hand, if R is any closed invariant equivalence relation and $\pi_R\colon K \to K/R$ the induced map, then $(x, y) \in R$ if and only if $f(\pi_R(x)) = f(\pi_R(y))$ for every $f \in \mathrm{C}(K/R)$. However, $T_{\pi_R} f = f \circ \pi_R \in \mathrm{fix}(T_\varphi)$ for every $f \in \mathrm{C}(K/R)$. This shows $R_{\mathrm{fix}} \subseteq \mathcal{E}_\mathrm{u}(R_\varphi)$ and consequently $R_{\mathrm{fix}} = \mathcal{E}_\mathrm{u}(R_\varphi)$. The claim now follows from Lemma 3.24. ☐

Corollary 3.32. *A groupoid action $(K, q, \mathcal{G}, \varphi)$ is topologically ergodic if and only if $\mathrm{fix}(T_\varphi)$ is one-dimensional.*

Corollary 3.33. *A groupoid $\mathcal{G}$ with compact unit space $\mathcal{G}^{(0)}$ is topologically ergodic if and only if $\mathrm{fix}(T_\mathcal{G})$ is one-dimensional.*

4 Representations of compact transitive groupoids

In this section we study the representation theory of compact transitive groupoids and apply it to the uniform enveloping (semi)groupoids of pseudoisometric groupoid actions. We start by recalling the following consequence of the Peter–Weyl theorem for representations of compact groups (see [EFHN15, Theorem 15.14]).

Theorem 4.1. *Let $T\colon G \to \mathscr{L}(E)$ be a strongly continuous representation of a compact group G on a Banach space E. Then the following assertions hold:*

(i) *The union of all finite-dimensional invariant subspaces of E is dense in E.*

(ii) *If G is abelian, then the union of all one-dimensional invariant subspaces of E is total in E.*

We prove a generalization of this result to representations of compact transitive groupoids in Theorem 4.8. We then apply this generalization to prove Theorem 4.14, the main result of this section that characterizes pseudoisometric groupoid actions. To perform this generalization, we need to start by replacing Banach spaces by Banach bundles (see, e.g., [DG83, Definition 1.1] or [Gie82, Section 1 and Theorem 3.2]).

Definition 4.2. Let L be a compact space. A **Banach bundle** over L is a topological space E together with a continuous open surjection $p\colon E \to L$ with the following properties.

(i) Every fiber E_l is a Banach space.

(ii) The mappings

$$+\colon E \times_L E \to E, \quad (e,f) \mapsto e+f$$
$$\cdot\colon \mathbb{C} \times E \to E, \quad (\lambda, e) \mapsto \lambda e$$

are continuous.

(iii) The norm mapping

$$\|\cdot\|\colon E \to [0,\infty), \quad e \mapsto \|e\|$$

is upper semicontinuous.

(iv) For each $l \in L$ the sets

$$\{e \in E \mid p(e) \in U, \|e\| < \varepsilon\}$$

for neighborhoods $U \subseteq L$ of l and $\varepsilon > 0$ define a neighborhood base of $0_l \in E_l$.

A Banach bundle E is

- **continuous** if the norm mapping $\|\cdot\|$ of (iii) is continuous,
- **of constant dimension** n for some $n \in \mathbb{N}_0$ if $\dim(E_l) = n$ for every $l \in L$,
- **of constant finite dimension** if it is of constant dimension n for some $n \in \mathbb{N}_0$,
- **locally trivial** if for each $l \in L$ there are a compact neighborhood W of l, $n \in \mathbb{N}_0$ and a homeomorphism $\Phi\colon p^{-1}(W) \to W \times \mathbb{C}^n$ with the following properties:
  - The diagram

$$\begin{tikzcd} p^{-1}(W) \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & W \times \mathbb{C}^n \arrow[dl, "\mathrm{pr}_1"] \\ & W & \end{tikzcd}$$

    commutes where $\mathrm{pr}_1\colon W \times \mathbb{C}^n \to W$ is the projection onto the first component.
  - $\Phi|_{E_l}\colon E_l \to \{l\} \times \mathbb{C}^n$ is an isomorphism of vector spaces for every $l \in W$.
  - There are constants $c_1, c_2 > 0$ such that

$$c_1 \cdot \|e\| \leqslant \|\mathrm{pr}_2(\Phi(e))\| \leqslant c_2 \cdot \|e\|$$

    for every $e \in p^{-1}(W)$ where $\mathrm{pr}_2\colon W \times \mathbb{C}^n \to \mathbb{C}^n$ is the projection onto the second component.

Moreover, we write

$$\Gamma(E) := \{\sigma \in \mathrm{C}(L, E) \mid p \circ \sigma = \mathrm{id}_L\}$$

for the **space of continuous sections of** E.

Remark 4.3. (i) If E is a Banach bundle over a compact space L, then $\Gamma(E)$ is canonically a module over $\mathrm{C}(L)$ and a Banach space with the norm defined by $\|\sigma\| := \sup_{l \in L} \|\sigma(l)\|_{E_l}$ for $\sigma \in \Gamma(E)$. Moreover, $\|f\sigma\| \leqslant \|f\| \cdot \|\sigma\|$ for all $f \in \mathrm{C}(L)$ and $\sigma \in \Gamma(E)$, i.e., $\Gamma(E)$ is a Banach module over $\mathrm{C}(L)$ (cf. [DG83, Chapter 2]).

(ii) If E is a continuous Banach bundle, then its total space is Hausdorff (see [Gie82, Proposition 16.4]).

(iii) A Banach bundle with finite-dimensional fibers which is locally trivial as a vector bundle (in the usual sense) is locally trivial as a Banach bundle since the required norm estimates follow from compactness and upper semicontinuity of the norm (see [Gie82, Proposition 10.9]).

(iv) By [Gie82, Theorem 18.5], a Banach bundle of constant finite dimension has a Hausdorff total space if and only if it is locally trivial.

We also recall the notion of subbundles.

Definition 4.4. A **subbundle** of a Banach bundle E is a subset F of E together with the restricted mapping $p|_F\colon F \to L$ such that the following conditions are satisfied.

- $F_l = F \cap E_l$ is a closed linear subspace of E_l for every $l \in L$.
- The restricted mapping $p|_F$ is still open.

Under these conditions, F together with $p|_F$ is again a Banach bundle (see [Gie82, Section 8]).

There are plenty of examples of Banach bundles coming from differential geometry. Here we are interested in Banach bundles arising from surjections between compact spaces.

Example 4.5. Let $q\colon K \to L$ be an open continuous surjection between compact spaces. Then a moment's thought reveals that the compact-open topology on $\mathrm{C}_q(K)$ agrees with the topology generated by the base

$$V(F, U, \varepsilon) := \{f \in \mathrm{C}_q(K) \mid s(f) \in U, \|f - F|_{K_{s(f)}}\| < \varepsilon\}$$

for $F \in \mathrm{C}(K)$, open $U \subseteq L$, and $\varepsilon > 0$ (considered, e.g., in [Kna67, p. 30]). Together with the canonical mapping $p := s\colon \mathrm{C}_q(K) \to L$, the space $\mathrm{C}_q(K)$ becomes a continuous Banach bundle over L. Moreover, the mapping

$$\mathrm{C}(K) \to \Gamma(\mathrm{C}_q(K)), \quad F \mapsto [l \mapsto F_l]$$

is an isometric isomorphism of Banach modules over $\mathrm{C}(L)$ by means of which we identify the continuous sections of $\mathrm{C}_q(K)$ with $\mathrm{C}(K)$.

Next, we introduce the notion of continuous representations for topological groupoids (cf. Definition 3.1 of [Bos11]). Note here that if E is a Banach bundle over a compact space L, then the space $\mathscr{G}(E)$ of all invertible bounded linear operators

$$T\colon E_l \to E_{\tilde{l}}$$

for $l, \tilde{l} \in L$ is a subsemigroupoid of $\mathrm{C}_p^p(E, E)$ and itself a groupoid.

Definition 4.6. Let $\mathcal{G}$ be a topological groupoid. A **continuous representation** of $\mathcal{G}$ on a Banach bundle E over $\mathcal{G}^{(0)}$ is a homomorphism

$$T\colon \mathcal{G} \to \mathscr{G}(E)$$

of groupoids such that

$$\mathcal{G} \times_{s,p} E \to E, \quad (g, v) \mapsto T(g)v$$

is continuous. Moreover, we call a subset F of E **T-invariant** if

$$T(g)(F \cap E_{s(g)}) \subseteq F$$

for every $g \in \mathcal{G}$.

An important class of examples of continuous groupoid representations are the Koopman representations we already considered in Definition 3.29.

Proposition 4.7. *Let $(K, q, \mathcal{G}, \varphi)$ be a groupoid action. Then the Koopman representation*

$$\mathcal{G} \to \mathscr{G}(\mathrm{C}_q(K)), \quad g \mapsto T_g$$

is a continuuous representation of $\mathcal{G}$.

Proof. We only check that the mapping

$$\mathcal{G} \times_{s,s} \mathrm{C}_q(K) \to \mathrm{C}_q(K), \quad (g, f) \mapsto T_g f$$

is continuous since the remaining assertions are obvious. Pick a net $((g_\alpha, f_\alpha))_{\alpha \in A}$ in $\mathcal{G} \times_{s,s} \mathrm{C}_q(K)$ converging to $(g, f) \in \mathcal{G} \times_{s,s} \mathrm{C}_q(K)$. We have to show that $T_{g_\alpha} f_\alpha$ converges to $T_g f$ with respect to the compact-open topology.

Let $((g_\beta, f_\beta))_{\beta \in B}$ be a subnet and $(x_\beta)_{\beta \in B}$ be a convergent net in K with limit $x \in K$ that satisfies $q(x_\beta) = s(g_\beta^{-1})$ for every $\beta \in B$. Then $\lim_\beta g_\beta^{-1}(x_\beta) = g^{-1}(x)$. Since $\lim_\beta f_\beta = f$,

$$\lim_\beta f_\beta(g_\beta^{-1}(x_\beta)) = f(g^{-1}(x)).$$

This shows that T_φ is continuous. □

We now state our first main result: a Peter–Weyl-type theorem for compact transitive groupoids. Here, a subset F of a Banach bundle E over a compact space L is called **fiberwise dense** if $F \cap E_l$ is dense in E_l for every $l \in L$. The notion of a **fiberwise total set** is defined analogously. Moreover, a groupoid $\mathcal{G}$ is abelian if all its isotropy groups $\mathcal{G}_u^u$ for $u \in \mathcal{G}^{(0)}$ are abelian.

Theorem 4.8. *For a continuous representation T of a compact transitive groupoid $\mathcal{G}$ on a Banach bundle E over the unit space $\mathcal{G}^{(0)}$ the following assertions hold.*

(i) *The union of all invariant subbundles of constant finite dimension is fiberwise dense in E.*
(ii) *If $\mathcal{G}$ is abelian, then the union of all invariant subbundles of constant dimension one is fiberwise total in E.*
(iii) *If $F_1, F_2 \subseteq E$ are two subbundles of constant finite dimension, then their sum $F_1 + F_2$ is again an invariant subbundle of constant finite dimension.*

Remark 4.9. Notice that if E has a Hausdorff total space (in particular, if E has continuous norm), then the subbundles in Theorem 4.8 are locally trivial (see Remark 4.3 (iv)).

The proof of Theorem 4.8 uses the following lemma which reduces the problem to a single isotropy group.

Lemma 4.10. *Let T be a continuous representation of a compact transitive groupoid on a Banach bundle E, $u \in \mathcal{G}^{(0)}$, and $V \subseteq E_u$ a closed $\mathcal{G}_u^u$-invariant subspace. Then setting $F_{r(g)} := T(g)V$ for every $g \in \mathcal{G}_u$ defines an invariant subbundle $F \subseteq E$.*

Proof. Note first that F is well-defined. In fact, if $g, h \in \mathcal{G}_u$ with $r(g) = r(h)$, then

$$T(h)V = T(g)T(g^{-1}h)V = T(g)V$$

since $g^{-1}h \in \mathcal{G}_u^u$ and V is $\mathcal{G}_u^u$-invariant. Clearly, F is $\mathcal{G}$-invariant and fiberwise closed.

To show that F is a subbundle of E, it suffices to check that $p|_F \colon F \to \mathcal{G}^{(0)}$ is open. So let $f \in F$ and $(v_\alpha)_{\alpha\in A}$ be a convergent net in $\mathcal{G}^{(0)}$ with limit

$$v := p(f) \in \mathcal{G}^{(0)}.$$

Since $(s, r) \colon \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is open by Proposition 3.18, we may assume, after passing to a subnet, that there is a net $(g_\alpha)_{\alpha\in A}$ with limit v such that

$$s(g_\alpha) = v \quad \text{and} \quad r(g_\alpha) = v_\alpha$$

for every $\alpha \in A$. The net $(T(g_\alpha)f)_{\alpha\in A}$ then is a net over $(v_\alpha)_{\alpha\in A}$ that converges to f, showing that $p|_F$ is open. □

Proof of Theorem 4.8. For fixed $u \in \mathcal{G}^{(0)}$, use Theorem 4.1 to see that the union of all finite-dimensional $\mathcal{G}_u^u$-invariant subspaces is dense in E_u. But by Lemma 4.10 each of these invariant subspaces defines a $\mathcal{G}$-invariant subbundle of constant finite dimension which implies (i). Likewise, a combination of Theorem 4.1 and Lemma 4.10 proves (ii) and (iii). □

We can now apply Theorem 4.8 to representations of uniform enveloping groupoids $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ of groupoid actions $(K, q, \mathcal{G})$ on the Banach bundle $\mathrm{C}_q(K)$. However, as in Knapp's article [Kna67], we are primarily interested in results formulated in terms of the Banach space $\mathrm{C}(K)$ (instead of the Banach bundle $\mathrm{C}_q(K)$). Since $\mathrm{C}(K)$ can be identified with the space $\Gamma(\mathrm{C}_q(K))$ of continuous sections of $\mathrm{C}_q(K)$, the following remark explains why this can be achieved.

Remark 4.11. Let $p\colon E \to L$ be a Banach bundle and $\Gamma(E)$ its space of continuous sections. As noted in Remark 4.3, $\Gamma(E)$ is a $\mathrm{C}(L)$-module and if $F \subseteq E$ is a subbundle, then $\Gamma(F)$ is a closed submodule of $\Gamma(E)$. Conversely, if $\Gamma \subseteq \Gamma(E)$ is a closed submodule, one can define a corresponding subbundle $F_\Gamma \subseteq E$ as follows: For $l \in L$, let $\mathrm{ev}_l\colon \Gamma(E) \to E_l$ denote the point evaluation in l and set

$$F_\Gamma := \bigcup_{l\in L} \mathrm{ev}_l(\Gamma) \subseteq E.$$

It is shown in [Gie82, Theorem 8.6 and Remark 8.7] that this does indeed define a subbundle of E and that the assignments $F \mapsto \Gamma(F)$ and $\Gamma \mapsto F_\Gamma$ are mutually inverse. In particular, there is a one-to-one correspondence between subbundles of E. Thus, it is natural to try to rephrase properties for subbundles in terms of their associated submodules.

We now translate the notions "invariant subbundle" and "locally trivial" to the language of modules.

Definition 4.12. Let T be a representation of a groupoid $\mathcal{G}$ on a Banach bundle E. Then a subset $M \subseteq \Gamma(E)$ is called **T-invariant** if $T_g M_{s(g)} \subseteq M_{r(g)}$ for every $g \in \mathcal{G}$.

It is easy to see that a subbundle $F \subseteq E$ is invariant if and only if the submodule $\Gamma(F) \subseteq \Gamma(E)$ is invariant.

The following result, which is a Banach bundle version of the classical Serre–Swan duality (see [Swa62]), characterizes local triviality. Recall here that a module Γ over a commutative unital ring R is **projective** if there is an R-module $\tilde{\Gamma}$ such that the module $\Gamma \oplus \tilde{\Gamma}$ is free, i.e., has a basis.

Proposition 4.13. *For a Banach bundle E over a compact space L the following assertions are equivalent:*

(a) *E is locally trivial.*

(b) *$\Gamma(E)$ is a finitely generated and projective $\mathrm{C}(L)$-module.*

Proof. The implication "(a) $\Rightarrow$ (b)" follows directly from the Serre–Swan theorem. Conversely, assume that (b) holds. Using the Serre–Swan theorem a second time, we find a locally trivial vector bundle F over L and a (not necessarily continuous) $\mathrm{C}(L)$-module isomorphism $T\colon \Gamma(F) \to \Gamma(E)$. We now construct a bundle morphism $\Phi\colon F \to E$ from T and show that it is continuous in order to prove that E and F are isomorphic.

Equip F with any map $\|\cdot\|: F \to [0,\infty)$ turning F into a Banach bundle (these always exist, see [Swa62, Lemma 2]). Then by Remark 4.3 (iii), F is also locally trivial as a Banach bundle. If $l \in L$ and $\sigma \in \Gamma(F)$ with $\sigma(l) = 0$, then we find $k \in \mathbb{N}$, $h_j \in \mathrm{C}(L)$ with $h_j(l) = 0$ and $\tau_j \in \Gamma(F)$ for $j = 1, \dots, k$ such that $\sigma = \sum_{j=1}^k h_j\tau_j$ by [Swa62, Lemma 4]. But then

$$(T\sigma)(l) = \sum_{j=1}^{k} h_j(l)(T\tau_j)(l) = 0$$

for every $l \in L$. We therefore obtain a well-defined linear map $\Phi_l\colon F_l \to E_l$ by setting $\Phi_l\sigma(l) := (T\sigma)(l)$ for $\sigma \in \Gamma(F)$ and $l \in L$. Moreover, since F_l is finite-dimensional, Φ_l is bounded for every $l \in L$. We show as in the proof of [Swa62, Theorem 1] that

$$\Phi\colon F \to E, \quad f \mapsto \Phi_{p(f)}f$$

is continuous. Pick $l \in L$. Since F is locally trivial, we find a neighborhood $V \in \mathcal{U}_L(l)$, sections $\sigma_1, \dots, \sigma_n \in \Gamma(F)$ such that $\sigma_1(\tilde{l}), \dots, \sigma_n(\tilde{l})$ define a base in $F_{\tilde{l}}$ for every $\tilde{l} \in V$, and continuous functions $h_1, \dots, h_n\colon p^{-1}(V) \to \mathbb{C}$ such that

$$f = \sum_{j=1}^{n} h_j(f)\sigma_j(p(f)) \quad \text{for every } f \in p^{-1}(V).$$

But then $\Phi(f) = \sum_{j=1}^n h_j(f)(T\sigma_j)(p(f))$ for every $f \in p^{-1}(V)$. Since $T\sigma_j$ is continuous for every $j \in \{1, \dots, n\}$ we obtain that Φ is continuous and hence a morphism of Banach bundles (see [Gie82, Definition 10.1, Proposition 10.2]). Moreover, $T\sigma = \Phi \circ \sigma$ for every $\sigma \in \Gamma(F)$, i.e., T is the operator induced by the morphism Φ between the spaces of continuous sections (see [Gie82, Section 10]). However, by the bounded inverse theorem, a morphism of Banach bundles is an isomorphism if and only if the induced operator between the spaces of continuous sections is bijective (see [Gie82, Remark 10.19 (b)]). Thus, the bundles E and F are isomorphic and E is locally trivial. □

With these translations we now formulate the main result of this section.

Theorem 4.14. *Let $(K, q, \mathcal{G}, \varphi)$ be a groupoid action by a topologically ergodic groupoid $\mathcal{G}$. Then the following assertions are equivalent:*

(a) *$(K, q, \mathcal{G}, \varphi)$ is pseudoisometric.*

(b) *$\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is a compact groupoid.*

(c) *The union of all locally trivial T_φ-invariant subbundles is fiberwise dense in $\mathrm{C}_q(K)$.*

(d) *The union of all finitely generated, projective, closed, T_φ-invariant $\mathrm{C}(\mathcal{G}^{(0)})$-submodules of $\mathrm{C}(K)$ is dense in $\mathrm{C}(K)$.*

(e) *The finitely generated, projective, closed, T_φ-invariant* $\mathrm{C}(\mathcal{G}^{(0)})$*-submodules of* $\mathrm{C}(K)$ *generate* $\mathrm{C}(K)$ *as a* C^**-algebra.*

Moreover, if these assertions hold, then all locally trivial T_φ-invariant subbundles of $\mathrm{C}_q(K)$ have constant finite dimension and the set of these subbundles is closed under finite sums.

We first prove some short and useful lemmas about locally trivial Banach bundles.

Lemma 4.15. *Let E be a Banach bundle over a compact space L which is locally trivial. If $M \subseteq E$ is a bounded subset, i.e.,* $\sup_{e\in M} \|e\|_{p(e)} < \infty$*, then it is precompact.*

Proof. We may assume M to be closed. Now pick a net $(e_\alpha)_{\alpha\in A}$ in M. Passing to a subnet, we may assume that $(p(e_\alpha))_{\alpha\in A}$ converges to some $l \in L$. By choosing a local trivialization as in Definition 4.2, the claim reduces to the case of a trivial Banach bundle $L \times \mathbb{C}^n$ for which it is obvious. □

Lemma 4.16. *Let E be a Hausdorff Banach bundle over a compact space L and $F \subseteq E$ a locally trivial subbundle. Then F is closed.*

Proof. Take a net $(e_\alpha)_{\alpha\in A}$ in F converging to some $e \in E$. There is an $\alpha_0 \in A$ such that $\sup_{\alpha\geqslant\alpha_0} \|e_\alpha\| < \infty$ and by Lemma 4.15 we find a subnet converging to an element of F. Since E is a Hausdorff space, we obtain that $e \in F$. □

Lemma 4.17. *Let $(K, q, \mathcal{G}, \varphi)$ be a groupoid action. If $F \subseteq \mathrm{C}_q(K)$ is a T_φ-invariant locally trivial subbundle, then F is also invariant under the induced representation of $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$.*

Proof. Let F be a locally trivial T_φ-invariant subbundle and consider the set

$$\mathcal{S} := \{\vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}) \mid F_{r(\vartheta)} \circ \vartheta \in F_{s(\vartheta)}\}.$$

It is clear that $\mathcal{S}$ is a subsemigroupoid of $\mathrm{C}_q^q(K, K)$ that contains $\mathcal{S}(K, q, \mathcal{G})$. We show that it is closed which implies $\mathcal{S} = \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$. So take a net $(\vartheta_\alpha)_{\alpha\in A}$ in $\mathcal{S}$ converging to $\vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ and $e \in F_{r(\vartheta)} = \mathrm{C}(K_{r(\vartheta)})$. Since F is a subbundle, we then find a continuous extension $f \in \mathrm{C}(K)$ with $f|_{K_{r(\vartheta)}} = e$ and $f|_{K_u} \in F_u$ for all $u \in \mathcal{G}^{(0)}$. Then

$$e \circ \vartheta = \lim_\alpha f|_{K_{r(\vartheta_\alpha)}} \circ \vartheta_\alpha \in F$$

by Lemma 4.16. □

Proof of Theorem 4.14. First recall that (a) and (b) are equivalent by Theorem 3.27. We now show that (b) implies (c) and (d) as well as the additional claims. So assume that $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is a compact groupoid. Then it is also transitive by Theorem 3.25 since $\mathcal{G}$ is topologically ergodic. Applying Theorem 4.8 to the Koopman representation of $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ on $\mathrm{C}_q(K)$ yields that the union of all $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$-invariant subbundles of constant finite dimension is fiber-wise dense in $\mathrm{C}_q(K)$ and that the set of these subbundles is closed under finite sums. Since $\mathrm{C}_q(K)$ is Hausdorff, these Banach bundles are locally trivial by Remark 4.3(iv). Conversely, since every T_φ-invariant locally trivial subbundle is invariant with respect to the representation of $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ by Lemma 4.17, all its fibers are isomorphic and it therefore has to be of constant finite dimension.

To show that (d) holds, observe that for each locally trivial invariant subbundle $F \subseteq \mathrm{C}_q(K)$, the set

$$\tilde{\Gamma}(F) := \{\sigma \in \Gamma(\mathrm{C}_q(K)) \mid \forall u \in \mathcal{G}^{(0)} \colon \sigma(u) \in F_u\} \subseteq \Gamma(\mathrm{C}_q(K))$$

is a $\mathrm{C}(\mathcal{G}^{(0)})$-submodule of $\Gamma(\mathrm{C}_q(K))$ which is isometrically isomorphic to $\Gamma(F)$ as a Banach module over $\mathrm{C}(\mathcal{G}^{(0)})$. In particular, $\tilde{\Gamma}(F)$ is finitely generated and projective as a $\mathrm{C}(\mathcal{G}^{(0)})$-module (see Remark 4.3 and Proposition 4.13) and closed in $\Gamma(\mathrm{C}_q(K))$. Let M be the union of all modules $\tilde{\Gamma}(F)$ where F is a locally trivial subbundle. Then M is a $\mathrm{C}(\mathcal{G}^{(0)})$-submodule since the sum $F = F_1 + F_2$ of two locally trivial invariant subbundles of F_1 and F_2 is again a locally trivial invariant subbundle and $\tilde{\Gamma}(F_1) + \tilde{\Gamma}(F_2) \subseteq \tilde{\Gamma}(F)$. Moreover, by Theorem 4.8, M is stalkwise dense in the sense of [Gie82, Definition 4.1] and via a Stone–Weierstraß theorem for bundles (see [Gie82, Corollary 4.3]), this implies that M is dense in $\Gamma(\mathrm{C}_q(K))$. Using the canonical isomorphism $\Gamma(\mathrm{C}_q(K)) \cong \mathrm{C}(K)$ noted in Example 4.5, we conclude that the union of all closed finitely generated and projective T_φ-invariant $\mathrm{C}(\mathcal{G}^{(0)})$-submodules is dense in $\mathrm{C}(K)$. Hence, (b) also implies (d).

Clearly, (d) implies (e). But also (c) implies (e): Assume that (c) holds. By the Stone–Weierstraß theorem it suffices to show that the elements of finitely generated, projective, closed, T_φ-invariant $\mathrm{C}(\mathcal{G}^{(0)})$-submodules separate the points of K. Let $x, y \in K$. If $q(x) \neq q(y)$, then we find $f \in \mathrm{C}(\mathcal{G}^{(0)})$ with $f(q(x)) \neq f(q(y))$ and therefore the elements of the submodule $\mathrm{C}(\mathcal{G}^{(0)}) \cdot \mathbb{1}$ separate x and y. On the other hand, if $u := q(x) = q(y)$, we find a locally trivial invariant subbundle $F \subseteq \mathrm{C}_q(K)$ and a section $\sigma \in \Gamma(F)$ with $\sigma(u)(x) \neq \sigma(u)(y)$. Since $\Gamma(F)$ defines a finitely generated, projective, closed, T_φ-invariant $\mathrm{C}(\mathcal{G}^{(0)})$-submodule of $\mathrm{C}(K)$, this shows the claim.

To finish the proof, we assume that e holds. We show that $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is compact which, by Proposition 3.11, already yields (b). By Theorem 3.13 it suffices to show that $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is equicontinuous. Recall from Corollary 3.14 that this is equivalent to $\{f \circ \vartheta \mid \vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})\}$ being equicontinuous or, equivalently, precompact in $\mathrm{C}_q(K)$ for every f in a set generating $\mathrm{C}(K)$ as a C*-algebra. In particular, by (e) we only have to show precompactness of this set for $f \in \mathrm{C}(K)$ contained in a finitely generated, projective, and closed T_φ-invariant $\mathrm{C}(\mathcal{G}^{(0)})$-submodule of $\mathrm{C}(K)$. Given such a submodule $\Gamma \subseteq \mathrm{C}(K)$, the subspaces $F_u := \Gamma|_{K_u}$ are finite-dimensional for $u \in \mathcal{G}^{(0)}$. As noted in Remark 4.11, they define a subbundle of the Banach bundle $\mathrm{C}_q(K)$ and $\Gamma(F)$ is isomorphic to Γ as a Banach module over $\mathrm{C}(\mathcal{G}^{(0)})$. Therefore F is locally trivial by Proposition 4.13. Clearly, it is T_φ-invariant. Since $\{f \circ \vartheta \mid \vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})\}$ is contained in F by Lemma 4.17, its precompactness follows using Lemma 4.15 since it is a bounded subset of $\mathrm{C}_q(K)$. We therefore obtain that $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is compact which finishes the proof. □

5 Haar systems and relatively invariant measures

Using the (uniform) enveloping semigroup, it can be shown that any equicontinuous, minimal system (K, G) has a unique invariant probability measure which is the pushforward of the Haar measure on the compact group $\mathrm{E}(K, G)$. A relative version of this result also holds in the sense that, given an equicontinuous extension $q\colon (K, G) \to (L, G)$ of minimal systems, there exists a unique relatively invariant measure for the extension (see [Gla75, Corllary 3.7]). We recall the definition and remind the reader of the notation introduced in the introduction.

Definition 5.1. Let $(K, q, \mathcal{G})$ be a groupoid action. A **relatively invariant measure** for $(K, q, \mathcal{G})$ is a continuous mapping

$$\mu\colon \mathcal{G}^{(0)} \to \mathrm{P}(K), \quad u \mapsto \mu_u$$

such that

- $q_*\mu_u = \delta_u$ for all $u \in \mathcal{G}^{(0)}$,
- $\varphi_g^*\mu_{r(g)} = \mu_{s(g)}$ for all $g \in \mathcal{G}$.

We call μ **fully supported** if $\operatorname{supp}\mu_u = K_u$ for every $u \in \mathcal{G}^{(0)}$.

Example 5.2. Let $q\colon (\mathbb{D}, \varphi) \to ([0,1], \mathrm{id}_{[0,1]})$ be the extension from Example 3.6 where φ is the rotation with varying velocity on $\mathbb{D}$. Then the groupoid action $(\mathbb{D}, q, \mathcal{S}(q))$ has a unique relatively invariant measure μ which is fully supported.

Relatively invariant measures have been studied systematically by Glasner in [Gla75]: They allow to lift measures along extensions; they serve as a topological version of the conditional expectations that ergodic theory relies on considerably; and as we discuss in Section 6, they are also essential for the Fourier analysis of pseudoisometric extensions. In this Section, we show that unique relatively invariant measures exist for pseudoisometric extensions of topologically ergodic systems. This extends previous results to a much larger class of nonminimal systems, including in particular all transitive systems. As above, it is essential to consider extensions as groupoid actions to carry out this generalization.

Since the unique invariant measure of a compact transitive group action is given by the pushforward of the Haar measure, we try to adapt this argument to the groupoid case. We therefore consider Haar systems, a generalization of Haar measures to group bundles and more generally groupoids; see [Ren80, Definition 2.2].

Definition 5.3. Let $\mathcal{G}$ be a compact group bundle and for $u \in \mathcal{G}^{(0)}$ let m_u be the Haar measure on the fiber group $\mathcal{G}_u$. Then $\mathcal{G}$ **has a continuous Haar system** if the mapping

$$\mathcal{G}^{(0)} \to \mathbb{C}, \quad u \mapsto \int f \,\mathrm{d}m_u$$

is continuous for each $f \in \mathrm{C}(\mathcal{G})$.

It is known that a compact group bundle $\mathcal{G}$ has a continuous Haar system if and only if the mapping $p \colon \mathcal{G} \to \mathcal{G}^{(0)}$ is open (see [Ren91, Lemma 1.3]). In particular, by Proposition 3.18 the isotropy bundle of every compact transitive groupoid has a continuous Haar system. With this knowledge, we can prove the first result of this section. Recall from Definition 1.10 that a groupoid action is called transitive if every orbit is the entire space.

Theorem 5.4. *Let $(K, q, \mathcal{G})$ be an action by a compact transitive groupoid $\mathcal{G}$. Then the following assertions hold.*

(i) $(K, q, \mathcal{G})$ *admits a relatively invariant measure.*

(ii) $(K, q, \mathcal{G})$ *admits a unique relatively invariant measure if and only if the action $(K, q, \mathcal{G})$ is transitive. In this case, the measure is fully supported.*

The proof requires the following continuity lemma.

Lemma 5.5. *Let $q \colon K \to L$ be a continuous open surjection between compact spaces and*

$$\mu \colon L \to \mathrm{P}(K), \quad l \mapsto \mu_l$$

a continuous map with $q_*\mu_l = \delta_l$ for every $l \in L$. Moreover, let $(f_\alpha)_{\alpha \in A}$ be a convergent net in $\mathrm{C}_q(K)$ with limit $f \in \mathrm{C}_q(K)$. Then

$$\lim_\alpha \int_{K_{s(f_\alpha)}} f_\alpha \,\mathrm{d}\mu_{s(f_\alpha)} = \int_{K_{s(f)}} f \,\mathrm{d}\mu_{s(f)}.$$

Proof. Choose $F \in \mathrm{C}(K)$ such that $F|_{K_{s(f)}} = f$. For each $\alpha \in A$ choose an $x_\alpha \in K_{s(f_\alpha)}$ such that

$$|f_\alpha(x_\alpha) - F(x_\alpha)| = \sup_{x \in K_{s(f_\alpha)}} |f_\alpha(x) - F(x)|.$$

For each subnet of $(f_\alpha)_{\alpha \in A}$ we then find a subnet $(f_\beta)_{\beta \in B}$ such that $x = \lim_\beta x_\beta$ exists in K. But then

$$\lim_\beta \sup_{x \in K_{s(f_\beta)}} |f_\beta(x) - F(x)| = \lim_\beta |f_\beta(x_\beta) - F(x_\beta)| = 0.$$

As a consequence,

$$\lim_\alpha \left| \int_{K_{s(f_\alpha)}} f_\alpha \,\mathrm{d}\mu_{s(f_\alpha)} - \int_{K_{s(f_\alpha)}} F \,\mathrm{d}\mu_{s(f_\alpha)} \right| \leqslant \lim_\alpha \sup_{x \in K_{s(f_\alpha)}} |f_\alpha(x) - F(x)| = 0,$$

which implies the claim. □

Proof of Theorem 5.4. As above, we denote the Haar measure on $\mathcal{G}_u^u$ by m_u for $u \in \mathcal{G}^{(0)}$. In order to prove (i), it suffices to consider the case that $(K, q, \mathcal{G})$ is transitive (in which case $\mathcal{G}$ is automatically transitive). To see this, note that for fixed $x \in K$, the orbit $\mathcal{G}x$ is a closed, $\mathcal{G}$-invariant subset and that q restricted to $\mathcal{G}x$ is again an open surjection (use Proposition 3.18).

Now suppose $(K, q, \mathcal{G})$ is transitive. For $x \in K$, denote by

$$\rho_x \colon \mathcal{G}_{q(x)}^{q(x)} \to K_{q(x)}, \quad g \mapsto gx$$

the induced map onto the orbit of x. Now pick a point $x_u \in K_u$ for each $u \in \mathcal{G}^{(0)}$ and set

$$\mu_u := (\rho_{x_u})_*(\mathrm{m}_u).$$

It is clear from the transitivity of the group action of $\mathcal{G}_u^u$ on K_u that μ_u does not depend on the choice of $x_u \in K_u$ and that $\operatorname{supp} \mu_u = K_u$ for every $u \in \mathcal{G}^{(0)}$. Moreover, $\varphi_g^* \mu_{s(g)} = \mu_{r(g)}$ for every $g \in \mathcal{G}$.

Now take $f \in \mathrm{C}(K)$. We show that $\lim_\alpha \mu_{u_\alpha}(f) = \mu_u(f)$ for every net $(u_\alpha)_{\alpha \in A}$ converging to some $u \in L$. By passing to a subnet, we may assume that there is a convergent net $(x_\alpha)_{\alpha \in A}$ in K with limit $x \in K$ that satisfies $q(x_\alpha) = u_\alpha$ for all $\alpha \in A$. Then $\rho_{x_\alpha} \to \rho_x$ with respect to the compact-open topology and so $f \circ \rho_{x_\alpha} \to f \circ \rho_x$ with respect to the compact-open topology. Therefore, Lemma 5.5 yields

$$\lim_{\alpha \in A} \langle f, \mu_{u_\alpha} \rangle = \lim_{\alpha \in A} \langle f \circ \rho_{x_\alpha}, \mathrm{m}_{u_\alpha} \rangle = \langle f \circ \rho_x, \mathrm{m}_u \rangle = \langle f, \mu_u \rangle.$$

Hence, $\mu \colon \mathcal{G}^{(0)} \to \mathrm{P}(K)$ is continuous. This shows (i) as well as the existence of a fully supported relatively invariant measure in case of a transitive action.

It remains to show that there is a unique relatively invariant measure if and only if $(K, q, \mathcal{G})$ is transitive. Since we have seen that any orbit of K carries a relatively invariant measure, the action must be transitive if there is only one relatively invariant measure. Conversely, suppose $(K, q, \mathcal{G})$ is transitive and let μ be the relatively invariant measure constructed above. Take any relatively invariant measure $\nu\colon \mathcal{G}^{(0)} \to \mathrm{P}(K)$ for $(K, q, \mathcal{G})$ and let $u \in \mathcal{G}^{(0)}$. Then ν is invariant under the action of $\mathcal{G}_u^u$. Since a transitive action of a compact group is equicontinuous and minimal and therefore uniquely ergodic, $\nu_u = \mu_u$. Since $u \in \mathcal{G}^{(0)}$ was arbitrary, μ is the unique relatively invariant measure for $(K, q, \mathcal{G})$. $\square$

In order to apply Theorem 5.4 to a pseudoisometric groupoid action $(K, q, \mathcal{G})$ via the uniform enveloping groupoid, we have to understand when the induced action of $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ is transitive. As noted in Remark 1.11, the transitivity of the groupoid action $(K, q, \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}))$ can be split into two transitivity properties:

- "Transitivity in direction of $\mathcal{G}^{(0)}$": The induced action $(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}))$ on the unit space $\mathcal{G}^{(0)}$ is transitive.
- "Transitivity in direction of q": The isotropy groups of $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ act transitively on the fibers of q, i.e., $(K, q, \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}))$ is fiberwise transitive.

We have already shown that transitivity in direction of $\mathcal{G}^{(0)}$ is equivalent to $\mathcal{G}$ being topologically ergodic, so it remains to find a useful characterization for fiberwise transitivity. To this end, we introduce a notion of relative topological ergodicity for groupoid actions and show that it yields the desired characterization.

Definition 5.6. A groupoid action $(K, q, \mathcal{G})$ is called **relatively topologically ergodic** if the canonical map $\alpha\colon \mathrm{fix}(K, q, \mathcal{G}) \to \mathrm{fix}(\mathcal{G}^{(0)}, \mathrm{id}, \mathcal{G})$ is an isomorphism of maximal trivial factors.

Remark 5.7. A groupoid action $(K, q, \mathcal{G})$ is relatively topologically ergodic if and only if the restricted operator

$$T_q|_{\mathrm{fix}(T_{\mathcal{G}})}\colon \mathrm{fix}(T_{\mathcal{G}}) \to \mathrm{fix}(T_{\varphi}), \quad f \mapsto f \circ q$$

is bijective.

Clearly, every topologically ergodic groupoid action is relatively topologically ergodic. However, there are groupoid actions which are only relatively topologically ergodic, but not topologically ergodic.

Example 5.8. If $\mathcal{G}$ is any groupoid with compact unit space which is not topologically ergodic, then the action $(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$ is still topologically ergodic. More concretely, let (K, G) be a topological dynamical system which is not topologically ergodic. Then the action of the action groupoid $G \ltimes K$ on K is relatively topologically ergodic but not topologically ergodic.

The following result now relates ergodicity of a pseudoisometric groupoid action with fiberwise transitivity of the action of the induced uniform enveloping groupoid.

Proposition 5.9. *For a pseudoisometric groupoid action* $(K, q, \mathcal{G})$ *the following assertions are equivalent.*

(a) *The action of* $\mathcal{G}$ *on* K *is relatively topologically ergodic.*

(b) *The action of* $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ *on* K *is fiberwise transitive.*

The proof follows from the following lemma which provides a more explicit characterization of topological ergodicity in terms of orbits of the uniform enveloping groupoid.

Lemma 5.10. *Let* $(K, q, \mathcal{G}, \varphi)$ *be a pseudoisometric groupoid action. Then the following assertions hold:*

(i) *The map*
$$\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}) \ltimes K \to \mathcal{E}_{\mathrm{u}}(R_\varphi), \quad (\vartheta, x) \mapsto (x, \vartheta(x))$$
is a surjective morphism of compact groupoids.

(ii) *The orbits of* $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ *on* K *are precisely the equivalence classes of* $\mathcal{E}_{\mathrm{u}}(R_\varphi)$.

(iii) *For each* $x \in K$
$$q\big(\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})x\big) = \mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})q(x).$$
Moreover, $(K, q, \mathcal{G})$ *is relatively topologically ergodic if and only if for each* $x \in K$
$$\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})x = q^{-1}(\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})q(x)),$$
i.e., if every $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$*-invariant subset* $A \subseteq K$ *is* q*-saturated.*

Proof. For i, notice that the set
$$\mathcal{S} := \{\vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}) \mid \forall x \in K_{s(\vartheta)} \colon (x, \vartheta(x)) \in \mathcal{E}_{\mathrm{u}}(R_\varphi)\}$$
is a closed subsemigroupoid of $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ that contains $\mathcal{S}(K, q, \mathcal{G})$ and therefore
$$\mathcal{S} = \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}).$$
Clearly, the mapping
$$\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}) \ltimes K \to \mathcal{E}_{\mathrm{u}}(R_\varphi), \quad (\vartheta, x) \mapsto (x, \vartheta(x))$$
is continuous and a morphism of groupoids. Since its image is a compact subsemigroupoid of $\mathcal{E}_{\mathrm{u}}(R_\varphi)$ that contains R_φ, (i) holds. Moreover, (ii) is a direct consequence of (i).

For part (iii), use Lemma 3.19 to see that the extension

$$q\colon (K, q, \mathcal{G}) \to (\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$$

extends to an extension

$$q\colon (K, q, \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})) \to (\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})).$$

Thus, q maps orbits of $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ onto orbits of $\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})$. Now, consider the following commutative diagram:

$$\begin{tikzcd}
 & (K, q, \mathcal{G}) \arrow[dl, "q"'] \arrow[dr, "q_{\mathrm{fix}}^{K}"] & \\
(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G}) \arrow[dr, "q_{\mathrm{fix}}^{\mathcal{G}^{(0)}}"'] & & \mathrm{fix}(K, q, \mathcal{G}) \arrow[dl, "\alpha"] \\
 & \mathrm{fix}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G}) &
\end{tikzcd}$$

Suppose that $(K, q, \mathcal{G})$ is relatively topologically ergodic, i.e., that α is an isomorphism. Then every $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$-invariant subset $A \subseteq K$ is q_{fix}^{K}-saturated by (ii) and since the above diagram commutes, it is also saturated with respect to $q_{\mathrm{fix}}^{\mathcal{G}^{(0)}} \circ q$ and hence with respect to q.

Conversely, suppose that every $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$-invariant subset $A \subseteq K$ is q-saturated and take such a set A. Then

$$q(A) = q(\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})A) = \mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})q(A).$$

In other words, $q(A)$ is also saturated with respect to $q_{\mathrm{fix}}^{\mathcal{G}^{(0)}}$. Hence, we conclude that q_{fix}^{K} and $q_{\mathrm{fix}}^{\mathcal{G}^{(0)}} \circ q = \alpha \circ q_{\mathrm{fix}}^{K}$ have the same saturated sets, meaning that α has to be injective. Therefore, α is an isomorphism. □

Proof of Proposition 5.9. If (a) holds and $x \in K$, then

$$q^{-1}(q(x)) \subseteq \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})x$$

by Lemma 5.10 (iii) which yields $q^{-1}(q(x)) = \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})_{q(x)}^{q(x)}x$.

Now assume that (b) holds. We take $x \in K$ and $y \in q^{-1}(\mathcal{E}_{\mathrm{u}}(\mathcal{G}^{(0)}, \mathrm{id}_{\mathcal{G}^{(0)}}, \mathcal{G})x)$. By Lemma 3.19 we find $\vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ with $s(\vartheta) = q(x)$ and $r(\vartheta) = q(y)$. But then we can apply (b) to find $\varrho \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ with $s(\varrho) = r(\varrho) = q(y)$ and $\vartheta(\varrho(x)) = y$. This shows that

$$y = (\vartheta \circ \varrho)(x) \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})x$$

and therefore (a) by Lemma 5.10 (iii). □

Corollary 5.11. *For a pseudoisometric groupoid action $(K, q, \mathcal{G})$ the following assertions are equivalent:*

(a) $(K, q, \mathcal{G})$ *is topologically ergodic.*

(b) $\mathcal{G}$ *is topologically ergodic and* $(K, q, \mathcal{G})$ *is relatively topologically ergodic.*

(c) $(K, q, \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}))$ *is transitive.*

For the groupoid actions satisfying the equivalent conditions of Corollary 5.11, we now prove the existence and uniqueness of relatively invariant measures.

Theorem 5.12. *Let $(K, q, \mathcal{G})$ be a pseudoisometric and topologically ergodic groupoid action. Then there is a unique relatively invariant measure for $(K, q, \mathcal{G})$. Moreover, this relatively invariant measure is fully supported.*

Proof. The existence of a fully supported relatively invariant measure follows by combining Corollary 5.11 and Theorem 5.4. To establish uniqueness, we need to know that any relatively invariant measure for $(K, q, \mathcal{G})$ also is a relatively invariant measure for $(K, q, \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}))$ to apply Theorem 5.4 again. This is done in Lemma 5.13 below. □

Lemma 5.13. *Let $(K, q, \mathcal{G})$ be a groupoid action with relatively invariant measure μ. Then μ also is a relatively invariant measure for the groupoid action $(K, q, \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}))$.*

Proof. We need to show that $\vartheta_* \mu_{s(\vartheta)} = \mu_{r(\vartheta)}$ for every $\vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$. The set

$$\mathcal{S} := \{\vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}) \mid \vartheta_* \mu_{s(\vartheta)} = \mu_{r(\vartheta)}\}$$

is a subsemigroupoid $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ that contains $\mathcal{S}(K, q, \mathcal{G})$. We only have to check that it is closed. If $(\vartheta_\alpha)_{\alpha\in A}$ is a net in $\mathcal{S}$ converging to $\vartheta \in \mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G})$ and $f \in \mathrm{C}(K)$, then $\lim_\alpha T_{\vartheta_\alpha} f = T_\vartheta f$ in $\mathrm{C}_q(K)$ and therefore $\lim_\alpha \langle T_{\vartheta_\alpha} f, \mu_{s(\vartheta_\alpha)}\rangle = \langle T_\vartheta f, \mu_{s(\vartheta)}\rangle$ by Lemma 5.5. Thus,

$$\langle f, \vartheta_* \mu_{s(\vartheta)}\rangle = \lim_\alpha \langle T_\vartheta f, \mu_{s(\vartheta)}\rangle = \lim_\alpha \langle f, (\vartheta_\alpha)_* \mu_{s(\vartheta_\alpha)}\rangle = \lim_\alpha \langle f, \mu_{r(\vartheta_\alpha)}\rangle = \langle f, \mu_{r(\vartheta)}\rangle.$$

This shows that $\vartheta \in \mathcal{S}$ and so $\mathcal{E}_{\mathrm{u}}(K, q, \mathcal{G}) = \mathcal{S}$. □

6 Fourier analysis

The classical Peter–Weyl theorem allows, given a compact group G with its Haar measure m, to decompose the Hilbert space $\mathrm{L}^2(G, m)$ into a canonical orthogonal sum of finite-dimensional G-invariant subspaces. These subspaces and the projections onto them are defined by means of the unitary dual $\hat{G}$ which consists of equivalence classes $[\pi]$ of irreducible unitary representations π of G. As we recall in Theorem 6.1 below, this Fourier-analytic result can easily be extended to a transitive action (K, G) of a compact group G, allowing to similarly decompose the space $\mathrm{L}^2(K, \mu)$ where μ denotes the unique invariant probability measure on K obtained as the pushforward of the Haar measure m. The goal of this section is to generalize this to a Fourier analysis result for actions of compact transitive groupoids which is of interest on its own but will also be applied to uniform enveloping semigroupoids in Section 7.

To understand the situation for a transitive action (K, G) of a compact group, let μ denote the above-mentioned unique G-invariant probability measure on K. In order to obtain prospective projection operators on $\mathrm{L}^2(K, \mu)$, define for $f \in \mathrm{L}^2(K, \mu)$, $[\pi] \in \hat{G}$, and μ-a.e. $x \in K$

$$(P_{[\pi]}f)(x) := \dim([\pi]) \int_G \mathrm{tr}([\pi])(g) f(g^{-1}x)\, \mathrm{d}m(g).$$

Here, $\dim([\pi])$ and $\mathrm{tr}([\pi])$ denote the dimension and trace of $[\pi]$, respectively. With these definitions, one obtains the following easy consequence of the Peter–Weyl theorem.

Theorem 6.1. *Let (K, G) be a transitive action of a compact group G and μ its unique invariant probability measure. Then the following assertions hold.*

(i) $P_{[\pi]} \in \mathscr{L}(\mathrm{L}^2(K, \mu))$ *is an orthogonal projection for every* $[\pi] \in \hat{G}$.

(ii) *The ranges* $\mathrm{rg}(P_{[\pi]})$ *for* $[\pi] \in \hat{G}$ *are finite-dimensional, invariant subspaces of* $\mathrm{C}(K)$ *and are pairwise orthogonal in* $\mathrm{L}^2(K, \mu)$.

(iii) *For every* $f \in \mathrm{L}^2(K, \mu)$ *and* $[\pi] \in \hat{G}$

$$\|P_{[\pi]}f\|_{\mathrm{C}(K)} \leqslant \|f\|_{\mathrm{L}^2(K,\mu)}.$$

(iv) *Each* $f \in \mathrm{C}(K)$ *is contained in*

$$\overline{\mathrm{lin}\{P_{[\pi]}f \mid [\pi] \in \hat{G}\}}^{\|\cdot\|_{\mathrm{C}(K)}} \subseteq \mathrm{C}(K).$$

(v) *For each* $f \in \mathrm{L}^2(K, \mu)$ *we have*

$$(f \mid f) = \sum_{[\pi] \in \hat{G}} (P_{[\pi]}f \mid P_{[\pi]}f).$$

(vi) *Each* $f \in \mathrm{L}^2(K, \mu)$ *can be decomposed into a series*

$$f = \sum_{[\pi] \in \hat{G}} P_{[\pi]}f$$

converging in $\mathrm{L}^2(K, \mu)$.

Proof. A simple application of the Cauchy–Schwarz inequality and Fubini's theorem shows that $P_{[\pi]} \in \mathscr{L}(\mathrm{L}^2(K,\mu))$ for every $[\pi] \in \hat{G}$. Moreover, if $g \in G$ and $T_g \in \mathscr{L}(\mathrm{L}^2(K,\mu))$ is the Koopman operator defined by $T_g f(x) := f(g^{-1}x)$ for $x \in K$ and $f \in \mathrm{L}^2(K,\mu)$, then $P_{[\pi]}T_g = T_g P_{[\pi]}$ since the trace of a representation is constant on conjugacy classes. In particular, $\mathrm{rg}(P_{[\pi]})$ is T_g-invariant for every $g \in G$.

Now, note that by Fourier analysis of compact groups (see, e.g., [Fol15, Sections 5.2 and 5.3]), the remaining assertions are clear if (K,G) is given by multiplication from the left on $K = G$. In this case, we denote the projections by $Q_{[\pi]}$ for $[\pi] \in \hat{G}$. If (K,G) is a general transitive action of G, we fix $x \in K$. The orbit map

$$\varrho_x \colon G \to K, \quad g \mapsto gx$$

then is a continuous surjection. It induces an isometry $T_{\varrho_x} \in \mathscr{L}(\mathrm{C}(K),\mathrm{C}(G))$ which then extends to an isometric embedding $T_{\varrho_x} \in \mathscr{L}(\mathrm{L}^2(K,\mu),\mathrm{L}^2(G,m))$. Since $T_{\varrho_x}P_{[\pi]} = Q_{[\pi]}T_{\varrho_x}$ for every $[\pi] \in \hat{G}$, the statements now readily extend to the general situation. □

In order to prove a version of THeorem 6.1 for transitive actions of compact groupoids, it is necessary to replace the unique invariant probability measure with the unique relatively invariant measure of Theorem 5.4, and the induced Hilbert space with the space of continuous sections of a (continuous) Hilbert bundle.

Definition 6.2. Let $q \colon K \to L$ be an open continuous surjection between compact spaces and $\mu \colon L \to \mathrm{P}(K)$, $l \mapsto \mu_l$ a weak*-continuous mapping with $q_*\mu_l = \delta_l$ and $\operatorname{supp}\mu_l = K_l$ for every $l \in L$. We consider the Banach bundle defined by

$$\mathrm{L}^2_q(K,\mu) := \bigcup_{l\in L} \mathrm{L}^2(K_l,\mu_l)$$

with the canonical mapping $p \colon \mathrm{L}^2_q(K,\mu) \to L$ and the topology defined by the sets

$$V(F,U,\varepsilon) := \{f \in \mathrm{L}^2_q(K,\mu) \mid p(f) \in U, \|f - F|_{K_{p(f)}}\|_{\mathrm{L}^2(K_{p(f)},\mu_{p(f)})} < \varepsilon\}$$

for $F \in \mathrm{C}(K)$, $U \subseteq L$ open, and $\varepsilon > 0$.

Remark 6.3. In the situation of Definition 6.2, it is standard to check that $\mathrm{L}^2_q(K,\mu)$ endowed with the natural norm mapping is a continuous Banach bundle. In fact, it is even a **Hilbert bundle**, i.e., the map

$$(\cdot \mid \cdot) \colon \mathrm{L}^2_q(K,\mu) \times_L \mathrm{L}^2_q(K,\mu) \to \mathbb{C}, \quad (f_1,f_2) \mapsto (f_1 \mid f_2)_{\mu_{p(f_1)}} := \int f_1\overline{f_2}\,\mathrm{d}\mu_{p(f_1)}$$

is continuous. Its space of sections $\Gamma(\mathrm{L}^2_q(K,\mu))$ equipped with the "vector-valued inner product"

$$(\cdot \mid \cdot)_\mu \colon \Gamma(\mathrm{L}^2(K,\mu)) \to \mathrm{C}(L), \quad \sigma \mapsto [l \mapsto (\sigma(l) \mid \sigma(l))_{\mu_l}]$$

is then a Hilbert C*-module over $\mathrm{C}(L)$ (see, e.g., [DG83] or [Lan95] for this concept). We indentify $\mathrm{C}(K)$ with a submodule of $\Gamma(\mathrm{L}^2_q(K,\mu))$ via the injective $\mathrm{C}(L)$-module homomorphism

$$\mathrm{C}(K) \to \Gamma(\mathrm{L}^2_q(K,\mu)), \quad f \mapsto [l \to f|_{K_l}].$$

By the Stone–Weierstraß theorem for bundles (see [Gie82, Corollary 4.3]), $\mathrm{C}(K)$ is dense in $\Gamma(\mathrm{L}^2_q(K,\mu))$.

Given a transitive action $(K, q, \mathcal{G})$ of a compact groupoid $\mathcal{G}$, we now obtain a Hilbert bundle in a canonical way by applying the conctruction of Definition 6.2 to the unique relatively invariant measure of Theorem 5.4. The space $\Gamma(\mathrm{L}^2_q(K,\mu))$ of continuous sections of this bundle will then take the role the Hilbert space $\mathrm{L}^2(K,\mu)$ played in the case of a group action.

Finally, in order to formulate a version of Theorem 6.1 for groupoid actions $(K, q, \mathcal{G})$, the last required ingredient is a generalization of the occurring projection operators $P_{[\pi]}$. Below, we first define them on each fiber of the Hilbert bundle $\mathrm{L}^2(K,\mu)$ using the irreducible representations of the isotropy bundle $\mathrm{Iso}(\mathcal{G})$ of $\mathcal{G}$. In order for the fiber operators to fit together to a well-defined projection operator on $\Gamma(\mathrm{L}^2(K,\mu))$ and to ensure its $\mathcal{G}$-invariance, we need to enforce a compatibility condition on the irreducible representations of $\mathrm{Iso}(\mathcal{G})$ that are employed.

Definition 6.4. Let $\mathcal{G}$ be a compact groupoid.

(i) If π is an irreducible unitary representation of $\mathcal{G}_u^u$ and $g \in \mathcal{G}_u$, we define $\pi^g(h) := \pi(g^{-1}hg)$ for $h \in \mathcal{G}_{r(g)}^{r(g)}$. Moreover, set

$$[\pi]^g := [\pi^g] \in \widehat{\mathcal{G}_{r(g)}^{r(g)}}.$$

(ii) We call a map

$$\gamma\colon \mathcal{G}^{(0)} \to \bigcup_{u\in\mathcal{G}^{(0)}} \widehat{\mathcal{G}_u^u}$$

an **invariant section** if

- $\gamma(u) \in \widehat{\mathcal{G}_u^u}$ for every $u \in \mathcal{G}^{(0)}$.
- $\gamma(u)^g = \gamma(gug^{-1})$ for all $u \in \mathcal{G}^{(0)}$ and $g \in \mathcal{G}_u$.

Moreover, $\Gamma_{\mathcal{G}}$ denotes the set of all such invariant sections.

Remark 6.5. If $\mathcal{G}$ is a transitive compact groupoid and $u \in \mathcal{G}^{(0)}$ is fixed, then every $[\pi] \in \widehat{\mathcal{G}_u^u}$ defines an invariant section via $\gamma_{[\pi]}(g^{-1}ug) := [\pi^g]$ for all $g \in \mathcal{G}^u$, and every invariant section is of this form. In this case, we therefore obtain a bijection

$$\widehat{\mathcal{G}_u^u} \to \Gamma_{\mathcal{G}}, \quad [\pi] \mapsto \gamma_{[\pi]},$$

i.e., up to choosing a base point $u \in \mathcal{G}^{(0)}$, the set $\Gamma_{\mathcal{G}}$ can simply be seen as one of the unitary duals of the isotropy groups of $\mathcal{G}$.

We can now define the projection operators defined by such invariant sections. Recall here that a compact groupoid acting transitively has to be transitive itself and therefore the observations of Remark 6.5 are valid in this context.

Definition 6.6. Let $(K, q, \mathcal{G})$ be a transitive groupoid action of a compact groupoid $\mathcal{G}$ with unique relatively invariant measure μ. For $\gamma \in \Gamma_{\mathcal{G}}$ the **associated projection** P_γ is defined by $(P_\gamma\sigma)(u) := P_{\gamma(u)}\sigma(u)$ for $u \in \mathcal{G}^{(0)}$ and $\sigma \in \Gamma(\mathrm{L}^2_q(K, \mu))$.

We now obtain our Fourier analytic result for transitive actions of compact groupoids extending results of Knapp (cf. [Kna67, Theorem 1.2]). Here, two subsets $M_1, M_2 \subseteq \Gamma(\mathrm{L}^2(K, \mu))$ are called **orthogonal** if

$$(\sigma_1 \mid \sigma_2)_\mu = 0$$

for all $\sigma_1 \in M_1$, $\sigma_2 \in M_2$. Moreover, $\mathrm{lin}_{\mathrm{C}(\mathcal{G}^{(0)})}$ denotes the linear hull with respect to the $\mathrm{C}(\mathcal{G}^{(0)})$-module structure on $\mathrm{C}(K)$. Recall also from Remark 6.3 that we identify $\mathrm{C}(K)$ with a dense submodule of $\Gamma(\mathrm{L}^2_q(K, \mu))$.

Theorem 6.7. *For a transitive action $(K, q, \mathcal{G}, \varphi)$ of a compact groupoid $\mathcal{G}$ with unique relatively invariant measure μ the following assertions hold:*

(i) *For every $\gamma \in \Gamma_{\mathcal{G}}$, $P_\gamma \in \mathscr{L}(\Gamma(\mathrm{L}^2_q(K, \mu)))$ is a projection and a $\mathrm{C}(\mathcal{G}^{(0)})$-module homomorphism.*

(ii) *For every $\sigma \in \Gamma(\mathrm{L}^2_q(K, \mu))$ and $\gamma \in \Gamma_{\mathcal{G}}$*

$$\|P_\gamma\sigma\|_{\mathrm{C}(K)} \leqslant \|\sigma\|_{\Gamma(\mathrm{L}^2_q(K,\mu))}.$$

(iii) *The ranges $\mathrm{rg}(P_\gamma)$ for $\gamma \in \Gamma_{\mathcal{G}}$ are closed, finitely generated, projective, T_φ-invariant $\mathrm{C}(\mathcal{G}^{(0)})$-submodules of $\mathrm{C}(K)$ and are pairwise orthogonal in $\Gamma(\mathrm{L}^2_q(K, \mu))$.*

(iv) *Each $f \in \mathrm{C}(K)$ is contained in*

$$\overline{\mathrm{lin}_{\mathrm{C}(\mathcal{G}^{(0)})}\{P_\gamma f \mid \gamma \in \Gamma_{\mathcal{G}}\}} \subseteq \mathrm{C}(K).$$

(v) *For each $\sigma \in \Gamma(\mathrm{L}^2_q(K, \mu))$*

$$(\sigma \mid \sigma)_\mu = \sum_{\gamma \in \Gamma_{\mathcal{G}}} (P_\gamma\sigma \mid P_\gamma\sigma)_\mu$$

with convergence in $\mathrm{C}(\mathcal{G}^{(0)})$.

(vi) *Each $\sigma \in \Gamma(\mathrm{L}^2_q(K, \mu))$ can be decomposed into a series*

$$\sigma = \sum_{\gamma \in \Gamma_{\mathcal{G}}} P_\gamma\sigma$$

converging in $\Gamma(\mathrm{L}^2_q(K, \mu))$.

Proof. We start with the proof of assertions (i), (ii), and (iii), so fix $\gamma \in \Gamma_{\mathcal{G}}$. We first show that $P_\gamma f \in \mathrm{C}(K)$ for every $f \in \mathrm{C}(K)$. So let $f \in \mathrm{C}(K)$ and define for every $x \in K$ the continuous function

$$F_x \colon \mathcal{G}_{q(x)}^{q(x)} \to \mathbb{C}, \quad g \mapsto \mathrm{tr}(\gamma(q(x))(g)f(g^{-1}x).$$

We claim that the map

$$F \colon K \to \mathrm{C}_p^p(\mathrm{Iso}(\mathcal{G})), \quad x \mapsto F_x$$

is continuous which would imply the continuity of $P_\gamma f$ via the integral continuity criterion from Lemma 5.5. To see that F is indeed continuous, we use the continuity characterization from Proposition 2.4. So let $(x_\alpha)_{\alpha\in A}$ be a net in K with limit $x \in K$, $(x_\beta)_{\beta\in B}$ be any subnet, and let $(g_\beta)_{\beta\in B}$ be a net in $\mathrm{Iso}(\mathcal{G})$ such that $p(g_\beta) = q(x_\beta)$ for every $\beta \in B$. Since $\mathcal{G}$ is transitive, there is an $h_\beta \in \mathcal{G}_{p(g)}^{p(g_\beta)}$ for every $\beta \in B$ and by the usual subnet arguments we may assume that $(h_\beta)_{\beta\in B}$ converges to the unit $p(g) = q(x)$ of $\mathcal{G}_{p(g)}^{p(g)}$. Since γ is an invariant section, we obtain for every $\beta \in B$

$$\begin{aligned}\mathrm{tr}(\gamma(q(x_\beta))(g_\beta)) &= \mathrm{tr}(\gamma(q(h_\beta^{-1}x_\beta h_\beta))^{h_\beta}(g_\beta)) = \mathrm{tr}(\gamma(q(x))^{h_\beta}(g_\beta))\\ &= \mathrm{tr}(\gamma(q(x))(h_\beta^{-1}g_\beta h_\beta).\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_\beta F_{x_\beta}(g_\beta) &= \lim_\beta \mathrm{tr}(\gamma(q(x_\beta))(g_\beta))f(g_\beta^{-1}x_\beta)\\ &= \lim_\beta \mathrm{tr}(\gamma(q(x))(h_\beta^{-1}g_\beta h_\beta))f(g_\beta^{-1}x_\beta)\\ &= \mathrm{tr}(\gamma(q(x))(g))f(g^{-1}x)\\ &= F_x(g).\end{aligned}$$

Hence, F is continuous and so $P_\gamma f \in \mathrm{C}(K)$. Moreover, for $f \in \mathrm{C}(K)$ it follows from Theorem 6.1 (iii) that

$$\|P_\gamma f\|_{\mathrm{C}(K)} \leqslant \|f\|_{\Gamma(\mathrm{L}^2(K,\mu))}.$$

For $\sigma \in \Gamma(\mathrm{L}_q^2(K,\mu))$ we already know from Theorem 6.1(ii) that $(P_\gamma\sigma)(u) \in \mathrm{C}(K_u)$ for every $u \in \mathcal{G}^{(0)}$. Moreover, for $\varepsilon > 0$ there is $f_\varepsilon \in \mathrm{C}(K)$ with

$$\|f_\varepsilon|_{K_u} - \sigma(u)\|_{\mathrm{L}^2(K_u,\mu_u)} \leqslant \varepsilon$$

for all $u \in \mathcal{G}^{(0)}$ since $\mathrm{C}(K)$ is dense in $\Gamma(\mathrm{L}_q^2(K,\mu))$. But then

$$\|(P_\gamma f_\varepsilon)|_{K_u} - (P_\gamma\sigma)(u)\|_{\mathrm{C}(K_u)} \leqslant \varepsilon$$

for every $u \in \mathcal{G}^{(0}$ and $\varepsilon > 0$ by Theorem 6.1 (iii). Using this observation and the fact that—by the above—$P_\gamma f_\varepsilon \in \mathrm{C}(K)$ for every $\varepsilon > 0$ it is easy to check that $P_\gamma \sigma \in \mathrm{C}(K)$. Moreover, since $\mathrm{C}(K)$ is dense in $\Gamma(\mathrm{L}^2(K,\mu))$, we obtain

$$\|P_\gamma \sigma\|_{\mathrm{C}(K)} \leqslant \|\sigma\|_{\Gamma(\mathrm{L}^2_q(K,\mu))}$$

for every $\sigma \in \Gamma(\mathrm{L}^2_q(K,\mu))$. This proves (ii). Due to the fiberwise definition of P_γ, one readily verifies that it is a projection and a $\mathrm{C}(\mathcal{G}^{(0)})$-module homomorphism, proving (i).

We now prove that $\mathrm{rg}(P_\gamma)$ is $\mathcal{G}$-invariant, i.e., for all $\hbar \in \mathcal{G}$ that

$$T_\hbar(\mathrm{rg}(P_\gamma)|_{K_{s(\hbar)}}) \subseteq \mathrm{rg}(P_\gamma)|_{K_{r(\hbar)}}.$$

To that end, let $\hbar \in \mathcal{G}$ and note that $\mathrm{rg}(P_\gamma)|_{K_{s(\hbar)}}$ is finite-dimensional, so since $P_\gamma(\mathrm{C}(K))$ is dense in $\mathrm{rg}(P_\gamma)$,

$$\mathrm{rg}(P_\gamma)|_{K_{s(\hbar)}} = P_\gamma(\mathrm{C}(K))|_{K_{s(\hbar)}}.$$

Therefore, an element of $\mathrm{rg}(P_\gamma)|_{K_{s(\hbar)}}$ can be written as $(P_\gamma f)|_{K_{s(\hbar)}}$ for some $f \in \mathrm{C}(K)$. If an element of $\mathrm{rg}(P_\gamma)|_{K_{s(\hbar)}}$ is presented in this way, then for every $x \in K_{r(\hbar)}$

$$\begin{aligned}(T_\hbar(P_\gamma f)|_{K_{s(\hbar)}})(x) &= \dim(\gamma(s(\hbar))) \int_{\mathcal{G}^{s(\hbar)}_{s(\hbar)}} \mathrm{tr}(\gamma(s(\hbar)))(g) f(g^{-1}\hbar^{-1}x)\,\mathrm{d}m_{s(\hbar)}(g)\\ &= \dim(\gamma(s(\hbar))^{\hbar^{-1}}) \int_{\mathcal{G}^{r(\hbar)}_{r(\hbar)}} \mathrm{tr}(\gamma(s(\hbar)))(\hbar g \hbar^{-1}) f(\hbar g^{-1}x)\,\mathrm{d}m_{r(\hbar)}(g)\\ &= \dim(\gamma(r(\hbar))) \int_{\mathcal{G}^{r(\hbar)}_{r(\hbar)}} \mathrm{tr}(\gamma(r(\hbar)))(g)(T_{\hbar^{-1}}f)(g^{-1}x)\,\mathrm{d}m_{r(\hbar)}(g)\\ &= (P_\gamma T_\hbar(f|_{K_{s(\hbar)}}))(x).\end{aligned}$$

Hence, $\mathrm{rg}(P_\gamma)$ is a $\mathcal{G}$-invariant submodule of $\mathrm{C}(K)$ and it is also closed since it is the range of a projection. Now, to prove the remaining claims in iii, consider the disjoint union F of the vector spaces $F_u := \mathrm{rg}(P_\gamma)|_{K_u}$ for $u \in \mathcal{G}^{(0)}$. Using the correspondence between submodules and subbundles noted in Remark 4.11, it follows that F is a subbundle of the Banach bundle $\mathrm{C}_q(K)$ and that

$$\mathrm{rg}(P_\gamma) \to \Gamma(F), \quad f \mapsto [u \mapsto f|_{K_u}]$$

is an isometric isomorphism of Banach modules over $\mathrm{C}(\mathcal{G}^{(0)})$. Since the fibers of F are finite-dimensional by Theorem 6.1 and all have the same dimension by invariance of F and transitivity of $\mathcal{G}$, we conclude that F is a locally trivial Banach bundle (see Remark 4.3 (iv)). By Proposition 4.13, $\Gamma(F)$ is finitely generated and projective. Thus

$$\mathrm{rg}(P_\gamma) \cong \Gamma(F)$$

is a closed, finitely generated, projective, T_φ-invariant $\mathrm{C}(\mathcal{G}^{(0)})$-submodule of $\mathrm{C}(K)$. Finally, it is a consequence of Theorem 6.1 that the ranges of P_{γ_1} and P_{γ_2} are orthogonal for distinct $\gamma_1, \gamma_2 \in \Gamma_\mathcal{G}$. This proves (iii).

The approximation property (iv) is clear on each fiber of $\mathrm{C}_q(K)$ by Theorem 6.1(iv), so we use the Stone–Weierstraß theorem for bundles to achieve uniform approximation: Take $f \in \mathrm{C}(K)$ and consider the closed submodule Γ of $\mathrm{C}(K)$ generated by f and every $P_\gamma f$ for $\gamma \in \Gamma_\mathcal{G}$. By Example 4.5 and the correspondence from Remark 4.11 we obtain a subbundle F of $\mathrm{C}_q(K)$ by $F_u := \Gamma|_{K_u}$ for $u \in \mathcal{G}^{(0)}$ and

$$\Gamma \to \Gamma(F), \quad f \mapsto [u \mapsto f|_{K_u}]$$

is an isometric isomorphism of Banach modules over $\mathrm{C}(\mathcal{G}^{(0)})$. By Theorem 6.1(iv)

$$\mathrm{lin}_{\mathrm{C}(\mathcal{G}^{(0)})}\{P_\gamma f \mid \gamma \in \Gamma_\mathcal{G}\}$$

defines a stalkwise dense subset of $\Gamma(F)$ in the sense of [Gie82, Definition 4.1]. By the Stone–Weierstraß theorem [Gie82, Corollary 4.3], this set is dense in $\Gamma(F)$ and therefore

$$\overline{\mathrm{lin}_{\mathrm{C}(\mathcal{G}^{(0)})}\{P_\gamma f \mid \gamma \in \Gamma_\mathcal{G}\}} = \Gamma.$$

In particular,

$$f \in \overline{\mathrm{lin}_{\mathrm{C}(\mathcal{G}^{(0)})}\{P_\gamma f \mid \gamma \in \Gamma_\mathcal{G}\}}.$$

Parts (v) and (vi) follow directly from the corresponding parts of Theorem 6.1 and Dini's theorem. □

7 Applications to extensions of topological dynamical systems

In this final section we translate our results on groupoid actions to extensions of topological dynamical systems. Recall from Example 1.13 that every open extension $q\colon (K, G) \to (L, G)$ can be equivalently described as a groupoid action $(K, q, G \ltimes L)$ and that the unit space of $G \ltimes L$ can be identified with L. In particular, we obtain a uniform enveloping semigroupoid $\mathcal{E}_\mathrm{u}(K, q, G \ltimes L)$ which we denote by $\mathcal{E}_\mathrm{u}(q)$ in the following.

We also remind the reader that the usual notions of structuredness of the extension (i.e., being stable, equicontinuous, pseudoisometric, or isometric) are equivalent to the corresponding concepts for the groupoid action. Finally, recall from the introduction that a topological dynamical system (K, G) is **topologically ergodic** if $\mathrm{fix}(T_\varphi)$ contains only the constant functions. This is the case if and only if the action groupoid $G \ltimes K$ is topologically ergodic and there are many examples for such systems (e.g., transitive systems) which are not minimal.

With these correspondences in mind, we restate our main results in the case of extensions of topological dynamical systems, starting with the characterization of equicontinuity via compactness from Corollary 3.14.

Theorem 7.1. *For an open extension* $q\colon (K, G, \varphi) \to (L, G, \psi)$ *of topological dynamical systems the following assertions are equivalent:*

(a) *q is equicontinuous.*

(b) $\{\varphi_g|_{K_l} \mid g \in G, l \in L\} \subseteq \mathrm{C}_q^q(K, K)$ *is precompact.*

(c) $\{(T_{\varphi_g} f)|_{K_l} \mid g \in G, l \in L\} \subseteq \mathrm{C}_q(K)$ *is equicontinuous for every* $f \in \mathrm{C}(K)$.

(d) $\{(T_{\varphi_g} f)|_{K_l} \mid g \in G, l \in L\} \subseteq \mathrm{C}_q(K)$ *is precompact for every* $f \in \mathrm{C}(K)$.

We have noted that—in contrast to extensions of minimal systems—there is a significant difference between equicontinuous and pseudoisometric extensions: There are examples of extensions of nonminimal systems which are equicontinuous but not pseudoisometric; see Example 3.15. The following result, combining Theorem 3.27, Remark 3.28 and Theorem 4.14, indicates that pseudoisometric extensions are the most "natural" generalizations of almost periodic systems.

Theorem 7.2. *For an open extension* $q\colon (K, G, \varphi) \to (L, G, \psi)$ *of topological dynamical systems such that* (L, G, ψ) *is topologically ergodic, the following assertions are equivalent.*

(a) *q is pseudoisometric.*

(b) *The uniform enveloping semigroupoid* $\mathcal{E}_{\mathrm{u}}(q)$ *is a compact groupoid.*

(c) *The union of all locally trivial* T_φ*-invariant subbundles is fiberwise dense in* $\mathrm{C}_q(K)$.

(d) *The union of all finitely generated, projective, closed* T_φ*-invariant* $\mathrm{C}(L)$*-submodules of* $\mathrm{C}(K)$ *is dense in* $\mathrm{C}(K)$.

If these assertions hold, then the locally trivial invariant subbundles in (c) *are of constant finite dimension. Moreover, if K is metrizable, then* (a) *can be replaced by*

(a′) *q is isometric.*

Theorem 7.2 shows that the known characterizations of almost periodic systems via the enveloping semigroup or the Koopman operator extend in a canonical way to extensions of dynamical systems. In particular, it provides a clear picture of (pseudo)isometric extensions from an operator theoretic point of view.

If we require both systems to be topologically ergodic, we even obtain the existence of relatively invariant measures, a result previously only known in the minimal case (see, e.g., [Gla75, Section 3] and Corollary 3.7 therein, or [Kna67, Proposition 5.5]). Given an extension $q\colon (K, G) \to (L, G)$ of dynamical systems, a map $\mu\colon L \to \mathrm{P}(K)$ is called a **relatively invariant measure for** q, if μ is weak*-continuous, $\mathrm{supp}(\mu_l) \subseteq K_l$, and $g_*\mu_l = \mu_{gl}$ for every $g \in G$ and $l \in L$. A relatively invariant measure is called **fully supported** if $\mathrm{supp}(\mu_l) = K_l$ for each $l \in L$. It is immediate that a map $\mu : L \to \mathrm{P}(K)$ is a relatively invariant measure for the extension $q\colon (K, G) \to (L, G)$ if and only if it is a relatively invariant measure for the groupoid action $(K, q, G \ltimes L)$. As a direct consequence of Theorem 5.12, we obtain the following existence result for relatively invariant measures.

Theorem 7.3. *Every open pseudoisometric extension of topologically ergodic topological dynamical systems has a unique and fully supported relatively invariant measure.*

Finally, applying Theorem 6.7 to the uniform enveloping groupoid of an open, pseudoisometric extension yields Fourier analytic results for such extensions (cf. [Kna67, Theorem 1.2]).

Theorem 7.4. *Let $q\colon (K, G, \varphi) \to (L, G, \psi)$ be an open pseudoisometric extension of ergodic topological dynamical systems, μ its unique relatively invariant measure and $\Gamma = \Gamma_{\mathcal{E}_{\mathrm{u}}(q)}$ the space of invariant sections into the unitary dual of the isotropy bundle of $\mathcal{E}_{\mathrm{u}}(q)$.*

(i) *For every $\gamma \in \Gamma$, $P_\gamma \in \mathscr{L}(\Gamma(\mathrm{L}^2_q(K, \mu)))$ is a projection and a $\mathrm{C}(L)$-module homomorphism.*

(ii) *The ranges $\mathrm{rg}(P_\gamma)$ for $\gamma \in \Gamma$ are closed, finitely generated, projective, invariant $\mathrm{C}(L)$-submodules of $\mathrm{C}(K)$ and are pairwise orthogonal in $\Gamma(\mathrm{L}^2_q(K, \mu))$.*

(iii) *The inequality*

$$\|P_\gamma\sigma\|_{\mathrm{C}(K)} \leqslant \|\sigma\|_{\Gamma(\mathrm{L}^2_q(K,\mu))}$$

holds for every $\sigma \in \Gamma(\mathrm{L}^2_q(K, \mu))$ and $\gamma \in \Gamma$.

(iv) *Each $f \in \mathrm{C}(K)$ is contained in*

$$\overline{\mathrm{lin}}_{\mathrm{C}(L)}\{P_\gamma f \mid \gamma \in \Gamma\}.$$

(v) *For each $\sigma \in \Gamma(\mathrm{L}^2_q(K, \mu))$*

$$(\sigma \mid \sigma)_\mu = \sum_{\gamma\in\Gamma}(P_\gamma\sigma \mid P_\gamma\sigma)_\mu$$

with convergence in $\mathrm{C}(L)$.

(vi) *Each $\sigma \in \Gamma(\mathrm{L}^2_q(K, \mu))$ can be decomposed into a series*

$$\sigma = \sum_{\gamma\in\Gamma} P_\gamma\sigma$$

that converges in $\Gamma(\mathrm{L}^2_q(K, \mu))$.

Thus, if both systems are ergodic, then a pseudoisometric extension can be decomposed functional analytically into "simple" parts. This yields a precise understanding of the extension in terms of its Koopman representation.

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